Focusing Gentzen's LK proof system

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Abstract Gentzen's sequent calculus **LK** is a landmark proof system. In particular, it identifies the structural rules of weakening and contraction as notable inference rules, and they allow for an elegant statement and proof of both cut-elimination and consistency for classical and intuitionistic logics. However, sequent calculi usually have many undesirable features, and leading among such features is the fact that their inferences rules are low-level, and they frequently permute over each other. As a result, large-scale structures within sequent calculus proofs are hard to identify. In this paper, we present a different approach to designing a sequent calculus for classical logic. Starting with Gentzen's **LK** proof system, we first examine the *proof search* meaning of his inference rules and classify those rules as involving either *don't care nondeterminism* or *don't know nondeterminism*. Based on that classification, we design the *focused* proof system **LKF** in which inference rules belong to one of two phases of proof construction depending on which flavor of nondeterminism they involve. We then prove that the cut-rule and the general form of the initial rule are admissible in **LKF**. Finally, by showing that the rules of inference for **LK** are all admissible in **LKF**, we can give a completeness proof for **LKF** provability with respect to **LK** provability. We shall also apply these properties of the **LKF** proof system to establish other meta-theoretic properties of classical logic, including Herbrand's theorem.

Key words: Sequent calculus, Gentzen's **LK**, focused proof system, **LKF**, polarization, cut-elimination

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1 Introduction

In his attempt to prove the *Hauptsatz* (cut-elimination) for both intuitionistic and classical logics, Gentzen (1935) moved away from natural deduction to the sequent calculus since the latter allowed for defining the structural rules of weakening and contraction on the right-hand side of sequents. If you are only interested in cut-elimination and consistency, then the sequent calculus, as Gentzen presented it, is a great tool. If, however, you are working on applying logic to, say, computation, the structure of proofs plays a central role in such applications. For example, proofs can be used to describe programs (in a functional programming setting), and cut-free proofs can be used to describe computation traces (in a logic programming setting). However, it is generally difficult to gleam from sequent calculus proofs useful structure since that calculus feels too low-level. In particular, one must usually make numerous and tedious arguments involving the permutabilities of inference rules (Kleene 1952) to extract structural information from sequent calculus proofs. For an example of reasoning with inference rule permutations in the sequent calculus, see the proofs in (Miller 1989, Miller et al. 1991) where sequent calculus proofs are used to describe a logic programming language.

In this paper, we examine a *focused* version of the **LK** sequent calculus proof system, called **LKF** (Liang and Miller 2009). The key properties of **LKF**—cut-elimination and relative completeness to **LK**—have been proved elsewhere (Liang and Miller 2009; 2011) by using complex and indirect arguments involving linear logic (Girard 1987), a focused proof system for linear logic due to Andreoli (1992), and a focused proof systems **LJF** for intuitionistic logic. Here, we present **LKF** from first principles: we make no use of intuitionistic or linear logics nor the meta-theory of other proof systems.

After presenting the **LK** inference rules, we describe some of the shortcomings of that proof system in Section 2. In Section 3, that criticism of **LK** motivates the design of **LKF**. We then prove the following results about **LKF**.

1. The four variants of the cut rule in **LKF** are all admissible in (cut-free) **LKF** (Section 4).
2. While the initial rule in **LKF** is limited to atomic formulas, the general form of the initial rule is admissible (Section 5).
3. The rules of **LK** are admissible in **LKF** (Section 7).

Taken together, these results can be used to prove that **LKF** is complete for **LK**. A similar proof outline for proving the completeness of focused proof systems has been used by Laurent (2004) for linear logic, by Chaudhuri et al. (2008b) for an intuitionistic version of linear logic, and by Simmons (2014) for a propositional intuitionistic logic. The proofs of these meta-theoretic results for **LKF** rely almost exclusively on tedious arguments about the permutability of inference rules. One of the design goals for **LKF** has been to build a calculus that can be used directly to prove other proof-theoretic results without the need to involve such tedious permutation arguments. We illustrate this principle by proving the admissibility of cut in cut-free

Structural rules

$$\frac{\Gamma, B, B \vdash \Delta}{\Gamma, B \vdash \Delta}\ cL \qquad \frac{\Gamma \vdash \Delta, B, B}{\Gamma \vdash \Delta, B}\ cR \qquad \frac{\Gamma \vdash \Delta}{\Gamma, B \vdash \Delta}\ wL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, B}\ wR$$

Identity rules

$$\frac{}{B \vdash B}\ init \qquad \frac{\Gamma \vdash \Delta, B \quad \Gamma, B \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

Introduction rules

$$\frac{\Gamma, B_i \vdash \Delta}{\Gamma, B_1 \wedge B_2 \vdash \Delta}\ \wedge_i L \qquad \frac{\Gamma \vdash \Delta, B \quad \Gamma \vdash \Delta, C}{\Gamma \vdash \Delta, B \wedge C}\ \wedge R \qquad \frac{}{\Gamma \vdash \Delta, t}\ tR$$

$$\frac{\Gamma, B \vdash \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, B \vee C \vdash \Delta}\ \vee L \qquad \frac{}{\Gamma, f \vdash \Delta}\ fL \qquad \frac{\Gamma \vdash \Delta, B_i}{\Gamma \vdash \Delta, B_1 \vee B_2}\ \vee_i R$$

$$\frac{\Gamma \vdash \Delta, B \quad \Gamma, C \vdash \Delta'}{\Gamma, \Gamma', B \supset C \vdash \Delta, \Delta'}\ \supset L \qquad \frac{\Gamma, B \vdash \Delta, C}{\Gamma \vdash \Delta, B \supset C}\ \supset R$$

$$\frac{\Gamma, [t/x]B \vdash \Delta}{\Gamma, \forall x.B \vdash \Delta}\ \forall L \qquad \frac{\Gamma \vdash \Delta, [y/x]B}{\Gamma \vdash \Delta, \forall x.B}\ \forall R \qquad \frac{\Gamma, [y/x]B \vdash \Delta}{\Gamma, \exists x.B \vdash \Delta}\ \exists L \qquad \frac{\Gamma \vdash \Delta, [t/x]B}{\Gamma \vdash \Delta, \exists x.B}\ \exists R$$

Fig. 1 The rules for **LK**. In the $\forall R$ and $\exists L$ rules, the variable y is not free in the conclusion. In the $\wedge_i L$ and $\vee_i R$ rules, $i \in \{1, 2\}$.

LK (Section 9.1) and by proving Herbrand's theorem (Section 9.3), both proofs do not explicitly involve permutation arguments.

2 The LK proof system

Formulas for first-order classical logic are defined as follows. Atomic formulas are of the form $P(t_1, \ldots, t_n)$, where $n \geq 0$, P is a predicate of arity n, and $t_1, \ldots, t_n$ is a list of first-order terms. Formulas are built from atomic formulas using both the logical connectives $\wedge$, t, $\vee$, f, $\supset$ as well as the two first-order quantifiers $\forall$ and $\exists$. We shall assume the usual treatment of bound variables and substitution: in particular, the expression $[t/x]B$ denotes the result of performing a capture-avoiding substitution of t for all occurrences of the variable x in the formula B.

Figure 1 presents the **LK** sequent proof calculus of Gentzen (1935). The main differences between the proof system in that figure and Gentzen's presentation of **LK** are the following.

1. In Gentzen's system, contexts are lists of formulas, and the exchange rule, which allowed two adjacent formulas to be swapped, was used. In Figure 1, contexts are multisets of formulas and the exchange rule is not used.
2. Gentzen did not have the logical units for true and false while here they are explicitly written as t and f: they also have associated inference rules.

3. Gentzen's system contained negation as a primitive connective while we shall treat it as an abbreviation: in particular, $\neg B$ is defined to be $B \supset f$.

For this paper, we shall make the following distinction between proof and derivation. By *proof*, we mean a tree structure of inference rules and sequents such that all premises are closed, in the sense that the inference rules at the leaves have zero premises (such as the initial rule). By *derivation*, we mean a similar tree structure of inference rules and sequents, but we do not assume that all leaves are closed: derivations can have unproven premises.

Gentzen's sequent calculus was designed to support the proof of cut-elimination (for both classical and intuitionistic logics). As we suggested in the introduction, sequent calculus is difficult to apply in a number of application areas. We describe four major shortcomings of the **LK** sequent calculus.

2.1 The collision of cut and the structural rules

Consider the following instance of the cut rule.

$$\frac{\Gamma \vdash C \qquad \Gamma', C \vdash B}{\Gamma, \Gamma' \vdash B}\ cut \tag{†}$$

If the right premise is proved by a left-contraction rule from the sequent $\Gamma', C, C \vdash B$, then cut-elimination proceeds by permuting the *cut* rule to the right premises, yielding the derivation

$$\cfrac{\cfrac{\Gamma \vdash C \qquad \cfrac{\Gamma \vdash C \qquad \Gamma', C, C \vdash B}{\Gamma, \Gamma', C \vdash B}\ cut}{\Gamma, \Gamma, \Gamma' \vdash B}\ cut}{\Gamma, \Gamma' \vdash B}\ cL.$$

(An inference figure written with double lines indicates possibly several applications of the rules listed as its justification.) In the intuitionistic variant of the sequent calculus, it is not possible for the occurrence of C in the left premise of (†) to be contracted since two formulas are not allowed on the right of the sequent arrow. If the cut inference in (†) takes place in the classical proof system **LK**, it is possible that the left premise is the conclusion of a contraction applied to $\Gamma \vdash C, C$. In that case, cut-elimination can also proceed by permuting the cut rule to the left premise.

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash C, C \quad \Gamma', C \vdash B}{\Gamma, \Gamma' \vdash C, B}\ cut \qquad \Gamma', C \vdash B}{\Gamma, \Gamma', \Gamma' \vdash B, B}\ cut}{\Gamma, \Gamma' \vdash B}\ cL,\ cR$$

Thus, in **LK**, it is possible for both occurrences of C in (†) to be contracted and, hence, the elimination of cut is non-deterministic since the cut rule can move to both the left and right premises.

Such nondeterminism in cut-elimination is even more pronounced when we consider the collision of the cut rule with weakening. Consider the derivation (taken from (Girard et al. 1989; Appendix B)).

$$\dfrac{\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash B}}{\vdash C, B}\ wR \qquad \dfrac{\dfrac{\Xi_2}{\vdash B}}{C \vdash B}\ wL}{\vdash B, B}\ cut}{\vdash B}\ cR$$

Cut-elimination here can yield either Ξ_1 or Ξ_2: thus, nondeterminism arising from weakening can lead to completely different proofs of B. This kind of example does not occur in the intuitionistic (single-sided) version of the sequent calculus.

These problems with cut-elimination and the structural rules were noted in Danos et al. (1997) and by Lafont in (Girard et al. 1989). Lafont concludes that in order to avoid this problem with cut-elimination, one could chose from among two solutions: either make the sequent calculus asymmetric (leading to intuitionistic logic where the structural rules are not available on the right) or forbid all structural rules (leading to linear logic where structural rules are not available on the left and right). It is possible, however, to remain in classical logic by employing a third solution that uses both *polarization* and *focused proof systems*. Such an approach was proposed by Girard (1991) in his **LC** proof system and by Danos et al. (1997) in their $\mathbf{LK}^{\eta}$ proof system. In this paper, we present the **LKF** proof system, which is also based on the notions of polarization and focusing. As we shall see, the problems with the nondeterminism in cut-elimination caused by the use of structural rules in classical logic disappears in **LKF** for two reasons. First, weakening will be allowed only in the initial rules of **LKF** where it cannot cause problems with cut-elimination. Second, a cut takes place between a positive and a negative formula (the cut-formula and its De Morgan dual) and, in **LKF**, contraction is only applied to positive formulas.

2.2 Permutations of inference rules

A dominating feature of sequent calculus proofs in **LK** is that many pairs of inference rules permute over each other. For example, when an occurrence of $\supset L$ is below $\forall R$, as in the derivation

$$\dfrac{\Gamma_1 \vdash B, \Delta_1 \qquad \dfrac{\Gamma_2, C \vdash [y/x]D, \Delta_2}{\Gamma_2, C \vdash \forall x.D, \Delta_2}\ \forall R}{\Gamma_1, \Gamma_2, B \supset C \vdash \forall x.D, \Delta_1, \Delta_2}\ \supset L,$$

the order of these two rules can be switched to form the derivation

$$\cfrac{\cfrac{\Gamma_1 \vdash B, \Delta_1 \quad \Gamma_2, C \vdash [y/x]D, \Delta_2}{\Gamma_1, \Gamma_2, B \supset C \vdash [y/x]D, \Delta_1, \Delta_2} \supset L}{\Gamma_1, \Gamma_2, B \supset C \vdash \forall x.D, \Delta_1, \Delta_2} \; \forall R.$$

Similarly, the following two deviations are such that permuting the inference rules in one derivation yields the other derivation.

$$\cfrac{\cfrac{\Gamma, B_i, C_j \vdash \Delta}{\Gamma, B_i, C_1 \wedge C_2 \vdash \Delta}}{\Gamma, B_1 \wedge B_2, C_1 \wedge C_2 \vdash \Delta} \qquad \cfrac{\cfrac{\Gamma, B_i, C_j \vdash \Delta}{\Gamma, B_1 \wedge B_2, C_j \vdash \Delta}}{\Gamma, B_1 \wedge B_2, C_1 \wedge C_2 \vdash \Delta}$$

The existence of such permutations of inference rules suggests that uncovering structures in proofs will always be disturbed by the possibilities of such shallow rearrangements of inference rules. For such reasons, people have often argued that the "essence" of proof structures are better captured in some radically different proof systems, such as, for example, expansion trees (Miller 1987), proof nets (Girard 1987, Laurent 2011), and deep inference (Guglielmi 2007). In this paper, we also replace Gentzen-style sequent calculus with something else, namely **LKF**, but this time, that replacement will still resemble sequent calculus but with more structure added to both sequents and inference rules.

An introduction rule of **LK** is *invertible* if whenever there is an **LK** proof of its conclusion, there are **LK** proofs of the premises. When attempting to build a proof from the bottom-up, invertible rules can always be applied without losing provability. If an introduction rule is not invertible, it is *non-invertible*. The **LK** introduction rules can be classified as follows: the invertible rules are $\wedge R$, tR, $\vee L$, fL, $\supset R$, $\forall R$, $\exists L$ while the non-invertible rules are $\wedge_i L$, $\vee_i R$, $\supset L$, $\forall L$, $\exists R$. Note that every connection has an invertible introduction rule on one side of the $\vdash$, and every occurrence of the corresponding introduction rule on the other side is non-invertible. (This last statement is vacuously true for t and f since they have zero introduction rules on the left and right, respectively.) Observing the invertibility of introduction rules allows us to give some structure to the permutation of inference rules. In particular, an invertible rule above any other rule can always be permuted down. Furthermore, two non-invertible rules, one above the other, can always be permuted as well.

We make one additional observation: if an occurrence of a non-atomic formula on the left or right of a sequent can be the consequence of an invertible rule, that formula occurrence never needs to have a structural rule applied to it. For example, the contraction-left rule never needs to be applied to a disjunction since the disjunction-left rule is invertible.

These three observations about invertible and non-invertible rules—the left-right duality regarding invertibility; the permutations involving invertible and non-invertible rules; and the connection between invertible rules and the structural rules—will all be made explicit of the design of the **LKF** proof system.

$$\frac{\Gamma, B_1, B_2 \vdash \Delta}{\Gamma, B_1 \wedge B_2 \vdash \Delta}\ \wedge L \qquad \frac{\Gamma \vdash \Delta}{\Gamma, t \vdash \Delta}\ tL \qquad \frac{\Gamma \vdash \Delta, B \quad \Gamma' \vdash \Delta', C}{\Gamma, \Gamma' \vdash \Delta, \Delta', B \wedge C}\ \wedge R \qquad \frac{}{\cdot \vdash t}\ tR$$

$$\frac{\Gamma, B \vdash \Delta \quad \Gamma', C \vdash \Delta'}{\Gamma, \Gamma' B \vee C \vdash \Delta, \Delta'}\ \vee L \qquad \frac{}{f \vdash \cdot}\ fL \qquad \frac{\Gamma \vdash \Delta, B_1, B_2}{\Gamma \vdash \Delta, B_1 \vee B_2}\ \vee R \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, f}\ fR$$

Fig. 2 The introduction rules for conjunction, disjunction, and their units using multiplicative instead of additive rules.

2.3 Additive and multiplicative rules and connectives

The **LK** rules that have two premises can be classified as either *additive*, in which case the side formulas (Γ, Δ) are the same in the conclusion as well as in both premises, or *multiplicative*, in which case the side formulas in the premises (Γ, Δ and Γ', Δ') are accumulated to form the side formulas in the conclusion. Of the four inference rules in Figure 1 with two premises, the *cut* rule and the implication-left rule are multiplicative while the disjunction-left rule and the conjunction-right rule are additive.

Consider the alternative inference rules in Figure 2 for conjunction and disjunction. The rules in that figure with two premises are multiplicative. We can make the following observations.

1. The rules above are inter-admissible with those for the same connectives given in Figure 1. Establishing that fact requires using the structural rules of weakening and contraction.
2. The $\wedge R$ rule in Figure 1 is invertible while the corresponding $\wedge R$ rule in Figure 2 is not invertible. Similarly, the $R\vee$ rule in Figure 1 is not invertible while the corresponding $R\vee$ rule in Figure 2 is invertible.
3. If we are keen to separate the roles of structural rules from cut-elimination, then we should not mix the various rules in Figures 1 and 2. For example, if we replace the $L\wedge$ in Figure 1 with the $L\wedge$ in Figure 2, then the proof that a cut of a conjunction can be eliminated will necessarily use a structural rule.

Although Gentzen used the additive rules for conjunction and disjunction, there are reasons to admit other choices. For example, it is a popular choice to select invertible right introduction rules for both conjunction and disjunction, which means selecting the additive conjunction and the multiplicative disjunction. Ketonen introduced such a variant of Gentzen's original calculus and used it to give "a strikingly elegant proof of completeness" (von Plato 2012). People working in automated theorem proving often use the invertible rules since it simplifies implementations of proof search. In particular, it is possible to define one-side sequent systems for classical logic in such a way that all (right) introduction rules are invertible except for the existential introduction rule. As a result, proof search algorithms can limit backtracking to only the treatment of existential quantifiers.

The **LKF** proof system contains both the additive and multiplicative versions of conjunction and disjunction (and their units).

2.4 The need for synthetic inference rules

Our final criticism of **LK** is that its inference rules are too small, especially for applications involving theories. For example, assume that we are working with a theory (a set of assumptions) that has an axiom that declares that the binary predicate *path* is transitive: that is, that the theory contains the formula

$$\forall x \forall y \forall z\ (path(x, y) \supset path(y, z) \supset path(x, z))$$

If that formula is invoked in an **LK** proof, there will be a minimal of five introduction rules involved in that invocation. That seems unfortunate since it is more natural to view that formula as denoting one of the following inference rules.

$$\frac{\Gamma \vdash \Delta,\ path(x, y) \quad \Gamma \vdash \Delta,\ path(y, z)}{\Gamma \vdash \Delta, path(x, z)} \ \text{ or } \ \frac{path(x, y),\ path(y, z),\ path(x, z), \Gamma \vdash \Delta}{path(x, y),\ path(y, z), \Gamma \vdash \Delta}\ .$$

These *synthetic rules* would be a more appropriate way to invoke the transitivity axiom. Such synthetic rules have been addressed before in the literature, particularly as a back-chaining inference rule (Hallnäs and Schroeder-Heister 1990, Miller et al. 1991) or as a forward-chaining inference rule (Negri and von Plato 1998). One of the immediate applications of **LKF** is as a formal framework for computing and justifying the addition of such synthetic inference rules to **LK**.

3 The LKF proof system

The **LKF** proof system does not deal with formulas but with *polarized formulas*: these are built from atomic formulas and negated atomic formulas (collectively called literals), and *polarized logical connectives* as well as the first-order quantifiers $\forall$ and $\exists$. The polarized logical connectives come in two flavors: the *positive connectives* are f^+, $\vee^+$, t^+, $\wedge^+$, and $\exists$ while the *negative connectives* are t^-, $\wedge^-$, f^-, $\vee^-$, and $\forall$.

Literals are also assigned a polarity as follows. An *atomic bias assignment* is a function $\delta(\cdot)$ that maps atomic formulas to the set of two tokens $\{+, -\}$: if $\delta(A)$ is $+$ then that atomic formula is positive and if $\delta(A)$ is $-$ then that atomic formula is negative. We extend $\delta(\cdot)$ to literals by setting $\delta(\neg A)$ to be the opposite polarity of $\delta(A)$. We may ask that all atomic formulas are positive, that they are all negative, or we can mix polarity assignments. In particular, the atomic bias assignment $\delta^+(\cdot)$ assigns all atoms the positive polarity while $\delta^-(\cdot)$ assigns all atoms the negative polarity. We shall often suppress explicit reference to atomic bias assignments, assuming that they have been specified and fixed. The only restriction we impose on atomic bias assignments is that they are stable under substitution: that is, for all atomic formulas A and all first-order substitutions, $\delta(A) = \delta(\theta A)$. This restriction is equivalent to saying that the value of $\delta(\cdot)$ is determined by the predicate that is the top-level symbol of A: that is, if A and A' are to atoms formed with the same predicate, then $\delta(A) = \delta(A')$.

A polarized formula is *positive* if it is a positive literal or has a top-level positive connective; similarly, a formula is *negative* it is a negative literal or has a top-level negative connective.

Polarized formulas are in *negation normal form* (nnf), meaning that there is no occurrences of implication $\supset$ and that the negation symbol $\neg$ has only atomic scope. When the negation symbol $\neg$ is used with the non-atomic polarized formulas of **LKF**, we shall view it as the following function that transforms that polarized formula to its De Morgan dual.

Definition 1 The negation symbol $\neg$ is defined as the following function when applied to non-atomic polarized formulas.

- $\neg\neg a = a$ for atomic formula a
- $\neg(A \wedge^+ B) = \neg A \vee^- \neg B, \quad \neg(A \vee^- B) = \neg A \wedge^+ \neg B$
- $\neg(A \vee^+ B) = \neg A \wedge^- \neg B, \quad \neg(A \wedge^- B) = \neg A \vee^+ \neg B$
- $\neg\exists x.A = \forall x.\neg A, \quad \neg\forall x.A = \exists x.\neg A$

Clearly, negation is treated differently between unpolarized formulas (where it is an abbreviation for "implies false") and unpolarized formulas (where it computes the De Morgan dual).

The sequent calculus **LKF** is presented in Figure 3: this presentation is a simplification of our original presentation given in (Liang and Miller 2009). This proof system uses one-sided sequents, but of two varieties, namely, $\vdash \Gamma \Uparrow \Theta$ and $\vdash A \Downarrow \Theta$, where Γ is a multiset of formulas, Θ is a set of formulas, and A is a single formula. In notation such as $\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'$, the multiset Γ, Γ' represents the multiset sum of Γ and Γ' while the set Θ, Θ' represents the union of the two sets Θ and Θ': it is, of course, possible for Θ and Θ' to share a non-empty intersection. When moving a collection of formulas from the left to the right of $\Uparrow$, we coerce multisets into sets in the obvious way. Note that by inspection, the set of formulas on the right of the double arrows ($\Uparrow$ and $\Downarrow$) in the conclusion of an inference rule is always a subset of formulas on the right of the double arrows in the premises. We say that the polarized formula B has an **LKF** proof if the sequent $\vdash B \Uparrow \cdot$ has a proof using the inference rules from Figure 3.

We borrow the terminology *asynchronous* and *synchronous* rules from Andreoli (1992). A derivation composed only of asynchronous rules and the *store* rule will be called an *asynchronous phase*, and a derivation composed only of synchronous rules and the *init* rule will be called a *synchronous phase*. The sequents in an asynchronous phase all involve $\Uparrow$-sequents while the sequents in a synchronous phase all involve $\Downarrow$-sequents. An **LKF** proof is composed of alternations of these two kinds of phases. In particular, the *decide* rule connects a synchronous phase above its premise with an asynchronous phase below its conclusion, and the *release* rule connects an asynchronous phase above its premise with a synchronous phase below its conclusion.

The asynchronous phase can be used to encapsulate what is often called *don't care nondeterminism*. That is, if we consider the asynchronous phase as a large scale

Asynchronous introduction rules

$$\frac{}{\vdash t^-, \Gamma \Uparrow \Theta}\ t^- \qquad \frac{\vdash A, \Gamma \Uparrow \Theta \quad \vdash B, \Gamma \Uparrow \Theta}{\vdash A \wedge^- B, \Gamma \Uparrow \Theta}\ \wedge^- \qquad \frac{\vdash \Gamma \Uparrow \Theta}{\vdash f^-, \Gamma \Uparrow \Theta}\ f^- \qquad \frac{\vdash A, B, \Gamma \Uparrow \Theta}{\vdash A \vee^- B, \Gamma \Uparrow \Theta}\ \vee^-$$

$$\frac{\vdash [y/x]B, \Gamma \Uparrow \Theta}{\vdash \forall x.B, \Gamma \Uparrow \Theta}\ \forall$$

Synchronous introduction rules

$$\frac{}{\vdash t^+ \Downarrow \Theta}\ t^+ \qquad \frac{\vdash A \Downarrow \Theta \quad \vdash B \Downarrow \Theta}{\vdash A \wedge^+ B \Downarrow \Theta}\ \wedge^+ \qquad \frac{\vdash B_i \Downarrow \Theta}{\vdash B_1 \vee^+ B_2 \Downarrow \Theta}\ \vee^+,\ i \in \{1, 2\} \qquad \frac{\vdash [t/x]B \Downarrow \Theta}{\vdash \exists x.B \Downarrow \Theta}\ \exists$$

Initial, store, release, and decide rules

$$\frac{}{\vdash p \Downarrow \neg p, \Theta}\ \mathit{init} \qquad \frac{\vdash \Gamma \Uparrow Q, \Theta}{\vdash Q, \Gamma \Uparrow \Theta}\ \mathit{store} \qquad \frac{\vdash N \Uparrow \Theta}{\vdash N \Downarrow \Theta}\ \mathit{release} \qquad \frac{\vdash P \Downarrow P, \Theta}{\vdash \cdot \Uparrow P, \Theta}\ \mathit{decide}$$

Fig. 3 The inference rules for **LKF**. Here, P is a positive formula and p is a positive literal; N is a negative formula and Q is a positive formula or negative literal. The rule for $\forall$ has the usual eigenvariable restriction: y is not free in any formula in the concluding sequent.

inference rule having a sequent of the form $\vdash N \Uparrow \Theta$ as its conclusion and sequents of the form $\vdash \cdot \Uparrow \Theta'$ as its premises, then that large scale rule is independent of the sequence of rule applications within the asynchronous phase (see Lemma 2). On the other hand, the synchronous phase is a sequence of applications of inference rules with choices (particularly for the $\vee^+$ and $\exists$ introduction rules), and different choices will yield different synchronous phases: such phases, therefore, capture *don't know nondeterminism*.

While the weakening and contraction rules are not explicitly given in **LKF**, both of these rules occur implicitly. The *decide* rule does an implicit contraction on the formula P: hence, the only formulas contracted in an **LKF** proof are positive formulas. The *init* and the t^+ rules do implicit weakening on the formulas in Θ: thus weakening is available for positive formulas and negative literals. Thus, a negative, non-literal formula is never weakened nor contracted: in that sense, such formulas are treated *linearly*, in the sense of linear logic (Girard 1987).

The four binary logical connectives of **LKF**—$\vee^+$, $\vee^-$, $\wedge^+$, $\wedge^-$—can be classified using three different attributes: positive or negative; additive or multiplicative; and conjunctive or disjunctive. By fixing any two of these attributes, the third attribute is uniquely determined. For example, a connective that is both additive and positive must be the disjunction $\vee^+$. Note also that the De Morgan dual of a logical connective flips its polarity and conjunctive/disjunctive status but does not change its additive/multiplicative status.

The proof system for **LKF** given in Figure 3 has no cut rule; thus the proofs built using the rules in Figure 3 are cut-free proofs. Cut-rules for **LKF** and a cut-elimination theorem will be presented in the next section.

Let B be a polarized formula and let $\check{B}$ be the *depolarized* version of B: that is, $\check{B}$ is the unpolarized formula that results from removing the superscript + and − from the logical connectives in B. Since B is in negation normal form, the formula $\check{B}$ will have occurrences of negations but might have implications of the form $A \supset f$ for atomic A. Depolarizing a multiset or set of polarized formulas Γ is the set $\check{\Gamma}$ resulting from depolarized the formulas in Γ.

Theorem 1 (Soundness of LKF) *Let B be a polarized formula and let Γ and Θ be a multiset and set, respectively, of polarized formulas. If $\vdash \Gamma \Uparrow \Theta$ is provable in **LKF** then $\vdash \check{\Gamma}, \check{\Theta}$ is provable in **LK**. If $\vdash B \Downarrow \Theta$ is provable in **LKF** then $\vdash \check{B}, \check{\Theta}$ is provable in **LK**.*

Proof This theorem can be proved by a straightforward mutual induction on the structure of (cut-free) **LKF** proofs. Most cases of this mutual induction are straightforward. For example, the introduction rule for $\vee^+$ in **LKF** corresponds to the introduction rule for $\vee$ in **LK**, while the introduction rule for $\vee^-$ in **LKF** corresponds to the multiplicative version of the introduction rule for $\vee$ in Figure 2. The *init* rule in **LKF** corresponds, however, to the following **LK** derivation.

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\ init}{p \vdash p, f, \check{\Theta}}\ wR}{\vdash p,\ p \supset f,\ \check{\Theta}}\ \supset R$$

Finally, *decide* in **LKF** corresponds to the *cR* rule, and *store* and *release* do not contribute to the **LK** proof. □

The converse of this soundness theorem is more challenging to prove: we shall state and prove such completeness as Theorem 8 in Section 8. In anticipation of that result, we state a version of that completeness theorem here. Let B be a first-order polarized formula and let $\delta(\cdot)$ be any atomic bias assignment and let C be the unpolarized formula $\check{B}$. If C is provable in **LK** then B is provable in **LKF**. A consequence of this completeness theorem is the following: if let C be an unpolarized formula is provable in **LK**, then for every polarized formula B (and atomic bias assignment) such that $\check{B}$ is C, then B has an **LKF** proof. Note that if there are n occurrences of propositional connectives in C, there are 2^n formulas B such that $\check{B} = C$. Clearly, polarization does not affect provability, but it can have a large impact on the structure of (focused) proofs.

We now state two properties about (cut-free) **LKF** proofs.

Lemma 1 (Admissibility of Weakening) *If $\vdash \Gamma \Uparrow \Theta$ and $\vdash A \Downarrow \Theta$ are (cut-free) provable and if Θ' is a set of positive formulas and negative literals then $\vdash \Gamma \Uparrow \Theta, \Theta'$ and $\vdash A \Downarrow \Theta, \Theta'$ are also provable.*

This lemma is proved easily by induction on the structure of proofs. The proof further shows that weakening also does not affect the structure of proofs in that the same inference rules are applied at each step.

The following lemma captures the fact that the asynchronous phase of inference rules can deal with don't-care-nondeterminism: any formula to the left of the $\Uparrow$ can be selected to be processed first.

Lemma 2 *If there is a (cut-free) proof of* $\vdash A, \Gamma \Uparrow \Theta$ *then there is a (cut-free) proof that ends with either an introduction of A or a store rule on A.*

Proof This lemma holds because the asynchronous introduction rules permuted over each other in such a way that the same premises remain. The formal proof of this lemma is by induction on the sum of the sizes of formulas in Γ. The size of a formula is the number of occurrences of literals, connectives, and quantifiers in the formula. In particular, A and $\neg A$ are of the same size. In the base case, Γ is empty, and the result is trivial. For the inductive case, let $\Gamma = B, \Gamma'$ and assume that the sequent $\vdash A, B, \Gamma' \Uparrow \Theta$ is the conclusion of an inference rule ρ which is either an introduction or *store* on B. We then proceed to show that the ρ rule can be permuted above the introduction or *store* of A. There are several cases to consider.

Case: A and B are both either positive formulas or negative literals. In this case, ρ is *store* on B with premise $\vdash A, \Gamma' \Uparrow \Theta, B$. By inductive hypothesis on the smaller Γ', the next rule above must be a *store* on A, with premise $\vdash \Gamma' \Uparrow \Theta, A, B$. But clearly we can switch the order of the two *store* rules:

$$\cfrac{\cfrac{\vdash \Gamma' \Uparrow \Theta, A, B}{\vdash B, \Gamma' \Uparrow \Theta, A}\ \mathit{store}}{\vdash A, B, \Gamma' \Uparrow \Theta}\ \mathit{store}$$

Case: A is a positive formula or negative literal and B is a non-literal negative formula. In this case, we consider the structure of B. For example, if B is $B_1 \vee^- B_2$, then the premise of ρ is $\vdash A, B_1, B_2, \Gamma' \Uparrow \Theta$. Since the size of B_1, B_2, Γ' is smaller than the size of $B_1 \vee^- B_2, \Gamma'$, the inductive hypothesis provides a proof where the rule above ρ is the *store* rule applied to A with premise $\vdash B_1, B_2, \Gamma' \Uparrow \Theta, A$. Starting from that sequent, we can switch the *store* and $\vee^-$ rules, resulting in

$$\cfrac{\cfrac{\vdash B_1, B_2, \Gamma' \Uparrow \Theta, A}{\vdash B_1 \vee^- B_2, \Gamma' \Uparrow \Theta, A}\ \vee^-}{\vdash A, B_1 \vee^- B_2, \Gamma' \Uparrow \Theta}\ \mathit{store}$$

The cases of B is t^-, $B_1 \wedge^- B_2$, $\forall x.B'$ and f^- are similar.

Case: B is a positive formula or negative literal and A is a non-literal negative formula. This case is analogous to the above case. We illustrate with the case that A is $A_1 \wedge^- A_2$. Since the ρ rule is *store* on B, its premise is $\vdash A_1 \wedge^- A_2, \Gamma' \Uparrow \Theta, B$. By the inductive hypothesis, the next rule above is the introduction for $\wedge^-$:

$$\cfrac{\cfrac{\vdash A_1, \Gamma' \Uparrow \Theta, B \quad \vdash A_2, \Gamma' \Uparrow \Theta, B}{\vdash A_1 \wedge^- A_2, \Gamma' \Uparrow \Theta, B}\ \wedge^-}{\vdash A_1 \wedge^- A_2, B, \Gamma' \Uparrow \Theta}\ \mathit{store}$$

These rules can be permuted to yield the desired form

$$\dfrac{\dfrac{\vdash A_1, \Gamma' \Uparrow \Theta, B}{\vdash A_1, B, \Gamma' \Uparrow \Theta}\ \mathit{store} \quad \dfrac{\vdash A_2, \Gamma' \Uparrow \Theta, B}{\vdash A_2, B, \Gamma' \Uparrow \Theta}\ \mathit{store}}{\vdash A_1 \wedge^- A_2, B, \Gamma' \Uparrow \Theta}\ \wedge^-$$

Case: A and B are both non-literal negative formulas. There are several cases to consider, but they are all similar. For example, if A and B are $A_1 \vee^- A_2$ and $B = B_1 \vee^- B_2$, respectively, and the last rule introduces B, we just need to show that the two $\vee^-$-introductions permute over each other, which follows easily from the fact that both proofs can be constructed from the common premise of $\vdash A_1, A_2, B_1, B_2, \Gamma' \Uparrow \Theta$. In the case where A is $A_1 \vee^- A_2$ and B is $B_1 \wedge^- B_2$, introducing $B_1 \wedge^- B_2$ results in the premises $\vdash A_1 \vee^- A_2, B_1, \Gamma' \Uparrow \Theta$ and $\vdash A_1 \vee^- A_2, B_2, \Gamma' \Uparrow \Theta$, both of which have a smaller inductive measure, which allows us to assume that the next rule above will introduce $A_1 \vee^- A_2$ and we can therefore build the proof

$$\dfrac{\dfrac{\vdash A_1, A_2, B_1, \Gamma' \Uparrow \Theta \quad \vdash A_1, A_2, B_2, \Gamma' \Uparrow \Theta}{\vdash A_1, A_2, B_1 \wedge^- B_2, \Gamma' \Uparrow \Theta}\ \wedge^-}{\vdash A_1 \vee^- A_2, B_1 \wedge^- B_2, \Gamma' \Uparrow \Theta}\ \vee^-$$

The remaining cases are treated in a similar fashion. □

Definition 2 We say that a (cut-free) proof of $\vdash A, \Gamma \Uparrow \Theta$ is *eager* with respect to A if the last inference rule introduces A or is a *store* rule on A. We say that the proof is *delayed* with respect to A if either

1. Γ is empty, or
2. the last inference rule does not introduce A, is not a *store* rule on A, and each immediate subproof above $\vdash A, \Gamma \Uparrow \Theta$ is also delayed with respect to A.

In other words, a proof is delayed with respect to A if A is only subject to an introduction or *store* rule on A when it appears in a conclusion of the form $\vdash A \Uparrow \Theta$. Note also that a proof of $\vdash A \Uparrow \Theta$ is both eager and delayed with respect to A.

Lemma 2 implies that a proof can be transformed into either the eager or the delayed form.

4 Cut Elimination for LKF

Given that **LKF** has two kinds of sequents and each of these has two zones for holding formulas, we introduce in Figure 4 a total of four cut rules in order to state and prove the cut-elimination theorem for **LKF**. The cut_u rule (called the *unfocused* cut rule) applies only to $\Uparrow$-sequents while the cut_f rule (called the *focused* cut rule) involves one $\Downarrow$-sequent. Both of those cut rules also have a "delayed" version in which one of the occurrences of the cut formula is "locked" on the right of a double arrow.

$$\frac{\vdash A, \Gamma \Uparrow \Theta \quad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ cut_u \qquad \frac{\vdash A \Downarrow \Theta \quad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma' \Uparrow \Theta, \Theta'}\ cut_f$$

$$\frac{\vdash \Gamma \Uparrow \Theta, P \quad \vdash \neg P, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u \qquad \frac{\vdash B \Downarrow \Theta, P \quad \vdash \neg P \Uparrow \Theta'}{\vdash B \Downarrow \Theta, \Theta'}\ dcut_f$$

Fig. 4 The Cut Rules of LKF. Here, A and B are arbitrary polarized formulas and P is a positive polarized formulas.

It is important to note that in the delayed cuts, the cut formula P is positive and not a negative literal: in particular, if P were a negative literal in the $dcut_f$ rule and if $B = \neg P$ then $dcut_f$ is not admissible since focusing on a positive literal requires the proof to end in an initial rule.

A simple observation shows that the cut-rules in Figure 4 do not suffer the collision problems mentioned in Section 2.1. As we noted in the previous section, only positive formulas are contracted (by the *decide* rule) in **LKF** proofs: as a result, exactly one of the pair of formulas A and $\neg A$ involved in a cut rule will be positive, and only one of them can be contracted. Similarly, weakening only appears within the *init* rule in **LKF** proofs and, as a result, the problematic case involving weakening also disappears.

The general strategy for proving cut-elimination in **LKF** extended with these cut rules is familiar: we reduce cuts to “key cases” in which the cut formula is principal in both premises. The proof proceeds by simultaneous induction over the permutabilities of all four cuts. The inductive measure is the lexicographical ordering consisting of the size of the cut formula followed by the sum of the heights of the subproofs above the cut. We apply the procedure to the topmost cuts first, thus assuming that the cuts to be reduced have cut-free subproofs.

Lemma 2 is used to simplify the cut-elimination proof. However, the application of this lemma for proof transformation may affect the height of proofs (because of the t^- rule). These transformations must be applied carefully in order to preserve the inductive measure. For the cut-elimination proof, we further require that the following conditions be placed on the cut rules.

1. In cut_u, the subproof of the premise with the positive cut formula must be *eager* with respect to the cut formula; the subproof of the premise with the negative cut formula must be *delayed* with respect to the cut formula.
2. In $dcut_u$, the subproof of the premise with the negative cut formula must be *delayed* with respect to the cut formula.
3. In cut_f, the subproof of the sequent $\vdash \neg A, \Gamma' \Uparrow \Theta'$, where $\neg A$ is the cut formula, must be *eager* with respect to $\neg A$ regardless of the polarity of A.

The third requirement may appear inconsistent with the others when $\neg A$ is negative in cut_f: however, the transition from cut_u or $dcut_u$ to a cut_f only occurs when the cut formula is decomposed into subformulas, which reduces the stronger inductive measure. For the $dcut_f$ rule, the subproof above the negative cut formula $\neg P$ can be considered both eager and delayed with respect to $\neg P$ because it is the only formula to the left of $\Uparrow$. By Lemma 2, any proof can be transformed into the required forms so that the reducibility of the restricted cuts also implies the reducibility of the unrestricted versions. In other words, before the application of any cut, we can always apply Lemma 2 to assume that the subproofs are in the required forms. The cut elimination arguments will show that all restrictions are preserved when any of the four cut rules are permuted to other cut rules.

We detail the permutation of each of the four cuts. We sometimes do not repeat cases that are obvious, and we generally ignore the quantifiers as the first-order quantifiers add nothing to the argument: their treatment is completely standard.

4.1 Permutations of cut_u

The cut_u rule has the general form, repeated here for convenience:

$$\frac{\vdash A, \Gamma \Uparrow \Theta \qquad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ cut_u$$

Assume without loss of generality that A is positive and, therefore, $\neg A$ is negative. It is also required that the left subproof above cut_u is *eager* with respect to the positive A, i.e., it ends in a *store* rule on the cut formula A. Furthermore, the right subproof above the negative cut formula $\neg A$, is required to be *delayed* with respect to $\neg A$. These assumptions mean that this cut can be transformed immediately into a $dcut_u$:

$$\frac{\dfrac{\vdash \Gamma \Uparrow \Theta, A}{\vdash A, \Gamma \Uparrow \Theta}\ store \qquad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ cut_u \quad \longrightarrow \quad \frac{\vdash \Gamma \Uparrow \Theta, A \qquad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u$$

Clearly the restriction on the *delayed* form of the subproof above the negative cut formula $\neg A$ is preserved for the $dcut_u$ rule. The inductive measure is reduced by the smaller height of the left subproofs above the cut.

4.2 Permutations of $dcut_u$

The delayed, unfocused $dcut_u$ rule has the form

$$\frac{\vdash \Gamma \Uparrow \Theta, P \qquad \vdash \neg P, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u$$

where the cut formula P is positive. It is required that the subproof above the right premise is *delayed* with respect to the cut formula $\neg P$. These cuts are permuted to the point where P is selected for focus, at which point the cut transforms into a combination of cut_f and $dcut_f$. In other words, the "goal" or "target" of all permutations of $dcut_u$ is to be able to apply the following transformation when the left premise of the $dcut_u$ is the *decide* rule.

$$\dfrac{\dfrac{\vdash P \Downarrow \Theta, P}{\vdash \cdot \Uparrow \Theta, P} \quad \vdash \neg P \Uparrow \Theta'}{\vdash \cdot \Uparrow \Theta, \Theta'}\ dcut_u \quad \longrightarrow \quad \dfrac{\dfrac{\vdash P \Downarrow \Theta, P \quad \vdash \neg P \Uparrow \Theta'}{\vdash P \Downarrow \Theta, \Theta'}\ dcut_f \quad \vdash \neg P \Uparrow \Theta'}{\vdash \cdot \Uparrow \Theta, \Theta'}\ cut_f$$

In the transformed proof, the upper $dcut_f$ has subproofs of lesser height measure, while the lower cut_u is a *key case* cut where the cut formula is principal in both subproofs. That is, cut-free proofs for $\vdash P \Downarrow \Theta, \Theta'$ and $\vdash \neg P \Uparrow \Theta'$ must both end with the cut formulas P and $\neg P$ subject to an inference rule. The key-case cuts immediately decompose into cuts on subformulas of a smaller size than P (or reduces completely by weakening in case of P being a positive literal). Thus, the inductive measure of both cuts is reduced.

Note that the *eager* restriction on the right subproof above cut_f is trivially preserved since $\neg P$ is the only formula on the left of $\Uparrow$.

All other permutations of $dcut_u$ make progress toward this case. We organize these permutations into two stages.

The first stage performs permutations over inference rules in the right subproof of $dcut_u$. The right subproof above $dcut_u$ ends in $\vdash \neg P, \Gamma' \Uparrow \Theta'$. We permute $dcut_f$ until it has such a right subproof with an empty Γ'. The fact that this subproof is *delayed* with respect to $\neg P$ means that if it ends in a conclusion $\vdash \neg P, B, \Gamma' \Uparrow \Theta'$ we can assume that the last rule either introduces B or is a *store* rule on B (and not on $\neg P$). There are many subcases depending on the form of B:

Case: B is a positive formula or negative literal. In this case, the rule above in a *store* on B, resulting in the following permutation.

$$\dfrac{\vdash \Gamma \Uparrow \Theta, P \quad \dfrac{\vdash \neg P, \Gamma' \Uparrow \Theta', B}{\vdash \neg P, B, \Gamma' \Uparrow \Theta'}\ store}{\vdash B, \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u \quad \longrightarrow \quad \dfrac{\dfrac{\vdash \Gamma \Uparrow \Theta, P \quad \vdash \neg P, \Gamma' \Uparrow \Theta', B}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta', B}\ dcut_u}{\vdash B, \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ store$$

The "delayed" restriction on the right subproof above $dcut_u$ is preserved by definition: an immediate subproof of a delayed proof is also delayed. This property applies similarly to all subsequent cases.

Case: B is $B_1 \vee^- B_2$. In this case, we can transform

$$\dfrac{\vdash \Gamma \Uparrow \Theta, P \quad \dfrac{\vdash \neg P, B_1, B_2, \Gamma' \Uparrow \Theta'}{\vdash \neg P, B_1 \vee^- B_2, \Gamma' \Uparrow \Theta'}\ \vee^-}{\vdash B_1 \vee^- B_2, \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u$$

into the following derivation.

$$\dfrac{\dfrac{\vdash \Gamma \Uparrow \Theta, P \quad \vdash \neg P, B_1, B_2, \Gamma' \Uparrow \Theta'}{\vdash B_1, B_2, \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u}{\vdash B_1 \vee^- B_2, \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ \vee^-$$

Case: B is $B_1 \wedge^- B_2$. In this case, we can transform

$$\dfrac{\vdash \Gamma \Uparrow \Theta, P \qquad \dfrac{\vdash \neg P, B_1, \Gamma' \Uparrow \Theta' \quad \vdash \neg P, B_1, \Gamma' \Uparrow \Theta'}{\vdash \neg P, B_1 \wedge^- B_2, \Gamma' \Uparrow \Theta'}\ \wedge^-}{\vdash B_1 \wedge^- B_2, \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u$$

into the following derivation.

$$\dfrac{\dfrac{\vdash \Gamma \Uparrow \Theta, P \quad \vdash \neg P, B_1, \Gamma' \Uparrow \Theta'}{\vdash B_1, \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u \quad \dfrac{\vdash \Gamma \Uparrow \Theta, P \quad \vdash \neg P, B_2, \Gamma' \Uparrow \Theta'}{\vdash B_2, \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u}{\vdash B_1 \wedge^- B_2, \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ \wedge^-$$

The other cases for B are proved similarly. This stage ends when the right subproof concludes with a sequent of the form $\vdash \neg P \Uparrow \Theta'$.

The second stage performs permutation over inference rules in the left subproof of $dcut_u$. The cases of asynchronous introduction rules are analogous to the cases demonstrated above and are equally straightforward. Generally speaking, the permutation of cut above introduction rules is always straightforward. The important cases to point out are the *decide*, *release*, and *store* rules. A *store* rule ending the left subproof is also a trivial case because it cannot affect the cut formula. The interesting case is when the left subproof ends in the form $\vdash \cdot \Uparrow \Theta, P$. The rule above this sequent must be *decide*. There are two cases depending on whether or not the formula selected for focus is the cut formula P or not. If it is not the cut formula but, say, another formula Q, then we can permute inference rules as follow.

$$\dfrac{\dfrac{\vdash Q \Downarrow Q, \Theta, P}{\vdash \cdot \Uparrow Q, \Theta, P}\ decide \quad \vdash \neg P \Uparrow \Theta'}{\vdash \cdot \Uparrow Q, \Theta, \Theta'}\ dcut_u \quad \longrightarrow \quad \dfrac{\dfrac{\vdash Q \Downarrow Q, \Theta, P \quad \vdash \neg P \Uparrow \Theta'}{\vdash Q \Downarrow Q, \Theta, \Theta'}\ dcut_f}{\vdash \cdot \Uparrow Q, \Theta, \Theta'}\ decide$$

If the formula selected for focus is P, then we have reached the targeted transition to key-case cuts as already described above.

4.3 Permutations of $dcut_f$

The general form of $dcut_f$ is

$$\dfrac{\vdash B \Downarrow \Theta, P \qquad \vdash \neg P \Uparrow \Theta'}{\vdash B \Downarrow \Theta, \Theta'}\ dcut_f$$

with P positive. This cut permutes over synchronous introduction rules until reaching an *init* or *release* rule on its left premise, at which point the cut will transition to a $dcut_u$ with lower subproofs:

$$\frac{\dfrac{\vdash B \Uparrow \Theta, P}{\vdash B \Downarrow \Theta, P}\ release \qquad \vdash \neg P \Uparrow \Theta'}{\vdash B \Downarrow \Theta, \Theta'}\ dcut_f \quad \longrightarrow \quad \dfrac{\dfrac{\vdash B \Uparrow \Theta, P \qquad \vdash \neg P \Uparrow \Theta'}{\vdash B \Uparrow \Theta, \Theta'}\ dcut_u}{\vdash B \Downarrow \Theta, \Theta'}\ release$$

Besides the cases of initial rules, all other permutations of $dcut_f$ make progress towards this case. Since $\neg P$ is the only formula to the left of $\Uparrow$, the "delayed" requirement of $dcut_u$ is trivially met. The right-side subproof with the negative cut formula stays intact during these permutations. We consider two cases where B is a positive formula: the other cases are treated similarly. If B is a positive literal, then $\vdash B \Downarrow \Theta, P$ must be the conclusion of an initial rule. Since P is also positive, it must be the case that $B \in \Theta$. Thus $\vdash b \Downarrow \Theta, \Theta'$ is also the conclusion of an initial rule. If B is $B_1 \vee^+ B_2$, then we have the following transformation (here, i is 1 or 2):

$$\frac{\dfrac{\vdash B_i \Downarrow \Theta, P}{\vdash B_1 \vee^+ B_2 \Downarrow \Theta, P}\ \vee^+ \qquad \vdash \neg P \Uparrow \Theta'}{\vdash B_1 \vee^+ B_2 \Downarrow \Theta, \Theta'}\ dcut_f \quad \longrightarrow \quad \dfrac{\dfrac{\vdash B_i \Downarrow \Theta, P \qquad \vdash \neg P \Uparrow \Theta'}{\vdash B_i \Downarrow \Theta, \Theta'}\ dcut_f}{\vdash B_1 \vee^+ B_2 \Downarrow \Theta, \Theta'}\ \vee^+$$

4.4 Permutations of cut_f

The cut_f rule has the general form

$$\frac{\vdash A \Downarrow \Theta \qquad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma' \Uparrow \Theta, \Theta'}\ cut_f$$

It is required that the subproof above the unfocused sequent $\vdash \neg A, \Gamma' \Uparrow \Theta'$ is *eager* with respect to $\neg A$.

If A is negative, then the left subproof above cut_f must be the conclusion of a *release* rule, and the cut permutes to a cut_u with shorter subproofs:

$$\frac{\vdash A \Uparrow \Theta \qquad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma' \Uparrow \Theta, \Theta'}\ cut_u$$

As for the restrictions on cut_u, $\neg A$ must be positive if A is negative, so the subproof above the positive cut formula stays *eager* with respect to that formula, and the subproof above $\vdash A \Uparrow \Theta$ is trivially *delayed* above the negative cut formula.

If A is positive, then the left subproof above cut_f must be either *init* or an introduction of the cut formula A. We illustrate three cases below: the other cases are similar.

1. If A is a positive literal p then the left premise of cut_f, $\vdash p \Downarrow \Theta,$ is the conclusion of an initial rule with $\neg p \in \Theta$. The other, *eager* subproof of $\vdash \neg p, \Gamma' \Uparrow \Theta'$, must

end in a *store* rule on $\neg p$, with premise $\vdash \Gamma' \Uparrow \Theta', \neg p$. But since $\neg p \in \Theta$, the provability of $\vdash \Gamma' \Uparrow \Theta, \Theta'$ follows from weakening.

2. If A is $A_1 \vee^+ A_2$, then $\neg A$ is $\neg A_1 \wedge^- \neg A_2$. This key case requires transforming the derivation

$$\dfrac{\dfrac{\vdash A_i \Downarrow \Theta}{\vdash A_1 \vee^+ A_2 \Downarrow \Theta}\ \vee^+ \qquad \dfrac{\vdash \neg A_1, \Gamma' \Uparrow \Theta' \quad \vdash \neg A_2, \Gamma' \Uparrow \Theta'}{\vdash \neg A_1 \wedge^- \neg A_2, \Gamma' \Uparrow \Theta'}\ \wedge^-}{\vdash \Gamma' \Uparrow \Theta, \Theta'}\ cut_f$$

into the derivation

$$\dfrac{\vdash A_i \Downarrow \Theta \qquad \vdash \neg A_i, \Gamma' \Uparrow \Theta'}{\vdash \Gamma' \Uparrow \Theta, \Theta'}\ cut_f$$

The inductive measure is reduced by the size of the cut formulas. Here we can apply Lemma 2 to the subproof above $\vdash \neg A_i, \Gamma' \Uparrow \Theta'$ so that it becomes *eager* with respect to (each) $\neg A_i$ without regard to how the transformation might affect the height of proofs, because the lexicographical inductive measure is still reduced. This argument similarly applies to the other key cases.

3. if $A = A_1 \wedge^+ A_2$ then $\neg A = \neg A_1 \vee^- \neg A_2$ and the proof is transformed as follows:

$$\dfrac{\dfrac{\vdash A_1 \Downarrow \Theta \quad \vdash A_2 \Downarrow \Theta}{\vdash A_1 \wedge^+ A_2 \Downarrow \Theta}\ \wedge^+ \qquad \dfrac{\vdash \neg A_1, \neg A_2, \Gamma' \Uparrow \Theta'}{\vdash \neg A_1 \vee^- \neg A_2, \Gamma' \Uparrow \Theta'}\ \vee^-}{\vdash \Gamma' \Uparrow \Theta, \Theta'}\ cut_f$$

$$\downarrow$$

$$\dfrac{\vdash A_2 \Downarrow \Theta \qquad \dfrac{\vdash A_1 \Downarrow \Theta \qquad \vdash \neg A_1, \neg A_2, \Gamma' \Uparrow \Theta'}{\vdash \neg A_2, \Gamma' \Uparrow \Theta, \Theta'}\ cut_f}{\vdash \Gamma' \Uparrow \Theta, \Theta'}\ cut_f$$

The two cuts introduced are both on smaller cut-formulas compared to the original cut: the inductive hypothesis is first applied to the upper cut to obtain a cut-free proof, then to the lower one.

With these permutation results in hand, we can now prove the cut-admissibility theorem for **LKF**.

Theorem 2 *The rules cut_u, cut_f, $dcut_u$ and $dcut_f$ are admissible in **LKF**.*

Proof The formal proof is a nested induction argument: first on the number of cuts in each proof, the second on the lexicographical measure for each cut. The corresponding procedure is: select a top-most cut with cut-free subproofs and apply Lemma 2 so that the subproofs satisfy the requirements concerning the *eager* and *delayed* properties. Then apply the transformations to reduce the cut. Apply this procedure repeated until all cuts are eliminated. □

5 Admissibility of the general *init* rule

The initial rule of **LKF** requires A to be a literal in order to prove the sequent $\vdash A \Downarrow \neg A, \Theta$. Just as important as the admissibility of cut is the admissibility of the more general form of *init*: that is, the sequent $\vdash A, \neg A \Uparrow \Theta$ is provable for every polarized formula A. For unfocused sequent calculus, the proof of this result is straightforward because of the perfect duality between the introduction rules for dual logical connectives. In particular, assuming that A is negative, apply its (invertible) introduction rule followed by the introduction rule for $\neg A$ (reading rules from conclusion to premises). The inductive hypothesis can then be applied directly to the premises. In a focused setting, however, the proof becomes more difficult since multiple asynchronous or synchronous connectives are introduced in a single phase. To solve this problem, we introduce the following relation (also used in (Liang and Miller 2011)).

Definition 3 Let $\uparrow$ be the binary relation between a polarized formula and multisets of polarized formulas defined inductively as follows:

- $A \uparrow \{A\}$ if A is a positive formula or negative literal.
- $f^- \uparrow \{\}$.
- $A \vee^- B \uparrow \Phi, \Phi'$ if $A \uparrow \Phi$ and $B \uparrow \Phi'$.
- $A \wedge^- B \uparrow \Phi$ if $A \uparrow \Phi$ or $B \uparrow \Phi$.
- $\forall x.A \uparrow \Phi$ if $A \uparrow \Phi$.

Clearly each such Φ contains only positive formulas and negative literals. Note that the formulas t^- and $A \vee^- t^-$ are not $\uparrow$-related to any multiset set of polarized formulas.

The following lemmas establish the properties of the asynchronous and synchronous phases in a form that allows us to derive the admissibility of the general *init* rule.

Lemma 3 *For all formulas A, multisets of formulas Γ, and sets of formulas Θ, if $\vdash \Phi, \Gamma \Uparrow \Theta$ is provable for all Φ such that $A \uparrow \Phi$, then $\vdash A, \Gamma \Uparrow \Theta$ is also provable.*

Proof The proof is by induction on the size of A. If a polarized formula A is not $\uparrow$-related to any multiset of polarized formulas then we say that $\uparrow$ is *undefined* for A. Note that if $\uparrow$ is undefined for A then the lemma implies that $\vdash A, \Gamma \Uparrow \Theta$ is provable.

- If A is a positive formula or negative literal, the property is trivial since only $A \uparrow \{A\}$ holds and Φ contains only A.
- If A is the constant f^-, then the property holds by the f^- rule.
- If A is the constant t^-, then $\vdash t^-, \Gamma \Uparrow \Theta$ is provable by the rule for t^-.
- Let A be the formula $B \wedge^- C$. If $\uparrow$ is undefined for A, then it is undefined for B and for C, and the inductive hypothesis states that $\vdash B, \Gamma \Uparrow \Theta$ and $\vdash C, \Gamma \Uparrow \Theta$ are provable. Otherwise, if $\vdash \Phi, \Gamma \Uparrow \Theta$ is provable for all Φ such that $A \uparrow \Phi$, then it is provable for all Φ such that $B \uparrow \Phi$ or $C \uparrow \Phi$. The inductive hypothesis yields the provability of both $\vdash B, \Gamma \Uparrow \Theta$ and $\vdash C, \Gamma \Uparrow \Theta$. In either case, the $\wedge^-$ rule yields a proof of $\vdash B \wedge^- C, \Gamma \Uparrow \Theta$.

- Let A be the formula $B \vee^- C$. Assume that $\vdash \Phi, \Gamma \Uparrow \Theta$ is provable for all Φ such that $B \vee^- C \uparrow \Phi$. This assumption is equivalent to assuming that $\vdash \Phi', \Phi'', \Gamma \Uparrow \Theta$ is provable for all Φ' and Φ'' such that $B \uparrow \Phi'$ and $C \uparrow \Phi''$. Now assume that $B \uparrow \Phi'$ and $C \uparrow \Phi''$ hold. By the above hypothesis, we have $\vdash \Phi', \Phi'', \Gamma \Uparrow \Theta$ is provable. By the inductive hypothesis applied to B, we know that $\vdash B, \Phi'', \Gamma \Uparrow \Theta$ is provable and by the inductive hypothesis applied to C, we know that $\vdash B, C, \Gamma \Uparrow \Theta$ is provable. The $\vee^-$ rule thus yields a proof of $\vdash B \vee^- C, \Gamma \Uparrow \Theta$.
- For A be the formula $\forall x.B$, we assume that x is not free in Γ, Θ. If $A \uparrow \Phi$ then $B \uparrow \Phi$. If $\uparrow$ is undefined for A then it is also undefined for B. In either case the inductive hypothesis states that if $\vdash \Phi, \Gamma \Uparrow \Theta$ is provable for all Φ such that $B \uparrow \Phi$, then $\vdash B, \Gamma \Uparrow \Theta$ is provable. The property is then established by applying the $\forall$ rule.

$\square$

The next lemma connects the synchronous phase with the $\uparrow$-relation.

Lemma 4 *For all polarized formulas A and multisets of polarized formulas Φ, if $A \uparrow \Phi$ then $\vdash \neg A \Downarrow \Phi$ is provable.*

Proof The proof proceeds by induction on the size of A, which is the same as the size of $\neg A$.

- If A is t^-, then the property holds vacuously.
- If A is a negative literal then the property holds by the initial rule *init*.
- If A is f^-, the property holds by the rule for t^+.
- If A is $B \wedge^- C$ then $\neg A$ is $\neg B \vee^+ \neg C$). Assuming that $A \uparrow \Phi$ then either $B \uparrow \Phi$ or $C \uparrow \Phi$. Assume without loss of generality that $B \uparrow \Phi$: by inductive hypothesis $\vdash \neg B \Downarrow \Phi$ is provable. Thus, $\vdash \neg B \vee^+ \neg C \Downarrow \Phi$ is provable using the $\vee^+$ rule.
- If A is $B \vee^- C$ then $\neg A$ is $\neg B \wedge^+ \neg C$). Assuming that $A \uparrow \Phi$ then there are multisets Φ' and Φ'' such that $B \uparrow \Phi'$ and $C \uparrow \Phi''$. By the inductive hypotheses, we know that $\vdash \neg B \Downarrow \Phi$ and $\vdash \neg C \Downarrow \Phi'$ are provable. Apply weakening (Lemma 1) to both sequents and we get that $\vdash \neg B \Downarrow \Phi, \Phi'$ and $\vdash \neg C \Downarrow \Phi, \Phi'$ are provable. Thus $\vdash \neg B \wedge^+ \neg C \Downarrow \Phi, \Phi'$ is provable using the $\wedge^+$ rule.
- If A is $\forall x.B$ the $\neg A$ is $\exists x.\neg B$. If $A \uparrow \Phi$ then $B \uparrow \Phi$. By inductive hypothesis we have $\vdash \neg B \Downarrow \Phi$ and by the $\exists$ rule, we have $\vdash \exists x.\neg B \Downarrow \Phi$.
- If A is a positive formula, then the inductive hypothesis also applies to the proper subformulas of $\neg A$, which is negative and of the same size as A. Thus if $\neg A \uparrow \Phi$ then the cases above show that $\vdash A \Downarrow \Phi$ is provable. By weakening $\vdash A \Downarrow A, \Phi$ is also provable, and we can form the derivation

$$\cfrac{\cfrac{\vdash A \Downarrow A, \Phi}{\vdash \cdot \Uparrow A, \Phi}\ \mathit{decide}}{\vdash \Phi \Uparrow A}\ \mathit{store}$$

is provable where a sequence of *store* rules are applied to the positive formulas and negative literals in Φ. This holds for all Φ such that $\neg A \uparrow \Phi$, so by Lemma 3, $\vdash \neg A \Uparrow A$ is provable, and by applying the *release* rule, we finally have a proof of $\vdash \neg A \Downarrow A$. This establishes the property for positive A for which only $A \uparrow \{A\}$ holds.

□

The following theorem states the admissibility of the general form of the *init* rule.

Theorem 3 $\vdash A, \neg A \Uparrow \cdot$ *is provable for all polarized formulas A.*

Proof Assume without loss of generality that A is positive. Then $A \uparrow \{A\}$ and Lemma 4 states that $\vdash \neg A \Downarrow A$ is provable. Since $\neg A$ is negative, this sequent must be the conclusion of a release rule in a cut-free proof, so $\vdash \neg A \Uparrow A$ is provable. Applying the store rule on A to this sequent gives a proof of $\vdash A, \neg A \Uparrow \cdot$. □

6 Generalized invertibility

The following results about the invertibility of the negative introduction rules is now easily proved using the admissibility of cut. The following corollary is the converse of Lemma 3.

Corollary 1 *If* $\vdash A, \Gamma \Uparrow \Theta$ *is provable and* $A \uparrow \Phi$*, then* $\vdash \Phi, \Gamma \Uparrow \Theta$ *is provable.*

Proof Given the assumption $A \uparrow \Phi$, Lemma 4 implies that the sequent $\vdash \neg A \Downarrow \Phi$ is provable. Using a cut rule, we therefore have the following proof.

$$\dfrac{\dfrac{\vdash A, \Gamma \Uparrow \Theta \quad \vdash \neg A \Downarrow \Phi}{\vdash \Gamma \Uparrow \Theta, \Phi}\ cut_f}{\vdash \Phi, \Gamma \Uparrow \Theta}\ store$$

The final result follows from the admissibility of cut (Theorem 2). □

From the generalized invertibility property and Lemma 3, we can derive the invertibility of the individual asynchronous introduction rules.

Lemma 5 *The introduction rules for the negative connectives are invertible; i.e., the provability of the conclusion of each rule implies the provability of all of its premises.*

Proof First, consider the case for $\vee^-$. Assume that $\vdash A, B, \Gamma \Uparrow \Theta$ is provable and assume that A is $\uparrow$-related to exactly the multisets $\Phi_A^1, \ldots, \Phi_A^n$ and that B is $\uparrow$-related to exactly $\Phi_B^1, \ldots, \Phi_B^m$, where $n, m \geq 0$. By the definition of $\uparrow$, we know that $A \vee^- B \uparrow \Phi_A^i \Phi_B^k$ for each i and k such that $1 \leq i \leq n$ and $1 \leq k \leq m$. (Note that if either n or m is 0 then this statement is vacuously true.) Corollary 1 implies that $\vdash \Phi_A^i \Phi_B^k, \Gamma \Uparrow \Theta$ is provable. By Lemma 3, this means that $\vdash A, B, \Gamma \Uparrow \Theta$ is provable.

To consider the case for $\wedge^-$ assume that $\vdash A \wedge B, \Gamma \Uparrow \Theta$ is provable and (as above) A is $\uparrow$-related to $\Phi_A^1, \ldots, \Phi_A^n$ and B is $\uparrow$-related to $\Phi_B^1, \ldots, \Phi_B^m$, where $n, m \geq 0$. Then $A \wedge B \uparrow \Phi_A^i$ for each i such that $1 \leq i \leq n$ and $A \wedge B \uparrow \Phi_B^k$ for each k such that $1 \leq k \leq m$. By Corollary 1, this implies that $\vdash \Phi_A^i, \Gamma \Uparrow \Theta$ is provable for each i such that $1 \leq i \leq n$ and $\vdash \Phi_B^k, \Gamma \Uparrow \Theta$ is provable for each k such that $1 \leq k \leq m$. By Lemma 3, $\vdash A, \Gamma \Uparrow \Theta$ and $\vdash B, \Gamma \Uparrow \Theta$ are provable.

The cases for t^- and $\forall$ are similar and omitted. □

Given Lemmas 3 and 4, we often use the following *argument schema* to establish the provability of $\vdash A_1, \ldots, A_n, \Gamma \Uparrow \Theta$: If $\uparrow$ is undefined for any A_i then Lemma 3 already shows that the sequent is provable. Otherwise, assume that for each $i \in \{1, \ldots, n\}$ there is an n_i greater than or equal to 1 such that A_i is $\uparrow$-related to exactly $\Phi_i^1, \ldots, \Phi_i^{n_i}$. Show that for each possible selection of $\Phi_1^{k_1}, \ldots, \Phi_n^{k_n}$, the sequent $\vdash \Gamma \Uparrow \Theta, \Phi_1^{k_1}, \ldots, \Phi_n^{k_n}$ is provable. Then $\vdash A_1, \ldots, A_n, \Gamma \Uparrow \Theta$ is provable by Lemma 3 plus enough applications of the *store* rule to move each member of $\Phi_i^{k_i}$ to the left side of $\Uparrow$. Furthermore, if Γ consists of a single positive formula P (P can also be in Θ with Γ empty) and $\vdash P \Downarrow P, \Theta, \Phi_1^{k_1}, \ldots, \Phi_n^{k_n}$ is provable, then using the *decide* rule

$$\frac{\vdash P \Downarrow P, \Theta, \Phi_1^{k_1}, \ldots, \Phi_n^{k_n}}{\vdash \cdot \Uparrow P, \Theta, \Phi_1^{k_1}, \ldots, \Phi_n^{k_n}}\ \textit{decide}$$

the provability $\vdash A_1, \ldots, A_n, P \Uparrow \Theta$ also follows from Lemma 3 and the *store* rule. The provability of the focused sequent above *decide* often follows from Lemma 4.

7 Returning to LK

In this section, we show how the unfocused **LK** proof system can be faithfully captured within **LKF**. We do this in three steps: (1) we translate the two-sided proof system **LK** into a one-sided system; (2) we show that a more general form of contraction is admissible in **LKF**; and (3) we prove that the unfocused introduction rules of (the one-side version of) **LK** are admissible in **LKF**. As a consequence, **LKF** is complete for **LK**.

Gentzen's original version of LK used the additive versions of conjunction and disjunction, namely $\wedge^-$ and $\vee^+$, while his implication $\supset$ was multiplicative. Gentzen himself noted (Gentzen (1935), Remark 2.4) that **LK** is 'dual' in the sense that the left and right inference rules are symmetrical with the exception of $\supset$. In **LKF**, the multiplicative connective $\vee^-$ can be used to decompose $A \supset B$ into $\neg A \vee^- B$. Hence, the dual of implication $\neg(A \supset B)$ can be written as $A \wedge^+ \neg B$. As a result, we can remove implications and negated implications by mapping them to these multiplicative connectives.

Definition 4 The **LK** *-polarization* $(\cdot)^\pm$ *of classical formulas* is defined as follows (recall that the negation of polarized formulas is given in Definition 1):

- For atomic a, $a^\pm = a$ and $(\neg a)^\pm = \neg a$.
- $(A \wedge B)^\pm = A^\pm \wedge^- B^\pm$; $(A \vee B)^\pm = A^\pm \vee^+ B^\pm$; $t^\pm = t^-$; $f^\pm = f^+$;
- $(A \supset B)^\pm = \neg A^\pm \vee^- B^\pm$

We also assume that all atomic formulas are polarized positively.

Structural rules and Identity rules

$$\frac{\vdash \Delta, B, B}{\vdash \Delta, B}\ cR \qquad \frac{\vdash \Delta}{\vdash \Delta, B}\ wR \qquad \frac{}{\vdash B, \neg B}\ init \qquad \frac{\vdash B, \Delta \quad \vdash \neg B, \Delta'}{\vdash \Delta, \Delta'}\ cut$$

Introduction rules

$$\frac{}{\vdash t^-, \Delta}\ t^- \qquad \frac{\vdash A, \Theta \quad \vdash B, \Theta}{\vdash A \wedge^- B, \Theta}\ \wedge^- \qquad \frac{}{\vdash t^+}\ t^+ \qquad \frac{\vdash A, \Theta \qquad \vdash B, \Theta'}{\vdash A \wedge^+ B, \Theta, \Theta'}\ \wedge^+$$

$$\frac{\vdash B_i, \Theta}{\vdash B_1 \vee^+ B_2, \Theta}\ \vee^+ \qquad \frac{\vdash \Theta}{\vdash f^-, \Theta}\ f^- \qquad \frac{\vdash A, B, \Theta}{\vdash A \vee^- B, \Theta}\ \vee^-$$

$$\frac{\vdash \Delta, [y/x]B}{\vdash \Delta, \forall x.B}\ \forall \qquad \frac{\vdash \Delta, [t/x]B}{\vdash \Delta, \exists x.B}\ \exists$$

Fig. 5 The rules for **LKi**. In the $\forall$ rule, the variable y is not free in the conclusion. In the $\vee^+$ rule, $i \in \{1, 2\}$.

Figure 5 contains the inference rules for **LKi**, a sequent calculus intermediate between **LK** and **LKF** in the sense that it is a one-sided sequent calculus that contains polarized formulas but it is not focused. An **LK** sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ is represented in this setting as $\vdash \neg A_1{}^{\pm}, \ldots, \neg A_n{}^{\pm}, B_1{}^{\pm}, \ldots, B_m{}^{\pm}$. Each inference rule of **LK** is translated directly into this setting: replace each sequent in the premises and conclusion of the rule with their one-sided, polarized versions. Left-introductions rules on A_i are thus represented as one-sided introduction rules on $\neg A_i{}^{\pm}$.

Theorem 4 *Let $n, m \geq 0$ and let $A_1, \ldots, A_n, B_1, \ldots, B_m$ be unpolarized formulas. If the sequent $\vdash \neg A_1{}^{\pm}, \ldots, \neg A_n{}^{\pm}, B_1{}^{\pm}, \ldots, B_m{}^{\pm} \Uparrow \cdot$ is provable in **LKF** then the sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ is provable in **LK**.*

Proof Note that an **LKF** proof of $\vdash \neg A_1{}^{\pm}, \ldots, \neg A_n{}^{\pm}, B_1{}^{\pm}, \ldots, B_m{}^{\pm} \Uparrow \cdot$ can easily be translated to an **LKi** proof of $\vdash \neg A_1{}^{\pm}, \ldots, \neg A_n{}^{\pm}, B_1{}^{\pm}, \ldots, B_m{}^{\pm}$. Such an **LKi** proof can then be converted to a proof of the two-sided sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ in **LK**. In this later transformation, when the multiplicative connectives $\vee^-$ and $\wedge^+$ are introduced in the **LKi** proof, implications are introduced on the right or left in the **LK** proof. □

We shall now proceed to prove that the rules of **LKi** are admissible in **LKF** by presenting new admissible **LKF** rules derived from the **LKi** rules. When naming the new admissible **LKF** rules, we will add parentheses around the name of the **LKi** rule. For example, the *init* rule of **LKi** yields the admissible **LKF** rule

$$\frac{}{\vdash B, \neg B, \Gamma \Uparrow \cdot}\ (init).$$

The admissibility of (*init*) follows immediately from Theorem 3. The admissibility of (*wR*), namely,

$$\frac{\vdash \Delta \Uparrow \Theta}{\vdash B, \Delta \Uparrow \Theta, \Theta'}\ (wR)$$

follows from Lemma 1 and a simple induction on the structure of B. We delay the proof of the admissibility of the **LKi** *cut* rule until Section 9.1. We now proceed to prove the admissibility of contraction and the introduction rules of **LKi**.

Unlike **LK** and **LKi**, **LKF** does not include explicit rules for contraction. In **LKF**, the rule of contraction is only applied to positive formulas and only within the *decide* rule. We now show that contraction for *all* polarized formulas is admissible in **LKF**.

Lemma 6 *The following rule is admissible in **LKF** for all formulas A.*

$$\frac{\vdash A, A, \Gamma \Uparrow \Theta}{\vdash A, \Gamma \Uparrow \Theta}\ (cR)$$

Proof Assume that $\vdash A, A, \Gamma \Uparrow \Theta$ has an **LKF** proof. Using Lemma 2, we can assume that this proof is eager for the first occurrence of A. If A is a positive formula or negative literal, then the only rule that can be applied to it is *store*, which means that the sequent $\vdash A, \Gamma \Uparrow A, \Theta$ has an **LKF** proof. Again, this sequent has a proof eager for A and, thus, must be proved by the *store* rule, which implies that $\vdash \Gamma \Uparrow A, \Theta$ has an **LKF** proof. By using that sequent as the premise of the *store* rule we have an **LKF** proof of $\vdash A, \Gamma \Uparrow \Theta$.

Consider the cases where A is a non-literal negative formula. The case where A is t^- is immediate. The case where A is f^- follows using Lemma 5 twice. If A is $B \vee^- C$ then, using Lemmas 2 and 5 twice, it is the case that $\vdash B, B, C, C, \Gamma \Uparrow \Theta$ is provable. The result follows by using the inductive assumption twice along with the $\vee^-$ rule. If A is $B \wedge^- C$ then, using Lemmas 2 and 5 twice, it is the case that both $\vdash B, B, \Gamma \Uparrow \Theta$ and $\vdash C, C, \Gamma \Uparrow \Theta$ are provable. The result follows by using the inductive assumption twice along with the $\wedge^-$ rule. Finally, the case where A is universally quantified is similar and omitted here. □

From results in the preceding sections, we can show the admissibility of the unfocused introduction rules (corresponding to the rules of **LKi**) in **LKF**.

Theorem 5 (Admissibility of unfocused introduction rules) *All the introduction of **LKi** are admissible in **LKF**.*

Proof Throughout this proof, we use the admissibility of cut combined with the argument schema outlined at the end of Section 6.

The $\vee^+$-introduction rule for **LKi** is admissible in **LKF** in the form

$$\frac{\vdash B_i, \Gamma \Uparrow \Theta}{\vdash B_1 \vee^+ B_2, \Gamma \Uparrow \Theta}\ (\vee^+)$$

for $i \in \{1, 2\}$. Admissibility follows from using the admissibility of the cut_u rule in the derivation

$$\frac{\vdash B_i, \Gamma \Uparrow \Theta \qquad \vdash \neg B_i,\ B_1 \vee^+ B_2 \Uparrow \cdot}{\vdash B_1 \vee^+ B_2, \Gamma \Uparrow \Theta}\ cut_u.$$

To show the provability of the right premise above the cut we apply the argument schema of Section 6. Let $\neg B_i \uparrow \Phi^1, \ldots, \neg B_i \uparrow \Phi^n$ be an exhaustive list of multisets of formulas $\uparrow$-related to $\neg B_i$, for $n \geq 0$. If $n = 0$ then the sequent is provable by Lemma 3. Otherwise, n is positive. For each Φ^k ($k \in 1 \ldots n$), construct the following subproof

$$\dfrac{\dfrac{\dfrac{\vdash B_i \Downarrow B_1 \vee^+ B_2, \Phi^k}{\vdash B_1 \vee^+ B_2 \Downarrow B_1 \vee^+ B_2, \Phi^k}\ \vee^+}{\vdash \cdot \Uparrow B_1 \vee^+ B_2, \Phi^k}\ \mathit{decide}}{\vdash B_1 \vee^+ B_2, \Phi^k \Uparrow \cdot}\ \mathit{store}$$

The provability of the top sequent follows from Lemma 4 and the provability of $\vdash \neg B_i,\ B_1 \vee^+ B_2 \Uparrow \cdot$ follows from all such subproofs by Lemma 3.

The $\wedge^+$-introduction rule for **LKi** is admissible in **LKF** in the form

$$\dfrac{\vdash A, \Gamma \Uparrow \Theta \quad \vdash B, \Gamma \Uparrow \Theta}{\vdash A \wedge^+ B, \Gamma \Uparrow \Theta}\ (\wedge^+).$$

This rule is also justified using the admissibility of cut_u as follows.

$$\dfrac{\dfrac{\vdash B, \Gamma \Uparrow \Theta \qquad \dfrac{\vdash A, \Gamma \Uparrow \Theta \quad \vdash \neg A, \neg B, A \wedge^+ B \Uparrow \cdot}{\vdash \neg B, A \wedge^+ B, \Gamma \Uparrow \Theta}\ cut_u}{\vdash A \wedge^+ B, \Gamma, \Gamma \Uparrow \Theta}\ cut_u}{\vdash A \wedge^+ B, \Gamma \Uparrow \Theta}\ (cR)$$

The provability of the top right sequent uses the argument schema described above: let $\neg A \uparrow \Phi^1_{\neg A}, \ldots, \neg A \uparrow \Phi^n_{\neg A}$ and $\neg B \uparrow \Phi^1_{\neg B}, \ldots, \neg B \uparrow \Phi^m_{\neg B}$ be exhaustive lists of multiset of set related to $\neg A$ and $\neg B$, respectively. If either n or m is 0, then the sequent is already provable. Otherwise for each pair $\Phi^i_{\neg A}, \Phi^k_{\neg B}$ construct the subproof

$$\dfrac{\dfrac{\dfrac{\vdash A \Downarrow A \wedge^+ B, \Phi^i_{\neg A}, \Phi^k_{\neg B} \quad \vdash B \Downarrow A \wedge^+ B, \Phi^i_{\neg A}, \Phi^k_{\neg B}}{\vdash A \wedge^+ B \Downarrow A \wedge^+ B, \Phi^i_{\neg A}, \Phi^k_{\neg B}}\ \wedge^+}{\vdash \cdot \Uparrow A \wedge^+ B, \Phi^i_{\neg A}, \Phi^k_{\neg B}}\ \mathit{decide}}{\vdash A \wedge^+ B, \Phi^i_{\neg A}, \Phi^k_{\neg B} \Uparrow \cdot}\ \mathit{store}$$

The provability of the top sequents follows from Lemma 4 and from these subproofs the provability of $\vdash \neg A, \neg B, A \wedge^+ B \Uparrow \cdot$ follows by Lemma 3.

To prove the admissibility of the introduction of $\exists$, we similarly rewrite

$$\dfrac{\vdash A[t/x], \Gamma \Uparrow \Theta}{\vdash \exists x.A, \Gamma \Uparrow \Theta}\ (\exists) \quad \longrightarrow \quad \dfrac{\vdash A[t/x], \Gamma \Uparrow \Theta \quad \vdash \neg A[t/x], \exists x.A \Uparrow \cdot}{\vdash \exists x.A, \Gamma \Uparrow \Theta}\ cut_u$$

The provability of the right premise again uses the argument schema of Section 6: let $\neg A[t/x] \uparrow \Phi^1, \ldots, \neg A[t/x] \uparrow \Phi^n$ be the exhaustive list of multisets that are $\uparrow$-related to $\neg A[t/x]$. If $n = 0$, then the premise is already provable. Otherwise, for each Φ^i we have

$$\dfrac{\dfrac{\dfrac{\vdash A[t/x] \Downarrow \exists x.A, \Phi^i}{\vdash \exists x.A \Downarrow \exists x.A, \Phi^i}\ \exists}{\vdash \cdot \Uparrow \exists x.A, \Phi^i}\ decide}{\vdash \exists x.A, \Phi^i \Uparrow \cdot}\ store$$

from which the provability of $\vdash \neg A[t/x], \exists x.A \Uparrow \cdot$ follows.

The **LKi** introduction rule for t^+ yields the following admissible rule, which can be justified by the associated **LKF** derivation.

$$\dfrac{}{\vdash t^+, \Gamma \Uparrow \Theta}\ (t^+) \qquad \longrightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash t^+ \Downarrow t^+}\ t^+}{\vdash \cdot \Uparrow t^+}\ decide}{\vdash t^+ \Uparrow \cdot}\ store}{\vdash t^+, \Gamma \Uparrow \Theta}\ (wR)$$

□

The negative introduction rules already apply on the left side of $\Uparrow$. Thus every unfocused inference rule can be emulated on the left side $\Uparrow$, and the completeness of **LKF** with respect to the intermediate **LKi**, and to the original **LK** is therefore established.

Theorem 6 (Weak completeness of LKF) *If the sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ is provable in **LK** then the sequent $\vdash \neg A_1{}^{\pm}, \ldots, \neg A_n{}^{\pm}, B_1{}^{\pm}, \ldots, B_m{}^{\pm} \Uparrow \cdot$ is provable in **LKF**.*

We have labeled this theorem as "weak completeness" since it states that if an unpolarized formula is provable in **LK**, then there is *some* polarization of that formula (namely $(\cdot)^{\pm}$) which is provable in **LKF**. Theorem 8 in the next section is a stronger version of the completeness theorem since it states that *every* polarization of an unpolarized theorem is provable in **LKF**.

8 Choosing the polarization of formulas

We are now able to prove that every polarization of a formula provable in **LK** is provable in **LKF**. Formally, we say that the polarized formula B (together with an atom bias assignment $\delta(\cdot)$) is a *polarization* of C if $\check{B}$ is C.

We write $A \equiv B$ to mean that both $\vdash \neg A, B \Uparrow \cdot$ and $\vdash \neg B, A \Uparrow \cdot$ are provable. We first show that the positive and negative versions of each connective are equivalent.

Lemma 7 *For every pair of polarized formulas A and B, it is the case that $A \vee^+ B \equiv A \vee^- B$ and $A \wedge^+ B \equiv A \wedge^- B$.*

Proof To prove the first equivalence, we need proofs of $\vdash \neg A \wedge^- \neg B,\ A \vee^- B \Uparrow \cdot$ and $\vdash \neg A \wedge^+ \neg B,\ A \vee^+ B \Uparrow \cdot$ The first of these is straightforward given the admissibility of the general initial rule. The provability of the second sequent is equally simple given the admissibility of the *unfocused* introduction rules shown in Section 7, as demonstrated by the following derivation.

$$\dfrac{\dfrac{\vdash \neg A,\ A \Uparrow \cdot}{\vdash \neg A,\ A \vee^+ B \Uparrow \cdot}\,(\vee^+) \qquad \dfrac{\vdash \neg B,\ B \Uparrow \cdot}{\vdash \neg B,\ A \vee^+ B \Uparrow \cdot}\,(\vee^+)}{\vdash \neg A \wedge^+ \neg B,\ A \vee^+ B \Uparrow \cdot}\,(\wedge^+)$$

Showing $A \wedge^+ B \equiv A \wedge^- B$ is similar, and the equivalences between the positive and negative versions of the units are straightforward. □

Definition 5 Let $\circ$ represent one of the binary connectives $\vee^-$, $\vee^+$, $\wedge^-$, or $\wedge^+$ and let F be a syntactic variable ranging over arbitrary polarized formulas. Let S range over *subformula contexts* which are defined inductively by

$$S \ = \ [\cdot] \mid S \circ F \mid F \circ S \mid \exists x.S \mid \forall x.S.$$

Here, $[\cdot]$ is a constant denoting a primitive subformula context. The notation $S[A]$ denotes the polarized formula that results from replacing $[\cdot]$ in S with A.

Theorem 7 *Let S be a subformula context. If $A \equiv B$ then $S[A] \equiv S[B]$.*

Proof We prove the general property: if $\vdash \neg A, B \Uparrow \cdot$ is provable then for any subformula context S, $\vdash \neg S[A], S[B] \Uparrow \cdot$ is also provable.

The proof of this property essentially repeats the arguments for eliminating the generalized initial rule. However, instead of replicating Lemmas 3 and 4, we can take advantage of the admissibility of unfocused rules for the positive connectives.

We proceed by induction on S. In the base case, $S = [\cdot]$ and the property is immediate. If, instead, $S = F \vee^- S'$ then $S[A] = F \vee^- S'[A]$, and $\neg S[A] = \neg F \wedge^+ \neg S'[A]$: we construct

$$\dfrac{\dfrac{\vdash \neg F, F, S'[B] \Uparrow \cdot \qquad \vdash \neg S'[A], S'[B], F \Uparrow \cdot}{\vdash \neg F \wedge^+ \neg S'[A],\ F, S'[B] \Uparrow \cdot}\,(\wedge^+)}{\vdash \neg F \wedge^+ \neg S'[A],\ F \vee^- S'[B] \Uparrow \cdot}\,\vee^-$$

The left premise follows from the general initial rule admissibility and the right premise is provable by inductive hypothesis (plus weakening). All the other cases are proved in a similar fashion. □

Theorem 8 (Strong completeness of LKF) *Let C be an unpolarized formula that is provable in **LK** and let B be a polarization of C. Then B is provable in **LKF**.*

Proof Let C be an unpolarized formula that is provable in **LK** and let B be a polarized version of C and let $\delta(\cdot)$ be any atomic bias assignment. By weak completeness (Theorem 6), we know that $C^\pm$ is provable in **LKF**. Since the only difference between $C^\pm$ and B are polarized formulas is that the + and − signs on logical connectives might be different and, by construction, the atoms in C are all given positive bias. Using the equivalences in Lemma 7 and Theorem 7, we can conclude that B is provable, assuming that all atoms are positively biased.

What remains to be shown is that provability is preserved by imposing the atomic bias assignment $\delta(\cdot)$. Translating a proof with a negative atom a into one where a is considered positive is the same as translating a proof with $\neg a$ considered positive to one where $\neg a$ is considered negative, so we only need to show one direction of the translation. Assume that a is considered negative in a proof. A strategy for reconstructing the proof where a is considered positive is to use *delays* together with cut. In particular, we define the polarized formula B^δ as the result of replacing every occurrence of a in B with $a \vee^- f^-$ (and therefore every occurrence of $\neg a$ by $\neg a \wedge^+ t^+$). The strategy is to show that every proof of $\vdash B \Uparrow \cdot$ with a considered negative corresponds to a proof of $\vdash B^\delta \Uparrow \cdot$ with a considered positive. Then by the cut rule

$$\frac{\vdash B^\delta \Uparrow \cdot \quad \vdash \neg B^\delta, B \Uparrow \cdot}{\vdash B \Uparrow \cdot}\ cut_u$$

we derive a proof of B without delays and with a considered positive. We can easily generalized the proof of a single formula to the proof of a sequent since (by invertibility) a multiset $\{B_1, \ldots, B_n\}$ is equivalent to $B_1 \vee^- B_2 \ldots \vee^- B_n$.

The rules that may have a literal as principal formula are *store*, *release*, *decide*, and *init*. We show how each rule is emulated in a proof of B^δ:

- Both a and $\neg a$ can be subject to a store, in which case the emulations are as follow.

$$\frac{\vdash \Gamma \Uparrow a, \Theta}{\vdash a, \Gamma \Uparrow \Theta}\ store \qquad \longrightarrow \qquad \cfrac{\cfrac{\cfrac{\vdash \Gamma \Uparrow a, \Theta}{\vdash a, \Gamma \Uparrow \Theta}\ store}{\vdash a, f^-, \Gamma \Uparrow \Theta}\ f^-}{\vdash a \vee^- f^-, \Gamma \Uparrow \Theta}\ \vee^-$$

$$\frac{\vdash \Gamma \Uparrow \neg a, \Theta}{\vdash \neg a, \Gamma \Uparrow \Theta}\ store \qquad \longrightarrow \qquad \frac{\vdash \Gamma \Uparrow \neg a \wedge^+ t^+, \Theta}{\vdash \neg a \wedge^+ t^+, \Gamma \Uparrow \Theta}\ \vee^-$$

  Thus, in a proof of B^δ, a will appear on the right side of $\Uparrow$ and $\Downarrow$ as a but $\neg a$ will appear as $\neg a \wedge^+ t^+$.
- The *release* rule is applicable when a is considered negative and is still applicable to $a \vee^- f^-$ when a is considered positive. Since a is a literal, the only rule that can apply above *release* is *store*.

$$\cfrac{\cfrac{\vdash \cdot \Uparrow a, \Theta}{\vdash a \Uparrow \Theta}\ store}{\vdash a \Downarrow \Theta}\ release \qquad \longrightarrow \qquad \cfrac{\cfrac{\cfrac{\cfrac{\vdash \cdot \Uparrow a, \Theta}{\vdash a \Uparrow \Theta}\ store}{\vdash a, f^- \Uparrow \Theta}\ f^-}{\vdash a \vee^- f^- \Uparrow \Theta}\ \vee^-}{\vdash a \vee^- f^- \Downarrow \Theta}\ release$$

- In the *init* rule, a is negative: it is emulated as indicated.

$$\dfrac{}{\vdash \neg a \Downarrow a, \Theta}\ \mathit{init} \qquad \longrightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a \Downarrow \neg a, a, \Theta}\ \mathit{init}}{\vdash \cdot \Uparrow \neg a, a, \Theta}\ \mathit{decide}}{\vdash \neg a \Uparrow a, \Theta}\ \mathit{store}}{\vdash \neg a \Downarrow a, \Theta}\ \mathit{release} \qquad \dfrac{}{\vdash t^+ \Downarrow a, \Theta}\ t^+}{\vdash \neg a \wedge^+ t^+ \Downarrow a, \Theta}\ \wedge^+$$

- Finally, when a is considered negative, the *decide* rule can only be applied to $\neg a$, and must be preceded from above by an *init*, and so is emulated as follows

$$\dfrac{\vdash \neg a \Downarrow \neg a, a, \Theta}{\vdash \cdot \Uparrow \neg a, a, \Theta}\ \mathit{decide} \qquad \longrightarrow \qquad \dfrac{\vdash \neg a \wedge^+ t^+ \Downarrow \neg a \wedge^+ t^+, a, \Theta}{\vdash \cdot \Uparrow \neg a \wedge^+ t^+, a, \Theta}\ \mathit{decide}$$

  The proof of the remaining premise is easily to find.

Finally, in order to show that $\vdash \neg B^\delta, B \Uparrow \cdot$ is provable with a considered positive, we induct on the structure of B:

- If B is a or $\neg a$, consider the following derivations.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a \Downarrow a, \neg a}\ \mathit{init}}{\vdash \cdot \Uparrow a, \neg a}\ \mathit{decide}}{\vdash a, \neg a \Uparrow \cdot}\ \mathit{store}}{\vdash a, f^-, \neg a \Uparrow \cdot}\ f^-}{\vdash a \vee^- f^-, \neg a \Uparrow \cdot}\ \vee^- \qquad \qquad \dfrac{\dfrac{\vdash \neg a \wedge^+ t^+ \Downarrow \neg a \wedge^+ t^+, a}{\vdash \cdot \Uparrow \neg a \wedge^+ t^+, a}\ \mathit{decide}}{\vdash \neg a \wedge^+ t^+, a \Uparrow \cdot}\ \mathit{store}$$

  The proof of $\vdash \neg(a \vee^- f^-), a \Uparrow \cdot$ on the right is preceded from above by the same subproof as in the imitation of *decide* for $\neg a \wedge^+ t^+$.
- If B is $C \vee^- D$, we apply the admissible unfocused rules to simplify the proof:

$$\dfrac{\dfrac{\vdash \neg C^\delta, C, D \Uparrow \cdot \qquad \vdash \neg D^\delta, C, D \Uparrow \cdot}{\vdash \neg C^\delta \wedge^+ \neg D^\delta, C, D \Uparrow \cdot}\ (\wedge^+)}{\vdash \neg C^\delta \wedge^+ \neg D^\delta, C \vee^- D \Uparrow \cdot}\ \vee^-$$

  The premises are provable by inductive hypotheses and by weakening.
- If B is $C \vee^+ D$:

$$\dfrac{\dfrac{\vdash \neg C^\delta, C \Uparrow \cdot}{\vdash \neg C^\delta, C \vee^+ D \Uparrow \cdot}\ (\vee^+) \qquad \dfrac{\vdash \neg D^\delta, D \Uparrow \cdot}{\vdash \neg D^\delta, C \vee^+ D \Uparrow \cdot}\ (\vee^+)}{\vdash \neg C^\delta \wedge^- \neg D^\delta, C \vee^+ D \Uparrow \cdot}\ \wedge^-$$

  The premises are provable by inductive hypotheses.
- The cases of $C \wedge^+ D$ and $C \wedge^- D$ are symmetrical to the above. The cases of $\exists$ and $\forall$ are also similar and cases where a does not appear in B follows directly from the admissibility of the general initial rule.

□

Pimentel et al. (2016) give a similar analysis of how changing the polarity of atoms within the intuitionistic focused proof system **LJF** (Liang and Miller 2009) affects the structure of such proofs.

9 Four applications of LKF

Part of the motivation for developing the **LKF** proof system is that its meta-theory should help in proving other proof-theoretic results about first-order classical logic. To support this claim, we present four applications of **LKF**.

9.1 The admissibility of *cut* in LK

We can prove that the admissibility of cut holds for **LK** given that we have proved cut-admissibility for the more complex proof system **LKF**. While it is no surprise that this can be done, it is reassuring to see that that result for **LK** follows so directly from the results for **LKF**.

Theorem 9 *The cut rule for* ***LK*** *is admissible in the cut-free fragment of* ***LK****.*

Proof Assume that the sequents $\Gamma \vdash \Delta, B$ and $\Gamma', B \vdash \Theta'$ have cut-free **LK**-proofs. By the weak completeness of **LKF** (Theorem 6), the sequents $\vdash \neg(\Gamma)^{\pm}, (\Delta)^{\pm}, B^{\pm} \Uparrow \cdot$ and $\vdash \neg(\Gamma')^{\pm}, \neg(B^{\pm}), (\Delta')^{\pm} \Uparrow \cdot$ both have (cut-free) **LKF** proofs. By the admissibility of cut for **LKF** (Theorem 2), we know that $\vdash \neg(\Gamma)^{\pm}, \neg(\Gamma')^{\pm}, (\Delta)^{\pm}, (\Delta')^{\pm} \Uparrow \cdot$ has a (cut-free) **LKF** proof. Finally, by Theorem 4, we know that $\Gamma, \Gamma' \vdash \Delta, \Delta'$ has a cut-free **LK** proof. □

9.2 Synthetic inference rules

Following up on the suggestion in Section 2.4, we show now how to define larger-scale, synthetic inference rules in using the **LKF** proof system.

A sequent of the form $\vdash \cdot \Uparrow \Theta$ a *border sequent*. The only **LKF** proof rule that can have a border sequent as a conclusion is the *decide* rule.

Definition 6 (Synthetic inference rule) A *synthetic inference rule* is an inference rule involving only border sequents. They are of the form

$$\frac{\vdash \cdot \Uparrow \Theta_1 \quad \dots \quad \vdash \cdot \Uparrow \Theta_n}{\vdash \cdot \Uparrow \Theta}$$

which is *justified* by a derivation of the form

$$\begin{array}{c}\vdash \cdot \Uparrow \Theta_1 \quad \ldots \quad \vdash \cdot \Uparrow \Theta_n \\ \Pi \\ \vdash \cdot \Uparrow \Theta\end{array}$$

Here, $n \geq 0$, and the derivation Π contains exactly one occurrence of the *decide* rule and that occurrence is the last inference rule (having the conclusion $\vdash \cdot \Uparrow \Theta$). If that *decide* rule selects as its focus the formula $B \in \Theta$, we say that this derivation is a *synthetic inference rule for B*.

Consider again using the formula (from Section 2.4)

$$\forall x \forall y \forall z.(path(x, y) \supset path(y, z) \supset path(x, z))$$

as an assumption in a given fixed theory. In the one-sided sequent setting of **LKF**, consider instead negating this assumption, namely,

$$\exists x \exists y \exists z.(path(x, y) \wedge^+ path(y, z) \wedge^+ \neg path(x, z))$$

and with moving it to the right-hand side of a border sequent. Assuming that this negative formula is a member of Θ, then consider the following derivation.

$$\dfrac{\dfrac{\dfrac{\begin{array}{ccc}\Xi_1 & \Xi_2 & \Xi_3 \\ \vdash path(r, s) \Downarrow \Theta & \vdash path(s, t) \Downarrow \Theta & \vdash \neg path(r, t) \Downarrow \Theta\end{array}}{\vdash path(r, s) \wedge^+ path(s, t) \wedge^+ \neg path(r, t) \Downarrow \Theta}\ \wedge^+ \times 2}{\vdash \exists x \exists y \exists z.(path(x, y) \wedge^+ path(y, z) \wedge^+ \neg path(x, z)) \Downarrow \Theta}\ \exists \times 3}{\vdash \cdot \Uparrow \Theta}\ decide$$

In order to determine the shape of the proofs Ξ_1, Ξ_2, and Ξ_3, we must declare the polarization given to atoms with the *path* predicate. If all such atoms have a negative polarity assigned to them, then both Ξ_1 and Ξ_2 end with the *release* and *store* rules while the proof Ξ_3 must be trivial (just containing the *init* rule) and $path(r, t)$ must be a member of Θ. We can write the synthetic rule justified by the above derivation as

$$\frac{\vdash \cdot \Uparrow path(r, s), \Theta \qquad \vdash \cdot \Uparrow path(s, t), \Theta}{\vdash \cdot \Uparrow path(r, t), \Theta}$$

However, if all *path*-atoms have a positive polarity assigned to them, then Ξ_3 end with the *release* and *store* rules while the proof Ξ_1 and Ξ_2 must be trivial and both $path(r, s)$ and $path(s, t)$ must be members of Θ. We can write the synthetic rule justified by the above derivation as

$$\frac{\vdash \cdot \Uparrow path(r, s), path(s, t), path(r, t), \Theta}{\vdash \cdot \Uparrow path(r, s), path(s, t), \Theta}$$

Note that these synthetic inference rules are the one-sided version of the back-chaining and forward-chaining synthetic inference rules for *path* displayed in Section 2.4.

The paper (Marin et al. 2020) develops the proof theory of synthetic inferences for both classical and intuitionistic logic by using the focused proof systems **LKF** and **LJF**. That paper also shows that cut and the general initial rule are both admissible in the **LK** and **LJ** proof system augmented with such synthetic inference rules based on *geometric formulas*.

9.3 Herbrand's theorem

The completeness of **LKF** proofs yields a surprisingly simple proof of Herbrand's theorem, particularly the variant of Herbrand's theorem based on formulas with only existential quantifiers in prefix position. A richer connection between a more general form of Herbrand's theorem, based on expansion trees (Miller 1987), and **LKF** proofs can be found in (Chaudhuri et al. 2016).

Theorem 10 (Herbrand's theorem) *Let B be an (unpolarized) quantifier-free formula of first-order classical logic, $n \geq 1$, and $x_1, \ldots, x_n$ be a list of first-order variables containing all free variable of B. The formula $\exists x_1 \ldots \exists x_n.B$ is provable in **LK** if and only if there is an $m \geq 1$ and substitutions $\theta_1, \ldots, \theta_m$ for the variables $x_1, \ldots, x_n$ such that $B\theta_1 \vee \cdots \vee B\theta_m$ is provable in **LK**.*

Proof Let $\hat{B}$ be a polarized version of B in which all logical connectives and units in B are polarized negatively. (For convenience, we abbreviate $\exists x_1 \ldots \exists x_n$ with $\exists \bar{x}$.) Since $\exists \bar{x}.B$ is provable in **LK**, the sequent $\vdash \exists \bar{x}.\hat{B} \Uparrow \cdot$ must have an **LKF**, say Ξ. Clearly, the last inference rule of Ξ is the *store* rule with premise $\vdash \cdot \Uparrow \exists \bar{x}.\hat{B}$. Given our choice of polarization, it is easy to show that every border sequent in Ξ is of the form $\vdash \cdot \Uparrow \exists \bar{x}.\hat{B}, \mathcal{L}$, where $\mathcal{L}$ is a set of literals. Thus, there are only two different ways that the *decide* rule is applied in Ξ. If the *decide* rule is used with a positive literal, the premise is immediately proved using the *init* rule. Otherwise, the *decide* rule starts the synchronous phase with the choice of $\exists \bar{x}.\hat{B}$ and the subproof determined by that occurrence of the *decide* rule ends with the following inference rules.

$$\dfrac{\dfrac{\vdash \hat{B}\theta \Uparrow \exists \bar{x}.\hat{B}, \mathcal{L}}{\vdash \hat{B}\theta \Downarrow \exists \bar{x}.\hat{B}, \mathcal{L}}\ \textit{release}}{\vdash \exists \bar{x}.\hat{B} \Downarrow \exists \bar{x}.\hat{B}, \mathcal{L}}\ \exists \times n$$

That is, every non-trivial synchronous phase encodes a substitution. Let $m \geq 1$ be the number of such non-trivial synchronous phases and let $\theta_1, \ldots, \theta_m$ be the substitutions that those phases encode.

Now let C be the polarized formula C equal to $\hat{B}\theta_1 \vee^+ \ldots \vee^+ \hat{B}\theta_m$ and consider building an **LKF** proof of $\vdash C \Uparrow \cdot$. In order to ensure that C is polarized positively, if $m = 1$, we take C to be $C \vee^+ f^+$ (essentially encoding a unary version of the binary $\vee^+$). It is now a simple matter to convert the proof Ξ of $\vdash \exists \bar{x}.\hat{B} \Uparrow \cdot$ into a proof of $\vdash \hat{B}\theta_1 \vee^+ \ldots \vee^+ \hat{B}\theta_m \Uparrow \cdot$ by copying the asynchronous phases directly and by replacing all the non-trivial synchronous phase in Ξ as follows.

$$\dfrac{\dfrac{\vdash \hat{B}\theta_i \Uparrow \exists\bar{x}.\hat{B}, \mathcal{L}}{\vdash \hat{B}\theta_i \Downarrow \exists\bar{x}.\hat{B}, \mathcal{L}}\ release}{\vdash \exists\bar{x}.\hat{B} \Downarrow \exists\bar{x}.\hat{B}, \mathcal{L}}\ \exists \times n \qquad \Longrightarrow \qquad \dfrac{\dfrac{\vdash \hat{B}\theta_i \Uparrow C, \mathcal{L}}{\vdash \hat{B}\theta_i \Downarrow C, \mathcal{L}}\ release}{\vdash \hat{B}\theta_1 \vee^+ \ldots \vee^+ \hat{B}\theta_m \Downarrow C, \mathcal{L}}\ \vee^+$$

In this way, the phase-by-phase structure of Ξ can be used to build an **LKF** proof for $\vdash \hat{B}\theta_1 \vee^+ \ldots \vee^+ \hat{B}\theta_m \Uparrow \cdot$. □

9.4 Hosting other focused proof systems

Proof systems with focusing-like behaviors can sometimes be hosted inside **LKF**. Such hosting is usually done by translating unpolarized classical logic formulas into polarized formulas in which *delays* have been inserted. These delays are written as $\partial_-(B)$ and $\partial_+(B)$ and are such that they are both logically equivalent to the formula B and are such that $\partial_-(B)$ is negative and $\partial_-(B)$ is positive. The expression $\partial_-(B)$ can be defined to be either $f^- \vee^- B$, $t^- \wedge^- B$, or $\forall x B$ (where x is not free in B). Similarly, the expression $\partial_+(B)$ can be defined to be either $f^+ \vee^+ B$, $t^+ \wedge^+ B$, or $\exists x B$ (where x is not free in B). The **LKQ** and **LKT** proof systems of (Danos et al. 1995) can be seen as **LKF** proofs in which the following polarization functions are used. Below we define the left and right translations of unpolarized formulas containing only implications and atoms to polarized formulas. Here, A ranges over atomic formulas.

	Atoms are negative		Atoms are positive
LKT	$(A)^l = \neg A$	**LKQ**	$(A)^l = \neg A$
	$(A)^r = A$		$(A)^r = A$
	$(B \supset C)^l = (B)^r \wedge^+ (C)^l$		$(B \supset C)^l = (B)^r \wedge^+ \partial_-((C)^l)$
	$(B \supset C)^r = (B)^l \vee^- \partial_+((C)^r)$		$(B \supset C)^r = \partial_+((B)^l \vee^- (C)^r)$

It is then the case that (cut-free) proofs in **LKT** of an unpolarized formula B using only implications correspond to **LKF** proofs of $(B)^r$ (using the **LKT** definition) and (cut-free) proofs in **LKQ** of an unpolarized formula B using only implications correspond to **LKF** proofs of $(B)^r$ (using the **LKQ** definition). **LKT** focuses only on the left and **LKQ** only on the right of two-sided sequents. These systems are also examples of "less aggressive" focused systems that designate a *"stoup"* formula: these systems impose fewer restrictions than the formula under focus in **LKF**. The delays emulate the one-sided focusing character of these system as well as adopt the stoup to a strongly focused system.

10 Other variants for focusing in classical logic

There have been several variations on focusing systems studied in the literature. The **LKF** proof system we have given here can be called a *strongly focused* system: the *decide* rule can only be invoked after *every* negative non-atomic formula has

SYNCHRONOUS INTRODUCTION RULES

$$\frac{}{\vdash t^+ \Downarrow \Gamma} \qquad \frac{\vdash B_1, \Theta_1 \Downarrow \Gamma \quad \vdash B_2, \Theta_2 \Downarrow \Gamma}{\vdash B_1 \wedge^+ B_2, \Theta_1, \Theta_2 \Downarrow \Gamma} \qquad \frac{\vdash B_i, \Theta \Downarrow \Gamma}{\vdash B_1 \vee^+ B_2, \Theta \Downarrow \Gamma}\ i \in \{1, 2\} \qquad \frac{\vdash [t/x]B, \Theta \Downarrow \Gamma}{\vdash \exists x.B, \Theta \Downarrow \Gamma}$$

RELEASE AND DECIDE RULES

$$\frac{\vdash \Delta \Uparrow \Gamma}{\vdash \Delta \Downarrow \Gamma}\ \mathit{release}^{\dagger} \qquad \frac{\vdash \Delta \Downarrow \bar{\Delta}, \Gamma}{\vdash \cdot \Uparrow \bar{\Delta}, \Gamma}\ \mathit{decide}^{\ddagger}$$

The † proviso requires that Δ consists of only negative formulas. In the *decide* rule, Δ is a non-empty multiset of positive formulas and $\bar{\Delta}$ is its underlying set of formula. The ‡ proviso is discussed in the text.

Fig. 6 Variations in some of the **LKF** inference rules.

been removed from sequent. If we do not insist that all negative formulas have been removed in this way, the resulting variant is called a *weakly focused* proof system following (Laurent 2004, Simmons and Pfenning 2011). Girard's LC proof system is an early example of a weakly focused proof system for classical logic (Girard 1991). A variant on strong focusing is a system where one chooses a predetermined *suspension criterion* and then allows explicitly suspected negative formulas to remain in the conclusion of the (suitably modified) *decide* rule: suspensions of this kind are useful when the logic contains fixed point expressions (Gérard and Miller 2017).

Let **LKFm** be the proof system that results from replacing the inference rules for **LKF** with the extended version of the synchronous introduction rules and the *release* and *decide* rules give in Figure 6. If the ‡ proviso on the *decide* rule requires that the multiset Δ contains exactly one positive formula, then **LKFm** is the same as **LKF**. It is for this reason that we say that **LKF** is *single focused*: in such proofs, the zone to the left of the $\Downarrow$ always contains exactly one formula (the focus of that sequent). If the ‡ proviso restricts Δ to be just a non-empty set of positive formulas, then the resulting proof system is *multifocused* and that proof system contains more proofs then the single conclusion system. These were first considered in (Delande and Miller 2008, Delande et al. 2010) (in the context of linear logic) and the notion of *maximal multifocused* proofs have been used to describe canonical proof system in linear logic (Chaudhuri et al. 2008a) and classical logic (Chaudhuri et al. 2016) and to relate sequent calculus proofs to natural deduction proofs (Pimentel et al. 2016).

Note that the version of the $\wedge^+$ introduction rule in **LKFm** is not necessarily invertible, while the version of that introduction rule in **LKF** is invertible: it appears that the true status of $\wedge^+$ introduction as belonging to the synchronous phase only becomes apparent in the multifocused setting. Note also that it is immediate to prove that the completeness of **LKFm** given the completeness of **LKF**.

Two simple changes to the **LKF** proof system yields a focused proof system for *multiplicative additive linear logic* **MALL** (Girard 1987). First, the set of formulas to the right of the double arrows must be changed to multisets. Second, the four following four inference rules must replace the corresponding inference rules from **LKF** (Figure 3).

$$\frac{A \text{ atomic}}{\vdash A \Downarrow \neg A}\ init \qquad \frac{\vdash P \Downarrow \Gamma}{\vdash \cdot \Uparrow P, \Gamma}\ decide \qquad \frac{}{\vdash t^+ \Downarrow \cdot}\ t^+ \qquad \frac{\vdash A \Downarrow \Theta_1 \quad \vdash B \Downarrow \Theta_2}{\vdash A \wedge^+ B \Downarrow \Theta_1, \Theta_2}\ \wedge^+$$

Here, the *init* and t^+ rules does not do an implicit weakening, the *decide* rule does not do an implicit contraction, and the side formulas of $\wedge^+$ are treated multiplicatively. The resulting proof system, called **MALLF** in (Liang and Miller 2011), is a focused proof system for **MALL**. Of course, the usual presentation of **MALL** results from replacing the logical connectives t^-, t^+, f^-, f^+, $\wedge^-$, $\wedge^+$, $\vee^+$, and $\vee^-$ need to be written as $\top$, $\mathbf{1}$, $\bot$, $\mathbf{0}$, &, $\otimes$, ⅋, and $\oplus$, respectively. The fact that this proof system is sound and complete for **MALL** immediately follows from the results about focusing in full linear logic given by Andreoli (1992).

Another variation uses a list, not a multiset, of formulas to the left of the $\Uparrow$: that is, the order by which the asynchronous inference rules are attempted is proscribed in a fixed fashion: this variation was used by Andreoli (1992) in his first focused proof system for linear logic and is useful for the actual implementation of proof search algorithms for focused proof systems.

The **LKF** proof system was designed to support automated proof checking and proof search (Chihani et al. 2017) as well as to provide new means to prove meta-theoretic results for first-order classical logic (see Section 9). Other researchers have been focused instead on supporting the Curry-Howard correspondence (proofs-as-programs) perspective, and they have designed still other variants of focusing for classical logic. In particular, see the LC proof system (Girard 1991), and the $\mathbf{LK}^{\eta}_{\rho}$ (Danos et al. 1995; 1997), and the proof system used to define the $\bar{\lambda}\mu\tilde{\mu}$-calculus (Curien and Herbelin 2000).

11 Conclusion

We have presented the proof system **LKF** and have proved that it is sound and complete for **LK** and that the cut rule and the initial rule are admissible. The proofs of these theorems were all done directly using permutation arguments. While the **LKF** system exhibits features from linear logic, the proofs here do not assume any background in linear logic or in intuitionistic logic. We expect that **LKF** will provide a convenient framework for proving many proof-theoretic proofs of first-order classical logic.

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