A Focused Approach to Combining Logics

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Abstract

We present a compact sequent calculus LKU for classical logic organized around the concept of *polarization.* Focused sequent calculi for classical, intuitionistic, and multiplicative-additive linear logics are derived as fragments of the host system by varying the sensitivity of specialized structural rules to polarity information. We identify a general set of criteria under which cut elimination holds in such fragments. From cut elimination we derive a unified proof of the completeness of focusing. Furthermore, each sublogic can interact with other fragments through cut. We examine certain circumstances, for example, in which a classical lemma can be used in an intuitionistic proof while preserving intuitionistic provability. We also examine the possibility of defining classical-linear *hybrid logics.*

Key words: focused proof systems, unity of logic, linear logic

1. Introduction

Gentzen presented natural deduction proof systems for both intuitionistic and classical logics in [1]. The natural deduction system NJ for intuitionistic logic contained *introduction* and *elimination* rules for each logical connective. The natural deduction system NK for classical logic contained the same introduction and elimination rules but added the external axiom for the *excluded middle*. This one addition broke the systematic treatment of the connectives via introduction and elimination rules and, as a result, Gentzen moved away from natural deduction in order to develop a different framework that could provide a uniform proof of the *Hauptsatz* for these two logics.

That alternative framework was, of course, the sequent calculus. Proofs in the sequent calculus are built from tree structures of inference rules involving *left-* and *right-introduction* rules (playing the role of the elimination and introduction rules of natural deduction) and *sequents*, which are hypothetical judgments of the form $\Gamma \longrightarrow \Delta$ for two lists of formulas Γ and Δ. Since sequents are more complex objects than the formulas that they generalized,

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Gentzen introduced the *structural* rules of exchange, weakening, and contraction to manipulate this additional structure. Gentzen presented two formally different sequent proof systems—LJ for intuitionistic logic and LK for classical logic—where again the inference rules for the logical connectives were identical. The difference between classical and intuitionistic proofs was not captured by an external axiom but by restrictions on a structural rule: in particular, contraction was not allowed on the right of the sequent arrow within LJ. It was within the framework of sequents that Gentzen stated the *Hauptsatz*—the admissibility of the cut rule—and provided a uniform cut-elimination procedure for both intuitionistic and classical logics.

The critical role of structural rules in the description of logics is strikingly apparent from Girard's sequent calculus presentation of linear logic [2]. In particular, linear logic allows the exchange rule but removes all occurrences of the weakening and contraction rules except for those formulas prefixed by the so-called "exponentials" (written as !, ?): these modal-like operators actually mix the introduction rules of promotion and dereliction with the structural rules of weakening and contraction. All other logical connectives are provided introduction rules only. The sequent calculus allowed a convenient proof of the admissibility of the cut rule for linear logic.

The sequent calculus thus provides a perspicuous framework where classical, intuitionistic, and linear logics can be separately described: central to such descriptions are different restrictions on the structural rules. A natural possibility thus presents itself: to what extent can the logical connectives of these logics be mixed and placed into new logics. Since the restrictions on the structural rules that are used for intuitionistic and linear logics are applied globally within proofs, such mixing is not immediately evident.

In this paper, we present the LKU proof system that allows the mixing of connectives from these logics to form synthetic connectives. Central to this system is a rich notion of *polarization.* We shall provide introduction rules that fit with Gentzen's strict use of the term: thus, introduction rules will not be sensitive to polarity. Polarity will be used exclusively by the "structural rules" of a focused proof system. Our proof systems for classical, intuitionistic, and linear logics will all be, in fact, *focused* proof systems in the tradition of Andreoli [3]. The relationship between focusing and the traditional "structural rules" of contraction and weakening is that those rules are only needed *in between* the synchronous and asynchronous phases of focused proofs. In a focusing context, it is natural to generalize the notion of a structural rule to be any non-introduction rule that is active on the borders of the focusing phases: in other words, structural rules are those that are sensitive to a *change in polarity.*

We shall say that a certain proof system is a *fragment* of LKU if it arises from imposing restrictions on only structural rules. By varying the polarity restrictions on the structural rules, we shall be able to describe intuitionistic logic as a classical-linear hybrid and to identify known focused proof systems for multiplicative-additive linear logic, intuitionistic logic, and classical logic as fragments of LKU. General conditions are also given that guarantee that a fragment of the full proof system satisfies cut-elimination.

Some of the characteristics of LKU resemble those of the LU system of Girard [4]. In particular, polarities were also used in LU in place of the exponential operators ! and ? of linear logic. However, LU remained an unfocused system. The differences between LU and LKU also go beyond focusing (see Section 8).

In Section 2, we provide an overview of focusing proof systems by presenting focusing systems for linear logic and for classical logic. In Section 3, we present the complete LKU proof system and, in Section 4, show how to view intuitionistic logic as a fragment of that system. Section 5 provides a set of sufficient conditions that guarantee cut-elimination: this result establishes cut-elimination for the focused proofs of classical logic, intuitionistic logic, and multiplicative-additive linear logic (MALL). Section 6 concerns completeness properties, including sufficient conditions that guarantee that, within a fragment, focused proofs are sound and complete with respect to unfocused proofs. One of the appealing possibilities of a logic that includes various fragments is that the cut-rule can be used to communicate between different fragments: examples of such "cross cuts" are presented in Section 7. Section 8 provides a high-level comparison between LKU and LU and Section 9 describes a second hybrid logic called HL1. Finally, in Section 10 we discuss some future work and we briefly conclude in Section 11. This paper is an extended version of [5].

2. The LLF and LKF Focused Proof Systems

There are many examples of proof systems in literature that exhibit characteristics of focusing to one degree or another. These include, for example, *uniform proofs* [6], "polarized" proof systems LJT/LJQ [7, 8] and LK_p^η [9], as well as the more recent "mixed polarization" proof system λRCC [10]. Andreoli [3] identified focusing as arising from a duality between invertible and non-invertible inference rules and presented the "bi-polar" proof system LLF presented in Figure 1.

A *literal* is an atomic formula or the negation of an atomic formula. Connectives of linear logic are either *asynchronous* ($\&$, $⅋$, $\forall$, $?$) or *synchronous* ($\oplus$, $\otimes$, $\exists$, $!$). Atoms are assigned arbitrary polarity: that is, they are either assigned a *negative* or *positive* polarity in a fixed but arbitrary fashion. The negated atom $A^\perp$ takes the dual polarity of A. A formula is *negative* if it is either a negative literal or its top-level logical connective is asynchronous. A formula is *positive* if it is either a positive literal or its top-level logical connective is synchronous. LLF uses two kinds of sequents. In the sequent $\vdash \Gamma : \Delta \Uparrow L$, the "zones" Γ and Δ are multisets. In the original system L is a list, but it is also valid to consider L as a multiset. This sequent encodes the usual one-sided sequent $\vdash ?\Gamma, \Delta, L$. The zone to the left of the colon is the classical or *unbounded* context and the zone to the right of the colon is the linear or *bounded* context. This sequent will also satisfy the invariant that Δ contains only literals and synchronous formulas. In the sequent $\vdash \Gamma : \Delta \Downarrow F$, the zone Γ is a multiset of formulas, Δ is a multiset of literals and synchronous formulas, and F is a single formula. The use of these two zones replaces the need for explicit weakening and contraction rules.

Asynchronous rules

$$\frac{\vdash \Gamma\colon \Delta \Uparrow L}{\vdash \Gamma\colon \Delta \Uparrow \bot, L}\ [\bot] \qquad \frac{\vdash \Gamma\colon \Delta \Uparrow F, G, L}{\vdash \Gamma\colon \Delta \Uparrow F \parr G, L}\ [\parr] \qquad \frac{\vdash \Gamma, F\colon \Delta \Uparrow L}{\vdash \Gamma\colon \Delta \Uparrow ?F, L}\ [?]$$

$$\frac{}{\vdash \Gamma\colon \Delta \Uparrow \top, L}\ [\top] \qquad \frac{\vdash \Gamma\colon \Delta \Uparrow F, L \quad \Gamma\colon \Delta \Uparrow G, L}{\vdash \Gamma\colon \Delta \Uparrow F \mathbin{\&} G, L}\ [\&] \qquad \frac{\vdash \Gamma\colon \Delta \Uparrow B[y/x], L}{\vdash \Gamma\colon \Delta \Uparrow \forall x.B, L}\ [\forall]$$

Synchronous rules

$$\frac{}{\vdash \Gamma\colon \cdot \Downarrow \mathbf{1}}\ [\mathbf{1}] \qquad \frac{\vdash \Gamma\colon \Delta_1 \Downarrow F \quad \Gamma\colon \Delta_2 \Downarrow G}{\vdash \Gamma\colon \Delta_1, \Delta_2 \Downarrow F \otimes G}\ [\otimes] \qquad \frac{\vdash \Gamma\colon \cdot \Uparrow F}{\vdash \Gamma\colon \cdot \Downarrow !F}\ [!]$$

$$\frac{\vdash \Gamma\colon \Delta \Downarrow F_1}{\vdash \Gamma\colon \Delta \Downarrow F_1 \oplus F_2}\ [\oplus_l] \qquad \frac{\vdash \Gamma\colon \Delta \Downarrow F_2}{\vdash \Gamma\colon \Delta \Downarrow F_1 \oplus F_2}\ [\oplus_r] \qquad \frac{\vdash \Gamma\colon \Delta \Downarrow B[t/x]}{\vdash \Gamma\colon \Delta \Downarrow \exists x.B}\ [\exists]$$

Initial, Reaction, and Decide rules

If K a positive literal: $\dfrac{}{\vdash \Gamma\colon K^\perp \Downarrow K}\ [I_1] \qquad \dfrac{}{\vdash \Gamma, K^\perp\colon \cdot \Downarrow K}\ [I_2]$

$\dfrac{\vdash \Gamma\colon \Delta, F \Uparrow L}{\vdash \Gamma\colon \Delta \Uparrow F, L}\ [R\Uparrow]$ provided F is not asynchronous

$\dfrac{\vdash \Gamma\colon \Delta \Uparrow F}{\vdash \Gamma\colon \Delta \Downarrow F}\ [R\Downarrow]$ provided F is either asynchronous or a negative literal

If F is not a negative literal: $\dfrac{\vdash \Gamma\colon \Delta \Downarrow F}{\vdash \Gamma\colon \Delta, F \Uparrow \cdot}\ [D_1] \qquad \dfrac{\vdash \Gamma, F\colon \Delta \Downarrow F}{\vdash \Gamma, F\colon \Delta \Uparrow \cdot}\ [D_2]$

Figure 1: The focused proof system LLF for linear logic

The inference rules of the LLF proof system (see Figure 1) are divided into three groups. Those introduction rules involving $\Uparrow$-sequents belong to the *asynchronous phase* and those introduction rules involving a $\Downarrow$-sequent in the conclusion belong to the *synchronous phase*. The remaining rules are the initial rules (I_1 and I_2) and the *structural rules*, which are further divided into the *decision* rules (D_1 and D_2) and the *reaction* rules ($R\Uparrow$ and $R\Downarrow$). Some formulations of focusing, *e.g.*, [9, 11], avoid a presentation with two arrows in favor of careful descriptions of when a sequent proof is actually focused.

In LLF, the structural rules and the initial rules are the rules that are directly sensitive to polarity information: these rules show that *it is polarity that drives focusing.* In fact, if the polarity-related side conditions for these rules are removed, we are left with a rather convoluted version of an *unfocused* sequent calculus for linear logic where one would be able to switch between the $\Downarrow$ and the $\Uparrow$ states without regard to change in polarity. Notice also that the rules for the exponential operators ? and ! behave less like other introduction rules and more like the reaction rules $R\Uparrow$ and $R\Downarrow$.

Structural rules in the style of LLF will play a critical role in our unified

Asynchronous rules

$$\frac{}{\vdash [\Theta], \Gamma, \neg\mathcal{F}}\ \mathit{absurd} \qquad \frac{\vdash [\Theta], \Gamma}{\vdash [\Theta], \Gamma, \neg\mathcal{T}}\ \mathit{trivial} \qquad \frac{\vdash [\Theta], \Gamma, A \quad \vdash [\Theta], \Gamma, B}{\vdash [\Theta], \Gamma, A \wedge^- B}\ \wedge^-$$

$$\frac{\vdash [\Theta], \Gamma, A, B}{\vdash [\Theta], \Gamma, A \vee^- B}\ \vee^- \qquad \frac{\vdash [\Theta], \Gamma, A}{\vdash [\Theta], \Gamma, \forall x A}\ \forall$$

Synchronous rules

$$\frac{}{\mapsto [\Theta], \mathcal{T}}\ \mathcal{T} \quad \frac{\mapsto [\Theta], A \quad \mapsto [\Theta], B}{\mapsto [\Theta], A \wedge^+ B}\ \wedge^+ \quad \frac{\mapsto [\Theta], A_i}{\mapsto [\Theta], A_1 \vee^+ A_2}\ \vee^+ \quad \frac{\mapsto [\Theta], A[t/x]}{\mapsto [\Theta], \exists x A}\ \exists$$

Initial, Reaction, Decision rules

$$\frac{}{\mapsto [\neg P, \Theta], P}\ \mathit{Id}\ (\text{literal } P) \qquad \frac{\vdash [\Theta, C], \Gamma}{\vdash [\Theta], \Gamma, C}\ []$$

$$\frac{\vdash [\Theta], N}{\mapsto [\Theta], N}\ \mathit{Release} \qquad \frac{\mapsto [P, \Theta], P}{\vdash [P, \Theta]}\ \mathit{Focus}$$

Figure 2: The Focused Classical Sequent Calculus LKF.

sequent calculus. In fact, our project here is first to present a rich set of LLF-like structural rules and then to investigate different subsets of those structural rules to see how they account for different proof systems (for example, intuitionistic or linear logics).

LLF-style focused systems have also been adapted to classical and intuitionistic logic. In [12, 13], the authors presented the focused intuitionistic sequent calculus LJF that can be seen as an LU-inspired translation of intuitionistic logic into linear logic. That paper also presented the focused classical sequent calculus LKF that was inspired by a double-negation translation into LJF (similar to Girard's LC [14]). The system LKF is given in Figure 2 (a one-sided presentation of LJF is given in Figure 4). Here, P is positive, N is negative, C is a positive formula or a negative literal, Θ consists of positive formulas and negative literals, and x is not free in Θ, Γ. Sequents containing a focus (similar to the $\Downarrow$-sequents of LLF) are written as $\mapsto [\Theta], A$ and sequents with no focus (corresponding to $\Uparrow$-sequents of LLF) are written as $\vdash [\Theta], \Gamma$.

Both the additive and multiplicative versions of conjunction and disjunction are available in LKF: $\wedge^-$ and $\vee^+$ are additive while $\wedge^+$ and $\vee^-$ are multiplicative. The difference between the two conjunctions and two disjunctions lies in the focused proofs that they admit: they are, however, provably equivalent. In contrast, the linear connectives $\otimes$ and & are not provably equivalent.

While LKF inherits the structural rules of LLF, the reaction rules *Release* and [] (pronounced "bracket") of LKF correspond not to $R\Downarrow$ and $R\Uparrow$ but to the ! and ? introduction rules of LLF. The decision rule D_2 in LLF corresponds

Asynchronous rules

$$\frac{\vdash \Gamma : \Delta \Uparrow A, \Theta \quad \vdash \Gamma : \Delta \Uparrow B, \Theta}{\vdash \Gamma : \Delta \Uparrow A\ [\&|\wedge^-]\ B, \Theta}\ [\&|\wedge^-] \qquad \frac{\vdash \Gamma : \Delta \Uparrow A, B, \Theta}{\vdash \Gamma : \Delta \Uparrow A\ [\invamp|\vee^-]\ B, \Theta}\ [\invamp|\vee^-]$$

$$\frac{}{\vdash \Gamma : \Delta \Uparrow \top, \Theta}\ \top \qquad \frac{\vdash \Gamma : \Delta \Uparrow A, \Theta}{\vdash \Gamma : \Delta \Uparrow [\Pi|\forall]x.A, \Theta}\ [\Pi|\forall] \qquad \frac{\vdash \Gamma : \Delta \Uparrow \Theta}{\vdash \Gamma : \Delta \Uparrow \bot, \Theta}\ \bot$$

provided x is not free in Γ, Δ, Θ

Synchronous rules

$$\frac{}{\vdash \Gamma : \Downarrow 1}\ 1 \quad \frac{\vdash \Gamma : \Delta_1 \Downarrow A \quad \vdash \Gamma : \Delta_2 \Downarrow B}{\vdash \Gamma : \Delta_1\Delta_2 \Downarrow A\ [\otimes|\wedge^+]\ B}\ [\otimes|\wedge^+] \quad \frac{\vdash \Gamma : \Delta \Downarrow A[t/y]}{\vdash \Gamma : \Delta \Downarrow [\Sigma|\exists]y.A}\ [\Sigma|\exists]$$

$$\frac{\vdash \Gamma : \Delta \Downarrow A_i}{\vdash \Gamma : \Delta \Downarrow A_1\ [\oplus|\vee^+]\ A_2}\ [\oplus|\vee^+],\ \text{provided } i = 1 \text{ or } = 2$$

Initial, Reaction, and Decision rules

$$\frac{}{\vdash \Gamma : P^\perp \Downarrow P}\ I_1 \quad \frac{\vdash \Gamma : \Delta, C \Uparrow \Theta}{\vdash \Gamma : \Delta \Uparrow C, \Theta}\ R_1\Uparrow \quad \frac{\vdash \Gamma : \Delta \Uparrow N}{\vdash \Gamma : \Delta \Downarrow N}\ R_1\Downarrow \quad \frac{\vdash \Gamma : \Delta \Downarrow P}{\vdash \Gamma : \Delta, P \Uparrow}\ D_1$$

$$\frac{}{\vdash \Gamma, P^\perp : \Downarrow P}\ I_2 \quad \frac{\vdash C, \Gamma : \Delta \Uparrow \Theta}{\vdash \Gamma : \Delta \Uparrow C, \Theta}\ R_2\Uparrow \quad \frac{\vdash \Gamma : \Uparrow N}{\vdash \Gamma : \Downarrow N}\ R_2\Downarrow \quad \frac{\vdash P, \Gamma : \Delta \Downarrow P}{\vdash P, \Gamma : \Delta \Uparrow}\ D_2$$

P positive ($+2$ or $+1$), N negative (-2 or -1), C positive formula or negative literal

Figure 3: The Unified Focusing Sequent Calculus LKU

directly to the LKF rule *Focus*: both embody an explicit contraction. There is, however, an important difference between these two proof systems regarding the formulas that are contracted. In LKF (and LJF), formulas selected for focus (and thus subjected to contraction) are always positive. In LLF, however, the ? introduction rule stops asynchronous decomposition and so asynchronous formulas are also subject to contraction. The restriction of contraction to only positive formulas is an important characteristic of LKF and prompts us to adopt this feature to our unified system. In fact, we exchange the ability to represent full linear logic for the benefits of a system that is better behaved with respect to focusing, and which can still accommodate classical, intuitionistic, and multiplicative-additive linear logic. This simplification of LKF also leads to a more direct proof of cut-elimination (without the need for Gentzen's *mix* rule [1]).

3. The LKU proof system

Central to the LKU proof system, found in Figure 3, are four polarities which are divided into two *levels:* $+1$, -1, $+2$, and -2. Atomic formulas are assigned polarities from this set. Other formulas derive their polarity from their top-level connective as follows: $\wedge^+$, $\vee^+$, $\exists$, 1, 0 are given polarity +2; $\otimes$, $\oplus$, Σ are given polarity +1; $\invamp$, &, Π are given polarity −1; and $\wedge^-$, $\vee^-$, $\forall$, $\top$, $\bot$ are given polarity −2. Negation ($A^\perp$) is defined by the following De Morgan dualities: $\otimes/\invamp$, $\oplus/\&$, $\wedge^+/\vee^-$, $\vee^+/\wedge^-$, Σ/Π, $\exists/\forall$, $1/\bot$, $\top/0$, $A/A^\perp$ for literals A. The dual polarity of +1 is −1 and the dual of +2 is −2. Formulas are assumed to be in negation normal form (*i.e.*, negations have only atomic scope). All formulas are polarized as positive or negative.

Although the symbols chosen for the connectives of LKU resemble those of linear and classical logics, their meaning is not fixed within the unified system. There are *enough* connectives to distinguish between each of the binary choices conjunction/disjunction, additive/multiplicative, and classical/linear. The introduction rules make no distinction between the linear and classical interpretations of each connective: the notation $[\otimes \,|\, \wedge^+]$ means that the rule is applicable to both connectives. LKU can be divided into two principal components: the introduction rules, which are *invariant for all fragments,* and the collection of initial, reaction, and decision rules, which can be restricted to define sublogics. For convenience we shall refer to the reaction and decide rules as well as the initial rules as the *structural rules* of LKU. This classification is justified in that, just as with contraction, these rules are only active in between the focusing phases. The structural rules are further divided between the "level-1" and "level-2" rules.

LKU sequents are of the forms $\vdash \Gamma : \Delta \Downarrow B$ and $\vdash \Gamma : \Delta \Uparrow \Theta$ and will always satisfy the invariant that Γ and Δ contain only positive formulas or negative literals. As a consequence, the only possible instances of the two initial rules I_1 and I_2 (Figure 3) will be such that P is a positive literal.

As given, LKU can only be called classical logic. The four connectives for conjunction, $\wedge^+$, $\wedge^-$, $\otimes$, and & are all provably equivalent, as are the four for disjunction and the pairs of quantifiers and units. The structural rules are only sensitive to the positive/negative distinction and not to the linear/classical distinction. If we removed even this basic level of sensitivity to polarity, then we are left with a verbose version of the unfocused LK. With the sensitivity to positive/negative polarity, *every fragment of LKU will naturally be focused.* Clearly, the inference rules of LKU are sound with respect to classical logic. The classical completeness of LKU follows from the completeness of LKF [12, 13], which it contains (another proof of completeness is given in Section 6.2).

When reading inferences rules bottom-up, a synchronous phase ends with either the 1 introduction rule or with a reaction rule ($R_1\Downarrow$ or $R_2\Downarrow$) or an initial rule (I_1 or I_2). The ? and ! rules of LLF reappear in LKU as the level-2 reaction rules $R_2\Uparrow$ and $R_2\Downarrow$. As given, $R_2\Downarrow$ is subsumed by $R_1\Downarrow$: their distinction will become clear when we consider fragments of LKU. Both the $R_1\Uparrow$ and $R_2\Uparrow$ rules exclude asynchronous formulas, as does the D_2 rule. This divergence from LLF means that we will not be able to represent full-linear logic for reasons explained in the previous section. These restrictions are similar to those of *polarized* linear logic [15]. In LKU the role of the exponential operators is replaced entirely by polarity information. If we relaxed these restrictions and allowed $R_2\Uparrow$ and D_2

to be applicable for asynchronous formulas, then clearly every LLF proof can be mimicked. Although a unified logic that accommodates full linear logic is certainly an interesting topic (see Section 10), the restriction that we adopt is also worthy of separate study.

Fragments of LKU are defined by restricting the structural rules and possibly also the forms of formulas used. Not all fragments, however, can be called "logics" (see Section 9). Assume that end-sequents of LKU all have the form $\vdash : \Uparrow \Gamma$. The following fragments are immediate.

MALLF: If we forbid all uses of the level-2 structural rules and only allow I_1, D_1, $R_1\Uparrow$, and $R_1\Downarrow$, then the resulting system is essentially the same as LLF restricted to the MALL fragment (but with quantifiers). We shall call this fragment MALLF. Note that "forbidding level-2 rules" is not the same as forbidding the $+2/-2$ polarities: the units 0, 1, $\bot$ and $\top$ are all still accounted for in MALLF. In fact, we still retain all the connectives of LKU, but symbols such as $\wedge^+$ and $\otimes$ will both be interpreted as linear connectives.

LKF: If we forbid all the level-1 rules and only allowed the level-2 structural rules then we arrive at a more conventional sequent calculus for classical logic, one that is similar to LKF. Symbols such as $\otimes$ and $\invamp$ are retained but they will have the same meaning as their classical counterparts.

Retaining seemingly redundant symbols facilitates the communication between different fragments of LKU through cut: such communication is difficult to formalize if the fragments use disjoint sets of connectives.

4. The Intuitionistic Fragment

Intuitionistic logic appears as a linear-classical hybrid fragment within LKU and, as a result, that fragment provides a focused proof system for intuitionistic logic. The LJF proof system of [12, 13] is reconstructed as a fragment of LKU in Figure 4. Since LJF is itself a framework for describing a range of focused proof systems (*e.g.*, LJT [7], LJQ′ [8], and λRCC [10]) and unfocused proof systems (*e.g.*, LJ [1]) for intuitionistic logic, describing LJF is a good test of LKU's expressiveness.

As originally presented, formulas in LJF are "annotated" intuitionistic formulas: that is, atomic formulas are assigned an arbitrary but fixed polarity (either positive or negative) and conjunctions are annotated as being either additive $\wedge^-$ or multiplicative $\wedge^+$. The original LJF proof system is a two-sided sequent calculus using sequents of the following styles. The premises and conclusion of invertible inference rules use sequents of the form $[\Gamma], \Theta \longrightarrow R$: such sequents lack a distinguished "focus". Dually, the premises and conclusion of non-invertible inference rules use sequents such as $[\Gamma] \xrightarrow{L} [R]$, which provides a "left-focus" formula L or sequents such as $[\Gamma] -_R\!\!\rightarrow$, which provides a "right-focus" formula R. A set of "structural rules" are provided in LJF that mix sequents of both kinds.

$$\frac{}{\vdash \Gamma : Q^\perp \Downarrow Q}\ I_1 \quad \frac{\vdash \Gamma : C \Uparrow \Theta}{\vdash \Gamma : \Uparrow C, \Theta}\ R_1\Uparrow \quad \frac{\vdash \Gamma : C \Uparrow \mathcal{N}}{\vdash \Gamma : C \Downarrow \mathcal{N}}\ R_1\Downarrow \quad \frac{\vdash \Gamma : \Delta \Downarrow \mathcal{P}}{\vdash \Gamma : \Delta, \mathcal{P} \Uparrow}\ D_1$$

Q: $+1$ atom, C: $+2$ formula or -1 atom, $\mathcal{N}$: -2 formula, $\mathcal{P}$: $+2$ formula.

$$\frac{}{\vdash \Gamma, \mathcal{Q}^\perp : \Downarrow \mathcal{Q}}\ I_2 \quad \frac{\vdash D, \Gamma : \Delta \Uparrow \Theta}{\vdash \Gamma : \Delta \Uparrow D, \Theta}\ R_2\Uparrow \quad \frac{\vdash \Gamma : \Uparrow N}{\vdash \Gamma : \Downarrow N}\ R_2\Downarrow \quad \frac{\vdash P, \Gamma : \Delta \Downarrow P}{\vdash P, \Gamma : \Delta \Uparrow}\ D_2$$

$\mathcal{Q}$: $+2$ atom, D: $+1$ formula or -2 literal, N: -1 formula, P: $+1$ formula

Figure 4: The focused intuitionistic sequent calculus LJF as a fragment of LKU

$$\begin{array}{ll}
[B \wedge^- C]^R = [B]^R \,\&\, [C]^R & [B \wedge^+ C]^R = [B]^R \wedge^+ [C]^R \\
[B \supset C]^R = [B]^L \parr [C]^R & [B \vee C]^R = [B]^R \vee^+ [C]^R \\
[\forall x.B]^R = \Pi x.[B]^R & [\exists x.B]^R = \exists x.[B]^R \\
[B \wedge^- C]^L = [B]^L \oplus [C]^L & [B \wedge^+ C]^L = [B]^L \vee^- [C]^L \\
[B \supset C]^L = [B]^R \otimes [C]^L & [B \vee C]^L = [B]^L \wedge^- [C]^L \\
[\forall x.B]^L = \Sigma x.[B]^L & [\exists x.B]^L = \forall x.[B]^L
\end{array}$$

For atomic A, $[A]^R = A$ and $[A]^L = A^\perp$.

Figure 5: Mapping LJF formulas into LKU.

Formulas of LJF are mapped into formulas of LKU using the two functions $[\cdot]^R$ and $[\cdot]^L$ defined in Figure 5. This is a shallow, syntactic mapping of intuitionistic connectives (whose proof rules are described using two-sided sequents) to classical connectives (whose proof rules are described using one-sided sequents). That is, a left-occurrence of the intuitionistic $\supset$ is exactly the same as (a right-occurrences of) the LKU connective $\otimes$. Positive LJF atoms are assigned polarity $+2$ in LKU while negative LJF atoms are assigned polarity -1 in LKU. Formulas in the range of $[\cdot]^R$ are called *essentially right intuitionistic formulas* (they have polarity $+2$ or -1) and formulas in the range of $[\cdot]^L$ are called *essentially left intuitionistic formulas* (they have polarity -2 or $+1$). Notice that R is an essentially right intuitionistic formula if and only if the negation normal form of $R^\perp$ is an essentially left intuitionistic formula.

The usual symbols of linear logic are used to define the negative intuitionistic connectives. The left-hand side of an essentially right implication is essentially left (and vice versa) and is given a classical treatment by the reaction rules, thus mimicking the usual linear-logic interpretation of intuitionistic implication as $!A \multimap B$. As with LKF, the LJF fragment contains both positive and negative connectives for conjunction: $\wedge^+$ and & respectively on the right ($\vee^-$ and $\oplus$ on the left). However, there is only the positive disjunction $\vee^+$ ($\wedge^-$ on the left), with $\parr$ only used in the representation of intuitionistic implication. Intuitionistic negation $\sim A$ is defined as $A^\perp \parr 0$ when appearing essentially right. (For a minimal logic treatment of negation, replace 0 in the language with some designated +2 atom.)

To illustrate how two sided inference rules for intuitionistic logic can be represented in the one-sided, focused setting of LKU, consider the additive and multiplicative versions of the conjunction-left rule in (unfocused) LJ:

$$\frac{A_i, \Gamma \vdash C}{A_1 \wedge A_2, \Gamma \vdash C} \quad \text{and} \quad \frac{A_1, A_2, \Gamma \vdash C}{A_1 \wedge A_2, \Gamma \vdash C}.$$

These inference rules correspond to the focused LKU rules

$$\frac{\vdash \Gamma^\perp : C \Downarrow A_i^\perp}{\vdash \Gamma^\perp : C \Downarrow A_1^\perp \oplus A_2^\perp} \quad \text{and} \quad \frac{\vdash \Gamma^\perp : C \Uparrow A_1^\perp, A_2^\perp}{\vdash \Gamma^\perp : C \Uparrow A_1^\perp \vee^- A_2^\perp}.$$

For the reader familiar with LJF [12, 13], the two-sided LJF sequents correspond to the one-sided LKU sequents as follows:

$$\begin{array}{rcl}
[\Gamma], \Theta \longrightarrow R & \longleftrightarrow & \vdash [\Gamma]^L :\Uparrow [\Theta]^L, [R]^R \\
[\Gamma], \Theta \longrightarrow [R] & \longleftrightarrow & \vdash [\Gamma]^L : [R]^R \Uparrow [\Theta]^L \\
[\Gamma] -_R\!\!\rightarrow & \longleftrightarrow & \vdash [\Gamma]^L : \cdot \Downarrow [R]^R \\
[\Gamma] \xrightarrow{L} [R] & \longleftrightarrow & \vdash [\Gamma]^L : [R]^R \Downarrow [L]^L
\end{array}$$

The original structural rules of LJF and those in Figure 4 correspond as follow: $Lf \leftrightarrow D_2$, $Rf \leftrightarrow D_1$, $R_l \leftrightarrow R_1\Downarrow$, $R_r \leftrightarrow R_2\Downarrow$, $[]_l \leftrightarrow R_2\Uparrow$, $[]_r \leftrightarrow R_1\Uparrow$.

In the following we only consider LJF in its form as a fragment of LKU. Of the structural rules of LJF, I_1, $R_1\Downarrow$, $R_2\Uparrow$, and D_2 can be called “left rules” while I_2, $R_2\Downarrow$, $R_1\Uparrow$, and D_1 are the right rules.

Observe that the $R_1\Uparrow$ and $R_1\Downarrow$ rules allow only one essentially right formula inside the linear context of an LJF sequent. If we are interested only in mapping *complete* LJF proofs to intuitionistic proofs, then this restriction is not necessary: the single-conclusion condition is already enforced by other rules such as $R_2\Downarrow$, I_1, and I_2. When building a proof from the bottom-up, malformed sequents, *i.e.*, those with multiple essentially right formulas, will be rejected by the initial rules if not sooner. In fact, Lemmas 8 and 9 of Section 5 show that the single-conclusion property is a natural consequence of the structure of intuitionistic formulas and sequents.

The stronger restrictions for the $R_1\Uparrow$ and $R_1\Downarrow$ rules allow us to establish the stronger correspondence between open proofs as well. In LJF, malformed sequents could appear as a consequence of splitting the context when applying the $\otimes$ rule. The essentially-left occurrence of an implication $A \supset B$ has the form $A \otimes B^\perp$ where A is essentially right and $B^\perp$ essentially left. The implication-left rule of LJ thus appears in the form

$$\frac{\vdash \Gamma^\perp :\Downarrow A \quad \vdash \Gamma^\perp : C \Downarrow B^\perp}{\vdash \Gamma^\perp : C \Downarrow A \otimes B^\perp} \otimes$$

But it is also possible to split the context so as to have $\vdash \Gamma^\perp : C \Downarrow A$, which is a sequent with two essentially right formulas. The reaction rules of LJF are designed, however, to reject such a malformed sequent at the end of a focusing ($\Downarrow$) phase. Such a phase must end in either a reaction or an initial rule. In an *incomplete* proof structure, there could be occurrences of malformed sequents inside the synchronous phases of proofs, but we shall only consider completed phases as marking the boundary of inference rules: what defines a focused proof is not what happens in the details of each synchronous or asynchronous phase but what happens at the *borders* of such phases. Each synchronous or asynchronous phase can be thought of as the introduction of a synthesized connective; that is to say, a single introduction rule. A border sequent of LJF will be either an axiom or have the form $\vdash \Gamma^\perp : C \Uparrow$, which corresponds to a well-formed intuitionistic sequent. Without the explicit restriction to one formula in the level-1 reaction rules, malformed sequents may survive across focusing phases.

Thus if we strictly use only polarity information in restricting the structural rules, we can achieve a weak form of full-completeness. With the stronger forms of the rules as presented, the local structure of even partial intuitionistic proofs are preserved.

There is, however, one scenario in which a malformed sequent may also appear as part of a *complete* LJF proof. When considering full intuitionistic logic, as opposed to minimal logic, the intuitionistic context may be inconsistent. That is to say, the $\top$ rule (0 on the left) may appear in a proof. This problem is likewise encountered by LU and several other works that encodes intuitionistic logic into linear logic (including LJF). To resolve this problem we must show that even in such situations there is a LJF proof that corresponds to a well-formed LJ proof. Such an argument relies on cut-elimination (see Section 5).

The Negative Intuitionistic Fragment. There is a significant fragment of LJF where the problem with context splitting in the $\otimes$ rule does not appear. We shall call this fragment the *negative intuitionistic fragment nLJF* and it corresponds to the neutral intuitionistic fragment of LU. The structural rules that correspond to nLJF are found in Figure 6. In this fragment, essentially right formulas have only polarity -1 and essentially left formulas have only polarity $+1$. In an essentially left implication $A \otimes B^\perp$, A will have -1 polarity, which means that the appearance of a malformed sequent $\vdash \Gamma^\perp : C \Downarrow A$ will immediately invoke the $R_2\Downarrow$ rule, which fails because the linear context is not empty.

5. Unified Cut Elimination

In order to claim that a fragment of LKU is, in fact, a logic, one needs to at least show that the result of eliminating a cut between two proofs in the given fragment yields a proof still in that fragment. Not all fragments of LKU can be expected to satisfy cut-elimination. However, it is possible in this generalized framework, with its extended set of structural rules, to identify a set of sufficient conditions for cut-elimination. These conditions are clearly satisfied by the principal fragments MALLF and LKF. For LJF, we note that the reducibility

$$\frac{}{\vdash \Gamma : Q^{\perp} \Downarrow Q}\ I_1 \qquad \frac{\vdash \Gamma : C \Uparrow \Theta}{\vdash \Gamma : \Uparrow C, \Theta}\ R_1{\Uparrow}$$

$$\frac{\vdash P, \Gamma : \Delta \Uparrow \Theta}{\vdash \Gamma : \Delta \Uparrow P, \Theta}\ R_2{\Uparrow} \quad \frac{\vdash \Gamma : \Uparrow N}{\vdash \Gamma : \Downarrow N}\ R_2{\Downarrow} \quad \frac{\vdash P, \Gamma : \Delta \Downarrow P}{\vdash P, \Gamma : \Delta \Uparrow}\ D_2$$

Q: $+1$ atom, C: -1 atom, P: $+1$ formula, N: -1 formula

Figure 6: The Negative Intuitionistic Fragment nLJF

of cuts is a property of *complete* proofs and thus does not require the special restrictions used to ensure full completeness. That is to say, we can disregard the special restriction to a single linear formula in the $R_1{\Uparrow}$ and $R_1{\Downarrow}$ rules, and only use polarity information in the LJF structural rules. Generalizing the criteria for cut-elimination will also help us to consider possible new logics that can be defined as fragments of LKU.

A generalized proof of cut-elimination, along with initial-elimination, will also lead to a generalized proof of the completeness of focusing calculi with respect to their unfocused versions.

5.1. Generalizing the Introduction Rules

Since the introduction rules are shared by all the fragments of LKU, the permutation of cut above introductions can be demonstrated just once. Furthermore, instead of considering individual rules, we can define the following relations to characterize the structure of complete synchronous and asynchronous phases. (Synthetic connectives are treated similarly in [16].) In order to focus our analysis of LKU on essential matters, we shall not concern ourselves with the first-order quantifiers $\forall$, $\exists$, Π, and Σ: generalizing our definitions and results to handle these quantifiers is rather straightforward. For convenience, we write $\Gamma\Gamma'$ to denote the multiset union of Γ and Γ'.

Definition 1. Let $\uparrow$ and $\downarrow$ represent relations between formulas and multisets of formulas defined as follows:

- $A \uparrow \{A\}$ if A is a negative literal or positive.
- $\perp \uparrow \{\}$.
- $(A\ [\parr\,|\vee^-]\ B) \uparrow \Phi\Phi'$ if $A \uparrow \Phi$ and $B \uparrow \Phi'$.
- $(A\ [\&|\wedge^-]\ B) \uparrow \Phi$ if $A \uparrow \Phi$.
- $(A\ [\&|\wedge^-]\ B) \uparrow \Phi'$ if $B \uparrow \Phi'$.
- $A \downarrow \{A\}$ if A is a positive literal or negative.
- $1 \downarrow \{\}$.
- $(A\ [\otimes|\wedge^+]\ B) \downarrow \Psi\Psi'$ if $A \downarrow \Psi$ and $B \downarrow \Psi'$.
- $(A\ [\oplus|\vee^+]\ B) \downarrow \Psi$ if $A \downarrow \Psi$.
- $(A\ [\oplus|\vee^+]\ B) \downarrow \Psi'$ if $B \downarrow \Psi'$.

Using these dual relations, we can study how cuts permute only where it matters the most: at the borders between positive and negative focusing phases where the rules of reaction and decision come into play.

In MALL, the distributive laws can be used to put the synthetic connectives into normal forms: in particular, a positive synthetic connective is equivalent to $\oplus_{i \in I}(\otimes_{j \in J_i} N_{ij})$ and a negative synthetic connective is equivalent to $\&_{i \in I}(⅋_{j \in J_i} P_{ij})$, where I and J_i (for $i \in I$) are finite set of indices and N_{ij} denotes a negative formula or a literal and P_{ij} denotes a positive formula or a literal. Using the notation above, the following are satisfied:

$$\oplus_{i \in I}(\otimes_{j \in J_i} N_{ij}) \downarrow \{N_{ij}\ j \in J_i\} \quad (i \in I)$$

$$\&_{i \in I}(⅋_{j \in J_i} P_{ij}) \uparrow \{P_{ij}\ j \in J_i\} \quad (i \in I)$$

Thus, the $\downarrow$ selects the premises for a possible introduction rule of a positive synthetic connective while the $\uparrow$ selects a possible premise for the introduction rule of a negative synthetic connective. While normal forms for synthetic connectives are equivalent to using the $\downarrow$ and $\uparrow$ within MALL, one does not expect that similar distributive laws hold for all fragments of LKU and, as a consequence, normal forms for synthetic connectives might be hard to write down. For this reason, we employ the notation using arrows since they provide natural and immediate descriptions of the introduction rules for synthetic connectives in all of LKU.

Lemmas 2 through 4 below are all proved by induction on the structure of formulas.

Lemma 2. *Given a formula R, let $\Phi_1, \ldots, \Phi_m$ be multisets such that $R \uparrow \Phi_1, \ldots, R \uparrow \Phi_m$ and if $R \uparrow \Phi$ then $\Phi = \Phi_i$ for some unique $1 \leq i \leq m$. Every cut-free proof of $\vdash \Gamma : \Delta \Uparrow R, \Theta$ is equal up to permutations of asynchronous introduction rules to a proof of the form*

$$\cfrac{\cfrac{\vdash \Gamma\Phi_1^2 : \Delta\Phi_1^1 \Uparrow \Theta}{\vdots} \quad \cdots \quad \cfrac{\vdash \Gamma\Phi_m^2 : \Delta\Phi_m^1 \Uparrow \Theta}{\vdots}}{\vdash \Gamma : \Delta \Uparrow R, \Theta}$$

such that $\Phi_i^2\Phi_i^1 = \Phi_i$ for each $1 \leq i \leq m$. Furthermore, if

$$\vdash \Gamma\Phi_1^2 : \Delta\Phi_1^1 \Uparrow \Theta, \quad \ldots \quad \vdash \Gamma\Phi_m^2 : \Delta\Phi_m^1 \Uparrow \Theta$$

are all cut-free provable, then $\vdash \Gamma : \Delta \Uparrow R, \Theta$ is also cut-free provable.

The splitting of Φ_i into Φ_i^1 and Φ_i^2 represents a choice between $R_1\Uparrow$ and $R_2\Uparrow$. The above lemma does not specify how Φ_i^1 and Φ_i^2 are split: *e.g.*, Φ_i^1 may be empty. Given a fragment of LKU, we can be more specific as to how the multiset is split between the linear and classical contexts. In the MALLF fragment, Φ_i^2 must be empty. In the intuitionistic LJF fragment, Φ_i^2 consists of essentially left formulas and Φ_i^1 consists of at most one essentially right formula (see lemma 8 below). For the generalized proof of cut-elimination, however, it will only be necessary that the splitting of Φ_i is deterministic (see criteria **C1** below).

The dual lemma for $\downarrow$ is the following.

Lemma 3. *Let $R \downarrow \{a_1, \ldots, a_n\}$ and assume that $\vdash \Gamma : \Delta_1 \Downarrow a_1, \ldots, \vdash \Gamma : \Delta_n \Downarrow a_n$ are all cut-free provable. Then $\vdash \Gamma : \Delta_1 \ldots \Delta_n \Downarrow R$ is also cut-free provable. Furthermore, every cut-free proof of $\vdash \Gamma : \Delta_1 \ldots \Delta_n \Downarrow R$ is of the form*

$$\dfrac{\begin{array}{ccc}\dfrac{\vdash \Gamma : \Delta_1 \Downarrow a_1}{\vdots} & \cdots & \dfrac{\vdash \Gamma : \Delta_n \Downarrow a_n}{\vdots}\end{array}}{\vdash \Gamma : \Delta_1 \ldots \Delta_n \Downarrow R}$$

where $R \downarrow \{a_1, \ldots, a_n\}$, for some $a_1, \ldots, a_n$.

The central result that leads to cut-elimination is the following lemma.

Lemma 4. *$R \uparrow \{a_1, \ldots, a_n\}$ if and only if $R^\perp \downarrow \{a_1^\perp, \ldots, a_n^\perp\}$.*

The generalized cut-elimination theorem (Theorem 6) requires showing that weakening and contraction of formulas in the "unbounded" context is admissible. The following lemma is provable by a straightforward induction on the structure of proofs.

Lemma 5. *Within each fragment of LKU, if $\vdash P, P, \Gamma : \Delta \Uparrow \Theta$ has a cut-free proof, then $\vdash P, \Gamma : \Delta \Uparrow \Theta$ has a cut-free proof of the same height. If $\vdash \Gamma : \Delta \Uparrow \Theta$ has a cut-free proof, then $\vdash P, \Gamma : \Delta \Uparrow \Theta$ has a cut-free proof of the same height.*

5.2. Sufficient Criteria for Cut Elimination

Since LKU combines linear and classical features, the cut rule comes as a pair of inference rules.

$$\dfrac{\vdash \Gamma : \Delta, A \Uparrow \Theta \quad \vdash \Gamma' : \Delta' \Uparrow A^\perp, \Theta}{\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow \Theta\Theta'}\ cut_1 \qquad \dfrac{\vdash \Gamma, \mathcal{A} : \Delta \Uparrow \Theta \quad \vdash \Gamma' :\Uparrow \mathcal{A}^\perp}{\vdash \Gamma\Gamma' : \Delta \Uparrow \Theta}\ cut_2$$

In the case of intuitionistic logic, the two cuts will merge into a common form: i.e., the cut_1 form will also have an empty linear context on one side.

We identify the following sufficient criteria for a fragment of LKU to simultaneously satisfy the elimination of both cut rules. We refer to the set of formulas for which a rule such as $R_1\Uparrow$ applies to as $R_1\Uparrow$*-formulas.*

C1 The $R_1\Uparrow$-formulas and the $R_2\Uparrow$-formulas are mutually exclusive.

C2 A is an $R_1\Uparrow$-formula if and only if

- if A is positive then $A^\perp$ is an $R_1\Downarrow$-formula.
- if A is a negative literal then $A^\perp$ is an I_1-formula.

C3 $\mathcal{A}$ is an $R_2\Uparrow$-formula if and only if

- if $\mathcal{A}$ is positive then $\mathcal{A}^\perp$ is an $R_2\Downarrow$-formula.
- if $\mathcal{A}$ is a negative literal then $\mathcal{A}^\perp$ is an I_2-formula.

It is easy to show that for the splitting of the context in Lemma 2, Φ_i^2 represents $R_2\Uparrow$-formulas and Φ_i^1 represents $R_1\Uparrow$-formulas.

Conditions **C1** and **C2** imply that cut_1 is only applicable to $R_1\Downarrow$ and $R_1\Uparrow$ formulas and conditions **C1** and **C3** imply that cut_2 is only applicable to $R_2\Downarrow$ and $R_2\Uparrow$ formulas.

5.3. Generalized Proof of Admissibility

By virtue of the following theorem, the criteria **C1-C3** allow cut-elimination in a given fragment of LKU to be verified by inspection.

Theorem 6. *For any fragment of LKU that satisfies criteria* **C1-C3**, *the rules cut_1 and cut_2 are admissible.*

Proof The inductive measure for the cut-elimination proof is the usual lexicographical ordering on the size of the cut formula and the heights of subproofs. In a focused proof, the height of a proof can be taken as the maximum number of alternating asynchronous-synchronous phases (*i.e.*, the number of D_1 and D_2 rules) along a path to a leaf. Instances of cut are divided into two categories. *Key-case cuts* are cuts where both cut formulas are principal in their immediate subproofs, *i.e.*, when the positive cut formula comes under focus (via D_1 or D_2) and the negative one is decomposed immediately. *Parametric cuts* refer to cuts when, in at least one subproof, the cut formula is not principal. The *parametric formula* can be a synchronous formula under focus or an asynchronous formula. As usual, we can assume that the two subproofs involved in a cut are cut-free, since we can apply the procedure to the lowest-height cuts first. The cut-elimination procedure permutes the cut above the introduction of parametric formulas until a key case is reached; that is, until one of the following configurations is reached:

$$\dfrac{\dfrac{\vdash \Gamma : \Delta \Downarrow A}{\vdash \Gamma : A, \Delta \Uparrow}\,D_1 \qquad \vdash \Gamma' : \Delta' \Uparrow A^\perp}{\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow}\,cut_1 \qquad \dfrac{\dfrac{\vdash A, \Gamma : \Delta \Downarrow A}{\vdash A, \Gamma : \Delta \Uparrow}\,D_2 \qquad \vdash \Gamma' : \Uparrow A^\perp}{\vdash \Gamma\Gamma' : \Delta \Uparrow}\,cut_2$$

Asynchronous Parametric Decomposition. For the case of asynchronous parametric formulas, this permutability is a direct consequence of Lemma 2. To illustrate this point, in a focused system the sequent $\vdash \Gamma : \Delta, A \Uparrow B \parr C$ has a cut-free proof *if and only if* $\vdash \Gamma : \Delta, A \Uparrow B, C$ has a cut-free proof (A is the non-principal cut formula). Lemma 2 is used to generalize this equivalence to all parametric asynchronous phases. In the following we shall simply omit mention of the context Θ.

If the selected cut rule is

$$\frac{\vdash \Gamma : \Delta, A \Uparrow \quad \vdash \Gamma' : \Delta' \Uparrow A^\perp}{\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow}\ cut_1 \qquad \text{or} \qquad \frac{\vdash A, \Gamma : \Delta \Uparrow \quad \vdash \Gamma' : \Uparrow A^\perp}{\vdash \Gamma\Gamma' : \Delta \Uparrow}\ cut_2,$$

then the left-side subproof must end in a decision rule (D_1 or D_2), which selects a formula for focus. If the formula selected for focus is the cut formula A, then we have a key-case cut. If some other formula in Δ or Γ is selected for focus, then we have a parametric case with a positive parametric formula.

It is also possible that both A and $A^\perp$ are literals, which means that the right-side subproof will also contain a proof of $\vdash \Gamma' : \Delta', A^\perp \Uparrow$ or of $\vdash A^\perp, \Gamma' : \Delta' \Uparrow$ and these will then also require a formula to be selected for focus. This configuration provides a critical choice-point in cut-elimination: in particular, we must permute the cut above the subproof that contains the *positive* cut formula (the positive cut formula is "attractive" in the terminology of [9]). Below, we assume that A is positive.

Parametric Focus. The argument for the positive parametric case does not depend on whether the parametric formula B is selected for focus from the classical or the linear context. It does depend on whether cut_1 or cut_2 is being used. We demonstrate one principal case:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \Gamma, B : \Delta^1 \Downarrow b_1}{\vdots} \quad \cdots \quad \dfrac{\vdash \Gamma, B : \Delta^n \Downarrow b_n}{\vdots}}{\vdash \Gamma, B : \Delta, A \Downarrow B}}{\vdash \Gamma, B : \Delta, A \Uparrow}\ D_2 \qquad \vdash \Gamma' : \Delta' \Uparrow A^\perp}{\vdash \Gamma'\Gamma, B : \Delta\Delta' \Uparrow}\ cut_1$$

where $B \downarrow \{b_1, \ldots, b_n\}$ and $\Delta, A = \Delta^1 \ldots \Delta^n$. This form is guaranteed by Lemma 3.

Exactly one of the Δ^i will contain the cut formula A. If b_i is a positive literal, *it cannot be the case that $b_i = A^\perp$ because A is assumed positive.* This critical fact relies on the choice to always permute the cut above the subproof with the positive cut formula. Thus b_i must be negative and by (necessarily) $R_1\Downarrow$, we have a subproof of $\vdash \Gamma, B : \Delta^i \Uparrow b_i$. The original cut is permuted to a cut between $\vdash \Gamma, B : \Delta^i \Uparrow b_i$ and $\vdash \Gamma' : \Delta' \Uparrow A^\perp$ with a lower proof-height measure. Again by lemma 3, we can then synthesize the conclusion $\vdash \Gamma\Gamma', B : \Delta\Delta' \Uparrow$.

In the case of cut_2, which means that A is a $R_2\Uparrow$-formula, the argument differs as follows. Each premise of the parametric phase is of the form $\vdash B, A, \Gamma : \Delta^i \Downarrow b_i$. It is possible that b_i is positive if $\Delta_i = \{b_i^\perp\}$ or if Δ^i is empty and $b_i^\perp \in \Gamma$. In either case we get by weakening in the form of Lemma 5 that $\vdash B, \Gamma\Gamma' : \Delta^i \Downarrow b_i$ is provable. If b_i is negative, then it must be preceded (from above) by $R_1\Downarrow$ or $R_2\Downarrow$. We then permute the cut to a cut_2 between $\vdash B, A, \Gamma : \Delta^i \Uparrow b_i$ and $\vdash \Gamma' : \Uparrow A^\perp$, which again gives $\vdash B, \Gamma\Gamma' : \Delta^i \Uparrow b_i$. Again by applying Lemma 3, we synthesize the conclusion $\vdash B, \Gamma\Gamma' : \Delta \Uparrow$.

It is worthwhile to note that criteria **C1-C3** are not required in the parametric cases.

Key Cases. The argument for the key case cuts differ in the cut_1 and cut_2 cases only in that the latter involves the permutation of the cut above a contraction. It is important to note the following invariants:

1. The explicit contraction in D_2 is restricted to positive formulas. Thus contraction can only occur on one subproof of the (key) cut.
2. Only the cut_2 form is valid when the cut formula is in the unbounded context (by **C1** and **C3**), which requires an empty linear context on the subproof opposite of the contraction. This ensures that we can stack multiple cuts without copying the linear context.

The key-case cut is preceded above by several parametric cuts. That is, for the sequent $\vdash A, \Gamma :\Downarrow A$, the occurrence of A under focus is erased by a key-case cut while the "copy" is erased by parametric cuts. The parametric cuts have lower proof-height measures while the key cut reduces to smaller cut formulas. This argument would fail if we cannot assume that the A is positive: if A is negative then there could be no key case.

With this difference, both the cut_1 case and the cut_2 case involve the same arguments: in either case the (asynchronous) cut formula decomposes into some $R_1\Uparrow$-formulas and some $R_2\Uparrow$-formulas (i.e., to some linearly and some classically oriented formulas). The $R_1\Uparrow$ subformulas are erased by cut_1 rules and the $R_2\Uparrow$ subformulas are erased by cut_2 rules. The linear context is necessarily "forced" onto the cut_1 subproofs.

We demonstrate the argument in the case of cut_1. By lemmas 2 and 3, the cut will have the form

$$\cfrac{\cfrac{\cfrac{\cfrac{\vdash \Gamma : \Delta_1 \Downarrow a_1}{\vdots} \quad \cdots \quad \cfrac{\vdash \Gamma : \Delta_n \Downarrow a_n}{\vdots}}{\vdash \Gamma : \Delta \Downarrow A}}{\vdash \Gamma : \Delta, A \Uparrow}\, D_1 \qquad \cfrac{\cfrac{\vdash \Gamma'\Phi_1^2 : \Delta'\Phi_1^1 \Uparrow}{\vdots} \quad \cdots \quad \cfrac{\vdash \Gamma'\Phi_m^2 : \Delta'\Phi_m^1 \Uparrow}{\vdots}}{\vdash \Gamma' : \Delta' \Uparrow A^\perp}}{\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow}\, cut_1$$

where $A \downarrow \{a_1, \dots, a_n\}$, $A^\perp \uparrow \Phi_i^2\Phi_i^1$ for each $1 \le i \le m$, and $\Delta = \Delta_1 \dots \Delta_n$.

By lemma 4, one of the $\Phi_k^2\Phi_k^1$ will have the form $\{a_1^\perp, \dots, a_n^\perp\}$ (recall that $\Phi_1, \dots, \Phi_m$ are exhaustive). The cut can be permuted into zero or more cuts involving formulas of smaller size (or to a single *multicut*) between $\vdash \Gamma'\Phi_k^2 : \Delta'\Phi_k^1 \Uparrow$ and the sequents bordering the positive phase on the left subproof. For clarity in presentation we describe the reduction in stages.

First, we remove negative literals from Φ_k^2. Let $a_i^\perp$ be a negative literal in Φ_k^2. By **C1**, $a_i^\perp$ is a $R_2\Uparrow$-formula and thus by **C3** a_i is a I_2-formula. This means Δ_i is empty and $a_i \in \Gamma$. We note that by weakening (in the form of Lemma 5) on the right-side subproof that $\vdash \Gamma\Gamma'\Phi_k^2 - \{a_i^\perp\} : \Delta'\Phi_k^1 \Uparrow$ has a cut-free proof. Let $\Phi_k^{'2}$ be Φ_k^2 without negative literals.

Second, we can remove the remaining formulas in Φ_k^2. Let $a_i^\perp$ be a positive formula in Φ_k^2. By **C3**, a_i is a $R_2\Downarrow$ formula. Again Δ_i is empty. We form a cut_2 of smaller degree between $\vdash \Gamma'\Phi_k^{'2} : \Delta'\Phi_k^1 \Uparrow$ and $\vdash \Gamma :\Uparrow a_i$ to again obtain a proof of the sequent $\vdash \Gamma\Gamma'\Phi_k'^2 - \{a_i^\perp\} : \Delta'\Phi_k^1 \Uparrow$. We can now assume that all members of Φ_k^2 are eliminated.

Third, let $a_i^\perp$ be a negative literal in Φ_k^1. By **C1**, $a_i^\perp$ is a $R_1\Uparrow$-formula and thus by **C2** a_i is a I_1-formula. This means $\Delta_i = \{a_i^\perp\}$. By weakening on the right-side subproof, we have a cut-free proof of the form $\vdash \Gamma\Gamma' : \Delta_i\Delta'\Phi_k^1 - \{a_i^\perp\} \Uparrow$. Now let $\Phi_k'^1$ be Φ_k^1 without negative literals.

Finally, let $a_i^\perp$ be a positive formula in Φ_k^1. By **C2** a_i is a $R_1\Downarrow$-formula and thus we have a proof of $\Gamma : \Delta_i \Uparrow a_i$, with which we form a cut_1 with $\vdash \Gamma' : \Delta'\Phi_k^{1'} \Uparrow$ to obtain a proof of the form $\vdash \Gamma\Gamma' : \Delta_i\Delta'\Phi_k^{1'} - \{a_i^\perp\} \Uparrow$.

At each step $a_i^\perp$ is replaced by a Δ_i. At the end we obtain the conclusion $\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow$. Implicit in the argument is also contraction in the form of Lemma 5 on the many copies of Γ that are created.

This concludes our generalized cut-elimination proof, which can also be extended to the quantifiers with the appropriate additional cases. A special case is also needed when the cut formulas are $\top$ and 0: one shows that if a 0 can persist in a provable sequent then the same sequent is provable with 0 replaced by anything else (since there is no introduction rule for 0). □

Cut-elimination in each of the principal fragments of LKU now follows.

Corollary 7. *The cut rule*

$$\frac{\vdash \Gamma :\Uparrow A, \Theta \quad \vdash \Gamma' :\Uparrow A^\perp, \Theta'}{\vdash \Gamma\Gamma' :\Uparrow \Theta\Theta'}\ cut$$

can be eliminated in LKF. Similarly, the cut rule

$$\frac{\vdash: \Delta \Uparrow A, \Theta \quad \vdash: \Delta' \Uparrow A^\perp, \Theta'}{\vdash: \Delta\Delta' \Uparrow \Theta\Theta'}\ cut$$

can be eliminated in MALLF. Finally, the cut rule

$$\frac{\vdash \Gamma^\perp :\Uparrow A \quad \vdash \Delta^\perp : \Omega \Uparrow A^\perp, \Theta}{\vdash \Gamma^\perp\Delta^\perp : \Omega \Uparrow \Theta}\ cut$$

can be eliminated in LJF and nLJF. In this latter case, Ω consists of at most one essentially right formula.

The LKF cut is an instance of cut_2 (because LKU does not use the bounded context) and the MALLF cut is an instance of cut_1. The intuitionistic case requires slightly more explanation. We use the notation $\Gamma^\perp$ simply to signify that all of $\Gamma^\perp$ consists of essentially left formulas. The cut formula A is essentially right, which means it has polarity -1 or $+2$. If it is a -1 formula, then $A^\perp$, which is $+1$, is a $R_2\Uparrow$ formula and thus by **C3** and **C1**, the cut is an instance of cut_2. If A is a $+2$ formula, then the cut is technically an instance cut_1, since A is then a $R_1\Uparrow$-formula. The linear context in the left subproof is empty because of the single-conclusion requirement of intuitionistic sequents.

The generalized proof of cut-elimination does not technically assume the extra restrictions of the LJF fragment as defined in Figure 4 (in $R_1\Uparrow$ and $R_1\Downarrow$), which were used to impose full-completeness. To show that cut-elimination for LJF stays within these restrictions, we present the following lemmas concerning the structure of intuitionistic formulas and proofs in the context of LKU.

Lemma 8. *Let A be an essentially right intuitionistic formula and B an essentially left formula. Let $\Phi_A, \Phi_B, \Psi_A, \Psi_B$, be multisets such that $A \uparrow \Phi_A$, $B \uparrow \Phi_B$, $A \downarrow \Psi_A$, and $B \downarrow \Psi_B$. Then:*

1. *Φ_A contains exactly one essentially right formula;*
2. *Ψ_B contains exactly one essentially left formula;*
3. *Φ_B consists of only essentially left formulas; and*
4. *Ψ_A consists of only essentially right formulas.*

This lemma is proved by simultaneous induction on the structure of formulas.

Lemma 9. *Let Γ, Δ, Θ consist of only essentially left intuitionistic formulas. There is no LKU proof of $\vdash \Gamma : \Delta \Uparrow \Theta$ that does not include an instance of the $\top$ rule.*

This lemma is proved by contradiction: there cannot be such a proof of minimum height. Specifically, the argument is made by examining the premises of each inference rule, besides $\top$, that is available in the construction of cut-free proofs.

These lemmas apply to *any* fragment of LKU since they're stated purely in terms of $\uparrow$ and $\downarrow$. They imply that well-formed intuitionistic sequents will stay intuitionistic across the focusing and decomposition phases (with the exception on $\top$ described in Lemma 9). For example, if $A^\perp$ is a negative essentially left (-2) formula then it will only decompose to other essentially left formulas, thus preserving the single-conclusion characteristic. Similarly, if A is a negative essentially right formula (-1), then decomposing A will yield only one essentially right formula. Lemmas 8 and 9 show that the generalized cut-elimination proof applies to LJF independently of the explicit restriction in $R_1\Uparrow$ and $R_1\Downarrow$ of a single formula in the linear context. In a *successful* proof, this invariant is naturally assured by the structure of intuitionistic formulas and proofs.

The proof of cut-elimination for LJF in [13] used a simultaneous induction on *seven* versions of cut. Clearly the unified framework of LKU offers a better alternative to such proofs.

5.4. Strong Cut

It should be noted that cut-elimination as presented does not mean that the following cut:

$$\frac{\vdash :\Uparrow A, \Gamma \quad \vdash :\Uparrow A^\perp, \Gamma'}{\vdash :\Uparrow \Gamma\Gamma'}\ Cut$$

is admissible in every LKU fragment. If A is a (positive) $R_2\Uparrow$-formula and the decomposition of Γ' contains $R_1\Uparrow$-formulas, then cut_2 cannot be applied.

In fact the context restriction on cut_2 can be explained in terms of linear logic as follows. In LLF, a formula of the form $!A$, where A is an asynchronous formula, will not be decomposed eagerly. Instead a $R\Uparrow$ (corresponding to $R_1\Uparrow$) will be applied. Only when this formula is selected for focus will the context be checked to be empty. Thus the cut of LLF does not require the restriction of cut_2 even when ! and ? formulas are involved. But in LKU, all asynchronous formulas will be decomposed eagerly, even if they are strictly $R_2\Downarrow$-formulas[1]. Observe, however, that the ! rule of LLF is in fact *invertible.* It would be valid to eagerly decompose $!A$ *if* the linear context is empty. The context restriction of cut_2 ensures, without the explicit exponential operators, that a cut is only admissible when this criteria is observed.

It is useful, therefore, to identify the *Cut* rule above as a "stronger" form of cut, as it applies to all formulas. Proving the admissibility of the strong cut in a given fragment requires an additional argument to Theorem 6.

Corollary 10. *The strong cut is admissible in LKF, MALLF, LJF, and nLJF.*

The argument is obvious in the cases of LKF and MALLF. The intuitionistic cases are also easily verified given Lemma 8.

The conditions **C1-C3** also do not represent *necessary* conditions for cut-elimination. Indeed LKU itself does not satisfy these criteria. In particular **C1** does not hold because of the non-deterministic choice between $R_1\Uparrow$ and $R_2\Uparrow$. The following example shows that cuts are not admissible without restrictions in LKU:

$$\dfrac{\dfrac{\vdash \Gamma, A :\Uparrow}{\vdash \Gamma :\Uparrow A}\ R_2\Uparrow \qquad \dfrac{}{\vdash : A \Uparrow A^{\perp}}\ I_1}{\vdash \Gamma : A \Uparrow}\ cut$$

There may be no proof of the conclusion that does not require contraction on A. The strong *Cut* rule is, however, admissible in LKU because it states no more than cut-elimination in the classical fragment: i.e., a LKU cut can collapse to a LKF cut. This is a consequence of the following general *dereliction lemma* for LKU.

Lemma 11. *If* $\vdash \Gamma : \Delta, A \Uparrow \Theta$ *is provable in LKU then there is also a proof of* $\vdash A, \Gamma : \Delta \Uparrow \Theta$ *of the same proof-height.*

The conditions **C1-C3** describe greater structure in cut-free proofs. The dereliction lemma describes how a proof (or subproof) in one fragment can shift to a proof in another fragment.

5.5. Sample Application of Generalized Cut Elimination

As an example of using the generalized criteria for cut elimination, we briefly mention the following application. In one proof theoretic account of tabled

[1]One can simulate the structure of LLF proofs by wrapping a $R_2\Downarrow$-formula A inside $R_1\Downarrow$-formula using a dummy connective, such as $A \otimes 1$ (assuming that $+1$ is a $R_1\Uparrow$ polarity). But this will not be the same as proving A.

deduction in intuitionistic logic [17], once an atom is placed in a "table" (say, because it has been proved), its polarity is switched from negative to positive. In order to guarantee that such an atom is not reproved, the "reaction-left-rule" should no longer be applicable to the positive version of the atom. In particular, the LJF rule (which corresponds to $R_1\Downarrow$ in LKU notation):

$$\frac{[\Gamma], P \longrightarrow [R]}{[\Gamma] \xrightarrow{P} [R]}\ R_l$$

must be restricted for such a positive atom P: completeness is still guaranteed since P is present in Γ. In order to maintain cut-elimination in this modified version of LJF, condition **C2** also requires that we restrict the rule (which corresponds to $R_1\Uparrow$ in LKU)

$$\frac{[\Gamma], \Theta \longrightarrow [P]}{[\Gamma], \Theta \longrightarrow P}\ []_r$$

so that it is not applicable to the same positive atoms that R_l cannot be applied to. Given the general cut-elimination result, we are guaranteed that cut-elimination holds for the resulting restricted proof system.

6. Unified Completeness of Focusing

Cut elimination provides the central mechanism for transforming proofs and thus can be used to prove a variety of completeness properties. Yet cut-elimination by itself is not enough. A criteria that's not included in **C1-C3** is that the reaction rules $R_1\Uparrow$ and $R_2\Uparrow$ are complete for all formulas. Cut-elimination allows us to transform proofs that already exist, but does not show us how to prove anything in particular. The first step in generalizing the completeness properties of LKU is to show that meaningful proofs in fact exist.

6.1. Initial Elimination

We can provide general criteria that imply that for all formulas A, the initial sequent $\vdash :\Uparrow A, A^\perp$ is provable. Proofs of such sequents will be called *eta-proofs*. The existence of eta-proofs can also be referred to as *initial elimination* since it allows us to write proof systems without the initial rule for non-literal formulas. More generally, we show that $\vdash \Gamma :\Uparrow A, A^\perp$ holds by induction on the structure of the formulas A.

Assume A is positive. The proof is the same whether A is a $R_1\Uparrow$ or $R_2\Uparrow$ formula: assume that it's a $R_1\Uparrow$-formula. The argument is again by induction on the height of proofs. We need to show that each premise (as required by Lemma 2) of the form $\vdash \Gamma\Phi^2 : \Phi^1, A \Uparrow$, where $A \uparrow \Phi^2\Phi^1$, is provable. By

Lemma 4 and Lemma 3, this holds if we can build a proof of

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma\Phi^2 : \Delta_1 \Downarrow a_1}{\vdots} \quad \cdots \quad \dfrac{\vdash \Gamma\Phi^2 : \Delta_n \Downarrow a_n}{\vdots}}{\vdash \Gamma\Phi^2 : \Phi^1 \Downarrow A}}{\vdash \Gamma\Phi^2 : \Phi^1, A \Uparrow} D_1$$

such that $A \downarrow \{a_1, \ldots, a_n\}$ and $\{a_1^\perp, \ldots, a_n^\perp\} = \Phi^2\Phi^1$. For each $a_i^\perp \in \Phi^2$, let Δ_i be empty, and for each $a_j^\perp \in \Phi^1$, let $\Delta_j = \{a_j^\perp\}$. Then for each a_i, if a_i is positive, then the subproof ends in an I_1 or I_2. If a_i is negative, then by the inductive hypothesis we must have a proof of $\vdash \Gamma\Phi^2 :\Uparrow a_i, a_i^\perp$ (which is preceded from above by a $R_1\Uparrow$ or $R_2\Uparrow$ rule).

Only conditions **C2** and **C3** are required in this proof. In particular, if $a_i^\perp$ is an $R_1\Uparrow$-formula, then a_i is either an I_1-formula or a $R_1\Downarrow$-formula. However, the proof does assume that the $R_1\Uparrow$ and the $R_2\Uparrow$ formulas are complete. This is technically not required in cut-elimination, but we wish to state the initial elimination theorem in a more general form, with all formulas to the right side of $\Uparrow$. We therefore introduce another criteria:

C4 All positive formulas and negative literals are either $R_1\Uparrow$-formulas or $R_2\Uparrow$-formulas.

We summarize initial elimination in the following theorem:

Theorem 12. *In all fragments of LKU that satisfy conditions* **C2-C4***, the sequent* $\vdash :\Uparrow A, A^\perp$ *is provable for all formulas A.*

6.2. Completeness with Respect to Unfocused Systems

We now use cut elimination and initial elimination to prove the completeness of focused proof systems with respect to the unfocused version: see [16] for a similar proof in an intuitionistic setting. Essentially, the technique is to ensure that the immediate subformulas of a positive formula are negative, and vice versa, by using formulas such as $A \parr \perp$ and $A \otimes 1$. Let A^δ be the modified version of A. It is easy to establish that if A is provable in an unfocused setting then $\vdash :\Uparrow A^\delta$ is provable in a focused setting. We then show that $\vdash :\Uparrow A^{\delta\perp}, A$ is provable. These proofs imitate the eta-proofs of Theorem 12. Then by the admissibility of the following cut:

$$\dfrac{\vdash :\Uparrow A^\delta \quad \vdash :\Uparrow A^{\delta\perp}, A}{\vdash :\Uparrow A} \; Cut$$

we derive a focused proof of A. We require the strong cut rule as it applies to end sequents. The cut can be repeatedly applied to transform any sequent.

In order to preserve the polarities of formulas and their relationship to the structural rules, we introduce yet another condition. Together with **C1-C4**, it ensures that the structural rules are consistent for formulas of the same polarity:

C5 If i is 1 or 2 and if A is an $R_i\Uparrow$ formula and B is either a positive formula or negative literal of the same polarity of A, then B is also an $R_i\Uparrow$ formula.

We define a transformation that will ensure that both the asynchronous and synchronous phases are only one-level deep. First we must choose operations $\partial^+(A)$ and $\partial^-(A)$ to force A into a positive or negative formula respectively. These operations, which may vary depending on the polarity of A, can be defined by any number of connectives and their corresponding units, or even by vacuous quantifiers. For example, $\partial^+(A)$ can be $A \otimes 1$ or $A \vee^+ 0$. The choice should be made so that:

1. if A is a negative formula of polarity $-n$ and $-n$ literals are $R_i\Uparrow$-formulas, then $\partial^+(A)$ is also a $R_i\Uparrow$-formula. Furthermore, $\partial^-(A) = A$.
2. If A is a positive $R_i\Uparrow$-formula and $\partial^-(A)$ has polarity $-n$, then $-n$ literals are also $R_i\Uparrow$-formulas. Furthermore, $\partial^+(A) = A$.

We can, in fact, further require that $\partial^-(A) = \partial^+(A^\perp)^\perp$. These invariants ensure, for example, that if A is an essentially right intuitionistic formula then $\partial^-(A)$ and $\partial^+(A)$ are also essentially right formulas[2]. We now define by mutual recursion the dual translations $(\cdot)^+$ and $(\cdot)^-$ as follows:

- For positive literal or unit A, $A^+ = A$, $A^- = \partial^-(A)$.
- For negative literal or unit A, $A^+ = \partial^+(A)$, $A^- = A$.
- For a negative connective such as $⅋$, $(A ⅋ B)^+ = \partial^+(A^+ ⅋ B^+)$, $(A ⅋ B)^- = A^+ ⅋ B^+$.
- For a positive connective such as $\otimes$, $(A \otimes B)^+ = A^- \otimes B^-$, $(A \otimes B)^- = \partial^-(A^- \otimes B^-)$.

The translation of the other connectives follow these patterns in the obvious way. It can be shown that that $A^{+\perp} = A^{\perp -}$ and $A^{-\perp} = A^{\perp +}$. For negative N, $N^+ = \partial^+(N^-)$ and the immediate subformulas of N^- are positive formulas of the form C^+. For positive P, $P^- = \partial^-(P^+)$ and the immediate subformulas of P^+ are negative formulas of the form C^-.

A focused proof can always be emulated by a technically unfocused proof by selecting the appropriate principal formula at each step. It can be shown, by a tedious but uninteresting inductive argument, that the decomposition of an asynchronous formula A as described in Lemma 2 can be emulated by a series of inferences on A^+. The application of consecutive asynchronous introduction rules is interrupted by a sequence of reaction and decision rules: i.e., we "decide" on the right formulas to simulate a focused proof. Similarly, the focusing phase that begins with a D_1 or D_2 rule on a positive formula A as described by Lemma

[2]for essentially right formula A, $\partial^-(A)$ can be $\perp ⅋ A$ $(1 \supset A)$ and $\partial^+(A)$ can be $A \vee^+ 0$. Their duals will form the operations for the essentially left formulas B: $\partial^-(B) = B \wedge^- \top$ and $\partial^+(B) = 1 \otimes B$.

3 can be emulated by a series of inferences on A^-. The eta-proofs of Theorem 12 can therefore be imitated in proofs of $\vdash :\Uparrow A^{+\perp}, A$ and $\vdash :\Uparrow A^{-\perp}, A$.

We therefore have a uniform procedure for showing how a focused system is complete with respect to an unfocused one. In particular, given the results of Section 5, we can state the following.

Theorem 13. *MALLF, LKF and LJF are sound and complete for MALL, LK and LJ, respectively.*

6.3. Changing the Polarity of Atoms

One of the outstanding characteristics of focused proof systems is that atomic formulas can be assigned positive or negative polarity without affecting provability. This property was already known for LLF and follows from the fact that the completeness proof for LLF does not depend on the polarity of atoms. In fact, if we apply the transformations A^+/A^- of the previous section to a sequent, then clearly the polarity of a atom will not affect a proof. That is, an initial rule can only be applied immediately after a decision rule. However, since it is possible to define fragments of LKU that do not have any well-known unfocused counterparts, it is meaningful to directly demonstrate this property. Although the same mechanisms for showing completeness can be used, we give a simpler procedure here to show more clearly how a proof is actually transformed after a polarity switch.

The desired property can be generalized into consistently changing the polarity of a literal *without changing its $R_i\Uparrow$ classification.* For example, in LJF we can change a $+2$ atom into a -1 atom since both are essentially right formulas (and their negations are both left formulas). It is clearly not possible to change a $R_2\Uparrow$ literal, which is subject to contraction, to a $R_1\Uparrow$ one.

First we show that changing a *positive literal A to a negative one* affects the structure of a proof minimally. In fact, only the D_1/D_2 and I_1/I_2 rules are affected. But choosing a positive literal for focus with a decision rule must immediately invoke an initial rule. The new proof chooses $A^\perp$ in place of A. If the original proof has a non-trivial focusing phase that ends in an initial rule on A, then the proof is transformed as follows:

$$\dfrac{}{\vdash \Gamma : A^\perp \Downarrow A}\ I_1 \qquad \longmapsto \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, \Gamma :\Downarrow A^\perp}\ I_2}{\vdash A, \Gamma : A^\perp \Uparrow}\ D_1}{\vdash \Gamma : A^\perp \Uparrow A}\ R_2\Uparrow}{\vdash \Gamma : A^\perp \Downarrow A}\ R_1\Downarrow$$

The figure assumes that A is a $R_2\Uparrow$ formula but the transformation is similar in other cases.

When a *negative literal A is replaced with a positive one,* the transformation is less straightforward. The $R_i\Downarrow$ reactions may not immediately be replaced with initial rules on the now-positive A. The transformation may require the permutation of some focusing phases below others. We can replace every occurrence of A with $\partial^+(\partial^-(A))$[3]. In MALL, for example, this can be $(A \parr \bot) \otimes 1$. The $\partial^-()$ and $\partial^+()$ operations preserve the applicability of $R_i\Downarrow$ and $R_i\Uparrow$ rules respectively. If F is a formula that contains A, let F^ν be F with every occurrence of A so replaced. We show that a proof of F can be transformed to a proof of F^ν. Then by applying

$$\frac{\vdash :\Uparrow F^\nu \qquad \vdash :\Uparrow {F^\nu}^\bot, F}{\vdash :\Uparrow F}\ Cut$$

we derive a proof of F with A now positive. The right premise of the strong cut again follows from the imitation of initial elimination. The proof of F^ν imitates the proof of F with the following sample transformation:

$$\cfrac{\cfrac{\vdash \Gamma : A \Uparrow}{\vdash \Gamma :\Uparrow A}\ R_1\Uparrow}{\vdash \Gamma :\Downarrow A}\ R_2\Downarrow \qquad \longmapsto \qquad \cfrac{\begin{array}{c}\cfrac{\vdash \Gamma : A \Uparrow}{\vdash \Gamma :\Uparrow A}\ R_1\Uparrow \\ \hline \vdots\end{array} \qquad \cfrac{}{\vdash \Gamma :\Downarrow 1}\ 1}{\vdash \Gamma :\Downarrow (\bot \parr A) \wedge^+ 1}\ \wedge^+$$

The above figures assume intuitionistic structural rules: variations are similar. The positive A in the transformed proof will only be selected for focus by a decision rule when an initial rule is to be emulated, as indicated by the positive-to-negative transformation above.

We can see how the new proof of F is constructed by dissecting the details of the cut reduction. In the original proof of F and of F^ν, a formula other than A may be selected for focus above the premise $\vdash \Gamma : A \Uparrow$. Such a selection represents a *parametric focus* in the cut elimination procedure. This means that the cut between A in the proof of F^ν and $A^\bot$ in the emulated eta-proof must be permuted above the selection of the parametric formula. In the resulting cut-free proof, the parametric selection occurs beneath the initial rule for A.

7. Communication Between Fragments

Since all the fragments of LKU share the same connectives and atoms, different fragments can interact using cuts. If we are only interested in cut-free *classical* proofs, then all cuts between fragments collapse to classical cuts. In certain circumstances, cut-elimination can preserve more structure. We give two such examples. A formula is *pure* with respect to a polarity if all of its subformulas, up to and including literals, have the same polarity. Focusing on purely positive formulas leads to constructive proofs.

[3] It's possible to just use $\partial^-(A)$ but the argument is slightly simpler this way.

Theorem 14. *Let A be a purely $+2$ formula and let Δ consist of purely -2 formulas. Given an LKF proof of $\vdash :\Uparrow A, \Delta$ and an LJF proof of $\vdash \Gamma^\perp : \Omega \Uparrow A^\perp, \Theta$, the following instance of the cut rule*

$$\frac{\vdash :\Uparrow A, \Delta \quad \vdash \Gamma^\perp : \Omega \Uparrow A^\perp, \Theta}{\Gamma^\perp : \Omega \Uparrow \Delta\Theta}\ cut$$

can be replaced by a cut-free proof in LJF.

Proof Since Δ is purely -2, after asynchronous decomposition of the classical sequent and the selection of A for focus, we must have a LKF proof of the following form by lemma 3:

$$\dfrac{\dfrac{\dfrac{\vdash \Delta_1 :\Downarrow a_1}{\vdots} \quad \cdots \quad \dfrac{\vdash \Delta_n :\Downarrow a_n}{\vdots}}{\vdash \Phi_\Delta, A :\Downarrow A}}{\vdash \Phi_\Delta, A :\Uparrow}\ D_2$$

such that Φ_Δ consists of -2 literals and $A \downarrow \{a_1, \ldots, a_n\}$. Since each a_i is a $+2$ literal, the initial rule I_2 must apply. It cannot be that $A = a_i^\perp$ since they are both of $+2$ polarity. Thus Φ_Δ contains $a_1^\perp, \ldots, a_n^\perp$. If A is hereditarily $+2$ then $A^\perp$ is hereditarily -2. By lemmas 2 and 4, the LJF sequent will have a subproof ending in

$$\vdash \Gamma^\perp, a_1^\perp, \ldots, a_n^\perp : \Omega \Uparrow \Theta$$

By weakening (lemma 5), we also have an LJF proof of

$$\vdash \Gamma^\perp, \Phi_\Delta : \Omega \Uparrow \Theta$$

Again by lemma 2, we have a proof of $\Gamma^\perp : \Omega \Uparrow \Delta\Theta$. □

Note that formulas such as $P \vee^+ P^\perp$ are excluded from the scope of the theorem because they cannot be purely of one polarity. The scope of the theorem is expanded when one considers that, except for the quantifiers, every classical connective has an equivalent one of the opposite polarity. Furthermore, by the transformations in Section 6.3, provability in LKF is not affected by the polarity of atomic formulas.

Now consider cutting between a MALLF proof and an LJF proof. It is not immediate that a MALLF proof of an intuitionistic end-sequent (all formulas on the right side of $\Uparrow$) can be transformed into an intuitionistic proof. MALLF proofs may "split the context" differently from an intuitionistic proof. However, with Lemma 9 of Section 5 we can show that a MALLF proof of an intuitionistic sequent will only involve sequents with exactly one essentially right formula. (except when $\top$ is used). Then by applying the dereliction lemma (lemma 11) to the essentially left formulas, every such MALLF proof can be transformed into an LJF proof. From cut-elimination in LJF, we also have the following admissible *cross-cut*.

Theorem 15. *Given an LJF proof of* $\vdash \Gamma^{\perp} :\Uparrow A$ *and a MALLF proof of* $\vdash :\Uparrow A^{\perp}, B$ *where* B *is an essentially right intuitionistic formula, the following cut*

$$\frac{\vdash \Gamma^{\perp} :\Uparrow A \quad \vdash :\Uparrow A^{\perp}, B}{\vdash \Gamma^{\perp} :\Uparrow B}\ cut$$

can be replaced by a cut-free proof in LJF.

8. Comparing LU and LKU

It is not practical to reproduce here the LU proof system of Girard [4]. For the benefit of readers already familiar with that system, we briefly compare it with LKU.

Central to LU is a classification of formulas according to one of three polarities that are used to identify the formulas on which structural rules apply. In LU, one must examine the polarities of the connective's arguments to determine the additive/multiplicative (and positive/negative) nature of that connective: as a result, the proof system is not a sequent calculus proof system in the strict sense used by Gentzen. While the polarity notions used in LU can be seen as being compatible with those used in focused proof systems, these polarities are not the same and LU is not, in fact, focused. Another basic difference between these two proof systems is that LU can be described by a translation to linear logic (except at the level of atoms), whereas there is no translation of LKU proofs into linear logic proofs: instead, each of its fragments may require a different translation.

The proof system LKU contains a rich set of logical connectives (a merging of the connectives in linear, intuitionistic, and classical logics) and each connective has one inference rule. This stands in sharp contrast to LU where several connectives have a large number of introduction rules. On the other hand, LU provides a fixed set of structural rules while LKU has an extended set of structural rules (being a focused proof system causes some growth in the number of these rules). In LKU, the meaning of a connective, such as $\oplus$ and $\parr$, is determined not only by their (usual) introduction rule but also by the sensitivity of the structural rules to their polarity. By adjusting this sensitivity we can use the various symbols of LKU to derive focusing systems for classical logic, intuitionistic logic, MALL, and other interesting fragments of these logics. Since these fragments are based on formulas containing the same set of connectives, it is possible for these fragments to interact through cut elimination.

The LU system allows a similar interaction. In fact, an important property of LU is that a cut-free proof of a sequent in a given fragment stays within that fragment. Thus by the sub-formula property it does not matter what fragments are involved in the proof before cut-elimination. While such a result might seem enticing, it does not mean that there are no limitations to the communication between different logics. In the LU scheme of polarization, classical logic uses "positive" (+2 in LKU) and "negative" polarities and intuitionistic logic uses positive and "neutral" (linear) polarities. One can only form a valid cut between a classical (two-sided) sequent and an intuitionistic one if the cut formula is positive and the concluding sequent can be intuitionistic only if the classical sequent is free of negatives. These restrictions are similar to the pre-conditions of the results in Section 7. There must be conditions under which a proof can incorporate classical arguments and yet can be transformed into a purely intuitionistic proof. A framework such as LU or LKU cannot alter these conditions. The unified framework can only enable and clarify the extent of the possible communication between logics.

The LKU approach of placing more emphasis on structural rules is valuable in general since much of the effort in designing focused proof systems is centered on what structural rules they should include. For example, one can have systems that focus on a unique formula or on multiple formulas [18]. One can insist that an asynchronous phase terminates when all asynchronous formulas are removed or allow it to terminate before they are all removed. The LKU approach is an example of studying a range of possible restrictions to the structural rules. While fragments of LKU have been defined by imposing restrictions on the structural rules, all fragments share the same set of nine introduction rules. Such uniformity simplifies and generalizes the cut-elimination proof, as we have shown.

Our treatment of intuitionistic logic in LKU is similar to that of LU with two differences. First, LU is a two-sided sequent calculus whereas LKU is one-sided. The richness of polarity information in LKU replaces the need for a two-sided system: the polarity of a formula unambiguously determines its essentially left or right status. (Of course, one may still prefer a two-sided system for readability.) Second, an alternative polarity is possible in LKU for capturing intuitionistic logic. In particular, it is also possible in LKU to use the $-2/+1$ polarities for essentially right formulas and $+2/-1$ for the left ones by altering the restrictions on the structural rules. The only problem with this alternative polarization would be the assignment of polarities to the units 0 and 1. We have chosen to use one set of four units for the eight propositional connectives. However, it is equally valid to consider a version of LKU with two copies of each unit, or by simply assigning the units -1 or $+1$ polarity.

We do not claim for LKU all that LU promises. In particular, although never fully explained or further studied, LU leaves open the possibility of allowing hybrid formulas that use connectives from multiple logics *without restriction,* e.g., $(A \otimes B) \wedge^{+} C$. While such a possibility is not within the scope of LKU, we consider limited classical-linear hybrid logics in Section 9 and as future work in Section 10.

9. Synthesizing a New Logic within LKU

The existence of intuitionistic logic as a hybrid logic with both linear and classical characteristics suggests that other such hybrids may also exist.

It is tempting to define such a logic by simply restricting the level-1 structural rules to $+1/-1$ formulas and the level-2 rules to $+2/-2$ formulas. This system, which we shall refer to as *UHL* or Unrestricted Hybrid Logic, satisfies cut-elimination as a result of Theorem 6, but it does not admit the strong cut. This limitation is to be expected in a generalized hybrid setting between classical and linear logics. A classical equivalence may be provable only in a purely classical context (i.e., no $R_1 \Uparrow$ formulas), in which case it would not be valid to substitute the equivalent formula into a mixed classical-linear context. For example, $\wedge^+$ becomes equivalent to the linear $\otimes$ in the presence of a non-empty linear context, and can no longer be considered equivalent to $\wedge^-$. This is the reason behind the context restriction of the cut_2 rule. Intuitionistic logic "side-steps" this problem with its restrictions on formulas and sequents. However, without further restrictions, it is difficult to identify meaningful invariances inside UHL. One indication of the problem with UHL is that there is no apparent translation into linear logic. The problem is related to the way in which linear and classical subformulas are interleaved. One may attempt a translation of UHL into linear logic following the principles of LU. A formula $(A \otimes B) \wedge^+ C$ might be translated into the form $!(A \otimes B) \otimes !C$ (as suggested by the LU tables). But focusing in linear logic *cannot continue past the* !. It would be valid to transfer from a linear focusing state to a classical one, but not vice versa. The structural rules of LKU are not sensitive to such a "lateral" change of polarity[4].

One solution is to restrict the interleaving of classical and linear formulas. In particular, we can specify that *classical formulas contain no linear subformulas.* We designate this system as simply HL1. Define two categories of formulas as follows:

- $H \ := \ 0 \mid 1 \mid \top \mid \bot \mid H \wedge^+ H \mid H \wedge^- H \mid H \vee^+ H \mid H \vee^- H \mid \exists x.H \mid \forall x.H \mid +2/-2$ literals
- $L \ := \ H \mid L \otimes L \mid L \& L \mid L \oplus L \mid L \parr L \mid \Sigma x.L \mid \Pi x.L \mid +1/-1$ literals

The HL1 fragment of LKU has the structural rules of Figure 7. End-sequents of HL1 have the form $\vdash :\Uparrow \Lambda$ where Λ consists of L-formulas.

Clearly, both classical logic and MALL are found as sub-fragments of HL1. Since the asynchronous decomposition of H-formulas will completely absorb the formula into the classical context, a formula such as $H_1 \parr H_2$ is in fact equivalent to $H_1 \vee^- H_2$. But meaningful distinctions between classical and linear provability are sustained. Consider $A \parr (A^\perp \otimes A^\perp)$, which is provable if A is a classical formula but not if A is linear.

It is possible to understand HL1 by a translation to linear logic. We preserve the linear connectives and translate the classical connectives as suggested by LU. For example, if A is $+2$ and B is -2 then $A \wedge^+ B$ is translated as $A \otimes !B$ and $A \wedge^- B$ is translated as $?A \,\&\, B$.

Cut elimination in HL1 is verified by observation as a result of Theorem 6. Note that cut_2 can also be seen as a *cross-cut* between an HL1 proof and a LKF

[4]It may be possible to explain the behavior of UHL if linear logic is extended with other exponential operators, in particular an operator that is *self-dual.* Examples of such operators exist, such as in the affine *Light Linear Logic.*

$$\frac{}{\vdash \Gamma : Q^{\perp} \Downarrow Q}\ I_1 \quad \frac{\vdash \Gamma : \Delta, C \Uparrow \Theta}{\vdash \Gamma : \Delta \Uparrow C, \Theta}\ R_1{\Uparrow} \quad \frac{\vdash \Gamma : \Delta \Uparrow N}{\vdash \Gamma : \Delta \Downarrow N}\ R_1{\Downarrow} \quad \frac{\vdash \Gamma : \Delta \Downarrow P}{\vdash \Gamma : \Delta, P \Uparrow}\ D_1$$

Q: $+1$ atom, C: $+1$ formula or -1 atom, N: -1 formula, P: $+1$ formula.

$$\frac{}{\vdash \Gamma, \mathcal{Q}^{\perp} :\Downarrow \mathcal{Q}}\ I_2 \quad \frac{\vdash \mathcal{D}, \Gamma : \Delta \Uparrow \Theta}{\vdash \Gamma : \Delta \Uparrow \mathcal{D}, \Theta}\ R_2{\Uparrow} \quad \frac{\vdash \Gamma :\Uparrow \mathcal{N}}{\vdash \Gamma :\Downarrow \mathcal{N}}\ R_2{\Downarrow} \quad \frac{\vdash \mathcal{P}, \Gamma : \Delta \Downarrow \mathcal{P}}{\vdash \mathcal{P}, \Gamma : \Delta \Uparrow}\ D_2$$

$\mathcal{Q}$: $+2$ atom, $\mathcal{D}$: $+2$ formula or -2 literal, $\mathcal{N}$: -2 formula, $\mathcal{P}$: $+2$ formula

Figure 7: The Classical-Linear Hybrid Logic HL1

proof.

Within HL1, the classical equivalences between the positive and negative versions of connectives, such as $\vee^-$ and $\vee^+$, *hold only in a purely classical context.* In a mixed linear-classical context, $\vee^-$ is equivalent to $\parr$. This apparent anomaly does not contradict cut-elimination because of the restriction of cut_2. Observe that one cannot replace a $\vee^-$ with a $\vee^+$ through cut except in a purely classical context. The cut_2 rule is not applicable on the sequents

$$\vdash A^{\perp} \wedge^+ A^{\perp} :\Uparrow A \vee^+ A \qquad \text{and} \qquad \vdash : A^{\perp} \otimes A^{\perp} \Uparrow A \vee^- A$$

because the linear context in the right sequent is not empty.

This issue can also be understood in the context of the linear logic translation. Assume that A is positive in the sense that $A \equiv\, !A$ and B is negative in the sense that $B \equiv\, ?B$. The formula $A \vee^- B$ is translated into the formula $?A \parr B$ and $A \vee^+ B$ is translated into the formula $A \oplus !B$. In what sense are they equivalent? In one direction, $A \oplus !B \multimap ?A \parr B$ is provable in linear logic. In fact we can always replace a positive classical connective with its negative version and preserve provability. However, in the other direction we can only prove $!(?A \parr B) \multimap ?(A \oplus !B)$. That is, the equivalence holds only in a purely classical context.

10. Future Work

We naturally seek richer hybrid logics with few or no restrictions on how formulas can interleave. Girard's LU system leaves open the possibility of mixing logics without restriction. However, designing a focused system that is entirely faithful to LU faces difficulties. For example, the De Morgan dualities fail when "neutral" formulas are mixed with classical ones.

In this paper, we have employed the different approach of carefully restricting the structure of formulas and sequents using polarity information. Extracting a focused proof system for intuitionistic logic is a powerful validation of this approach. Extracting the logic HL1 is another example.

Still another approach to developing hybrid logics is to extend LKU with new polarities and structural rules. Restrictions on formulas are replaced by even greater sensitivity to polarity information. Focusing can be separated into distinct levels. For example, classical focusing can be represented by $\Downarrow^2$ and linear focusing by $\Downarrow^1$. Transition between focusing modes can be formulated by *lateral* reaction rules such as

$$\frac{\vdash \Gamma : \Delta \Downarrow^2 A}{\vdash \Gamma : \Delta \Downarrow^1 A}\ L\Downarrow$$

where A is a classical formula. More flexible variants of $R\Uparrow$ may be needed, including those that insert asynchronous formulas into the classical context, as one would expect from a system with the full power of linear logic. We are, in fact, currently studying a system with three distinct types of $\Downarrow$ and three of $\Uparrow$, corresponding to six distinct polarities.

Exploring the possible applications of cross cuts between MALL and intuitionistic and classical logics is an appealing topic to pursue. For example, if we add least fixed points, equality, and first-order terms to LKU (much as they have been added to MALL in [19, 20]), the resulting proof system should provide a novel setting to study the extent to which classical principles, such as the excluded middle or Markov's principle, can be safely used within intuitionistic arithmetic.

11. Conclusion

The system LKU that we have introduced can be described as a *kernel* of focused proof systems. In its barest form it can only be called classical logic. By adjusting the sensitivity of its structural rules, we can derive focusing systems for MALL and intuitionistic logic as well as explore the possibility for new logical systems. Cut-elimination and its important consequences can be generalized in this system. We have also shown how this general approach to cut-elimination can be applied to intuitionistic logic (LJF) and to tabled deduction (Section 5.5).

As logics, LK and MALL mirror each other. When considering combinations of these "perfect" logics, there appears to be several alternatives.

1. Introduce exponential operators such as, but not limited to, ! and ?.
2. Carefully restrict the form of sequents and formulas that are allowed.
3. Recognize polarities and use structural rules that are sensitive to polarity information.

One can also use a combination of these techniques. The first approach is that of linear logic. It can be described as "low-level": one would have to place the exponential operators carefully. The LU translation tables show that this is not trivial. In this paper we have principally followed the third approach, with some reliance on the second. Gentzen followed a similar approach: he defined intuitionistic logic by imposing a distinction between left and right-side formulas and making the structural rules of contraction and weakening sensitive to that polarization of formulas. Girard aggressively generalized this concept in the LU system. Polarity is also the central concept behind focusing. Our approach in this paper is to extend the range of what can be considered *structural* rules so as to increase their sensitivity to polarity information. These structural rules are most active when there is a *change* in polarity. This is the contribution of focusing to the polarized analysis of logical systems. Focusing imposes a structure on proofs that clarifies polarity information.

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