Focusing, Axioms and Synthetic Inference Rules (Extended Abstract)

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Proving a sequent in sequent-based systems often involves many choices. For example, at every node of a tree-derivation one could: apply an introduction rule over a non-atomic formula; apply a structural rule; introduce a lemma; apply initial rules, etc. Hence, there is a need for *discipline* in structuring such choices and taming proof-search. One such discipline is *focusing* [1].

Focused proof systems combines two devices: *polarization* and *focused rule application*. In classical/intuitionistic first order systems, polarized formulas are built using atomic formulas and polarized versions of logical connectives and constants. The positive and negative versions of connectives and constants have identical truth conditions but have different inference rules inside the polarized proof systems. For example, left introduction rules for positive connectives are invertible while left introduction rules for negative connectives are not necessarily invertible. The polarity of a non-atomic formula is determined by its top-level connective. Since every polarized formula is classified as positive or negative, a polarity to atomic formulas must also be provided. As it turns out, this assignment of polarity to atomic formulas can be arbitrary [1].

When *focusing* on a formula, the focus is transferred to the active formulas in the premises (focused rule applications). This process goes on in all branches of the derivation, until: an initial rule/introduction rule on constants is applied (and the derivation ends at that branch); either the polarity of the focused formula changes *or* the side (left/right) of the focus flips (but not both). In this case, focus is *released* and the formula is eagerly decomposed into its negative-left, positive-right and/or atomic subformulas, that are *stored* in the context. Reading derivations from the root upwards, this forces a sequent derivation to be organized into *focused phases*, each of them corresponding to an application of a *synthetic inference rule* [2], where the focused formula is rewritten into (some of) its subformulas.

There is a class of formulas corresponding to particularly interesting synthetic rules: the *bipolars*. Bipolars are negative formulas in which polarity can change at most once among its subformulas. This means that left-focusing on a bipolar

A gives rise to (possibly many) synthetic inference rules having simple shape, with leaves involving only atomic subformulas of A. We call a synthetic inference rule corresponding to the bipolar A a *bipole* for A.

In this talk, we will present a careful study of bipoles, giving a fresh view to an old problem: how to incorporate inference rules encoding axioms into proof systems for classical and intuitionistic logics.

We start by considering *LKF* and *LJF* [6,7] as the basic focused proof systems for classical and intuitionistic logics, respectively. In such systems, leafs of focused phases can be composed of either: (i) a conclusion-sequent of the application of introduction rule on constants; (ii) a (focused) conclusion-sequent of the application of the initial rule; (iii) an (unfocused) sequent after the storage of the remaining formulas. As an example, consider the following first order formula, that relates the subset and membership predicates in set theory:

$$A = \forall yz.(\forall x.x \in y \supset x \in z) \supset y \subseteq z.$$

Assuming that the predicate $\subseteq$ is given *negative* polarity, in the focused phase given by (left) focusing on A

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, x \in y \Uparrow \cdot \vdash \cdot \Uparrow x \in z, \Delta}{\Gamma \Uparrow x \in y \vdash x \in z \Uparrow \Delta}\ \mathsf{store}_l, \mathsf{store}_r}{\Gamma \Uparrow \cdot \vdash \forall x.x \in y \supset x \in z \Uparrow \Delta}\ \forall_r, \supset_r}{\Gamma \vdash \forall x.x \in y \supset x \in z \Downarrow \Delta}\ \mathsf{release}_r \qquad \dfrac{}{\Gamma \Downarrow y \subseteq z \vdash \Delta}\ \mathsf{init}_l}{\Gamma \Downarrow (\forall x.x \in y \supset x \in z) \supset y \subseteq z \vdash \Delta}\ \supset_l}{\Gamma \Downarrow \forall yz.(\forall x.x \in y \supset x \in z) \supset y \subseteq z \vdash \Delta}\ \forall_l$$

the right leaf has shape (ii) while the left one is of the form (iii). The formula between the $\Downarrow$ and $\vdash$ is the *focus* of that sequent.

Observe that it must be the case that $y \subseteq z \in \Delta$ (since $y \subseteq z$ is atomic, negative and under focus), while $x \in y, x \in z$ end-up being stored into contexts. This is not by chance: restricted to bipoles, leaves of the shape (ii) *forces atoms to belong to the context*, while leaves of the shape (iii) *adds atoms to the context*. This implies that principal and active formulas in bipoles for A (if any) are atomic formulas. That is: bipoles can be seen, in a sense, as introduction rules for *atoms*. For example, the bipolar above corresponds to the (unpolarized) synthetic rule

$$\frac{x \in y, \Gamma \vdash x \in z, \Delta}{\Gamma \vdash y \subseteq z, \Delta}\ .$$

which introduces $y \subseteq z$ from $x \in y$ and $x \in z$, where x is an eigenvariable.

Using such synthetic inference rules is one method for systematically generating proof systems for axiomatic extensions of classical/intuitionistic logics: focusing on a bipolar axiom yields a bipole.

A key step in transforming a formula into synthetic inference rules involves attaching a polarity to atomic formulas and to some logical connectives. Since there are different choices for assigning polarities, it is possible to produce different synthetic inference rules for the same formula. Indeed, if in our running example the predicate $\subseteq$ is given *positive* polarity, the corresponding (unpolarized) synthetic rule is

$$\frac{x \in y, \Gamma \vdash x \in z, \Delta \qquad y \subseteq z, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \ .$$

with x an eigenvariable.

We show that this flexibility allows for the generalization of different approaches for transforming axioms into sequent rules present in the literature, such as the works in [11,12,4,5]. In particular, bipolars correspond to (the first-order version of) the $\mathcal{N}_2$ class presented in [3], which subsumes the class of geometric axioms studied in [11,10].

We finish the talk by showing how to emulate precisely rules for modalities in labeled modal systems as synthetic connectives [9,8]. Such tight emulation means that proof search/proof checking on the focused version of the translated formulas models exactly proof search/proof checking in the correspondent labeled system. As a result, we are able to show that we can use focused proofs to precisely emulate modal proofs whenever Kripke frames are characterized by bipolar properties.

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