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A framework for proof systems

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Abstract Meta-logics and type systems based on intuitionistic logic are commonly used for specifying natural deduction proof systems. We shall show here that linear logic can be used as a meta-logic to specify a range of object-level proof systems. In particular, we show that by providing different *polarizations* within a *focused proof system* for linear logic, one can account for natural deduction (normal and non-normal), sequent proofs (with and without cut), and tableaux proofs. Armed with just a few, simple variations to the linear logic encodings, more proof systems can be accommodated, including proof system using generalized elimination and generalized introduction rules. In general, most of these proof systems are developed for both classical and intuitionistic logics. By using simple results about linear logic, we can also give simple and modular proofs of the soundness and relative completeness of all the proof systems we consider.

1 Introduction

Logics and type systems have been exploited in recent years as frameworks for the specification of deduction in a number of logics. The most common such *meta-logics* and *logical frameworks* have been based on intuitionistic logic (see, for example, [FM88, Pau89]) or dependent types (see [HHP93, Pfe89]). Such intuitionistic logics can be used to directly encode natural deduction style proof systems.

In a series of papers [Mil96, Pim01, MP02, MP04, PM05], Miller & Pimentel used classical linear logic as a meta-logic to specify and reason about a variety of sequent calculus proof systems. Since the encodings of such logical systems are natural and direct, the meta-theory of linear logic can be used to draw conclusions about the object-level proof systems. For example, in [MP02], a decision procedure was presented for determining if one encoded proof system is derivable from another. In the same paper, necessary conditions were presented (together with a decision procedure) for assuring that an encoded proof system satisfies cut-elimination. This last result used linear

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logic's dualities to formalize the fact that if the left and right introduction rules are suitable duals of each other then non-atomic cuts can be eliminated.

In this paper, we again use linear logic as a meta-logic but make critical use of the completeness of *focused proofs* for linear logic. Roughly speaking, focused proofs in linear logic divide cut-free, sequent calculus proofs into two different phases: the *negative* phase involves rules that are invertible while the *positive* phase involves the focused application of dual rules. In linear logic, it is clear to which phase each linear logic connective appears but it is completely arbitrary how atomic formulas can be assigned to these different phases. For example, all atomic formulas can be assigned a *negative polarity* or a *positive polarity* or, in fact, atomic formulas can be split with some being positive and the rest negative. The completeness of focused proofs then states that if a formula B is provable in linear logic and we fix on any polarity assignment to atomic formulas, then B will have a focused proof. Thus, while polarity assignment does not affect provability, it can result in strikingly different proofs. The earlier works of Miller & Pimentel assumed that all atoms were given negative polarity: this assignment resulted in an encoding of object-level sequent calculus. As we shall show here, if we vary that polarity assignment, we can get other object-level proof systems represented. Thus, while provability is not affected, different meta-level focused proofs are built and these encode different object-level proof systems.

Our main contribution in this paper is illustrating how a range of proof systems can be seen as different focusing disciplines on the same or (meta-logically) equivalent sets of linear logic specifications. Soundness and relative completeness of the encoded proof systems are generally derived via simple arguments about the structure of linear logic proofs. In particular, we present examples based on sequent calculus and natural deduction [Gen69], Generalized Elimination Rules [vP01], Free Deduction [Par92], the tableaux system KE [DM94], and Smullyan's Analytic Cut [Smu68a]. The adequacy of a given specification of inference rules requires first assigning polarity to meta-level atoms using in the specification: then adequacy is generally an immediate consequence of the focusing theorem of linear logic.

Comparing two proof systems can be done at three different levels of "adequacy": *relative completeness* claims simply that the provable sets of formulas are the same, *full completeness of proofs* claims that the completed proofs are in one-to-one correspondence, and *full completeness of derivations* claims that (open) derivations (such as inference rules themselves) are also in one-to-one correspondence. All the proof systems that we shall encode will be done with this third, most refined level of adequacy.

This paper is an extended and improved version of the conference paper [NM08a].

2 Preliminaries

2.1 A focusing proof system for linear logic

We shall assume that the reader is familiar with the basics of linear logic: we review a few specific points of the logic here. *Literals* are either atomic formulas or their negations. A formula is in *negation normal form* if negations have only atomic scope: the negation normal form of a formula is computed by using the de Morgan dualities to move negations deeper into formulas. If F is a linear logic formula, then we write $\neg F$ to denote the *negation normal form* of the negation of F. The connectives $\otimes$ and $⅋$ and their units 1 and $\bot$ are *multiplicative*; the connectives $\oplus$ and & and their units 0 and $\top$ are *additive*; $\forall$ and $\exists$ are quantifiers; and the operators ! and ? are the *exponentials.*

In general, we shall present *theories* in the linear meta-logic as appearing on the right-hand side of sequents. Thus, if $\mathcal{X}$ is a set of closed formulas then we say that the formula B is derived using theory $\mathcal{X}$ if $\vdash B, \mathcal{X}$ is provable in linear logic. We shall also write $B \equiv C$ to denote the formula $(\neg B \mathbin{⅋} C) \mathbin{\&} (\neg C \mathbin{⅋} B)$.

In [And92], Andreoli proved the completeness of the focused proof system for linear logic given in Figure 1. Focusing proof systems involve applying inference rules in alternating *polarities* or phases. In particular, formulas are *negative* if their top-level connective is either $⅋, \bot, \&, \top, ?,$ or $\forall$; formulas are *positive* if their top-level connective is $\oplus, 0, \otimes, 1, !,$ or $\exists$. This polarity assignment is rather natural in the sense that all right introduction rules for negative formulas are invertible while such introduction rules for positive formulas are not necessarily invertible. Atomic formulas must also belong to a phase, but here they are assigned to the positive or negative phase arbitrarily. The polarity of a negated atom is, of course, the flip of the atom's polarity. In the negative phase, represented by the judgment $\vdash \Theta : \Gamma \Uparrow L$, rules are applied only to negative formulas appearing in L, while positive formulas are moved to one of the multisets, Θ or Γ, on the left of the $\Uparrow$, by using the $[R \Uparrow]$ or $[?]$ rules. (We usually describe the dynamics of an inference rule by reading their effects on sequents when moving from the conclusion to the premises.) When L is empty, the positive phase begins by using one of the decide rules $[D_1]$ or $[D_2]$ to select a single formula on which to "focus": the judgment $\vdash \Theta : \Gamma \Downarrow F$ denotes such a sequent which is focused on F. Rules are then applied hereditarily to subformulas of F until a negative subformula is encountered, at which time, the reaction rule $[R \Downarrow]$ is used and another negative phase begins. We often refer to the context Θ as the *unbounded context* and the context Γ as the *linear* or *bounded context.*

We write $\vdash_{llf} \Theta : \Gamma \Uparrow$ to indicate that the sequent $\vdash \Theta : \Gamma \Uparrow$ has a proof in LLF; $\vdash_{llf} \Theta : \Gamma \Downarrow$ to indicate that the sequent $\vdash \Theta : \Gamma \Downarrow$ has a proof in LLF; and $\vdash_{ll} \Gamma$ to indicate that the sequent $\vdash \Gamma$ is provable in linear logic.

The following proposition can be proved by a simple induction on the structure of focused proofs.

Proposition 1 *Let Θ, Γ, and Δ be multisets of formulas and let L be a list of formulas and F a formula. If $\vdash \Theta : \Gamma \Uparrow L$ has a proof then $\vdash \Theta, \Delta : \Gamma \Uparrow L$ has a proof of the same height. If $\vdash \Theta : \Gamma \Downarrow F$ has a proof then $\vdash \Theta, \Delta : \Gamma \Downarrow F$ has a proof of the same height.*

The two-phase structure of LLF proofs allows us to collect introduction rules into "macro-rules" that can be seen as introducing "synthetic connectives." For example, if the formulas A_1, A_2, A_3 are negative formulas then we can view the positive formula $A_1 \oplus (A_2 \otimes A_3)$ as a synthetic connective with the following two "macro-rule":

$$\frac{\vdash \Theta : \Gamma \Uparrow A_1}{\vdash \Theta : \Gamma \Downarrow A_1 \oplus (A_2 \otimes A_3)} \qquad \frac{\vdash \Theta : \Gamma_1 \Uparrow A_2 \quad \vdash \Theta : \Gamma_2 \Uparrow A_3}{\vdash \Theta : \Gamma_1, \Gamma_2 \Downarrow A_1 \oplus (A_2 \otimes A_3)}$$

That is, within the LLF proof system, there are only these two ways to focus on this formula and there is no possibility to interleave other introduction rules ("micro-rules") with those that comprise these two macro rules.

The role of atoms and their polarity plays a special role in this paper. Andreoli's completeness theorem states that, for *any assignment of polarities* to atoms, a formula

Introduction Rules

$$\frac{\vdash \Theta : \Gamma \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, \bot}\ [\bot] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F, G}{\vdash \Theta : \Gamma \Uparrow L, F \parr G}\ [\parr] \qquad \frac{\vdash \Theta, F : \Gamma \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, ?F}\ [?]$$

$$\frac{}{\vdash \Theta : \Gamma \Uparrow L, \top}\ [\top] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F \quad \vdash \Theta : \Gamma \Uparrow L, G}{\vdash \Theta : \Gamma \Uparrow L, F \,\&\, G}\ [\&] \qquad \frac{\vdash \Theta : \Gamma \Uparrow L, F[c/x]}{\vdash \Theta : \Gamma \Uparrow L, \forall x\, F}\ [\forall]$$

$$\frac{}{\vdash \Theta : \Downarrow 1}\ [1] \qquad \frac{\vdash \Theta : \Gamma \Downarrow F \quad \vdash \Theta : \Gamma' \Downarrow G}{\vdash \Theta : \Gamma, \Gamma' \Downarrow F \otimes G}\ [\otimes] \qquad \frac{\vdash \Theta : \Uparrow F}{\vdash \Theta : \Downarrow\, !\, F}\ [!]$$

$$\frac{\vdash \Theta : \Gamma \Downarrow F}{\vdash \Theta : \Gamma \Downarrow F \oplus G}\ [\oplus_l] \qquad \frac{\vdash \Theta : \Gamma \Downarrow G}{\vdash \Theta : \Gamma \Downarrow F \oplus G}\ [\oplus_r] \qquad \frac{\vdash \Theta, F : \Gamma \Downarrow F[t/x]}{\vdash \Theta : \Gamma \Downarrow \exists x\, F}\ [\exists]$$

Identity, Reaction, and Decide rules

$$\frac{}{\vdash \Theta : A_p^\perp \Downarrow A_p}\ [I_1] \qquad \frac{}{\vdash \Theta, A_p^\perp : \Downarrow A_p}\ [I_2] \qquad \frac{\vdash \Theta : \Gamma, S \Uparrow L}{\vdash \Theta : \Gamma \Uparrow L, S}\ [R \Uparrow]$$

$$\frac{\vdash \Theta : \Gamma \Downarrow P}{\vdash \Theta : \Gamma, P \Uparrow}\ [D_1] \qquad \frac{\vdash \Theta, P : \Gamma \Downarrow P}{\vdash \Theta, P : \Gamma \Uparrow}\ [D_2] \qquad \frac{\vdash \Theta : \Gamma \Uparrow N}{\vdash \Theta : \Gamma \Downarrow N}\ [R \Downarrow]$$

Fig. 1 The focused proof system, LLF, for linear logic [And92]. Here, L is a list of formulas, Θ is a multiset of formulas, Γ is a multiset of literals and positive formulas, A_p is a positive literal, N is a negative formula, P is not a negative literal, and S is a positive formula or a negated atom.

F is provable in LLF if and only if it is provable in linear logic. Although the polarity assignment of literals does not affect provability, it does affect what synthetic connectives are available and, therefore, the shape and size of focused proofs. The polarity of atoms affects the structure of proofs because the rules $[I_1]$ and $[I_2]$ explicitly refer to the polarity assigned to literals. Consider, for example, focusing on the positive formula $A^\perp \otimes N$ where formula N and atom A are both negative: this leads to the construction of two macro-rules for this synthetic connective

$$\frac{\dfrac{}{\vdash \Theta, A : \cdot \Downarrow A^\perp}\ [I_1] \quad \dfrac{\vdash \Theta, A : \Gamma \Uparrow N}{\vdash \Theta, A : \Gamma \Downarrow N}\ [R \Downarrow]}{\vdash \Theta, A : \Gamma \Downarrow A^\perp \otimes N}\ [\otimes] \qquad \frac{\dfrac{}{\vdash \Theta : A \Downarrow A^\perp}\ [I_2] \quad \dfrac{\vdash \Theta : \Gamma \Uparrow N}{\vdash \Theta : \Gamma \Downarrow N}\ [R \Downarrow]}{\vdash \Theta : \Gamma, A \Downarrow A^\perp \otimes N}\ [\otimes]$$

Thus, in order for focusing on the formula $A^\perp \otimes N$ to yield a successful derivation, it must be the case that the formula A is present in either the unbounded or bounded context. On the other hand, if the atom A is assigned the positive polarity then the synthetic connective of $A^\perp \otimes N$ is introduced by a derivation of the form:

$$\frac{\dfrac{\vdash \Theta : \Gamma_1 \Uparrow A^\perp}{\vdash \Theta : \Gamma_1 \Downarrow A^\perp}\ [R \Downarrow] \quad \dfrac{\vdash \Theta : \Gamma_2 \Uparrow N}{\vdash \Theta : \Gamma_2 \Downarrow N}\ [R \Downarrow]}{\vdash \Theta : \Gamma_1, \Gamma_2 \Downarrow A^\perp \otimes N}\ [\otimes]$$

Here, there is no restriction imposed on A occurring in either the bounded or unbounded contexts.

Changes in the polarity assignment to atomic formulas allows one to switch from top-down (goal-directed) proofs to bottom-up (program-directed) proofs or some combinations of both [LM07, MN07] within a logic programming setting. As we shall see in this paper, differing atomic polarity assignments (in a linear meta-logic) are an important ingredient to deriving different proof systems (in an object-logic).

2.2 Encoding object-logic formulas and proof contexts

We shall assume that our meta-logic is a multi-sorted version of linear logic that results from imposing on linear logic Church's approach to representing terms and formulas as simply typed λ-terms [Chu40]. In particular, we use the type o for the type of meta-level formulas, the type *form* for object-level formulas, and the type i for object-level terms. The object-level quantifiers $\forall$ and $\exists$ are given the type $(i \rightarrow \mathit{form}) \rightarrow \mathit{form}$ and the expressions $\forall(\lambda x.B)$ and $\exists(\lambda x.B)$ are written, respectively, as $\forall x.B$ and $\exists x.B$. To deal with quantified object-level formulas, our meta-logic will quantify over variables of types $i \rightarrow \cdots \rightarrow i \rightarrow \mathit{form}$ (for 0 or more occurrences of i).

The proof systems that we encode have partial proofs that involve formulas in two *sense*. For example, in the process of building a natural deduction proof, some formulas are hypothesis (one argues *from* such formulas) and some formulas are conclusions (one argues *to* such formulas). In the process of building a sequent calculus proofs, some formulas are on the left of the sequent arrow and some are on the right. Tableaux proofs similarly use signed formulas (with either a **T** or **F** sign [Smu68b]) or places formulas on the left or right of a turnstile [DM94].

Informally, we will think of a *proof context* as being a collection of object-level formulas that are each present in these two senses. Thus, when encoding natural deduction, this collection can be a set or a multiset of object-level formulas marked as either being an hypothesis or the conclusion. In order to provide a consistent presentation of proof contexts throughout the range of proof systems, we introduce the two meta-level predicates $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ of type $\mathit{form} \rightarrow o$: the meta-level atomic formulas $\lfloor B\rfloor$ and $\lceil B\rceil$ are then used to denote these two different senses of how the object-level formula B is used within a proof context. The meta-level focused sequent $\Theta : \Gamma \Uparrow \cdot$ can then be used to collect together atomic formulas into a set via the unbounded context Θ or into a multiset via the bounded context Γ. Thus, the object-level sequent $B_1, \ldots, B_n \vdash C_1, \ldots, C_m$ can be encoded as the LLF sequent $\cdot : \lfloor B_1\rfloor, \ldots, \lfloor B_n\rfloor, \lceil C_1\rceil, \ldots, \lceil C_m\rceil \Uparrow \cdot$ if both the left and right side of the object-level sequent are multisets. If, say, the left side is a set and the right side is a multiset, then this sequent could be represented as $\lfloor B_1\rfloor, \ldots, \lfloor B_n\rfloor : \lceil C_1\rceil, \ldots, \lceil C_m\rceil \Uparrow \cdot$. Here, formulas on the left of the object-level sequent are marked using $\lfloor\cdot\rfloor$ and formulas on the right of the object-level sequent are marked using $\lceil\cdot\rceil$. For convenience, if Γ is a (multi)set of formulas, $\lfloor\Gamma\rfloor$ (resp. $\lceil\Gamma\rceil$) denotes the multiset of atoms $\{\lfloor F\rfloor \mid F \in \Gamma\}$ (resp. $\{\lceil F\rceil \mid F \in \Gamma\}$).

The theory $\mathcal{L}$ given in Figure 2 will be used throughout this paper in order to axiomatize the two senses for all the connectives in both intuitionistic and classical logic. For example, the conjunction connective appears in two formulas: once in the scope of $\lfloor\cdot\rfloor$ and once in the scope of $\lceil\cdot\rceil$. Notice that this axiomization is independent of the proof systems that this theory is used to describe. When we display formulas in this manner, we intend that the named formula is actually the result of applying ? to the existential closure of the formula. Thus, the formula named $(\wedge_L)$ is actually $?\exists A\exists B[\lfloor A \wedge B\rfloor^\perp \otimes (\lfloor A\rfloor \oplus \lfloor B\rfloor)]$. Furthermore, for intuitionistic and minimal logics, we use the following variant for the $(\Rightarrow_L)$ formula that contains a bang.

$$(\Rightarrow'_L) \qquad \lfloor A \Rightarrow B\rfloor^\perp \otimes (!\lceil A\rceil \otimes \lfloor B\rfloor)$$

$(\Rightarrow_L)$	$\lfloor A \Rightarrow B \rfloor^\perp \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$	$(\Rightarrow_R)$	$\lceil A \Rightarrow B \rceil^\perp \otimes (\lfloor A \rfloor \parr \lceil B \rceil)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^\perp \otimes (\lfloor A \rfloor \oplus \lfloor B \rfloor)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \mathbin{\&} \lceil B \rceil)$
$(\vee_L)$	$\lfloor A \vee B \rfloor^\perp \otimes (\lfloor A \rfloor \mathbin{\&} \lfloor B \rfloor)$	$(\vee_R)$	$\lceil A \vee B \rceil^\perp \otimes (\lceil A \rceil \oplus \lceil B \rceil)$
$(\forall_L)$	$\lfloor \forall B \rfloor^\perp \otimes \lfloor Bx \rfloor$	$(\forall_R)$	$\lceil \forall B \rceil^\perp \otimes \forall x \lceil Bx \rceil$
$(\exists_L)$	$\lfloor \exists B \rfloor^\perp \otimes \forall x \lfloor Bx \rfloor$	$(\exists_R)$	$\lceil \exists B \rceil^\perp \otimes \lceil Bx \rceil$
$(\perp_L)$	$\lfloor \perp \rfloor^\perp$	(t_R)	$\lceil t \rceil^\perp \otimes \top$

Fig. 2 The theory $\mathcal{L}$ used to encode various proof systems for minimal, intuitionistic, and classical logics.

(Id_1)	$\lfloor B \rfloor^\perp \otimes \lceil B \rceil^\perp$	(Id_2)	$\lfloor B \rfloor \otimes \lceil B \rceil$	$(Id_2{}')$	$\lfloor B \rfloor \otimes !\lceil B \rceil$
(Str_L)	$\lfloor B \rfloor^\perp \otimes ?\lfloor B \rfloor$	(Str_R)	$\lceil B \rceil^\perp \otimes ?\lceil B \rceil$	(W_R)	$\lceil C \rceil^\perp \otimes \perp$

Fig. 3 Specification of the identity rules (cut and initial) and of the structural rules (weakening and contraction).

This bang will be important to correctly encode the structural restriction for these logics, where sequents contain at most one formula in their right-hand-side. We denote by $\mathcal{L}_J$ the set obtained from $\mathcal{L}$ by replacing the formula $(\Rightarrow_L)$ by $(\Rightarrow_L')$ and by $\mathcal{L}_M$ the set obtained by removing the formula $(\perp_L)$ from $\mathcal{L}_J$.

The formulas in Figure 3 also play a central role in presenting proof systems. The Id_1 and Id_2 formulas can prove the duality of the $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ predicates: in particular, one can prove in linear logic that

$$\vdash \forall B(\lceil B \rceil \equiv \lfloor B \rfloor^\perp) \mathbin{\&} \forall B(\lfloor B \rfloor \equiv \lceil B \rceil^\perp), Id_1, Id_2$$

These two formulas are used, for example, to encode the initial and cut rules when we shall encode object-level sequent calculi (Section 3). To correctly encode the structural restrictions of intuitionistic and minimal logics, we use the clause $Id_2{}'$, instead of Id_2. The formulas Str_L and Str_R allow us to prove the equivalences $\lfloor B \rfloor \equiv ?\lfloor B \rfloor$ and $\lceil B \rceil \equiv ?\lceil B \rceil$. The last two equivalences allows the weakening and contraction of formulas at both the meta-level and object-level. For instance, in the encoding of minimal logics, where structural rules are only allowed in the left-hand-side, one should include only the Str_L formula; while in the encoding of classical logics, where structural rules are allowed in both sides of a sequent, one should include both Str_L and Str_R formulas. The formula W_R encodes the weakening right rule and is used to encode intuitionistic logics, where weakening, but not contraction, is allowed on formulas on the right-hand-side of a sequent.

From the Str_L clause we can derive the equivalence $\lfloor B \rfloor^\perp \equiv !\lfloor B \rfloor^\perp$ by negating the equivalence $\lfloor B \rfloor \equiv ?\lfloor B \rfloor$ obtained from this clause. This equivalence allows us to insert the ! before negative occurrences of $\lfloor\cdot\rfloor$. The presence of bangs in theories will play an important role in encoding correctly the structural rules of logics, such as minimal and intuitionistic logics, which require that right-hand-sides of sequents do not contain more than one formula. Although these equivalences do not affect provability, applying them can change focusing behavior significantly.

2.3 Adequacy levels for encodings

When comparing deductive systems, one can easily identify several "levels of adequacy." For example, Girard in [Gir06, Chapter 7] proposes three levels of adequacy based on semantical notions: the level of *truth*, the level of *functions*, and the level of *actions*. Here, we also identify three levels of adequacy but from a proof-theoretical point-of-view. The weakest level of adequacy is *relative completeness* which considers only *provability*: a formula has a proof in one system if it has a proof in another system. A stronger level of adequacy is of *full completeness of proofs*: the proofs of a given formula are in one-to-one correspondence with proofs in another system. If one uses the term "derivation" for possibly incomplete proofs (proofs that may have open premises), we can consider a even stronger level of adequacy. We use the term *full completeness of derivations* if the derivations (such as inference rules themselves) in one system are in one-to-one correspondence with those in another system.

For each of the object-logic proof systems that we consider here, we propose a meta-level theory, say $\mathcal{L}'$, that can be used to encode that system at the strongest level of adequacy. In all cases, we obtain $\mathcal{L}'$ from the formulas in Figures 2 and 3 by some combination of the following steps.

1) **Applying equivalences.** As we have shown, some equivalences are derivable from the identity and structural rules. Hence, we will at times replace occurrences of, for example, $\lfloor F \rfloor^{\perp}$ with $\lceil F \rceil$.

2) **Incorporating structural rules into introduction rules.** Although the formulas Str_L and Str_R provide an elegant specification of the weakening and contraction structural rules for the two difference senses for object-level formulas, they do not provide a good focusing behavior since the equivalences they imply can yield loops in a specification. Therefore, we incorporate the structural rules into a theory by adding ? and ! in its formulas. This transformation to a theory is usually formally justified using an induction of the height of proofs.

3) **Switching between multiplicative and additive introduction rules.** Given the presence of ? and ! within the specification of inference rules and the linear logic equivalences $?(A \oplus B) \equiv ?A \parr ?B$ and $!(A \,\&\, B) \equiv !A \otimes !B$ it is possible to replace, for example, the "additive" version of the rules $\wedge_L, \wedge_R, \vee_L, \vee_R$ in $\mathcal{L}$ with their "multiplicative" version, namely with

$$\begin{array}{ll} \lceil A \wedge B \rceil^{\perp} \otimes (\lceil A \rceil \otimes \lceil B \rceil) & \lfloor A \wedge B \rfloor^{\perp} \otimes (\lfloor A \rfloor \parr \lfloor B \rfloor) \\ \lfloor A \vee B \rfloor^{\perp} \otimes (\lceil A \rceil \otimes \lceil B \rceil) & \lceil A \vee B \rceil^{\perp} \otimes (\lceil A \rceil \parr \lfloor B \rfloor). \end{array}$$

Formal justification of this step will also be done using an induction on the height of proofs.

When we build $\mathcal{L}'$ from $\mathcal{L}$ and the rules in Figure 3 based on these steps, it will be a simple matter to prove that the new theory $\mathcal{L}'$ proves exactly the same formulas as the original theory. However, before we can formally say that a theory $\mathcal{L}'$ describes a proof system, we must assign polarity to the meta-level atomic formulas $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$. Only then can we claim that the "macro-rules" that result from focusing on formulas in that theory match exactly the inference rules of the corresponding encoded object-logic proof system. This polarity assignment may differ between different proof system encodings.

Although we concentrate on obtaining encodings of proof systems at the highest levels of adequacy, it is worth noticing that one might still be interested in theories that are adequate only at the level of (complete) proofs. For example, following the Curry-Howard isomorphism, functional programs are complete proofs and their execution involves the construction of cut-free proofs out of these programs. In that domain, one may not require adequacy at the level of (open) derivations.

$$\frac{\Gamma, A \Rightarrow B \vdash A \quad \Gamma, A \Rightarrow B, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}\ [\Rightarrow L] \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ [\Rightarrow R]$$

$$\frac{\Gamma, A_1 \wedge A_2, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}\ [\wedge Li] \qquad \frac{\Gamma \vdash A, \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ [\wedge R]$$

$$\frac{\Gamma, A \vee B, A \vdash C \quad \Gamma, A \vee B, B \vdash C}{\Gamma, A \vee B \vdash \Delta}\ [\vee L] \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ [\vee R_i]$$

$$\frac{\Gamma, \forall x\, A, A\{t/x\} \vdash C}{\Gamma, \forall x\, A \vdash C}\ [\forall L] \qquad \frac{\Gamma \vdash A\{c/x\}}{\Gamma \vdash \forall x\, A}\ [\forall R]$$

$$\frac{\Gamma, \exists x\, A, A\{c/x\} \vdash C}{\Gamma, \exists x\, A \vdash C}\ [\exists L] \qquad \frac{\Gamma \vdash A\{t/x\}}{\Gamma \vdash \exists x\, A}\ [\exists R]$$

$$\frac{\Gamma, A \vdash C \quad \Gamma \vdash A}{\Gamma \vdash C}\ [\mathrm{Cut}] \qquad \frac{}{\Gamma, A \vdash A}\ [\mathrm{I}] \qquad \frac{}{\Gamma \vdash t}\ [tR]$$

Fig. 4 The sequent calculus, LM, for minimal logic. Here, c is not free in $\Gamma \cup \{C\}$ and $i \in \{1, 2\}$.

$$\frac{}{\Gamma, \bot \vdash \cdot}\ [\bot L] \qquad \frac{\Gamma \vdash \cdot}{\Gamma \vdash C}\ [\mathrm{WR}]$$

Fig. 5 The rules to add to LM to obtain the sequent calculus, LJ, for intuitionistic logic.

$$\frac{\Gamma, A \Rightarrow B \vdash A, \Delta \quad \Gamma, A \Rightarrow B, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ [\Rightarrow L] \quad \frac{\Gamma, A \vdash A \Rightarrow B, B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ [\Rightarrow R]$$

$$\frac{\Gamma, A_1 \wedge A_2, A_i \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta}\ [\wedge Li] \quad \frac{\Gamma \vdash A \wedge B, A, \Delta \quad \Gamma \vdash A \wedge B, B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ [\wedge R]$$

$$\frac{\Gamma, A \vee B, A \vdash \Delta \quad \Gamma, A \vee B, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ [\vee L] \quad \frac{\Gamma \vdash A_1 \vee A_2, A_i, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta}\ [\vee R_i]$$

$$\frac{\Gamma, \forall x\, A, A\{t/x\} \vdash \Delta}{\Gamma, \forall x\, A \vdash \Delta}\ [\forall L] \quad \frac{\Gamma \vdash \forall x\, A, A\{c/x\}, \Delta}{\Gamma \vdash \forall x\, A, \Delta}\ [\forall R]$$

$$\frac{\Gamma, \exists x\, A, A\{c/x\} \vdash \Delta}{\Gamma, \exists x\, A \vdash \Delta}\ [\exists L] \quad \frac{\Gamma \vdash \exists x\, A, A\{t/x\}, \Delta}{\Gamma \vdash \exists x\, A, \Delta}\ [\exists R]$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}\ [\mathrm{Cut}] \quad \frac{}{\Gamma, A \vdash A, \Delta}\ [\mathrm{I}] \quad \frac{}{\Gamma \vdash t, \Delta}\ [tR] \quad \frac{}{\Gamma, \bot \vdash \Delta}\ [\bot L]$$

Fig. 6 The sequent calculus, LK, for classical logic. Here, c is not free in $\Gamma \cup \{C\}$ and $i \in \{1, 2\}$.

3 Sequent Calculus

Figures 4, 5, and 6, respectively, contain three sequent calculi for minimal (LM), intuitionistic (LJ), and classical logic (LK). A linear logic encoding for these systems is given by the theories, $\mathcal{L}_{lm}$, $\mathcal{L}_{lj}$ and $\mathcal{L}_{lk}$ shown in Figures 7, 8 and 9. These sets differ in the presence or absence of ? in front of $\lceil \cdot \rceil$, in the presence or absence of the formula $(\bot_L)$, and in the formula encoding the left introduction for implication. In particular, in the LM encoding, no structural rule is allowed for right-hand-side formulas; in the LJ encoding, the right-hand-side formulas can be weakened; and in the LK encoding, contraction is also allowed (using the exponential ?). The formula $(\bot_L)$ only appears in the encodings of LJ and LK. In the theories for LM and LJ, the formulas encoding the left introduction rule for implication and the formula $Id_2{}'$ contain a ! before a pos-

$(\Rightarrow_L)$	$\lfloor A \Rightarrow B \rfloor^\perp \otimes (!\lceil A \rceil \otimes ?\lfloor B \rfloor)$	$(\Rightarrow_R)$	$\lceil A \Rightarrow B \rceil^\perp \otimes (?\lfloor A \rfloor \parr \lceil B \rceil)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^\perp \otimes (?\lfloor A \rfloor \oplus ?\lfloor B \rfloor)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \mathbin{\&} \lceil B \rceil)$
$(\vee_L)$	$\lfloor A \vee B \rfloor^\perp \otimes (?\lfloor A \rfloor \mathbin{\&} ?\lfloor B \rfloor)$	$(\vee_R)$	$\lceil A \vee B \rceil^\perp \otimes (\lceil A \rceil \oplus \lceil B \rceil)$
$(\forall_L)$	$\lfloor \forall B \rfloor^\perp \otimes ?\lfloor Bx \rfloor$	$(\forall_R)$	$\lceil \forall B \rceil^\perp \otimes \forall x \lceil Bx \rceil$
$(\exists_L)$	$\lfloor \exists B \rfloor^\perp \otimes \forall x ?\lfloor Bx \rfloor$	$(\exists_R)$	$\lceil \exists B \rceil^\perp \otimes \lceil Bx \rceil$
		(t_R)	$\lceil t \rceil^\perp \otimes \top$
(Id_1)	$\lfloor B \rfloor^\perp \otimes \lceil B \rceil^\perp$	$(Id_2{}')$	$?\lfloor B \rfloor \otimes !\lceil B \rceil$

Fig. 7 The theory $\mathcal{L}_{lm}$ encodes the sequent calculus proof system LM.

$$(\perp_L) \quad \lfloor \perp \rfloor^\perp \qquad (W_R) \quad \lceil C \rceil^\perp \otimes \perp$$

Fig. 8 Adding these two clauses to $\mathcal{L}_{lm}$ yields $\mathcal{L}_{lj}$, which is used to encode the sequent calculus proof system LJ.

$(\Rightarrow_L)$	$\lfloor A \Rightarrow B \rfloor^\perp \otimes (?\lceil A \rceil \otimes ?\lfloor B \rfloor)$	$(\Rightarrow_R)$	$\lceil A \Rightarrow B \rceil^\perp \otimes (?\lfloor A \rfloor \parr ?\lceil B \rceil)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^\perp \otimes (?\lfloor A \rfloor \oplus ?\lfloor B \rfloor)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^\perp \otimes (?\lceil A \rceil \mathbin{\&} ?\lceil B \rceil)$
$(\vee_L)$	$\lfloor A \vee B \rfloor^\perp \otimes (?\lfloor A \rfloor \mathbin{\&} ?\lfloor B \rfloor)$	$(\vee_R)$	$\lceil A \vee B \rceil^\perp \otimes (?\lceil A \rceil \oplus ?\lceil B \rceil)$
$(\forall_L)$	$\lfloor \forall B \rfloor^\perp \otimes ?\lfloor Bx \rfloor$	$(\forall_R)$	$\lceil \forall B \rceil^\perp \otimes \forall x ?\lceil Bx \rceil$
$(\exists_L)$	$\lfloor \exists B \rfloor^\perp \otimes \forall x ?\lfloor Bx \rfloor$	$(\exists_R)$	$\lceil \exists B \rceil^\perp \otimes ?\lceil Bx \rceil$
$(\perp_L)$	$\lfloor \perp \rfloor^\perp$	(t_R)	$\lceil t \rceil^\perp \otimes \top$
(Id_1)	$\lfloor B \rfloor^\perp \otimes \lceil B \rceil^\perp$	(Id_2)	$?\lfloor B \rfloor \otimes ?\lceil B \rceil$

Fig. 9 The theory $\mathcal{L}_{lk}$ encodes the sequent calculus proof system LK.

$$\dfrac{\dfrac{}{\vdash \mathcal{K} :\Downarrow \lfloor A \Rightarrow B \rfloor^\perp}\,[I_2] \qquad \dfrac{\dfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lceil A \rceil}\,[!, R\Uparrow] \qquad \dfrac{\vdash \mathcal{K}, \lfloor B \rfloor : \lceil C \rceil \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow ?\lfloor B \rfloor}\,[R\Downarrow, ?]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow !\lceil A \rceil \otimes ?\lfloor B \rfloor}\,[\otimes]}{\dfrac{\vdash \mathcal{K} : \lceil C \rceil \Downarrow F}{\vdash \mathcal{K} : \lceil C \rceil \Uparrow \cdot}\,[D_2]}\,[2 \times \exists, \otimes]$$

Fig. 10 Here, the formula $A \Rightarrow B \in \Gamma$ and $\mathcal{K}$ denotes the set $\mathcal{L}_{lm}, \lfloor \Gamma \rfloor$.

itive occurrence of $\lceil \cdot \rceil$ atom. As we shall see, these occurrences of ! are necessary for preserving the invariant that in minimal and intuitionistic logics the right-hand-side of sequents do not contain more than one formula.

A key ingredient in capturing object-level sequent calculus inferences in a focused linear meta-logic is the assignment of negative polarity to all meta-level atomic formulas. To illustrate why focusing is relevant, consider the encoding of the left introduction rule for $\Rightarrow$: selecting this rule at the object-level corresponds to focusing on the formula $F = \exists A \exists B [\lfloor A \Rightarrow B \rfloor^\perp \otimes (!\lceil A \rceil \otimes \lfloor B \rfloor)]$ (which is a member of $\mathcal{L}_{lm}$). The focused derivation in Figure 10 is then forced once F is selected for the focus: for example, the left-hand-side subproof must be an application of initial – nothing else will work with the focusing discipline. Notice that this meta-level derivation directly encodes the usual left introduction rule for $\Rightarrow$: the object-level sequents $\Gamma, B \vdash C$ and $\Gamma \vdash A$ yields $\Gamma, A \Rightarrow B \vdash C$. Moreover, the ! enforces that in all branches there is at most one $\lceil \cdot \rceil$ atom.

If we fix the polarity of all meta-level atoms to be negative, then focused proofs using $\mathcal{L}_{lm}$, $\mathcal{L}_{lj}$, and $\mathcal{L}_{lk}$ yield encodings of the object-level proofs in LM, LJ, and LK. We use the judgments $\vdash_{lm}, \vdash_{lj}$, and $\vdash_{lk}$ to denote provability in LM, LJ, and LK.

Proposition 2 *Let $\Gamma \cup \Delta \cup \{C\}$ be a set of object-level formulas. Assume that all meta-level atomic formulas are given a negative polarity. Then*

$$1)\ \Gamma \vdash_{lm} C \text{ iff } \vdash_{llf} \mathcal{L}_{lm}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$$
$$2)\ \Gamma \vdash_{lj} C \text{ iff } \vdash_{llf} \mathcal{L}_{lj}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$$
$$3)\ \Gamma \vdash_{lk} \Delta \text{ iff } \vdash_{llf} \mathcal{L}_{lk}, \lfloor\Gamma\rfloor, \lceil\Delta\rceil : \cdot \Uparrow$$

Furthermore, adequacy for derivations also holds between the respective proof systems.

Proof First, one shows that focusing (deciding) on formulas within the linear logic theories $\mathcal{L}_{lm}$, $\mathcal{L}_{lj}$, and $\mathcal{L}_{lk}$ encodes exactly the corresponding sequent calculus inference rule. In all cases, this correspondence is shown with steps similar to the one offered above for the left-introduction of $\Rightarrow$. Once this level of adequacy for the encoding is established, the other results concerning the equivalences of provability follow immediately. See also [MP02, Pim01] for similar proofs related to the encoding of sequent calculus proofs. $\square$

If one removes the formula Id_2 and $Id_2{}'$ from the sets $\mathcal{L}_{lm}$, $\mathcal{L}_{lj}$, and $\mathcal{L}_{lk}$, obtaining the sets $\mathcal{L}^f_{lm}$, $\mathcal{L}^f_{lj}$, and $\mathcal{L}^f_{lk}$, respectively, one can restrict the encoded proofs to cut free (object-level) proofs, represented by the judgments $\vdash^f_{lm}$ for minimal logic, $\vdash^f_{lj}$ for intuitionistic logic, and $\vdash^f_{lk}$ for classical logic. The following proposition is an immediate consequence of the proof of Proposition 2.

Proposition 3 *Let $\Gamma \cup \Delta \cup \{C\}$ be a set of object-level formulas. Then*

$$1)\ \Gamma \vdash^f_{lm} C \text{ iff } \vdash_{llf} \mathcal{L}^f_{lm}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow \qquad 2)\ \Gamma \vdash^f_{lj} C \text{ iff } \vdash_{llf} \mathcal{L}^f_{lj}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$$
$$3)\ \Gamma \vdash^f_{lk} \Delta \text{ iff } \vdash_{llf} \mathcal{L}^f_{lk}, \lfloor\Gamma\rfloor, \lceil\Delta\rceil : \Uparrow$$

Furthermore, adequacy for derivations also holds between the respective proof systems.

Now that we have succeeded to find linear logic theories that encode the sequent calculus inference rules for minimal, intuitionistic, and classical logics at our strongest level of adequacy, we turn to showing how these theories are related back to the more elementary and modular sets of formulas shown in Figures 2 and 3. The equivalences that appear in the following three propositions are all at the most shallow level of adequacy: the equivalence of provability.

Proposition 4 *Let Γ and Δ be sets of object logic formulas. Then*

$$\vdash_{ll} \mathcal{L}, Id_1, Id_2, Str_L, Str_R, ?\lfloor\Gamma\rfloor, ?\lceil\Delta\rceil \text{ iff } \vdash_{ll} \mathcal{L}_{lk}, ?\lfloor\Gamma\rfloor, ?\lceil\Delta\rceil.$$

Proof From the structural rules, Str_L and Str_R, we know that $\lfloor C\rfloor \equiv ?\lfloor C\rfloor$ and $\lceil C\rceil \equiv ?\lceil C\rceil$. Since the only difference between $\mathcal{L}_{lk}$ and $\mathcal{L} \cup \{Id_1, Id_2\}$ is that the former has ? before positive occurrences of $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$, it is the case that $\mathcal{L}_{lk}$ is a consequence of $\mathcal{L} \cup \{Id_1, Id_2, Str_L, Str_R\}$, proving the $\Leftarrow$ direction.

For the $\Rightarrow$ direction, we need to show that the structural rules are admissible. We use focusing to help. In particular, we show that if $\vdash_{llf} \mathcal{L}, Id_1, Id_2, Str_L, Str_R, \mathcal{F}_1 : \mathcal{F}_2 \Uparrow$ then $\vdash_{llf} \mathcal{L}_{lk}, \mathcal{F}_1, \mathcal{F}_2 : \cdot \Uparrow$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are multisets of meta-level atoms (of which all are given a negative polarity). This is proved by induction on the height of focused proofs (the proof follows the same lines as in [MP04, Proposition 4.2]). We show the inductive case for $(\Rightarrow_L)$: all the others cases are done similarly. Thus, assume that our proof ends with a decide rule that selects an instance of the $(\Rightarrow_L)$

formula from Figure 2. Thus, the proof ends with the following derivation, where $\mathcal{K} = \mathcal{L}, Id_1, Id_2, Str_L, Str_R, \mathcal{F}_1$ and $\mathcal{F}_2 = \mathcal{F}_2^1 \cup \mathcal{F}_2^2$ (here, $\mathcal{F}_1$ and $\mathcal{F}_2$ are multisets of atomic formulas).

$$\dfrac{\dfrac{}{\vdash \mathcal{K} : \cdot \Downarrow \lfloor A \Rightarrow B \rfloor^{\perp}}\,[I_2] \quad \dfrac{\vdash \mathcal{K} : \mathcal{F}_2^1, \lceil A \rceil \Uparrow}{\vdash \mathcal{K} : \mathcal{F}_2^1, \Downarrow \lceil A \rceil}\,[R\Downarrow, R\Uparrow] \quad \dfrac{\vdash \mathcal{K} : \mathcal{F}_2^2, \lfloor B \rfloor \Uparrow}{\vdash \mathcal{K} : \mathcal{F}_2^2 \Downarrow \lfloor B \rfloor}\,[R\Downarrow, R\Uparrow]}{\dfrac{\vdash \mathcal{K} : \cdot \Downarrow \lfloor A \Rightarrow B \rfloor^{\perp} \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)}{\vdash \mathcal{K} : \mathcal{F}_2 \Uparrow}\,[D_2, 2 \times \exists]}\,[2 \times \otimes]$$

Thus, $\lfloor A \Rightarrow B \rfloor \in \mathcal{F}_1$ and by the inductive hypothesis, we have proofs of the sequents $\vdash \mathcal{L}_{lk}, \mathcal{F}_1 : \mathcal{F}_2^1, \lceil A \rceil \Uparrow$ and $\vdash \mathcal{L}_{lk}, \mathcal{F}_1 : \mathcal{F}_2^2, \lfloor B \rfloor \Uparrow$. By Proposition 1, the sequents $\vdash \mathcal{K}', \lceil A \rceil : \cdot \Uparrow$ and $\vdash \mathcal{K}', \lfloor B \rfloor : \cdot \Uparrow$ are also provable, where $\mathcal{K}' = \mathcal{L}_{lk}, \mathcal{F}_1, \mathcal{F}_2$. Thus, the desired proof using the theory $\mathcal{L}_{lk}$ but with focusing on the $(\Rightarrow_L)$ formula in $\mathcal{L}_{lk}$ is

$$\dfrac{\dfrac{}{\vdash \mathcal{K}' : \cdot \Downarrow \lfloor A \Rightarrow B \rfloor^{\perp}}\,[I_2] \quad \dfrac{\vdash \mathcal{K}', \lceil A \rceil : \cdot \Uparrow}{\vdash \mathcal{K}' : \cdot \Downarrow ?\lceil A \rceil}\,[R\Downarrow, ?] \quad \dfrac{\vdash \mathcal{K}', \lfloor B \rfloor : \cdot \Uparrow}{\vdash \mathcal{K}' : \cdot \Downarrow ?\lfloor B \rfloor}\,[R\Downarrow, ?]}{\dfrac{\vdash \mathcal{K}' : \cdot \Downarrow \lfloor A \Rightarrow B \rfloor^{\perp} \otimes (?\lceil A \rceil \otimes ?\lfloor B \rfloor)}{\vdash \mathcal{K}' : \cdot \Uparrow}\,[D_2, 2 \times \exists]}\,[2 \times \otimes]$$

The $\Rightarrow$ direction is a direct consequence of this intermediate result and the focusing theorem. $\square$

Proposition 5 *Let $\Gamma \cup \{C\}$ be a set of object logic formulas. Then*

1) $\vdash_{ll} \mathcal{L}_M, Id_1, Id_2{}', Str_L, ?\lfloor \Gamma \rfloor, \lceil C \rceil$ *iff* $\vdash_{ll} \mathcal{L}_{lm}, ?\lfloor \Gamma \rfloor, \lceil C \rceil$.
2) $\vdash_{ll} \mathcal{L}_J, Id_1, Id_2{}', Str_L, W_R, ?\lfloor \Gamma \rfloor, \lceil C \rceil$ *iff* $\vdash_{ll} \mathcal{L}_{lj}, ?\lfloor \Gamma \rfloor, \lceil C \rceil$.

Proof In the $\Rightarrow$ direction, we proceed in the same fashion as in Proposition 4. We prove that, for say minimal logic, if $\vdash_{ll} \mathcal{L}_M, Id_1, Id_2{}', Str_L, \mathcal{F}_1 : \mathcal{F}_2, \lceil C \rceil \Uparrow$ then $\vdash_{ll} \mathcal{L}_{lm}, \mathcal{F}_1, \mathcal{F}_2 : \lceil C \rceil \Uparrow$, where $\mathcal{F}_1 \cup \mathcal{F}_2$ is a multiset of $\lfloor \cdot \rfloor$ meta-level atoms and C is any object-logic formula. The main interesting case is when the proof of $\vdash \mathcal{K} : \mathcal{F}_2, \lceil C \rceil \Uparrow$ starts by focusing on $(\Rightarrow_L')$, where $\mathcal{K} = \mathcal{L}_M, Id_1, Id_2{}', Str_L, \mathcal{F}_1$. There is only one resulting focused derivation, due to the presence of the bang in $(\Rightarrow_L')$, and it has two open premises of the form $\vdash \mathcal{K} : \mathcal{F}_2^1, \lfloor B \rfloor, \lceil C \rceil \Uparrow$ and $\vdash \mathcal{K} : \mathcal{F}_2^2, \lceil A \rceil \Uparrow$, in which case the proof proceeds the same as in Proposition 4. $\square$

Proposition 6 *Let $\Gamma \cup \Delta \cup \{C\}$ be a set of object logic formulas. Then*

1) $\vdash_{ll} \mathcal{L}_M, Id_1, Str_L, ?\lfloor \Gamma \rfloor, \lceil C \rceil$ *iff* $\vdash_{ll} \mathcal{L}_{lm}^f, ?\lfloor \Gamma \rfloor, \lceil C \rceil$
2) $\vdash_{ll} \mathcal{L}_J, Id_1, Str_L, W_R, ?\lfloor \Gamma \rfloor, \lceil C \rceil$ *iff* $\vdash_{ll} \mathcal{L}_{lj}^f, ?\lfloor \Gamma \rfloor, \lceil C \rceil$
3) $\vdash_{ll} \mathcal{L}, Id_1, Str_L, Str_R, ?\lfloor \Gamma \rfloor, ?\lceil \Delta \rceil$ *iff* $\vdash_{ll} \mathcal{L}_{lk}^f, ?\lfloor \Gamma \rfloor, ?\lceil \Delta \rceil$.

Proof This proposition is proved in a similar way as the Propositions 4 and 5. $\square$

It is well known that for the sequent calculus systems LM, LJ, and LK the cut-elimination theorem holds. A direct consequence is the admissibility of the Id_2 rule in the theories considered for these sequent calculus systems, as states the following proposition.

Corollary 1 *Let $\Gamma \cup \Delta \cup \{C\}$ be a set of object logic formulas. Then*

1) $\vdash_{ll} \mathcal{L}_M, Id_1, Str_L, ?\lfloor\Gamma\rfloor, \lceil C\rceil$ *iff* $\vdash_{ll} \mathcal{L}_M, Id_1, Id_2', Str_L, ?\lfloor\Gamma\rfloor, \lceil C\rceil$
2) $\vdash_{ll} \mathcal{L}_J, Id_1, Str_L, W_R, ?\lfloor\Gamma\rfloor, \lceil C\rceil$ *iff* $\vdash_{ll} \mathcal{L}_J, Id_1, Id_2', Str_L, W_R, ?\lfloor\Gamma\rfloor, \lceil C\rceil$
3) $\vdash_{ll} \mathcal{L}, Id_1, Str_L, Str_R, ?\lfloor\Gamma\rfloor, ?\lceil\Delta\rceil$ *iff* $\vdash_{ll} \mathcal{L}, Id_1, Id_2, Str_L, Str_R, ?\lfloor\Gamma\rfloor, ?\lceil\Delta\rceil$.

The proof of this corollary follows from the admissibility of the cut rule [Gen69] and the encoding of the cut-free sequent calculus (Proposition 3). To see a setting in which the admissibility of the cut can be shown by directly considering the linear logic specification of inference rules, see [MP02, PM05].

4 Natural Deduction

The proof system depicted in Figure 11 is the $\forall$, $\wedge$, and $\Rightarrow$ intuitionistic fragment of the classical system in [SB98], presenting natural deduction using a sequent-style notation: sequents of the form $\Gamma \vdash C \uparrow$ are obtained from the conclusion by a derivation (reading bottom-up) where C is not the major premise of an elimination rule; and sequents of the form $\Gamma \vdash C \downarrow$ are obtained from the set of hypotheses by a derivation (from top-down) where C is extracted from the major premise of an elimination rule. These two types of derivations meet with either the match rule $[M]$ or the switch rule $[S]$. These two types of sequents can be used to distinguish general natural deduction proofs from *normal* form proofs [Pra65]: normal proofs are those in which the major premise of an elimination rule is not the conclusion of an introduction rule. Within the proof system in Figure 11, such proofs are exactly those that do not allow occurrences of the switch rule $[S]$. To the rules in Figure 11 we can add the introduction and elimination rules for $\vee$ and $\exists$ given in Figure 12. In those rules, occurrences of $\uparrow(\downarrow)$ denote either $\uparrow$ or $\downarrow$ with the proviso that all occurrences of $\uparrow(\downarrow)$ in a given inference rule are resolved the same way. Characterizing normal form proofs involving $\vee$ and $\exists$ is more involved to describe and we shall not consider such normal forms here.

We write $\Gamma \vdash_{nj} C$ to indicate that the natural deduction sequent $\Gamma \vdash C \uparrow$ has a proof in NJ and write $\Gamma \vdash^n_{nj} C$ to indicate that the natural deduction sequent $\Gamma \vdash C \uparrow$ has a normal proof in NJ: in this latter case, we shall restrict the formulas in $\Gamma \cup \{C\}$ to have no occurrences of $\vee$ and $\exists$.

The theory $\mathcal{L}_{nj}$ in Figure 13 encodes natural deduction for intuitionistic logic. The formula Str_L is incorporated in the theory by adding ? to some positive occurrences of $\lfloor\cdot\rfloor$ atoms and, to maintain the invariant that there is always at most one formula in the right-hand-side of sequents, we add ! to negative occurrences of $\lfloor\cdot\rfloor^\perp$. The judgment $\Gamma \vdash C \uparrow$ is encoded as the meta-level sequent $\vdash \mathcal{L}_{nj}, \lfloor\Gamma\rfloor : \lceil C\rceil$ and the judgment $\Gamma \vdash C \downarrow$ is encoded as the sequent $\vdash \mathcal{L}_{nj}, \lfloor\Gamma\rfloor : \lfloor C\rfloor^\perp$. In order for this encoding to be adequate at the level of derivations, we simply change the polarity assignment from what was used with sequent calculus: in particular, we assign atoms of the form $\lfloor\cdot\rfloor$ with positive polarity and atoms of the form $\lceil\cdot\rceil$ with negative polarity. This change in polarity changes left-introduction rules (within the sequent calculus) to elimination rules (within natural deduction). For example, the formula $(\Rightarrow_L)$ now encodes the im-

$$\frac{\Gamma \vdash A \Rightarrow B \downarrow \quad \Gamma \vdash A \uparrow}{\Gamma \vdash B \downarrow}\,[\Rightarrow E] \qquad \frac{\Gamma, A \vdash B \uparrow}{\Gamma \vdash A \Rightarrow B \uparrow}\,[\Rightarrow I]$$

$$\frac{\Gamma \vdash F \wedge G \downarrow}{\Gamma \vdash F \downarrow}\,[\wedge E] \qquad \frac{\Gamma \vdash F \uparrow \quad \Gamma \vdash G \uparrow}{\Gamma \vdash F \wedge G \uparrow}\,[\wedge I]$$

$$\frac{\Gamma \vdash \forall x\, A \downarrow}{\Gamma \vdash A\{t/x\} \downarrow}\,[\forall E] \qquad \frac{\Gamma \vdash A\{c/x\} \uparrow}{\Gamma \vdash \forall x\, A \uparrow}\,[\forall I]$$

$$\frac{}{\Gamma, A \vdash A \downarrow}\,[\mathrm{I}] \quad \frac{\Gamma \vdash A \downarrow}{\Gamma \vdash A \uparrow}\,[\mathrm{M}] \quad \frac{\Gamma \vdash A \uparrow}{\Gamma \vdash A \downarrow}\,[\mathrm{S}] \quad \frac{}{\Gamma \vdash t \uparrow}\,[tI] \quad \frac{\Gamma \vdash \bot \downarrow}{\Gamma \vdash C \uparrow}\,[\bot E]$$

Fig. 11 The rules for the $\Rightarrow$, $\forall$, and $\wedge$ fragment of intuitionistic natural deduction NJ.

$$\frac{\Gamma \vdash A \vee B \downarrow \;\; \Gamma, A \vdash C \uparrow\!(\downarrow) \;\; \Gamma, B \vdash C \uparrow\!(\downarrow)}{\Gamma \vdash C \uparrow\!(\downarrow)}\,[\vee E] \qquad \frac{\Gamma \vdash A_i \uparrow}{\Gamma \vdash A_1 \vee A_2 \uparrow}\,[\vee I]$$

$$\frac{\Gamma \vdash \exists x\, A \downarrow \quad \Gamma, A\{c/x\} \vdash C \uparrow\!(\downarrow)}{\Gamma \vdash C \uparrow\!(\downarrow)}\,[\exists E] \qquad \frac{\Gamma \vdash A\{t/x\} \uparrow}{\Gamma \vdash \exists x\, A \uparrow}\,[\exists I]$$

Fig. 12 The rules for $\vee$ and $\exists$ for intuitionistic natural deduction. In $[\vee L]$, $i \in \{1, 2\}$.

$$\begin{array}{llll}
(\Rightarrow_E) & \lfloor A \Rightarrow B \rfloor^\perp \otimes (!\lceil A \rceil \otimes \lfloor B \rfloor) & (\Rightarrow_I) & \lceil A \Rightarrow B \rceil^\perp \otimes (?\lfloor A \rfloor \invamp \lceil B \rceil) \\
(\wedge_E) & \lfloor A \wedge B \rfloor^\perp \otimes (\lfloor A \rfloor \oplus \lfloor B \rfloor) & (\wedge_I) & \lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \,\&\, \lceil B \rceil) \\
(\vee_E) & !\lfloor A \vee B \rfloor^\perp \otimes (?\lfloor A \rfloor \,\&\, ?\lfloor B \rfloor) & (\vee_I) & \lceil A \vee B \rceil^\perp \otimes (\lceil A \rceil \oplus \lceil B \rceil) \\
(\forall_E) & \lfloor \forall B \rfloor^\perp \otimes \lfloor Bx \rfloor & (\forall_I) & \lceil \forall B \rceil^\perp \otimes \forall x \lceil Bx \rceil \\
(\exists_E) & !\lfloor \exists B \rfloor^\perp \otimes \forall x ?\lfloor Bx \rfloor & (\exists_I) & \lceil \exists B \rceil^\perp \otimes \lceil Bx \rceil \\
(\bot) & \lfloor \bot \rfloor^\perp & (t_I) & \lceil t \rceil^\perp \otimes \top \\
(\bot_E) & \lceil C \rceil^\perp \otimes \bot & & \\
(Id_1) & \lfloor B \rfloor^\perp \otimes \lceil B \rceil^\perp & (Id_2) & \lfloor B \rfloor \otimes !\lceil B \rceil
\end{array}$$

Fig. 13 The specification $\mathcal{L}_{nj}$ for intuitionistic natural deduction.

plication elimination rule as is illustrated by the following derivation (here, $(\Rightarrow_L) \in \mathcal{K}$):

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{K} : \lfloor A \Rightarrow B \rfloor^\perp \Uparrow}{\vdash \mathcal{K} :\Downarrow \lfloor A \Rightarrow B \rfloor^\perp}\,[R\Downarrow, R\Uparrow] \quad \dfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lceil A \rceil}\,[!, R\Uparrow] \quad \dfrac{}{\vdash \mathcal{K} : \lfloor B \rfloor^\perp \Downarrow \lfloor B \rfloor}\,[I_1]}{\vdash \mathcal{K} : \lfloor B \rfloor^\perp \Downarrow \lfloor A \Rightarrow B \rfloor^\perp \otimes (!\lceil A \rceil \otimes \lfloor B \rfloor)}\,[2 \times \otimes]}{\vdash \mathcal{K} : \lfloor B \rfloor^\perp \Uparrow}\,[D_2, 2 \times \exists]$$

The change in the assignment of polarity also causes the formula $Id_2{}'$, which behaved like the cut rule in sequent calculus, to now behave like the switch rule, as illustrated by the following derivation, where $Id_2{}' \in \Sigma$.

$$\dfrac{\dfrac{\dfrac{}{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^\perp \Downarrow \lfloor C \rfloor}\,[I_1] \quad \dfrac{\vdash \Sigma, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow}{\vdash \Sigma, \lfloor \Gamma \rfloor :\Downarrow !\lceil C \rceil}\,[!, R\Uparrow]}{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^\perp \Downarrow \lfloor C \rfloor \otimes !\lceil C \rceil}\,[\otimes]}{\vdash \Sigma, \lfloor \Gamma \rfloor : \lfloor C \rfloor^\perp \Uparrow}\,[D_2, \exists]$$

These two examples can be developed for all inference rules in Figures 11 and 12 and for focusing on all formulas in Figure 13. (Most of the missing cases are included in the Appendix to further illustrate how these encodings work.) As the last example above suggests, we can capture normal natural deduction proofs if we remove instances of Id_2' from $\mathcal{L}_{nj}$. More specifically, let $\mathcal{L}^f_{nj}$ be the set of formulas $\mathcal{L}_{nj}$ except that we drop Id_2' and the formulas encoding the introduction rules for $\vee$ and $\exists$. As a result, it is an easy matter to prove the following proposition.

Proposition 7 *Let $\Gamma \cup \{C\}$ be a set of object-level formulas and assume that all $\lceil\cdot\rceil$ atomic formulas are given a negative polarity and that all $\lfloor\cdot\rfloor$ atomic formulas are given a positive polarity. Then $\Gamma \vdash_{nj} C$ if and only if $\vdash_{llf} \mathcal{L}_{nj}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$. Also, if the formulas in $\Gamma \cup \{C\}$ contain neither $\vee$ nor $\exists$, then $\Gamma \vdash^n_{nj} C$ if and only if $\vdash_{llf} \mathcal{L}^f_{nj}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$.*

Now that we have adequately encoded natural deduction derivations via the theory $\mathcal{L}_{nj}$, we can show how some (known) meta-theory results of intuitionistic logic can be achieved using these encodings. For example, we show in Proposition 8 below that sequent calculus proofs and natural deduction proofs prove the same formulas. First, the next two lemmas relate $\mathcal{L}_{nj}$ and $\mathcal{L}^f_{nj}$ with the formulas in Figure 2 and 3.

Lemma 1 *Let $\Gamma \cup \{C\}$ be a set of object logic formulas. Then*

$$\vdash_{ll} \mathcal{L}_J, Id_1, Id_2', Str_L, W_R, ?\lfloor\Gamma\rfloor, \lceil C\rceil \textit{ iff } \vdash_{ll} \mathcal{L}_{nj}, ?\lfloor\Gamma\rfloor, \lceil C\rceil.$$

Proof The proof follows the same lines as the proof of the Proposition 4. The main difference in the $\Leftarrow$ direction is that we also use the equivalence $\lfloor C\rfloor^\perp \equiv !\lfloor C\rfloor^\perp$ obtained from the Str_L.

In the $\Rightarrow$ direction, we first prove the following equivalence, by induction on the height of the proof and by assigning negative polarity to all $\lceil\cdot\rceil$ atoms and positive polarity to all $\lfloor\cdot\rfloor$ atoms:

$$\vdash_{llf} \mathcal{L}_{nj}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow \text{ iff } \vdash_{llf} \mathcal{L}_{nj}, Str_L, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$$

The case for when Str_L is focused on is the most interesting one. There are two cases, either (1) the resulting premises are of the form $\vdash \mathcal{L}_{nj}, Str_L, \lfloor\Gamma\rfloor : \lfloor B\rfloor^\perp \Uparrow$ and $\vdash \mathcal{L}_{nj}, Str_L, \lfloor\Gamma, B\rfloor : \lceil C\rceil \Uparrow$, for which case we can use a linear-logic cut rule with cut formula $?\lfloor B\rfloor$: one premise is provable due to the inductive hypothesis, and the other is provable also by the inductive hypothesis, but by first introducing the ! in the cut formula $!\lfloor B\rfloor^\perp$; or (2) the premises are of the form $\vdash \mathcal{L}_{nj}, Str_L, \lfloor\Gamma\rfloor : \lfloor B\rfloor^\perp, \lceil C\rceil \Uparrow$ and $\vdash \mathcal{L}_{nj}, Str_L, \lfloor\Gamma, B\rfloor : \cdot \Uparrow$. In this case, because the elimination rules permute over introduction rules in natural deduction, we can assume that the proof of $\vdash \mathcal{L}_{nj}, Str_L, \lfloor\Gamma\rfloor : \lfloor B\rfloor^\perp, \lceil C\rceil \Uparrow$ finishes with a derivation that focuses only on formulas encoding right introduction rules and has premises of the form $\vdash \mathcal{L}_{nj}, Str_L, \lfloor\Gamma'\rfloor : \lfloor B\rfloor^\perp \Uparrow$. In this case, we proceed as in the first case, but with the difference that we postpone the introduction of the bang of the cut formula, $!\lfloor B\rfloor^\perp$, when these premises are reached.

From the Str_L formula we derive the equivalence $\lfloor C\rfloor^\perp \equiv !\lfloor C\rfloor^\perp$, which allows us to obtain the equivalent theory, $\mathcal{L}'_{nj}$, from $\mathcal{L}_{nj}$ by replacing all occurrences of $!\lfloor C\rfloor^\perp$ by $\lfloor C\rfloor^\perp$. Now, we show the following intermediate result by induction on the height of proofs and using the same polarity assignment as before:

$$\vdash_{llf} \mathcal{L}'_{nj}, Str_L, \mathcal{F}_1, \mathcal{F}_2 : \lceil C\rceil \Uparrow \text{ iff } \vdash_{llf} \mathcal{L}_J, Id_1, Id_2', Str_L, \mathcal{F}_1 : \mathcal{F}_2, \lceil C\rceil \Uparrow$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are sets of $\lfloor\cdot\rfloor$ atoms and C an object-logic formula. This direction follows immediately from this intermediate result and the focusing theorem. $\square$

The proof of the following lemma is similar to the proof of Lemma 1.

$$\frac{\Gamma \vdash [A \Rightarrow B] \quad \Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ [\Rightarrow GE] \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ [\Rightarrow I]$$

$$\frac{\Gamma \vdash [A \wedge B] \quad \Gamma, A, B \vdash C}{\Gamma \vdash C}\ [\wedge GE] \qquad \frac{\Gamma \vdash F \quad \Gamma \vdash G}{\Gamma \vdash F \wedge G}\ [\wedge I]$$

$$\frac{\Gamma \vdash [A \vee B] \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ [\vee GE] \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ [\vee I]$$

$$\frac{\Gamma \vdash [\forall x\, A] \quad \Gamma, A\{t/x\} \vdash C}{\Gamma \vdash C}\ [\forall GE] \qquad \frac{\Gamma \vdash A\{c/x\}}{\Gamma \vdash \forall x\, A}\ [\forall I]$$

$$\frac{\Gamma \vdash [\exists x\, A] \quad \Gamma, A\{c/x\} \vdash C}{\Gamma \vdash C}\ [\exists GE] \qquad \frac{\Gamma \vdash A\{t/x\}}{\Gamma \vdash \exists x\, A}\ [\exists I]$$

$$\frac{}{\Gamma, A \vdash A}\ [\mathrm{I}] \qquad \frac{}{\Gamma \vdash t}\ [tI] \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash C}\ [\bot E]$$

Fig. 14 The rules for intuitionistic natural deduction system with generalized elimination rules, GE. The major premises of elimination rules is marked with brackets.

$(\Rightarrow_E)$	$![A \Rightarrow B] \otimes (![A] \otimes ?\lfloor B \rfloor)$	$(\Rightarrow_I)$	$\lceil A \Rightarrow B \rceil^\perp \otimes (?\lfloor A \rfloor \invamp \lceil B \rceil)$
$(\wedge_E)$	$!\lceil A \wedge B \rceil \otimes (?\lfloor A \rfloor \invamp ?\lfloor B \rfloor)$	$(\wedge_I)$	$\lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \,\&\, \lceil B \rceil)$
$(\vee_E)$	$!\lceil A \vee B \rceil \otimes (?\lfloor A \rfloor \,\&\, ?\lfloor B \rfloor)$	$(\vee_I)$	$\lceil A \vee B \rceil^\perp \otimes (\lceil A \rceil \oplus \lceil B \rceil)$
$(\forall_E)$	$!\lceil \forall B \rceil \otimes ?\lfloor Bx \rfloor$	$(\forall_I)$	$\lceil \forall B \rceil^\perp \otimes \forall x \lceil Bx \rceil$
$(\exists_E)$	$!\lfloor \exists B \rfloor^\perp \otimes \forall x ?\lfloor Bx \rfloor$	$(\exists_I)$	$\lceil \exists B \rceil^\perp \otimes \lceil Bx \rceil$
$(\bot)$	$\lceil \bot \rceil$	(t_I)	$\lceil t \rceil^\perp \otimes \top$
$(\bot_E)$	$\lceil C \rceil^\perp \otimes \bot$		
(Id_1)	$\lfloor B \rfloor^\perp \otimes \lceil B \rceil^\perp$		

Fig. 15 The specification $\mathcal{L}_{ge}$ for intuitionistic natural deduction with generalized elimination rules.

Lemma 2 *Let $\Gamma \cup \{C\}$ be a set of object logic formulas that do not contain occurrences of $\vee$ and $\exists$. Then*

$$\vdash_{ll} \mathcal{L}_J, Id_1, Str_L, W_R, ?\lfloor \Gamma \rfloor, \lceil C \rceil \text{ iff } \vdash_{ll} \mathcal{L}^f_{nj}, ?\lfloor \Gamma \rfloor, \lceil C \rceil.$$

From Propositions 5 and 6, Lemmas 1 and 2, and Propositions 2, 3, and 7, we obtain the following relative completeness result between LJ and NJ.

Proposition 8 *If $\Gamma \cup \{C\}$ be a set of object-level formulas, then $\Gamma \vdash_{lj} C$ if and only if $\Gamma \vdash_{nj} C$. Furthermore, if the formulas in $\Gamma \cup \{C\}$ contain neither $\vee$ nor $\exists$ then $\Gamma \vdash^f_{lj} C$ if and only if $\Gamma \vdash^n_{nj} C$.*

Treating negation (in particular, falsity) in natural deduction presentations of intuitionistic and classical logics is not straightforward. We show in [NM08b] that extra meta-logic formulas are needed to encode these systems. Since the treatment of negation in natural deduction is not one about focusing in the meta-level, we do not discuss this issue further here.

5 Natural Deduction with Generalized Elimination Rules

Schroeder-Heister [SH84] considered a form of natural deduction where the indirect style of elimination rules used for $\vee$ and $\exists$ (see Figure 12) were also applied to conjunction. Von Plato [vP01] used that style of elimination rule for all connectives. In

Figure 14 we present an additive version of a natural deduction system with generalized elimination inspired by one found in [NP01, page 167]. The bracketed formula in an elimination rule is called the *major premise*. To encode proofs in natural deduction using generalized elimination, we use the theory $\mathcal{L}_{ge}$ shown in Figure 15. Intuitively, $\mathcal{L}_{ge}$ is obtained from $\mathcal{L}$ by using the formula Str_L to insert ! and ? connectives, and by using the identity rules to replace negative literals $\lfloor C \rfloor^\perp$ by the positive atoms $\lceil C \rceil$.

In order to match focused proofs using $\mathcal{L}_{ge}$ with the proofs in Figure 14, we assign negative polarity to all $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ meta-level atomic formulas. For example, focusing on the formula $(\Rightarrow_E)$ in Figure 15 yields the following derivation, where $\mathcal{K} = \mathcal{L}_{ge} \cup \lfloor \Gamma \rfloor$:

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{K} : \lceil A \Rightarrow B \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lceil A \Rightarrow B \rceil}\ [!, R\Uparrow] \quad \dfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lceil A \rceil}\ [!, R\Uparrow] \quad \dfrac{\vdash \mathcal{K}, \lfloor B \rfloor : \lceil C \rceil \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow ?\lfloor B \rfloor}\ [R\Downarrow, ?]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow !\lceil A \Rightarrow B \rceil \otimes (!\lceil A \rceil \otimes ?\lfloor B \rfloor)}\ [2\times\otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Uparrow}\ [D_2, 2\times\exists]$$

We can repeat this computation for all formulas in $\mathcal{L}_{ge}$ and, in the process, prove the following proposition.

Proposition 9 *Let $\Gamma \cup \{C\}$ be a set of object-level formulas and assume that all meta-level atomic formulas are given a negative polarity. The sequent $\Gamma \vdash C$ is provable in GE if and only if $\vdash \mathcal{L}_{ge}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$ is provable in LLF. Furthermore, adequacy for derivations also holds between the respective proof systems.*

Given this linear logic theory, which encodes natural deduction with generalized elimination rules at our strongest level of adequacy, we turn to showing how $\mathcal{L}_{ge}$ relates back to the sets of formulas shown in Figures 2 and 3.

Proposition 10 *Let $\Gamma \cup \{C\}$ be a set of object logic formulas. Then, if $\vdash_{ll} \mathcal{L}_{ge}, ?\lfloor \Gamma \rfloor, \lceil C \rceil$ then $\vdash_{ll} \mathcal{L}_J, Id_1, Id_2{}', Str_L, ?\lfloor \Gamma \rfloor, \lceil C \rceil$. Furthermore, if $\vdash_{ll} \mathcal{L}_J, Id_1, Str_L, ?\lfloor \Gamma \rfloor, \lceil C \rceil$ then $\vdash_{ll} \mathcal{L}_{ge}, ?\lfloor \Gamma \rfloor, \lceil C \rceil$.*

Proof The second statement is proved in the same lines as in the proof of Proposition 4. For the first statement, we use a theory $\mathcal{L}'_J$, equivalent to $\mathcal{L}_J$, that is obtained by replacing literals of the form $\lfloor C \rfloor^\perp$ by the formula $\lfloor C \rfloor^\perp \parr \perp$, in the clauses $(\vee_L), (\wedge_L), (\Rightarrow_L)$, and $(\forall_L)$ in $\mathcal{L}_J$. Although $\lfloor C \rfloor^\perp$ and $\lfloor C \rfloor^\perp \parr \perp$ are logically equivalent, they have different focusing behaviors, as the latter has negative polarity regardless of the polarity given to $\lfloor C \rfloor$. Now, we assign negative polarity to all meta-level atoms and prove, by induction on the height of proofs, that if $\vdash_{llf} \mathcal{L}_{ge}, \mathcal{F}_1, \mathcal{F}_2 : \lceil C \rceil \Uparrow$ then $\vdash_{llf} \mathcal{L}'_J, Id_1, Id_2{}', Str_L, \mathcal{F}_1 : \mathcal{F}_2, \lceil C \rceil \Uparrow$, where $\mathcal{F}_1 \cup \mathcal{F}_2$ is a multiset of $\lfloor \cdot \rfloor$ meta-level atoms. In this proof, when necessary, we use the formulas $Id_2{}'$ and Str_L in $\mathcal{L}'_J$ to obtain a derivation for a sequent of the form $\vdash \mathcal{L}'_J, Id_1, Id_2{}', Str_L, \mathcal{F}_1 : \mathcal{F}_2, \lfloor C \rfloor^\perp \Uparrow$ with open premise of the form $\vdash \mathcal{L}'_J, Id_1, Id_2{}', Str_L, \mathcal{F}_1, \mathcal{F}_2 : \lceil C \rceil \Uparrow$. The statement follows directly from this intermediate result and the focusing theorem. □

Notice that from the lemma above, $\mathcal{L}_{ge}$'s expressiveness lies between a theory that does not contain $Id_2{}'$ and that theory with $Id_2{}'$. From Corollary 1, however, we know that the $Id_2{}'$ clause is admissible, so the following corollary holds.

Corollary 2 *Let $\Gamma \cup \{C\}$ be a set of object logic formulas. Then*

$$\vdash_{ll} \mathcal{L}_J, Id_1, Id_2{}', Str_L, ?\lfloor \Gamma \rfloor, \lceil C \rceil \ \textit{iff}\ \vdash_{ll} \mathcal{L}_{ge}, ?\lfloor \Gamma \rfloor, \lceil C \rceil$$

$$\frac{\Gamma \vdash A \Rightarrow B \downarrow \quad \Gamma \vdash A \quad \Gamma, B \vdash C \uparrow(\downarrow)}{\Gamma \vdash C \uparrow(\downarrow)} [\Rightarrow GE] \qquad \frac{\Gamma, A \vdash B \uparrow}{\Gamma \vdash A \Rightarrow B \uparrow} [\Rightarrow I]$$

$$\frac{\Gamma \vdash A \wedge B \downarrow \quad \Gamma, A, B \vdash C \uparrow(\downarrow)}{\Gamma \vdash C \uparrow(\downarrow)} [\wedge GE] \qquad \frac{\Gamma \vdash F \uparrow \quad \Gamma \vdash G \uparrow}{\Gamma \vdash F \wedge G \uparrow} [\wedge I]$$

$$\frac{\Gamma \vdash A \vee B \downarrow \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C \uparrow(\downarrow)}{\Gamma \vdash C \uparrow(\downarrow)} [\vee GE] \qquad \frac{\Gamma \vdash A_i \uparrow}{\Gamma \vdash A_1 \vee A_2 \uparrow} [\vee I]$$

$$\frac{\Gamma \vdash \forall x\, A \downarrow \quad \Gamma, A\{t/x\} \vdash C \uparrow(\downarrow)}{\Gamma \vdash C \uparrow(\downarrow)} [\forall GE] \qquad \frac{\Gamma \vdash A\{c/x\} \uparrow}{\Gamma \vdash \forall x\, A \uparrow} [\forall I]$$

$$\frac{\Gamma \vdash \exists x\, A \downarrow \quad \Gamma, A\{c/x\} \vdash C \uparrow(\downarrow)}{\Gamma \vdash C \uparrow(\downarrow)} [\exists GE] \qquad \frac{\Gamma \vdash A\{t/x\} \uparrow}{\Gamma \vdash \exists x\, A \uparrow} [\exists I]$$

$$\frac{}{\Gamma, A \vdash A \downarrow} [\mathrm{I}] \quad \frac{\Gamma \vdash A \downarrow}{\Gamma \vdash A \uparrow} [\mathrm{M}] \quad \frac{\Gamma \vdash A \uparrow}{\Gamma \vdash A \downarrow} [\mathrm{S}] \quad \frac{}{\Gamma \vdash t \uparrow} [tI] \quad \frac{\Gamma \vdash \bot \downarrow}{\Gamma \vdash C \uparrow} [\bot E]$$

Fig. 16 The rules for the natural deduction with generalized elimination rules and with annotated sequents, GEA.

$(\Rightarrow_E)$	$!\lfloor A \Rightarrow B\rfloor^\perp \otimes (!\lceil A\rceil \otimes ?\lfloor B\rfloor)$	$(\Rightarrow_I)$	$\lceil A \Rightarrow B\rceil^\perp \otimes (?\lfloor A\rfloor \mathbin{\wp} \lceil B\rceil)$
$(\wedge_E)$	$!\lfloor A \wedge B\rfloor^\perp \otimes (?\lfloor A\rfloor \mathbin{\wp} ?\lfloor B\rfloor)$	$(\wedge_I)$	$\lceil A \wedge B\rceil^\perp \otimes (\lceil A\rceil \mathbin{\&} \lceil B\rceil)$
$(\vee_E)$	$!\lfloor A \vee B\rfloor^\perp \otimes (?\lfloor A\rfloor \mathbin{\&} ?\lfloor B\rfloor)$	$(\vee_I)$	$\lceil A \vee B\rceil^\perp \otimes (\lceil A\rceil \oplus \lceil B\rceil)$
$(\forall_E)$	$!\lfloor \forall B\rfloor^\perp \otimes ?\lfloor Bx\rfloor$	$(\forall_I)$	$\lceil \forall B\rceil^\perp \otimes \forall x \lceil Bx\rceil$
$(\exists_E)$	$!\lfloor \exists B\rfloor^\perp \otimes \forall x ?\lfloor Bx\rfloor$	$(\exists_I)$	$\lceil \exists B\rceil^\perp \otimes \lceil Bx\rceil$
$(\bot)$	$\lfloor \bot\rfloor^\perp$	(t_I)	$\lceil t\rceil^\perp \otimes \top$
$(\bot_E)$	$\lceil C\rceil^\perp \otimes \bot$		
(Id_1)	$\lfloor B\rfloor^\perp \otimes \lceil B\rceil^\perp$	(Id_2)	$\lfloor B\rfloor \otimes !\lceil B\rceil$

Fig. 17 The specification $\mathcal{L}_{gea}$ for intuitionistic natural deduction with generalized elimination rules.

Although we obtain a theory that encodes GE with the strongest level of adequacy, we find it odd that $\mathcal{L}_{ge}$ does not relate so easily with other intuitionistic/minimal theories, since we used cut-elimination in the object-logic to establish the formal connection. We believe that the system as it is written does not pinpoint exactly where the clause $Id_2{}'$ is needed. A similar problem happens in traditional presentations of natural deductions that do not use annotated sequents and do not contain the $[M]$ and $[S]$ rules (Figure 11). The $[S]$ rule allows a natural deduction proof to have the major premise of an elimination rule be the conclusion of an introduction rule. Negri and von Plato in [NP01] call such pairs of inference rules *detour cuts* and it is these pairs that correspond to the cut rule in sequent calculus. We present a variant of GE, called GEA (Figure 16), that makes these detour cuts apparent by using two types of annotated sequents: $\Gamma \vdash C \uparrow$ and $\Gamma \vdash C \downarrow$. We denote by the judgment $\vdash_{gea}$ provability in GEA (possibly containing the inference rule $[S]$ and, hence, detour cuts) and we denote by the judgment $\vdash_{gea}^d$, provability from GEA without the inference rule $[S]$.

To encode GEA, we use the theory, $\mathcal{L}_{gea}$, shown in Figure 17, and we assign negative polarity to all $\lceil\cdot\rceil$ meta-level atoms and positive polarity to all $\lfloor\cdot\rfloor$ meta-level atoms. As before with natural deduction, the sequents $\Gamma \vdash C \uparrow$ and $\Gamma \vdash C \downarrow$ are encoded by meta-level sequents of the form $\vdash \mathcal{L}_{gea}, \lfloor\Gamma\rfloor : \lceil C\rceil \Uparrow$ and $\vdash \mathcal{L}_{gea}, \lfloor\Gamma\rfloor : \lfloor C\rfloor^\perp \Uparrow$, respectively. Now, the formula $(\Rightarrow_E)$ in $\mathcal{L}_{gea}$ encodes the generalized elimination rule for implication in GEA, as illustrated by the following derivation, where $\mathcal{K} = \mathcal{L}_{gea} \cup \lfloor\Gamma\rfloor$

and F is either $\lceil C \rceil$ or $\lfloor C \rfloor^{\perp}$:

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{K} : \lfloor A \Rightarrow B \rfloor^{\perp} \Uparrow}{\vdash \mathcal{K} : \cdot \Downarrow !\lfloor A \Rightarrow B \rfloor^{\perp}}\ [!, R\Uparrow] \quad \dfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} : \cdot \Downarrow !\lceil A \rceil}\ [!, R\Uparrow] \quad \dfrac{\vdash \mathcal{K}, \lfloor B \rfloor : F \Uparrow}{\vdash \mathcal{K} : F \Downarrow ?\lfloor B \rfloor}\ [R\Downarrow, ?]}{\vdash \mathcal{K} : F \Downarrow !\lfloor A \Rightarrow B \rfloor^{\perp} \otimes (!\lceil A \rceil \otimes ?\lfloor B \rfloor)}\ [2 \times \otimes]}{\vdash \mathcal{K} : F \Uparrow}\ [D_2, 2 \times \exists]$$

We can repeat this style computation of focused derivation for every formula of $\mathcal{L}^d_{gea}$, thereby proving the following proposition.

Proposition 11 *Let $\Gamma \cup \{C\}$ be a set of object-level formulas and let $\mathcal{L}^d_{gea} = \mathcal{L}_{gea} \setminus \{Id_2\}$. Assume that all $\lceil \cdot \rceil$ atomic formulas are given a negative polarity and that all $\lfloor \cdot \rfloor$ atomic formulas are given a positive polarity. Then*

1) $\Gamma \vdash_{gea} C\uparrow$ *iff* $\vdash_{llf} \mathcal{L}_{gea}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$ 2) $\Gamma \vdash^d_{gea} C\uparrow$ *iff* $\vdash_{llf} \mathcal{L}^d_{gea}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$
3) $\Gamma \vdash^d_{gea} C\downarrow$ *iff* $\vdash_{llf} \mathcal{L}^d_{gea}, \lfloor \Gamma \rfloor : \lfloor C \rfloor^{\perp} \Uparrow$.

The following proposition can be proved similarly to the proof of the Lemma 1. This proposition provides the more careful placement of the $Id_2{}'$ meta-level axiom that motivated our introduction of the annotated proof system.

Proposition 12 *Let $\Gamma \cup \{C\}$ be a set of object logic formulas and let $\mathcal{L}^d_{gea} = \mathcal{L}_{gea} \setminus \{Id_2\}$. Then*

1) $\vdash_{ll} \mathcal{L}_J, Id_1, Str_L, W_R, ?\lfloor \Gamma \rfloor, \lceil C \rceil$ *iff* $\vdash_{ll} \mathcal{L}^d_{gea}, ?\lfloor \Gamma \rfloor, \lceil C \rceil$
2) $\vdash_{ll} \mathcal{L}_J, Id_1, Id_2{}', Str_L, W_R, ?\lfloor \Gamma \rfloor, ?\lceil C \rceil$ *iff* $\vdash_{ll} \mathcal{L}_{gea}, ?\lfloor \Gamma \rfloor, \lceil C \rceil$.

Negri and von Plato in [NP01] identify another type of cut, called *permutation cuts*, which occurs whenever the major premise of an elimination rule is the conclusion of another elimination rule. They also propose a different notion of normal proofs, called *general normal form*, for proofs in natural deduction with generalized elimination rules where both detour and permutation cuts do not appear. In particular, derivations in general normal form are such that the major premise of elimination rules are assumptions. In other words, the major premises in the generalized elimination rules shown in Figure 16, are discharged assumptions. We write $\Gamma \vdash^n C$ to denote that there is a general normal form proof of C from assumptions Γ. In our framework, this amounts to enforcing, by the use of polarity assignment to meta-level atoms, that the major premises are present in the set of assumptions. We use the theory $\mathcal{L}^n_{ge}$ obtained from $\mathcal{L}^d_{gea}$, by replacing formulas of the form $!\lfloor C \rfloor^{\perp}$ by $\lfloor C \rfloor^{\perp}$, and assign negative polarity to all atoms of the form $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$, to encode general normal form proofs, represented by the judgment $\vdash^n$.

Proposition 13 *Let $\Gamma \cup \{C\}$ be a set of object-level formulas. Assume that all meta-level atomic formulas are given a negative polarity. Then $\Gamma \vdash^n C$ if and only if $\vdash_{llf} \mathcal{L}^n_{ge}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow$. Furthermore, adequacy for derivations also holds between the respective proof systems.*

Proof Proof by structural induction on the height of derivations. □

Proposition 14 *Let $\Gamma \cup \{C\}$ be a set of object logic formulas. Then*

$$\vdash_{ll} \mathcal{L}_J, Id_1, Str_L, W_R, ?\lfloor \Gamma \rfloor, \lceil C \rceil \text{ iff } \vdash_{ll} \mathcal{L}^n_{ge}, ?\lfloor \Gamma \rfloor, \lceil C \rceil$$

$$\frac{\Gamma \vdash \Delta, A \Rightarrow B \quad \Gamma \vdash \Delta, A \quad \Gamma, B \vdash \Delta}{\Gamma \vdash \Delta}\ [\Rightarrow GE]$$

$$\frac{\Gamma, A \Rightarrow B \vdash \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ [\Rightarrow GI_1] \qquad \frac{\Gamma, A \Rightarrow B \vdash \Delta \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta}\ [\Rightarrow GI_2]$$

$$\frac{\Gamma \vdash \Delta, A_1 \wedge A_2 \quad \Gamma, A_i \vdash \Delta}{\Gamma \vdash \Delta}\ [\wedge GE_i] \qquad \frac{\Gamma, A \wedge B \vdash \Delta \quad \Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta}\ [\wedge GI]$$

$$\frac{\Gamma \vdash \Delta, A \vee B \quad \Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma \vdash \Delta}\ [\vee GE] \qquad \frac{\Gamma, A_1 \vee A_2 \vdash \Delta \quad \Gamma \vdash \Delta, A_i}{\Gamma \vdash \Delta}\ [\vee GI_i]$$

$$\frac{}{\Gamma, A \vdash \Delta, A}\ [\mathrm{I}] \qquad \frac{\Gamma, \neg A \vdash \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ [\neg GI_1] \qquad \frac{\Gamma \vdash \Delta, \neg A \quad \Gamma \vdash \Delta, A}{\Gamma \vdash \Delta}\ [\neg GI_2]$$

Fig. 18 The rules for free deduction, FD.

$$\begin{array}{rlrl}
(\Rightarrow_E) & ?\lceil A \Rightarrow B \rceil \otimes (?\lceil A \rceil \otimes ?\lfloor B \rfloor) & (\Rightarrow_I) & ?\lfloor A \Rightarrow B \rfloor \otimes (?\lfloor A \rfloor \oplus ?\lceil B \rceil) \\
(\wedge_E) & ?\lceil A \wedge B \rceil \otimes (?\lfloor A \rfloor \oplus ?\lfloor B \rfloor) & (\wedge_I) & ?\lfloor A \wedge B \rfloor \otimes (?\lceil A \rceil \,\&\, ?\lceil B \rceil) \\
(\vee_E) & ?\lceil A \vee B \rceil \otimes (?\lfloor A \rfloor \,\&\, ?\lfloor B \rfloor) & (\vee_I) & ?\lfloor A \vee B \rfloor \otimes (?\lceil A \rceil \oplus ?\lceil B \rceil) \\
(\neg GI_1) & ?\lfloor \neg A \rfloor \otimes ?\lfloor A \rfloor & (\neg GI_2) & ?\lceil \neg A \rceil \otimes ?\lceil A \rceil \\
(Id_1) & \lfloor B \rfloor^{\perp} \otimes \lceil B \rceil^{\perp} & &
\end{array}$$

Fig. 19 The specification $\mathcal{L}_{fd}$ for free deduction.

Proof This proposition is proved in a similar way as Proposition 5. □

The following corollary is a direct consequence of Propositions 3, 6, 13, and 14.

Corollary 3 *Let $\Gamma \cup \{C\}$ be a set of formulas. Then $\Gamma \vdash^n C$ if and only if $\Gamma \vdash^f_{lj} C$.*

6 Free Deduction

In [Par92], Parigot introduced the *free deduction* proof system for propositional classical logic that employed both the generalized elimination rules of the previous section and generalized introduction rules[1]. The inference rules for free deduction proof system are given in Figure 18. In order to treat classical negation here, we introduce the negation $\neg B$ directly here and do not treat it as an abbreviation for $B \Rightarrow \perp$.

We use the theory $\mathcal{L}_{fd}$ in Figure 19 to encode free deduction. To obtain the strongest level of adequacy, we assign negative polarity to all meta-level atoms. For example, the formula $(\neg GI_2)$ encodes the inference rule $[\neg GI_2]$, as is illustrated in the following derivation, where $\mathcal{K} = \mathcal{L}_{fd} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil$:

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{K}, \lceil \neg A \rceil :\Uparrow}{\vdash \mathcal{K} :\Downarrow ?\lceil \neg A \rceil}\ [R\Uparrow, ?] \quad \dfrac{\vdash \mathcal{K}, \lceil A \rceil :\Uparrow}{\vdash \mathcal{K} :\Downarrow ?\lceil A \rceil}\ [R\Uparrow, ?]}{\vdash \mathcal{K} :\Downarrow ?\lceil \neg A \rceil \otimes ?\lceil A \rceil}\ [2 \times \otimes]}{\vdash \mathcal{K} :\Uparrow}\ [D_2, \exists]$$

We can repeat this computation for all formulas in $\mathcal{L}_{fd}$ and, in the process, prove the following proposition.

[1] Later and independently, Negri and von Plato also introduced generalized introduction rules in [NP01, p. 214].

Proposition 15 *Let $\Gamma \cup \Delta$ be a set of object-level formulas. Assume that all meta-level atomic formulas are given a negative polarity. Then $\Gamma \vdash \Delta$ is provable in FD iff $\vdash \mathcal{L}_{fd}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil :\Uparrow$ is provable in LLF. Furthermore, adequacy for derivations also holds between the respective proof systems.*

In order to relate the theory $\mathcal{L}_{fd}$ back to other theories, we must first replace $\neg$ by "implies false." We do this by using the operator ϕ inductively on propositional formulas as follows: $\phi(F \blacktriangle G) = \phi(F) \blacktriangle \phi(G)$; for all binary connectives $\blacktriangle$, $\phi(\neg F) = \phi(F) \Rightarrow \bot$; and $\phi(A) = A$ if A is an atom. Moreover, $\phi(\Gamma) = \{\phi(F) \mid F \in \Gamma\}$, where Γ is a multiset of formulas. We offer the following theorem as a means to related the provable formulas of $\mathcal{L}_{fd}$ with those in other classical theories.

Proposition 16 *Let $\Gamma \cup \Delta$ be a set of object logic, propositional classical formulas. Then*

$$\vdash_{ll} \mathcal{L}, Id_1, Id_2, Str_L, ?\lfloor \phi(\Gamma) \rfloor, ?\lceil \phi(\Delta) \rceil \quad \textit{iff} \quad \vdash_{ll} \mathcal{L}_{fd}, ?\lfloor \Gamma \rfloor, ?\lceil \Delta \rceil.$$

Proof The $\Leftarrow$ direction is proved in similar way as Proposition 4, by using the equivalences obtained from the structural and identity rules.

The $\Rightarrow$ direction is proved in similar way as in Proposition 4, by assigning negative polarity to the meta-level atoms. However, for the inductive case when the clause Id_2 is focused on, we use Parigot's observation that any instance of a sequent calculus cut-rule is translated in Free Deduction to a sequence of elimination and introduction rules whose main premises is the cut-formula. $\square$

From Propositions 4 and 16, we have the following relationship between sequents provable in free deduction and those provable in the LK sequent calculus.

Corollary 4 *Let Γ and Δ be sets of propositional, classical formulas. Then $\Gamma \vdash \Delta$ is provable in FD if and only if $\phi(\Gamma) \vdash \phi(\Delta)$ is provable in LK.*

Parigot notes that if one of the premises of the generalized rules is "killed", *i.e.*, it is always the conclusion of an initial rule, then one can obtain either sequent calculus or natural deduction proofs with multiple conclusions. The "killing" of a premise is accounted for in our framework by the use of polarities to enforce the presence of a formula in the context of the sequent. Our encoding of the LK calculus could be explained by just such a focusing restriction. A presentation of a natural deduction with multiple conclusions could be obtained in a similar way as for the natural deduction with single conclusion but with the main difference being that one has to also incorporate the Str_R rule in the theory by adding ? to positive occurrences of $\lceil\cdot\rceil$ atoms and negative occurrences of $\lfloor\cdot\rfloor$ atoms.

7 The Tableaux Proof System KE

In the previous sections, we dealt with systems that contained rules with more premises than the corresponding rules in sequent calculus or natural deduction. Now, we move to the other direction and deal with systems that contain rules with fewer premises.

In [DM94], D'Agostino and Mondadori proposed the propositional tableaux system KE displayed in Figure 20. Here, the only rule that has more than one premise is the cut rule. In the original system, the cut inference rule appears with a side condition limiting cuts to be analytical cuts: since that condition does not seem to be treated naturally in our context, we consider only the unrestricted cut rule.

$$\frac{\Gamma, A, A \Rightarrow B, B \vdash \Delta}{\Gamma, A, A \Rightarrow B \vdash \Delta}\ [\Rightarrow_{L1}] \quad \frac{\Gamma, A \Rightarrow B \vdash A, B, \Delta}{\Gamma, A \Rightarrow B \vdash B, \Delta}\ [\Rightarrow_{L2}] \quad \frac{\Gamma, A \vdash A \Rightarrow B, B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ [\Rightarrow_R]$$

$$\frac{\Gamma, A \wedge B, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ [\wedge_L] \quad \frac{\Gamma, A \vdash A \wedge B, B, \Delta}{\Gamma, A \vdash A \wedge B, \Delta}\ [\wedge_{R1}] \quad \frac{\Gamma, B \vdash A \wedge B, A, \Delta}{\Gamma, B \vdash A \wedge B, \Delta}\ [\wedge_{R1}]$$

$$\frac{\Gamma, A \vee B, B \vdash A, \Delta}{\Gamma, A \vee B \vdash A, \Delta}\ [\vee_{L1}] \quad \frac{\Gamma, A \vee B, A \vdash B, \Delta}{\Gamma, A \vee B \vdash B, \Delta}\ [\vee_{L2}] \quad \frac{\Gamma \vdash A, B, A \vee B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ [\vee_R]$$

$$\frac{\Gamma, \neg A \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ [\neg_L] \quad \frac{\Gamma, A \vdash \neg A, \Delta}{\Gamma \vdash \neg A, \Delta}\ [\neg_R]$$

$$\frac{}{\Gamma, A \vdash A, \Delta}\ [I] \quad \frac{\Gamma, A \vdash \Delta \quad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}\ [Cut]$$

Fig. 20 The rules for the classical propositional logic KE.

$(\Rightarrow_{L1})$	$\lfloor A \Rightarrow B \rfloor^{\perp} \otimes (?\lceil A \rceil \otimes \lceil B \rceil^{\perp})$	$(\Rightarrow_R)$	$\lceil A \Rightarrow B \rceil^{\perp} \otimes (?\lfloor A \rfloor \mathbin{⅋} ?\lceil B \rceil)$
$(\Rightarrow_{L2})$	$\lfloor A \Rightarrow B \rfloor^{\perp} \otimes (\lfloor A \rfloor^{\perp} \otimes ?\lfloor B \rfloor)$	$(\wedge_{R1})$	$\lceil A \wedge B \rceil^{\perp} \otimes (\lfloor A \rfloor^{\perp} \otimes ?\lceil B \rceil)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^{\perp} \otimes (?\lfloor A \rfloor \mathbin{⅋} ?\lfloor B \rfloor)$	$(\wedge_{R2})$	$\lceil A \wedge B \rceil^{\perp} \otimes (?\lceil A \rceil \otimes \lfloor B \rfloor^{\perp})$
$(\vee_{L1})$	$\lfloor A \vee B \rfloor^{\perp} \otimes (\lceil A \rceil^{\perp} \otimes ?\lfloor B \rfloor)$	$(\vee_R)$	$\lceil A \vee B \rceil^{\perp} \otimes (?\lceil A \rceil \mathbin{⅋} ?\lceil B \rceil)$
$(\vee_{L2})$	$\lfloor A \vee B \rfloor^{\perp} \otimes (\lfloor A \rfloor \otimes \lceil B \rceil^{\perp})$		
$(\neg_L)$	$\lfloor \neg A \rfloor^{\perp} \otimes \lceil A \rceil$	$(\neg_R)$	$\lceil \neg A \rceil^{\perp} \otimes \lfloor A \rfloor$
(Id_1)	$\lfloor B \rfloor^{\perp} \otimes \lceil B \rceil^{\perp}$	(Id_2)	$?\lfloor B \rfloor \otimes ?\lceil B \rceil$

Fig. 21 The specification $\mathcal{L}_{ke}$ for the system KE.

To encode KE, we use the theory $\mathcal{L}_{ke}$ in Figure 21. To obtain an adequacy on the level of derivations from $\mathcal{L}_{ke}$, we assign negative polarity to all atoms $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$. As before, the negative occurrences of $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ enforce the presence of formulas in the sequent, but now, $\mathcal{L}_{ke}$ contains formulas with two negative occurrences of meta-level atoms. These formulas encode the KE rules that contain only one premise. For example, the clause $(\Rightarrow_{L2})$ encodes KE's inference rule $[\Rightarrow_{L2}]$, as illustrates the following derivation, where $\mathcal{K} = \mathcal{L}_{ke} \cup \lfloor \Gamma, A \vee B \rfloor \cup \lceil \Delta, A \rceil$:

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathcal{K} :\Downarrow \lfloor A \vee B \rfloor^{\perp}}\ [I_2] \quad \dfrac{}{\vdash \mathcal{K} :\Downarrow \lceil A \rceil^{\perp}}\ [I_2] \quad \dfrac{\vdash \mathcal{K}, \lfloor B \rfloor : \cdot \Uparrow}{\vdash \mathcal{K} :\Downarrow ?\lfloor B \rfloor}\ [R\Downarrow, ?]}{\vdash \mathcal{K} :\Downarrow \lfloor A \vee B \rfloor^{\perp} \otimes (\lceil A \rceil^{\perp} \otimes ?\lfloor B \rfloor)}\ [2 \times \otimes]}{\vdash \mathcal{K} : \cdot \Uparrow}\ [D_2, 2 \times \exists]$$

By checking all the other the inference rules generated by focusing on formulas in $\mathcal{L}_{ke}$, we can conclude with the following proposition.

Proposition 17 *Let $\Gamma \cup \Delta$ be a set of object-level formulas. Assume that all meta-level atomic formulas are given a negative polarity. Then $\Gamma \vdash \Delta$ is provable in KE iff $\vdash \mathcal{L}_{ke}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil :\Uparrow$ is provable in LLF.*

The following proposition is proved by induction on the height of proofs, by taking into consideration the equivalences obtained by the identity and structural rules, and by using the operator ϕ to replace $\neg$ in formulas by its "implies false" meaning.

Proposition 18 *Let $\Gamma \cup \Delta$ be a set of object logic, classical, propositional formulas. Then*

$$\vdash_{ll} \mathcal{L}, Id_1, Id_2, Str_L, Str_R, ?\lfloor \phi(\Gamma) \rfloor, ?\lceil \phi(\Delta) \rceil \quad \textit{iff} \quad \vdash_{ll} \mathcal{L}_{ke}, ?\lfloor \Gamma \rfloor, ?\lceil \Delta \rceil$$

$$\frac{}{\Gamma, A \vee B \vdash A, B, \Delta}\ [\vee_L] \qquad \frac{}{\Gamma, A \vdash A \vee B, \Delta}\ [\vee_{R1}] \qquad \frac{}{\Gamma, B \vdash A \vee B, \Delta}\ [\vee_{R2}]$$

$$\frac{}{\Gamma, A \wedge B \vdash A, \Delta}\ [\wedge_{L1}] \qquad \frac{}{\Gamma, A \wedge B \vdash B, \Delta}\ [\wedge_{L2}] \qquad \frac{}{\Gamma, A, B \vdash A \wedge B, \Delta}\ [\wedge_R]$$

$$\frac{}{\Gamma, A, A \Rightarrow B \vdash B, \Delta}\ [\Rightarrow_L] \qquad \frac{}{\Gamma \vdash A, A \Rightarrow B, \Delta}\ [\Rightarrow_{R1}] \qquad \frac{}{\Gamma, B \vdash A \Rightarrow B, \Delta}\ [\Rightarrow_{R2}]$$

$$\frac{}{\Gamma, \neg A, A \vdash \Delta}\ [\neg_L] \qquad \frac{}{\Gamma \vdash A, \neg A, \Delta}\ [\neg_R]$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}\ [Cut] \qquad \frac{}{\Gamma, A \vdash A, \Delta}\ [I]$$

Fig. 22 Smullyan's Analytic Cut System for classical propositional logic, AC, except that the cut rule is not restricted.

$(\Rightarrow_L)$	$\lfloor A \Rightarrow B \rfloor^\perp \otimes (\lfloor A \rfloor^\perp \otimes \lceil B \rceil^\perp)$	$(\Rightarrow_R)$	$\lceil A \Rightarrow B \rceil^\perp \otimes (\lceil A \rceil^\perp \oplus \lfloor B \rfloor^\perp)$
$(\wedge_L)$	$\lfloor A \wedge B \rfloor^\perp \otimes (\lceil A \rceil^\perp \oplus \lceil B \rceil^\perp)$	$(\wedge_R)$	$\lceil A \wedge B \rceil^\perp \otimes (\lfloor A \rfloor^\perp \otimes \lfloor B \rfloor^\perp)$
$(\vee_L)$	$\lfloor A \vee B \rfloor^\perp \otimes (\lceil A \rceil^\perp \otimes \lceil B \rceil^\perp)$	$(\vee_R)$	$\lceil A \vee B \rceil^\perp \otimes (\lfloor A \rfloor^\perp \oplus \lfloor B \rfloor^\perp)$
$(\neg_L)$	$\lceil \neg A \rceil^\perp \otimes \lceil A \rceil^\perp$	$(\neg_R)$	$\lfloor \neg A \rfloor^\perp \otimes \lfloor A \rfloor^\perp$
(Id_1)	$\lfloor B \rfloor^\perp \otimes \lceil B \rceil^\perp$	(Id_2)	$?\lfloor B \rfloor \otimes ?\lceil B \rceil$

Fig. 23 The theory $\mathcal{L}_{ac}$ used to encode Smullyan's Analytic Cut System AC.

Proof The proof is similar to the proof of Lemma 4. □

The following result, establishing the equivalence between KE and propositional LK, is a direct consequence of Propositions 2, 4, 17 and 18.

Corollary 5 *Let Γ and Δ be a set of propositional formulas. Then $\Gamma \vdash \Delta$ is provable in KE if and only if $\phi(\Gamma) \vdash_{lk} \phi(\Delta)$ is provable in the propositional fragment of LK.*

8 Smullyan's Analytic Cut System

To illustrate how one can capture another extreme in proof systems, we consider Smullyan's proof system for analytic cut (AC) [Smu68a], which is depicted in Figure 22. Here, all rules except the cut rule have no premises. As the name of the system suggests, Smullyan also assigned a side condition to the cut rule, allowing only analytical cuts. As in the previous section, we shall drop this restriction as it is not directly captured in our framework.

We again assign negative polarity to $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ atoms and use the theory $\mathcal{L}_{ac}$, shown in Figure 23, to obtain the strongest level of adequacy. For example, the formula $(\Rightarrow_L)$ corresponds to the inference rule $\Rightarrow_L$ in AC, as illustrates the following derivation, where $\mathcal{K} = \mathcal{L}_{ac} \cup \lfloor \Gamma \rfloor \cup \lceil \Delta \rceil$ such that $A \Rightarrow B, A \in \Gamma$ and $B \in \Delta$:

$$\cfrac{\cfrac{\cfrac{}{\vdash \mathcal{K} : \cdot \Downarrow \lfloor A \Rightarrow B \rfloor^\perp}\ [I_2] \quad \cfrac{}{\vdash \mathcal{K} : \cdot \Downarrow \lfloor A \rfloor^\perp}\ [I_2] \quad \cfrac{}{\vdash \mathcal{K} : \cdot \Downarrow \lceil B \rceil^\perp}\ [I_2]}{\vdash \mathcal{K} : \cdot \Downarrow \lfloor A \Rightarrow B \rfloor^\perp \otimes (\lfloor A \rfloor^\perp \otimes \lceil B \rceil^\perp)}\ [2 \times \otimes]}{\vdash \mathcal{K} : \cdot \Uparrow}\ [D_2, 2 \times \exists]$$

Again, the following proposition follows from repeating such constructions for all formulas in $\mathcal{L}_{ac}$.

Proposition 19 *Let $\Gamma \cup \Delta$ be a set of object-level, classical propositional formulas. Assume that all meta-level atomic formulas are given a negative polarity. Then $\Gamma \vdash \Delta$ is provable in AC iff $\vdash \mathcal{L}_{\mathrm{ac}}, \lfloor \Gamma \rfloor, \lceil \Delta \rceil :\Uparrow$ is provable in LLF. Furthermore, adequacy for derivations also holds between the respective proof systems.*

Again by using the equalities obtained from the identity and structural rules and the operator ϕ, we obtain the following proposition.

Proposition 20 *Let $\Gamma \cup \Delta$ be a set of object logic, classical propositional formulas. Then*

$$\vdash_{ll} \mathcal{L}, Id_1, Id_2, Str_L, Str_R, ?\lfloor \phi(\Gamma) \rfloor, ?\lceil \phi(\Delta) \rceil \quad \textit{iff} \quad \vdash_{ll} \mathcal{L}_{\mathrm{ac}}, ?\lfloor \Gamma \rfloor, ?\lceil \Delta \rceil$$

Proof The proof is similar to the proof of Proposition 4. $\square$

The following result follows directly from the Propositions 2, 4, 19, and 20.

Corollary 6 *Let Γ and Δ be a set of classical, propositional formulas. Then $\Gamma \vdash \Delta$ is provable in AC if and only if $\phi(\Gamma) \vdash \phi(\Delta)$ is provable in the propositional fragment of LK.*

9 Related Work

A number of logical frameworks have been proposed to represent object-level proof systems. Many of these frameworks, as used in [FM88, HHP93, Pfe89], are based on intuitionistic (minimal) logic principles. In such settings, the dualities that we employ here, for example, $\lfloor B \rfloor \equiv \lceil B \rceil^{\perp}$, are not available within the logic and this makes reasoning about the relative completeness between object-level proof systems harder. Also, since minimal logic sequents must have a single conclusion, the storage of object-level formulas is generally done on the left-hand side of meta-level sequents (see [HM94, Pfe00]) with some kind of "marker" for the right-hand side (such as the non-logical "refutation" marker # in [Pfe00]). The flexibility of having the four meta-level literals $\lfloor B \rfloor$, $\lceil B \rceil$, $\lfloor B \rfloor^{\perp}$, and $\lceil B \rceil^{\perp}$ is not generally available in such intuitionistic systems. While it is natural in classical linear logic to consider having some atoms assigned negative and some positive polarities, most intuitionistic systems consider only uniform assignments of polarities to meta-level atoms (usually negative in order to support goal-directed proof search): the ability to mix polarity assignments for different meta-level atoms can only be achieved in more indirect fashions in such settings.

The abstract logic programming presentation of linear logic called Forum [Mil96] has been used to specify sequent calculus proof systems in a style similar to that used here. That presentation of linear logic was, however, also limited in that negation was not a primitive connective and that all atomic formulas were assumed to have negative polarity. The range of encodings contained in this paper are not directly available using Forum.

In [CGT08], Ciabattoni *et al.* consider a general approach to the specification of structural rules in sequent calculus which differs from our approach of specifying structural rules. In particular, their method would not use the exponentials of linear logic, as we do in the clauses Str_L and Str_R, but would rather treat structural rules more explicitly by having rules of the form

$$\lfloor B \rfloor^{\perp} \otimes (\lfloor B \rfloor \parr \lfloor B \rfloor)$$

to encode the contraction-left rules (of the sequent calculus). It is worth noting that while the Str_R formula allows for *both* weakening and contraction on the right, there is no corresponding modal operator in linear logic that allows for just weakening: hence, we also must also use the explicit weakening rule W_R when we only want weakening. It is possible to extend linear logic with *subexponentials* [NM09] that include exponential-like operators that allow, for example, formulas to be weaken but not contracted. One could imagine using such a subexponential, instead of the rule W_R, to specify the structural rules for intuitionistic logics. Exploring the use of subexponentials to specify proof systems is left for future work.

10 Conclusions and Further Remarks

We have shown that by employing different focusing annotations or using different sets of formulas that are (meta-logically) equivalent to $\mathcal{L}$, a range of sound and (relatively) complete object-level proof systems can be encoded. We have illustrated this principle by showing how linear logic focusing and logical equivalences can account for object-level proof systems based on sequent calculus, natural deduction, generalized introduction and elimination rules, free deduction, the tableaux system KE, and Smullyan's AC system employing only axioms and the cut rule.

Logical frameworks aim at allowing proof systems to be specified using compact and declarative specifications of inference rules. It now seems that a much broader range of possible proof systems can be further specified by allowing flexible assignment of polarity to meta-logical atoms (instead of making the usual assignment of some fixed, global polarity assignment). A natural next step would be to see what insights might be carried from this setting of linear-intuitionistic-classical logic to other, say, intermediate or sub-structural logics.

While focusing at the meta-level clearly provides a powerful normal form of proof, we have not described how to use the techniques presented in this paper to derive object-level focusing proof systems. Finding a means to derive such object-level normal form proofs is an interesting challenge that we plan to develop next.

Another interesting line of future research would be to consider differences in the *sizes* of proofs in these different paradigms since these differences can be related to the topic of comparing bottom-up and top-down deduction. Thus, it might be possible to flexibly change polarity assignments that would result in different and, hopefully, more compact presentations of proofs.

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11 Appendix: some inference rules and their linear logic encodings

We list below several examples of how natural deduction rules are accounted for by focused deduction in linear logic. The following correspondences can be used to prove Proposition 7. In the derivations below, $\mathcal{K} = \mathcal{L} \cup \{Str_L, Id_1, Id_2\} \cup \lfloor \Gamma \rfloor$ and all $\lceil \cdot \rceil$ given negative polarity and all $\lfloor \cdot \rfloor$ are given positive polarity.

$$\frac{}{\Gamma, C \vdash C \downarrow}\,[\mathrm{Ax}] \quad \leftrightsquigarrow \quad \cfrac{\cfrac{}{\vdash \mathcal{K}, \lfloor C \rfloor : \lfloor C \rfloor^{\perp} \Downarrow \lfloor C \rfloor}\,[I_1]}{\vdash \mathcal{K}, \lfloor C \rfloor : \lfloor C \rfloor^{\perp} \Uparrow}\,[D_2]$$

$$\frac{\Gamma \vdash A \downarrow}{\Gamma \vdash A \uparrow}\,[\mathrm{M}] \quad \leftrightsquigarrow \quad \cfrac{\cfrac{\cfrac{\vdash \mathcal{K} : \lfloor C \rfloor^{\perp} \Uparrow}{\vdash \mathcal{K} : \Downarrow \lfloor C \rfloor^{\perp}}\,[R\Downarrow, R\Uparrow] \qquad \cfrac{}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow \lceil C \rceil^{\perp}}\,[I_1]}{\vdash \mathcal{K} : \lceil C \rceil \Downarrow \lfloor C \rfloor^{\perp} \otimes \lceil C \rceil^{\perp}}\,[\otimes]}{\vdash \mathcal{K} : \lceil C \rceil \Uparrow}\,[D_2, \exists]$$

$$\frac{\Gamma \vdash A \uparrow}{\Gamma \vdash A \downarrow}\,[\mathrm{S}] \quad \leftrightsquigarrow \quad \cfrac{\cfrac{\cfrac{}{\vdash \mathcal{K} : \lfloor C \rfloor^{\perp} \Downarrow \lfloor C \rfloor}\,[I_1] \qquad \cfrac{\vdash \mathcal{K} : \lceil C \rceil \Uparrow}{\vdash \mathcal{K} : \Downarrow !\lceil C \rceil}\,[!, R\Uparrow]}{\vdash \mathcal{K} : \lfloor C \rfloor^{\perp} \Downarrow \lfloor C \rfloor \otimes !\lceil C \rceil}\,[\otimes]}{\vdash \mathcal{K} : \lfloor C \rfloor^{\perp} \Uparrow}\,[D_2, \exists]$$

$$\frac{\Gamma \vdash F \uparrow \quad \Gamma \vdash G \uparrow}{\Gamma \vdash F \wedge G \uparrow}\,[\wedge I] \quad \leftrightsquigarrow$$

$$\cfrac{\cfrac{\cfrac{}{\vdash \mathcal{K} : \lceil F \wedge G \rceil \Downarrow \lceil F \wedge G \rceil^{\perp}}\,[I_1] \qquad \cfrac{\vdash \mathcal{K} : \lceil F \rceil \Uparrow}{\vdash \mathcal{K} : \Downarrow \lceil F \rceil}\,[R\Downarrow, R\Uparrow] \qquad \cfrac{\vdash \mathcal{K} : \lceil G \rceil \Uparrow}{\vdash \mathcal{K} : \Downarrow \lceil G \rceil}\,[R\Downarrow, R\Uparrow]}{\vdash \mathcal{K} : \lceil F \wedge G \rceil \Downarrow \lceil F \wedge G \rceil^{\perp} \otimes \lceil F \rceil \otimes \lceil G \rceil}\,[2 \times \otimes]}{\vdash \mathcal{K} : \lceil F \wedge G \rceil \Uparrow}\,[D_2, 2 \times \exists]$$

$$\frac{\Gamma \vdash F \wedge G \downarrow}{\Gamma \vdash F \downarrow}\,[\wedge E] \quad \leftrightsquigarrow \quad \cfrac{\cfrac{\cfrac{\vdash \mathcal{K} : \lfloor F \wedge G \rfloor^{\perp} \Uparrow}{\vdash \mathcal{K} : \Downarrow \lfloor F \wedge G \rfloor^{\perp}}\,[R\Downarrow, R\Uparrow] \qquad \cfrac{}{\vdash \mathcal{K} : \lfloor F \rfloor^{\perp} \Downarrow \lfloor F \rfloor}\,[I_1]}{\vdash \mathcal{K} : \lceil F \wedge G \rceil \Downarrow \lfloor F \wedge G \rfloor^{\perp} \otimes (\lfloor F \rfloor \oplus \lfloor G \rfloor)}\,[\otimes, \oplus_l]}{\vdash \mathcal{K} : \lfloor F \rfloor^{\perp} \Uparrow}\,[D_2, 2 \times \exists]$$

$$\frac{\Gamma \vdash A_i \uparrow}{\Gamma \vdash A_1 \vee A_2 \uparrow}\,[\vee I], i \in \{1, 2\} \quad \leftrightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathcal{K} : \lceil A_1 \vee A_2 \rceil \Downarrow \lceil A_1 \vee A_2 \rceil^{\perp}}\,[I_1] \qquad \dfrac{\vdash \mathcal{K} : \lceil A_i \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow \lfloor A_1 \rfloor \oplus \lceil A_2 \rceil}\,[\oplus_{lr}, R\Downarrow, R\Uparrow]}{\vdash \mathcal{K} : \lceil A_1 \vee A_2 \rceil \Downarrow \lceil A_1 \vee A_2 \rceil^{\perp} \otimes (\lfloor A_1 \rfloor \oplus \lceil A_2 \rceil)}\,[2 \times \otimes]}{\vdash \mathcal{K} : \lceil A_1 \vee A_2 \rceil \Uparrow}\,[D_2, 2 \times \exists]$$

$$\dfrac{\Gamma \vdash A \vee B \downarrow \quad \Gamma, A \vdash C \uparrow\!(\downarrow) \quad \Gamma, A \vdash C \uparrow\!(\downarrow)}{\Gamma \vdash C \uparrow\!(\downarrow)}\,[\vee E] \qquad \leadsto$$

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{K} : \lfloor A \vee B \rfloor^{\perp} \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lfloor A \vee B \rfloor^{\perp}}\,[!, R\Uparrow] \quad \dfrac{\dfrac{\vdash \mathcal{K}, \lfloor A \rfloor : \lceil C \rceil\, (\lfloor C \rfloor^{\perp}) \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil\, (\lfloor C \rfloor^{\perp}) \Uparrow ?\lfloor A \rfloor}\,[?] \quad \dfrac{\vdash \mathcal{K}, \lfloor B \rfloor : \lceil C \rceil\, (\lfloor C \rfloor^{\perp}) \Uparrow}{\vdash \mathcal{K} : \lceil C \rceil\, (\lfloor C \rfloor^{\perp}) \Uparrow ?\lfloor B \rfloor}\,[?]}{\vdash \mathcal{K} : \lceil C \rceil\, (\lfloor C \rfloor^{\perp}) \Downarrow ?\lfloor A \rfloor \,\&\, ?\lfloor B \rfloor}\,[R\Downarrow, \&]}{\vdash \mathcal{K} : \lceil C \rceil\, (\lfloor C \rfloor^{\perp}) \Downarrow !\lfloor A \vee B \rfloor^{\perp} \otimes (?\lfloor A \rfloor \,\&\, ?\lfloor B \rfloor)}\,[\otimes]}{\vdash \mathcal{K} : \lceil C \rceil\, (\lfloor C \rfloor^{\perp}) \Uparrow}\,[D_2, 2 \times \exists]$$

$$\dfrac{\Gamma, A \vdash B \uparrow}{\Gamma \vdash A \Rightarrow B \uparrow}\,[\Rightarrow I] \qquad \leadsto$$

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathcal{K} : \lceil A \Rightarrow B \rceil \Downarrow \lceil A \Rightarrow B \rceil^{\perp}}\,[I_1] \quad \dfrac{\dfrac{\vdash \mathcal{K}, \lfloor A \rfloor : \lceil B \rceil \Uparrow}{\vdash \mathcal{K} :\Uparrow ?\lfloor A \rfloor, \lceil B \rceil}\,[?, R\Uparrow]}{\vdash \mathcal{K} :\Downarrow ?\lfloor A \rfloor \parr \lceil B \rceil}\,[R\Downarrow, \parr]}{\vdash \mathcal{K} : \lceil A \Rightarrow B \rceil \Downarrow \lceil A \Rightarrow B \rceil^{\perp} \otimes (?\lfloor A \rfloor \parr \lceil B \rceil)}\,[\otimes]}{\vdash \mathcal{K} : \lceil A \Rightarrow B \rceil \Uparrow}\,[D_2, 2 \times \exists]$$

$$\dfrac{\Gamma \vdash A \Rightarrow B \downarrow \quad \Gamma \vdash A \uparrow}{\Gamma \vdash B \downarrow}\,[\Rightarrow E] \qquad \leadsto$$

$$\dfrac{\dfrac{\dfrac{\vdash \mathcal{K} : \lfloor A \Rightarrow B \rfloor^{\perp} \Uparrow}{\vdash \mathcal{K} :\Downarrow \lfloor A \Rightarrow B \rfloor^{\perp}}\,[R\Downarrow, R\Uparrow] \quad \dfrac{\vdash \mathcal{K} : \lceil A \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow !\lceil A \rceil}\,[!, R\Uparrow] \quad \dfrac{}{\vdash \mathcal{K} : \lfloor B \rfloor^{\perp} \Downarrow \lfloor B \rfloor}\,[I_1]}{\vdash \mathcal{K} : \lfloor B \rfloor^{\perp} \Downarrow \lfloor A \Rightarrow B \rfloor^{\perp} \otimes (!\lceil A \rceil \otimes \lfloor B \rfloor)}\,[2 \times \otimes]}{\vdash \mathcal{K} : \lfloor B \rfloor^{\perp} \Uparrow}\,[D_2, 2 \times \exists]$$

$$\dfrac{}{\Gamma \vdash t \uparrow}\,[tI] \qquad \leadsto \qquad \dfrac{\dfrac{\dfrac{}{\vdash \mathcal{K} : \lceil t \rceil \Downarrow \lceil t \rceil^{\perp}}\,[I_1] \quad \dfrac{}{\vdash \mathcal{K} :\Downarrow \top}\,[R\Downarrow, \top]}{\vdash \mathcal{K} : \lceil t \rceil \Downarrow \lceil t \rceil^{\perp} \otimes \top}\,[\otimes]}{\vdash \mathcal{K} : \lceil t \rceil \Uparrow}\,[D_2]$$

$$\dfrac{\Gamma \vdash \bot \downarrow}{\Gamma \vdash C \uparrow}\,[\bot E] \qquad \leadsto \qquad \dfrac{\dfrac{\dfrac{}{\vdash \mathcal{K}, \lfloor \Gamma \rfloor : \lceil C \rceil \Downarrow \lceil C \rceil^{\perp}}\,[I_1] \quad \dfrac{\vdash \mathcal{K}, \lfloor \Gamma \rfloor : \cdot \Uparrow}{\vdash \mathcal{K}, \lfloor \Gamma \rfloor :\Downarrow \bot}\,[R\Downarrow, \bot]}{\vdash \mathcal{K}, \lfloor \Gamma \rfloor : \lceil C \rceil \Downarrow \lceil C \rceil^{\perp} \otimes \bot}\,[\otimes]}{\vdash \mathcal{K}, \lfloor \Gamma \rfloor : \lceil C \rceil \Uparrow}\,[D_2, \exists]$$

$$\frac{\Gamma \vdash A\{c/x\} \uparrow}{\Gamma \vdash \forall x\, A \uparrow}\ [\forall I] \quad \leftrightsquigarrow \quad \dfrac{\dfrac{}{\vdash \mathcal{K} : \lceil \forall x\, A \rceil \Downarrow \lceil \forall x\, A \rceil^{\perp}}\ [I_1] \quad \dfrac{\vdash \mathcal{K} : \lceil A\{c/x\} \rceil \Uparrow}{\vdash \mathcal{K} :\Downarrow \forall x\, \lceil A \rceil}\ [R\Downarrow, \forall, R\Uparrow]}{\dfrac{\vdash \mathcal{K} : \lceil \forall x\, A \rceil \Downarrow \lceil \forall x\, A \rceil^{\perp} \otimes \forall x\, \lceil A \rceil}{\vdash \mathcal{K} : \lceil \forall x\, A \rceil \Uparrow}\ [D_2, \exists]}\ [\otimes]$$

$$\frac{\Gamma \vdash \forall x\, A \downarrow}{\Gamma \vdash A\{t/x\} \downarrow}\ [\forall E] \quad \leftrightsquigarrow$$

$$\dfrac{\dfrac{\vdash \mathcal{K} : \lfloor \forall x\, A \rfloor^{\perp} \Uparrow}{\vdash \mathcal{K} :\Downarrow \lfloor \forall x\, A \rfloor^{\perp}}\ [R\Downarrow, \forall, R\Uparrow] \quad \dfrac{}{\vdash \mathcal{K} : \lfloor A\{t/x\} \rfloor^{\perp} \Downarrow \lfloor A\{t/x\} \rfloor}\ [I_1]}{\dfrac{\vdash \mathcal{K} : \lfloor A\{t/x\} \rfloor^{\perp} \Downarrow \lfloor \forall x\, A \rfloor^{\perp} \otimes \lfloor A\{t/x\} \rfloor}{\vdash \mathcal{K} : \lfloor A\{t/x\} \rfloor^{\perp} \Uparrow}\ [D_2, \exists]}\ [\otimes]$$

The pairing for the $\exists I$ and $\exists E$ rules are similar.