GIL: A Generalization of Intuitionistic Logic

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Abstract. We introduce GIL, a new logic inspired by linear logic, focusing and polarization. GIL is a *unified* logic in which connectives from intuitionistic, classical and linear logic can mix with few restrictions. Systems that resemble it include Girard's logic *LU,* although GIL is best considered a generalization of intuitionistic principles. Intuitionistic logic is seen as a less-complete version of GIL. The logic is organized around three axes of polarization which correspond to three distinct *layers* of focalization. The need for additional polarities is explained in the context of cut-elimination. We demonstrate the strength of this approach relative to linear logic and related systems. An extensive set of structural rules allow for the uniform presentation of GIL as a focused sequent calculus.

1 Introduction

With the advent of linear logic, it has become possible to analyze the differences between intuitionistic and classical logic in a wider context. The separation of connectives into additive and multiplicative has revealed that each logic is but a sub-system of a larger set of possibilities. The use of *polarization* and the introduction of *focusing* (or *focalization*) by Girard [4] and Andreoli [1] have revealed even more subtle characteristics of these logics. With the recognition of the role of polarities in classical logic, Girard introduced LC. A more aggressive use of polarization is found in the unified logic LU [5]. Concepts found in LU are clearly related to focusing (but LU itself is not focused). They have impacted several subsequent studies such as [2, 9], as well our own work in developing LJF and LKF, focused proofs systems for intuitionistic and classical logics [10]. In [11] we also introduced a unified system based on focusing using a large number of connectives, but which had a new notion of polarities and their relationship to structural rules. The system, called LKU, was successful in generalizing characteristics of classical, intuitionistic and multiplicative-additive linear logic (MALL), including a set of generalized criteria for cut-elimination and completeness. But in investigating ways to define new logics that combined classical and linear characteristics beyond intuitionistic logic, the system met with limitations.

We found that existing analyses of polarization, including those found in LU and LKU, were incomplete. In short, *there were not enough polarities.* The new dimension of polarization that we introduce here, called left-versus-right, is a generalization of the familiar intuitionistic principle due to Gentzen [3] (see also [7]). We consider the new logic as appropriately called *Generalized Intuitionistic Logic.* GIL is a *unified* logic with a focused proof system. Along with the polarization of classical logic identified by Girard, and of linear logic identified by Andreoli, we develop a system based on three axes of polarization. The rationale for our new polarities is best explained alongside cut-elimination and we do in Section 7. Each axis holds a significant logic: respectively MALL, classical logic, and the purely negative fragment of intuitionistic logic. In this context we see that full intuitionistic logic, with $\vee$, $\exists$ and *false,* is already a *hybrid* or *mixed* system that incorporates elements from other axes. In this broader arena it is possible to relax the restrictions of intuitionistic logic and allow it to be more thoroughly combined with other logics. MALL is found as a simple fragment of GIL. Classical logic does not require a double-negation translation to be embedded: instead, classical logic results from simply selecting the appropriate connectives and the form of *end-sequent.* Furthermore, GIL is much more than the disjoint union of classical, linear and intuitionistic logics. Formulas of any polarity can mix with few restrictions.

As an example of the application of such a mixing of formulas, consider a logic programming language based on intuitionistic logic (such as λProlog). A set of definite clauses of such a language may appear in the form $(G_1 \supset D_1) \wedge (G_2 \supset D_2) \wedge \ldots$. We can mix linear connectives into such a program so as to provide greater control over the execution of the logic program. For example, to impose that only one of these clauses can be selected, we can rewrite the program as: $(G_1 \supset (D_1 \parr a)) \wedge (G_2 \supset (D_2 \parr a)) \wedge \ldots$. Here, a is a linear atomic formula which serves as a token that gives permission for the clause to run. The execution of a "query" or goal Q can be replaced by $Q \parr a$, which asserts the consumable token. Moreover, we are assured that the rest of the computation is entirely intuitionistic.

Although full linear logic has also been used for similar purposes, GIL enjoys greater invariants. Comparing GIL to (full) linear logic is analogous to comparing a high-level programming language to assembly language. Instead of the operators ! and ?, GIL relies on polarity information. We show that its richness in polarity information not only adequately replaces the exponential operators but also represents a greater expressiveness. In particular, we will formulate a stronger representation of intuitionistic logic, one that not only preserves *provability* but which adequately captures focused *proofs,* including *partial* proofs (see Section 5). As we show in Section 6, full linear logic is still important in the analysis of GIL since we present a translation of GIL into linear logic. This translation allows GIL to inherit some of the semantics of linear logic.

2 Synthetic Connectives, Structural Rules, and Focusing

A core principle behind focused proof systems is that it enables the formulation of *synthetic logical connectives.* In particular, linear logic, with its binary connectives $\oplus, \otimes, \parr, \&$, their units, and the exponentials $!, ?$, provides a rich framework for studying this concept. We will use linear logic to help guide the construction of synthetic connectives for classical and intuitionistic logics.

To what extent can connectives be combined to form new ones? As remarked by Girard [6], introduction rules for synthetic connectives should support *initial elimination,* i.e., the principle that initial sequents $\vdash A, A^\perp$ (or $A \vdash A$) can be derived from atomic instances of initial sequents. To see how this principle can fail, consider the combination

$\otimes\&$ as a "synthetic" connective: *e.g.,* the "synthetic introduction rule" for the formula $A \otimes (B \mathbin{\&} C)$ should result from combining the usual rules for $\otimes$ and $\&$ as:

$$\frac{\vdash A, \Delta_1 \quad \vdash B, \Delta_2 \quad \vdash C, \Delta_2}{\vdash A \otimes (B \mathbin{\&} C), \Delta_1 \Delta_2} \ \otimes\&$$

Such a rule may seem valid, but one must also consider its dual, $A \parr (B \oplus C)$, for which the following introduction rules are immediate:

$$\frac{\vdash A, B, \Delta'}{\vdash A \parr (B \oplus C), \Delta'} \ \parr\oplus \qquad \frac{\vdash A, C, \Delta'}{\vdash A \parr (B \oplus C), \Delta'} \ \parr\oplus$$

These rules are clearly sound, and cut-elimination is preserved by them since a cut between the introduced formulas can be reduced to cuts on their sub-formulas. But initial-elimination fails: one cannot prove $\vdash A \otimes (B \mathbin{\&} C), A^{\perp} \parr (B^{\perp} \oplus C^{\perp})$ using *these* introduction rules.

The four combinations of $\otimes\&$, $\otimes\parr$, $\oplus\&$ and $\oplus\parr$ (and their duals) all fail the initial-elimination test since they and their duals cannot be assigned acceptable introduction rules. Fortunately, the *focused proof system* of Andreoli [1] provides a general recipe for the construction of synthetic connectives: in that system, connectives are divided into *negatives* (invertible right-introduction rules) $\parr, \&, \forall$ and *positives* (non-invertible right-introduction rules) $\oplus, \otimes, \exists$. These two sets of connectives are De Morgan duals of each other and any collection of connectives of the *same* polarity forms a proper synthetic connective. We *focus* on a positive formula and maintain that focus on its immediate positive subformulas: focus is then broken when negative subformulas are encountered. A positive synthetic connective is then described as the choice of positive formula and the extent to which focus is maintained on it. We shall also use the adjectives *asynchronous* and *synchronous* instead of negative and positive, respectively, when they are applied to connectives, inference rules, and phases of a focused proof.

Significantly, the above analysis extends to formulas containing the exponential operators ! and ?. Here, focalization reveals further subtleties. Focus must terminate with formulas such as $!(A \oplus B)$, *i.e.*, a ! before a positive formula. The initial-elimination test also shows why forms such as $!(A \oplus B)$ and $?(A \parr B)$ as well as $?!(A \mathbin{\&} B)$ and $!?(A \oplus B)$ cannot be considered synthetic connectives.

The connectives of classical and intuitionistic logic are, in fact, equivalent to valid synthetic connectives of linear logic. The connectives of GIL will follow this principle.

Focusing must stop when structural rules are needed, which, in linear logic, means that an exponential is encountered. For example, $\vdash A, B, B^{\perp}, A^{\perp} \otimes ?(C \oplus D)$ is only provable by weakening on $?(C \oplus D)$. *How far can focus continue* is bounded by *when are structural rules necessary.* Focalization is just as valid in classical and intuitionistic logic as it is in linear logic precisely because structural rules can be confined to the boundaries between the synchronous and asynchronous phases. Focusing clarifies the distinction between introduction rules and structural rules. The equivalence $?(?A \parr ?B) \equiv ?A \parr ?B$ (and similarly for $\&$) explains focusing in classical logic: structural rules can be *delayed* until subformulas of a different polarity are encountered. Similarly, the equivalence $?(?A \oplus ?B) \equiv ?(A \oplus B)$ suggests that structural rules can be applied *early,* to the formula at the outset as opposed to its subformulas. These equivalences also show how different modes of focusing can interact when formulas from different logics mix: for example, it is valid to shift from a linear focusing mode to a classical one, but not vice versa.

In this context, it also becomes possible to extend the notion of "structural" rules to include more than just contraction and weakening: other rules are also at work at the borders of the focusing phases. Andreoli's system, which we refer to as LLF, contains a rich set of such rules that govern the classification of formulas and the termination of focus. In fact, the rules for ! and ? in LLF have more in common with these rules than other introduction rules. We have already shown in [11] how the introduction rules can be fixed and variations on the extended structural rules alone can be used to define logics. In this style of focused sequent calculi, structural rules are rules that mark the boundaries between positive and negative phases: they *react* to changes in polarity.

In GIL, the most important rules are not the introduction rules, which are mostly obvious, but an extended set of structural rules and their provisos based on polarity information. GIL is thus naturally presented as a focused sequent calculus.

3 Axes of Polarization

While it may be possible to identify several dimensions of polarization, we find three sufficient. One of the first important uses of "polarity" can be attributed to Gentzen. The left-right polarization is crucial for intuitionistic logic, and we have chosen to name our *left* and *right* polarities in a way that is consistent with intuitionistic logic. We now introduce the several logical connectives for GIL along with their *polarity classifications*.

Positive Left: $\boxplus$, $\boxtimes$, $\exists^l$, 0_l, positive-left literals. "$+L$" polarity. Right-permeable.
Positive Right: $\vee^+$, $\wedge^+$, $\exists$, 0_r, 1_r, literals. "$+R$" polarity. Left-permeable.
Positive Linear: $\otimes$, $\oplus$, Σ, 0, 1, literals. "$+1$" polarity. non-permeable.
Negative Linear: $\parr$, $\&$, Π, $\top$, $\bot$, literals. "-1" polarity. non-permeable.
Negative Left: $\vee^-$, $\wedge^-$, $\forall$, $\bot_l$, $\top_l$, literals. "$-L$" polarity. Right-permeable.
Negative Right: $\sqcup$, $\sqcap$, $\forall^r$, $\top_r$, literals. "$-R$" polarity. Left-permeable.

The term *literal* above refers to an atomic formula A or its negation $A^\bot$. All formulas are written in negation normal form. The negation of non-atomic formulas is defined by the following De Morgan duals: $\otimes/\parr$, $\oplus/\&$, Σ/Π, $1/\bot$, $0/\top$, $\boxplus/\sqcap$, $\boxtimes/\sqcup$, $\exists^l/\forall^r$, $0_l/\top_r$, $\vee^+/\wedge^-$, $\wedge^+/\vee^-$, $\exists/\forall$, $0_r/\top_l$, $1_r/\bot_l$, $A/A^\bot$ for all literals A.

The six poles form three axes of polarization by De Morgan negation. The polarization scheme is also illustrated in Figure 1. The polarity of a formula is determined entirely by its top-level connective. For example, $A \parr B$, when both A and B are Left-formulas, is still considered to be -1, although it is provably equivalent to $A \vee^- B$.

We have included three sets of constants or "units," one on each axis, only for the convenience of having constants of every polarity. However, only the linear constants 1, $\bot$, $\top$ and 0 are "identities" for their respective connectives in the sense that, for example, $A \equiv A \otimes 1$ is provable in all contexts: here $P \equiv Q$ is defined to be $(P^\bot \parr Q) \;\&\; (Q^\bot \parr P)$. Polarity restrictions exist for their copies: 1_r, 0_r and $\top_r$ are only identities in this sense for Right-formulas and 0_l, $\top_l$ and $\bot_l$ are only identities for Left-formulas. However, it still holds, for example, that any single formula A is provable if and only if $A \vee^+ 0_r$ is provable. An alternative formulation would be to just use the four linear units, but that would make the presentation of the classical and intuitionistic

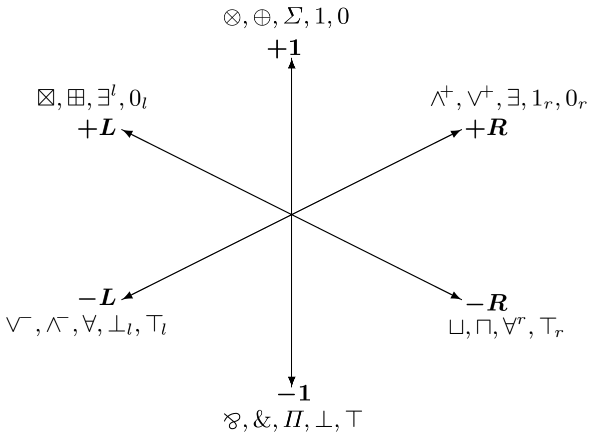

Fig. 1. Polarization in GIL

fragments somewhat awkward. Note that there is no 1_l or $\bot_r$: they would correspond to the connectives $\boxtimes$ and $\sqcup$ which are generalized forms of intuitionistic implication.

A Classification of Connectives. While there may appear to be a large number of propositional connectives and units in GIL, they all fit into a simple scheme involving four *attributes* that we can call *junctive*, *bias*, *realm*, and *permeability*. (By analogy, some elementary particles are similarly classified by the values they take for attributes such as mass, charge, and spin.) The *junctive* attribute is either set to *conjunctive* or *disjunctive*. This attribute describes, using game semantics terminology, how a proponent views a choice: if the proponent makes the choice, it is seen as a disjunction; if the opponent makes the choice, it is seen as a conjunction. The *bias* attribute is either *negative* or *positive* and is used to assigns a connective to one of the focusing phases. The *realm* attribute is either *linear* or *classical* and declares whether or not the connective yields formulas subjected to structural rules or not. Finally, a connective whose *realm* value is *classical* is also allowed an additional attribute of *permeability* that is either *left* or *right*: this attributes specifies on which side of a two-sided sequent the structure rules of weakening and contraction are to be applied. In addition, we can use a fifth attribute for *arity* which can take the value of any natural number: the value 0 yields a propositional constant and the value 2 yields a binary connective. Computing the De Morgan dual of a connective requires leaving the realm (and arity) attribute unchanged but flipping both junctive and bias attributes: in addition, if the realm attribute is *classical* then permeability is flipped as well. A connective is *multiplicative* if it is either conjunctive and positive or disjunctive and negative and *additive* if it is either disjunctive and positive or conjunctive and negative.

Restrictions on Formulas. A design goal of GIL is that *contraction on asynchronous formulas is never required* (a feature also found in *polarized linear logic* [8]). Thus, we will need to remove formulas that resemble the forms $?(A \parr B)$ and $!(A \oplus B)$, since these not only compromise focusing but can also wreck havoc with cut-elimination. To achieve this invariant, formulas are restricted so that whenever they contain a subformulas of the form $A \vee^- B$, $A \wedge^- B$ or $\forall x.A$, where A (or B) is negative then A (resp., B) must have the polarity $-L$. Dually, for $A \wedge^+ B$, $A \vee^+ B$ and $\exists x.A$, if A (or B) is positive then A (resp., B) must have polarity $+R$. A coupled restriction is imposed on sequents (introduced in the next section) of the form $\vdash \Gamma : \Delta \Uparrow^{\bullet} \Theta$: the multiset Θ may not contain negative formulas except $-L$ formulas. This invariant does not compromise the expressiveness of the logic since one can always switch polarities using any number of unary operations such as $A \otimes 1$.

In order to give GIL the character of intuitionistic logic, we need to impose another restriction that is related to Gentzen's single-conclusion characterization of intuitionistic logic. This restriction states that in a formula of the form $A \sqcup B$, at least one of A or B must be a Left-formula ($+L$ or $-L$). Dually, in $A \boxtimes B$, either A or B must be a Right-formula. As we shall see, these two connectives are used to model intuitionistic implication.

4 The Focused Sequent Calculus of GIL

The presentation of GIL is based on the focused proof systems LLF [1] and LKU [11]. The introduction rules are kept as uniform as possible while a set of expanded structural rules, which are active between synchronous and asynchronous phases, take center stage. The proof system for GIL is the first with multiple layers of focusing. Focusing and asynchronous decomposition along the linear +1/-1 axis are represented by $\Downarrow^1$/ $\Uparrow^1$, along the $+R$/$-L$ axis by $\Downarrow^{\bullet}$/ $\Uparrow^{\bullet}$, and along the $+L$/$-R$ axis by $\Downarrow^{\circ}$/ $\Uparrow^{\circ}$. Sequents of GIL have the form $\vdash \Gamma : \Delta \Uparrow^n \Theta$ or $\vdash \Gamma : \Delta \Downarrow^n A$, where $\Uparrow^n$ denotes $\Uparrow^1$, $\Uparrow^{\bullet}$ or $\Uparrow^{\circ}$ and likewise for $\Downarrow^n$. The multiset Γ is the classical context that admits contraction and weakening and Δ is the linear context. Θ is a unclassified multiset of formulas and A is a single formula under focus. *End sequents* of GIL have the form $\vdash : \Uparrow^1 \Theta$. The choice of this designation is principally due to cut-elimination (see Section 7). However, purely classical and intuitionistic end sequents may use $\Uparrow^{\bullet}$ and $\Uparrow^{\circ}$ as well. The structural rules are found in Figure 2. GIL is presented as a one-sided sequent calculus. The richness of polarity information replaces the need for two-sided sequents and allows for a more uniform and compact presentation.

The rules $R_1 \Downarrow^1$ and $R_2 \Downarrow^{\bullet}$ are also called *release* rules since they terminate focus. Rules $R_1 \Uparrow^1$ and $R_2 \Uparrow^{\bullet}$ *classify* formulas to be treated linearly or classically. Rules D_1 and D_2 are *decision* rules as they select formulas for focus. D_2, which embodies an explicit contraction, can only select a positive formula. The initial rule I_1 can be seen as the "missing case" for $R_1 \Downarrow^1$ and likewise for I_2 in relation to $R_2 \Downarrow^{\bullet}$. The lateral rules allow one-directional transition of focusing modes. The directions of these transitions are dictated by when structural rules are required between focusing phases (e.g., before a synchronous $\Downarrow^{\circ}$ phase). The number of positive and negative structural rules are kept small, which is made possible by the lateral rules. For example, when D_1 selects a -R formula for focus (the only case where a negative formula can be selected for focus), it will immediately trigger a lateral $L \Downarrow^1$ followed by a $R_2 \Downarrow^{\bullet}$.

Invariants. Sequents and formulas of GIL observe the following invariants:

Lateral Reactions

$$\frac{\vdash \Gamma :\Uparrow^1 E, \Upsilon}{\vdash \Gamma :\Uparrow^\circ E, \Upsilon}\ L\Uparrow^\circ \qquad \frac{\vdash \Gamma : \Delta \Uparrow^\bullet \Upsilon}{\vdash \Gamma : \Delta \Uparrow^1 \Upsilon}\ L\Uparrow^1 \qquad \frac{\vdash \Gamma : \Delta \Downarrow^1 F}{\vdash \Gamma : \Delta \Downarrow^\circ F}\ L\Downarrow^\circ \qquad \frac{\vdash \Gamma :\Downarrow^\bullet G}{\vdash \Gamma :\Downarrow^1 G}\ L\Downarrow^1$$

Negative Reactions

$$\frac{\vdash \Gamma : \Delta, C \Uparrow^1 \Theta}{\vdash \Gamma : \Delta \Uparrow^1 C, \Theta}\ R_1\Uparrow^1 \qquad \frac{\vdash D, \Gamma : \Delta \Uparrow^\bullet \Theta}{\vdash \Gamma : \Delta \Uparrow^\bullet D, \Theta}\ R_2\Uparrow^\bullet \qquad \frac{\vdash \Gamma : \Delta \Downarrow^1 S}{\vdash \Gamma : \Delta, S \Uparrow^n}\ D_1 \qquad \frac{\vdash T, \Gamma : \Delta \Downarrow^\circ T}{\vdash T, \Gamma : \Delta \Uparrow^n}\ D_2$$

Positive Reactions

$$\frac{\vdash \Gamma : \Delta \Uparrow^1 N}{\vdash \Gamma : \Delta \Downarrow^1 N}\ R_1\Downarrow^1 \qquad \frac{\vdash \Gamma :\Uparrow^\circ M}{\vdash \Gamma :\Downarrow^\bullet M}\ R_2\Downarrow^\bullet \qquad \frac{}{\vdash \Gamma : P^\perp \Downarrow^n P}\ I_1 \qquad \frac{}{\vdash Q^\perp, \Gamma :\Downarrow^n Q}\ I_2$$

E: not a $-R$ non-literal formula; Υ: all $-L$ and $+L$ formulas; F: not a $+L$ formula; G: $+R$ or $-R$ formula; C: $+1$, $+R$, $-R$ formula or -1 literal; D: $+1$, $+R$, $+L$ formulas and $-L$ literals; S: $+1$, $+R$, or non-literal $-R$ formula; T: $+R$, $+L$ or $+1$ formula; N: -1, $-L$ or $+L$ formula; M: -1, $-L$ or $-R$ formula; P, Q: positive literals.

Fig. 2. GIL Structural Rules

- The classical context Γ will only contain positive formulas and -L literals.
- The linear context Δ will only contain Right-formulas (+R and -R), +1 formulas and -1 literals.
- In $\Gamma : \Delta \Uparrow^\circ \Theta$, the multiset Θ consists of left formulas and at most one other formula. If this formula is not -R, then a lateral transition will be made to another mode.
- In $\Gamma : \Delta \Uparrow^\bullet \Theta$, the multiset Θ contains only positive formulas and -L formulas.
- The rule $R_1\Uparrow^1$ applies to a formula A (i.e., when A is to the right of $\Uparrow^1$) if and only if $R_1\Downarrow^1$ or I_1 applies to $A^\perp$.
- The rule $R_2\Uparrow^\bullet$ applies to A if and only if either $R_2\Downarrow^\bullet$ or I_2 applies to $A^\perp$.

The last invariant (also found in LKU) parallels the duality of ! and ? in linear logic.

The introduction rules for GIL are presented in Figures 3, 4, and 5. The number of rules is much less than an LU-style of presentation. Given six polarities, if such a style was used there could be as many as 36 introduction rules for a single binary connective. There is only one introduction rule for each GIL connective (taking into account the efficient presentation of $\oplus$).

The introduction rules that are most sensitive to polarity information are the rules for $\boxtimes$ and $\sqcup$. In $A \sqcup B$, we will assume, without loss of generality, that A is a Left-formula ($+L$ or $-L$). Similarly, in $A \boxtimes B$, we assume that A is a Right-formula. Because focusing behaves in an asymmetric way here, these introduction rules incorporate structural rules. The intuitionistic mode of focus can be kept only on B, the head of the implication. For A, the $\Uparrow^\circ / \Downarrow^\circ$ mode must be terminated. This behavior is consistent with the linear logic translation of these formulas.

$$\frac{\vdash \Gamma : \Delta \Uparrow^1 \Theta}{\vdash \Gamma : \Delta \Uparrow^1 \bot, \Theta}\ \bot \qquad \frac{}{\vdash \Gamma : \Delta \Uparrow^1 \top, \Theta}\ \top \qquad \frac{}{\vdash \Gamma :\Downarrow^1 1}\ 1$$

$$\frac{\vdash \Gamma : \Delta \Uparrow^\bullet \Theta}{\vdash \Gamma : \Delta \Uparrow^\bullet \bot_l, \Theta}\ \bot_l \qquad \frac{}{\vdash \Gamma : \Delta \Uparrow^\bullet \top_l, \Theta}\ \top_l \qquad \frac{}{\vdash \Gamma :\Downarrow^\bullet 1_r}\ 1_r \qquad \frac{}{\vdash \Gamma :\Uparrow^\circ \top_r, \Upsilon}\ \top_r$$

Fig. 3. GIL introduction rules for constants. Here, Υ contains only $-L$ or $+L$ formulas.

$$\frac{\vdash \Gamma : \Delta \Uparrow^1 A, B, \Theta}{\vdash \Gamma : \Delta \Uparrow^1 A \parr B, \Theta}\ \parr \qquad \frac{\vdash \Gamma : \Delta \Uparrow^1 A, \Theta \quad \vdash \Gamma : \Delta \Uparrow^1 B, \Theta}{\vdash \Gamma : \Delta \Uparrow^1 A \,\&\, B, \Theta}\ \& \qquad \frac{\vdash \Gamma : \Delta \Uparrow^1 A, \Theta}{\vdash \Gamma : \Delta \Uparrow^1 \Pi x.A, \Theta}\ \Pi$$

$$\frac{\vdash \Gamma : \Delta \Uparrow^\bullet A, B, \Theta}{\vdash \Gamma : \Delta \Uparrow^\bullet A \vee^- B, \Theta}\ \vee^- \qquad \frac{\vdash \Gamma : \Delta \Uparrow^\bullet A, \Theta \quad \vdash \Gamma : \Delta \Uparrow^\bullet B, \Theta}{\vdash \Gamma : \Delta \Uparrow^\bullet A \wedge^- B, \Theta}\ \wedge^- \qquad \frac{\vdash \Gamma : \Delta \Uparrow^\bullet A, \Theta}{\vdash \Gamma : \Delta \Uparrow^\bullet \forall x.A, \Theta}\ \forall$$

$$\frac{\vdash \Gamma :\Uparrow^\circ A, \Upsilon}{\vdash \Gamma :\Uparrow^\circ \forall^r x.A, \Upsilon}\ \forall^r \qquad \frac{\vdash \Gamma :\Uparrow^\circ A, \Upsilon \quad \vdash \Gamma :\Uparrow^\circ B, \Upsilon}{\vdash \Gamma :\Uparrow^\circ A \sqcap B, \Upsilon}\ \sqcap \qquad \frac{\vdash \Gamma :\Uparrow^\circ B, C, \Upsilon}{\vdash \Gamma :\Uparrow^\circ C \sqcup B, \Upsilon}\ \sqcup (\supset R)$$

Fig. 4. GIL introduction rules for the negatives. Here, x is not free in Γ, Δ, Θ; C is $+L$ or $-L$; and Υ contains only $-L$ or $+L$ formulas.

$$\frac{\vdash \Gamma : \Delta_1 \Downarrow^1 A \quad \vdash \Gamma : \Delta_2 \Downarrow^1 B}{\vdash \Gamma : \Delta_1 \Delta_2 \Downarrow^1 A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma : \Delta \Downarrow^1 A_i}{\vdash \Gamma : \Delta \Downarrow^1 A_1 \oplus A_2}\ \oplus \qquad \frac{\vdash \Gamma : \Delta \Downarrow^1 A[t/y]}{\vdash \Gamma : \Delta \Downarrow^1 \Sigma y.A}\ \Sigma$$

$$\frac{\vdash \Gamma :\Downarrow^\bullet A \quad \vdash \Gamma :\Downarrow^\bullet B}{\vdash \Gamma :\Downarrow^\bullet A \wedge^+ B}\ \wedge^+ \qquad \frac{\vdash \Gamma :\Downarrow^\bullet A_i}{\vdash \Gamma :\Downarrow^\bullet A_1 \vee^+ A_2}\ \vee^+ \qquad \frac{\vdash \Gamma :\Downarrow^\bullet A[t/y]}{\vdash \Gamma :\Downarrow^\bullet \exists y.A}\ \exists$$

$$\frac{\vdash \Gamma : \Delta \Downarrow^\circ A[t/y]}{\vdash \Gamma : \Delta \Downarrow^\circ \exists^l y.A}\ \exists^l \qquad \frac{\vdash \Gamma : \Delta \Downarrow^\circ A_i}{\vdash \Gamma : \Delta \Downarrow^\circ A_1 \boxplus A_2}\ \boxplus \qquad \frac{\vdash \Gamma :\Downarrow^1 D \quad \vdash \Gamma : \Delta \Downarrow^\circ B}{\vdash \Gamma : \Delta \Downarrow^\circ D \boxtimes B}\ \boxtimes (\supset L)$$

Fig. 5. GIL introduction rules for the positive connectives. Here, D is either $+R$ or $-R$.

As in classical logic, some additive and multiplicative versions of disjunction and conjunction are provability-equivalent in GIL. In particular, $\boxplus$ is equivalent to $\vee^-$ and $\sqcap$ is equivalent to $\wedge^+$. The differences they bring are in the structure of focused proofs. For example, one may implement a forward-chaining proof strategy by choosing the positive versions of the connectives.

We sometimes write implication as $A \supset B$ when its left/right status is clear. The polarity restriction on A allows us to prove, for example, the distributivity of $\boxtimes$ over $\boxplus$ (and of $\sqcup$ over $\sqcap$), which in terms of intuitionistic implication represents the equivalence between $A \supset (B \wedge C)$ and $(A \supset B) \wedge (A \supset C)$[1].

It is also possible to identify *pseudo-intuitionistic* implications as formulas of the form $A^\perp \parr B$ (or $A \multimap B$) where $A^\perp$ is a Left-formula. Technically, such a formula is a -1 linear formula. It should not be identified with the stronger form $A^\perp \sqcup B$. For example, an assumption of $A \multimap B$ (on the left as $A \otimes B^\perp$) would not be subject

[1] Moreover, this distributivity should hold at the denotational level since it reduces to the distributivity of $\otimes$ over $\oplus$ in linear logic (see the translation of Section 6).

to contraction. However, the identification of such forms can still be very useful in specifying computations.

5 Basic Fragments

We detail below how the focused versions of MALL, classical logic and intuitionistic logic are identified as fragments of GIL.

MALLF : Restrict to only +1/-1 formulas and use $\vdash:\Uparrow^1\Theta$ for the end sequent. The only applicable structural rules are $R_1\Uparrow^1$, $R_1\Downarrow^1$, D_1, and I_1. The resulting proof system is the MALL subset of Andreoli's proof system [1].

It is also easy to show that *GIL has the full power of linear logic* (see the translation to linear logic in Section 6). Given a MALL formula A, $?A$ can be recovered with $A \boxplus 0$, $(A \otimes 1) \vee^- \bot_l$ or any number of other forms. Similarly, $!A$ is recovered from the duals of these forms.

LKF : Restrict to only $+R$/$-$L formulas and to end sequents of the form $\vdash:\Uparrow^\bullet\Theta$ or $\vdash:\Uparrow^1 A_1 \vee^- \ldots \vee^- A_n \vee^- \bot_l$. The only applicable structural rules are $R_2\Uparrow^\bullet$, $R_2\Downarrow^\bullet$, D_2, I_2, and the lateral reactions for transition after a decide/release rule.

LJF : LJF formulas, as they originally appeared in [10] are mapped into GIL formulas using the two functions $[\cdot]^R$ (right) and $[\cdot]^L$ (left) defined in Figure 6. Thus, LJF formulas only employ connectives of polarity $+L$, $-L$, $+R$ and $-R$. Atoms are also restricted to $+R$ and $-R$. Intuitionistic negation is represented by $A \supset 0_r$ For minimal logic, replace 0_r with some designated $-R$ or $+R$ atom.

$$\begin{array}{rclrcl}
[B \wedge^- C]^R &=& [B]^R \sqcap [C]^R & [B \wedge^+ C]^R &=& [B]^R \wedge^+ [C]^R \\
[B \supset C]^R &=& [B]^L \sqcup [C]^R & [B \vee C]^R &=& [B]^R \vee^+ [C]^R \\
[\forall x.B]^R &=& \forall^r x.[B]^R & [\exists x.B]^R &=& \exists x.[B]^R \\
[B \wedge^- C]^L &=& [B]^L \boxplus [C]^L & [B \wedge^+ C]^L &=& [B]^L \vee^- [C]^L \\
[B \supset C]^L &=& [B]^R \boxtimes [C]^L & [B \vee C]^L &=& [B]^L \wedge^- [C]^L \\
[\forall x.B]^L &=& \exists^l x.[B]^L & [\exists x.B]^L &=& \forall x.[B]^L
\end{array}$$

For atomic A, $[A]^R = A$ and $[A]^L = A^\perp$.

Fig. 6. Left and Right Intuitionistic Formulas in GIL.

A formula in the range of $[\cdot]^R$ will be called *essentially right* and a formula in the range of $[\cdot]^L$ will be called *essentially left* [11]. End-sequents of LJF have the form $\vdash:\Uparrow^\circ\Gamma, A$ where Γ consists of essentially left formulas and A is an essentially right formula[2].

[2] If the end sequent was of the form $\vdash:\Uparrow^1\Gamma, A$ and A is a -R formula, then the proof would unnecessarily delay the asynchronous decomposition of A.

Well-formed intuitionistic formulas and sequents observe strong invariants. It can be shown (by a simultaneous induction) that essentially right formulas always asynchronously decompose to Υ, A, where A is a single right formula and Υ is a multiset of left formulas. On the other hand, right formulas synchronously decompose to only right formulas. Dual properties hold for left formulas. These invariants are consequences of the restrictions on the formulas of LJF. The formula restrictions of GIL (see the end of Section 3) are more relaxed, which allow the polarities to mix more freely.

The rules covering intuitionistic implication in GIL use polarity information when splitting the context Δ as a result of applying the $\boxtimes$ rule. In contrast to formulations of intuitionistic logic in linear logic and in LU (and LKU), there is no loss of "full completeness" with respect to intuitionistic implication or to focusing. Specifically, if implication (on the left) is represented with a multiplicative conjunction, then splitting the context may leave two right-formulas in the same sequent. But since one of the immediate subformulas of the conjunct is always a right-formula, for which the rule $L\Downarrow^1$ will enforce an empty linear context, Δ must be moved completely to the subproof containing the (possibly) non-right formula. In linear logic, the ! operator conveys this information. However, to preserve *focused proofs* this operator must be strategically removed *along with the polarity information it carries.* A linear logic translation of LJF cannot preserve valid proofs when the $\top$ rule is used, and it cannot preserve partial proofs. The novelty of GIL is in decomposing the ! into two polarities, $+R$ and $-R$. A GIL formula hence carries more information than a linear logic formula. A partial LJF proof is exactly a partial GIL proof.

nLJF : Restrict LJF to only $-R$ and $+L$ connectives: *i.e.*, only to the left-side of Figure 6 and only with -R literals. This fragment of intuitionistic logic is traditionally referred to as the "negative" fragment. The structural rules of nLJF are I_1, D_2, $R_1 \Uparrow^1$ (on -R literals), $R_2 \Uparrow^\bullet$ (on $+L$ formulas) and $R_2 \Downarrow^\bullet$. nLJF fits completely within one axis of polarization, using a single pair of arrows, $\Uparrow^\circ / \Downarrow^\circ$, except when vacuous laterals are needed to invoke the appropriate structural rules.

Within the LKF fragment, $\vee^+$ and $\vee^-$ are provably equivalent, as are $\wedge^-$ and $\wedge^+$. In the intuitionistic fragment, $\sqcap$ and $\wedge^+$ are provably equivalent: the negative version of conjunction in intuitionistic logic is not the same as in classical logic.

Except for the trivial use of the lateral reactions for purely bureaucratic reasons, there is in fact no need for lateral reaction rules shifting one focusing or decomposition mode to another in any of the basic fragments. In LJF, the forms of sequents involved are $\Downarrow^\bullet$ and $\Uparrow^\circ$ on essentially right formulas and $\Uparrow^\bullet$ and $\Downarrow^\circ$ on left formulas. The only lateral transition is from $\Downarrow^\circ$ to $\Downarrow^\bullet$ in the $\boxtimes$-rule ($\supset$-Left in traditional presentations of intuitionistic logic), and the corresponding transition from $\Uparrow^\circ$ to $\Uparrow^\bullet$ for $\sqcup$. However, the restricted form of intuitionistic formulas makes other transitions unnecessary. The basic fragments above can each be independently represented using a single pair of $\Uparrow / \Downarrow$, and thus do not represent the full potential of GIL. Other fragments of GIL can be identified, such as the following.

ACMALL: Additive Classical Logic with MALL This fragment is based on all of MALL plus the connectives $\sqcap$, $\boxplus$, $\forall^r$ and $\exists^l$. All formulas can mix freely among these polarities. However, for purely classical reasoning one must use the form

$$\vdash : \Uparrow^1 A_1 \boxplus \ldots \boxplus A_n \boxplus 0_l$$

for classical end-sequents. The modes $\Uparrow^1 / \Downarrow^1$ and $\Uparrow^\circ / \Downarrow^\circ$ are both used in ACMALL.

6 Translation of GIL into Linear Logic

Atoms can be considered second-order variables that represent arbitrary formulas. In this sense atoms can also be polarized. To assign polarity to atoms, we admit into linear logic, as was done in LU, atoms that are naturally *permeable,* i.e., $A \equiv ?A$ (right permeable) and $A \equiv !A$ (left permeable). However, we do not fix the *focusing* polarity of these atoms. That is, both $+L$ and $-L$ atoms are right permeable. Similarly, left permeable atoms can be assigned either $-R$ or $+R$ polarity.

Informally, the translation of GIL is based on the following correspondences:

$A \wedge^+ B$	$!A \otimes !B$	$A \boxplus B$	$?(A \oplus B)$
$A \vee^+ B$	$!A \oplus !B$	$A \sqcap B$	$!(A \,\&\, B)$
$A \wedge^- B$	$?A \,\&\, ?B$	$A \sqcup B$	$!(A \parr B)$
$A \vee^- B$	$?A \parr ?B$	$A \boxtimes B$	$?(A \otimes B)$

The polarities $+R$ and $-L$ represent formulas that Girard identified as naturally left or right permeable. We can see that focusing is valid for these formulas from equivalences such as $!(!A \otimes !B) \equiv !A \otimes !B$. That is, the ! can be dropped on subformulas of the same polarity. These equivalences are valid for both the multiplicatives and the additives. These polarities are enough to account for classical logic (LKF). It is interesting to note that focalization is indifferent to the additive/multiplicative distinction in both MALL and classical logic. With the polarities $+L$ and $-R$, a distinction appears, which in fact gives us a generalization of intuitionistic implication in a special pair of multiplicative connectives ($\sqcup$ and $\boxtimes$). With the additives it is easy to see that $?(A \oplus B) \equiv ?(?A \oplus ?B)$. The internal ?s can be dropped so focusing can continue, even when switching from an intuitionistic context to a linear one. There is no equivalence, however, between $!(!A \parr !B)$ and $!(A \parr B)$. With the multiplicatives, we only have the equivalence $!(?A \parr !B) \equiv !(?A \parr B)$ and its dual in terms of $\otimes$. But this form, called $\sqcup$, is equivalent to what is used in intuitionistic logic, where $A \supset B$ is translated as $!(?A^\perp \parr B)$. The external ! cannot be dropped if we wish to use these formulas in a unified setting, where intuitionistic, classical and linear formulas can exist in the same sequent. This equivalence explains how it is possible to "keep the focus" on formulas such as $A \supset B \supset C$ (on the left) in intuitionistic logic.

The formal translation is based on the possible polarities for each formula and is given by Table 1. To minimize the number of cases that need to presented, we note that the translation of the two subformulas of binary connectives are independent of each other. Thus we shall only display cases where the polarities of the two subformulas are different. For the cases of the classical/intuitionistic connectives, we also do not show the cases that can be inferred by duality.

The linear connectives (such as $\parr$) in GIL do *not* always translate to themselves in linear logic. When the linear connectives join $-R$ and $+L$ formulas, we "impart" the appropriate exponential operator onto them.

A	B	$(A \parr B)'$	$(A \& B)'$	$(A \otimes B)'$	$(A \oplus B)'$	$(\Pi x.A)'$	$(\Sigma x.B)'$
+1	–1	$A' \parr B'$	$A' \& B'$	$A' \otimes B'$	$A' \oplus B'$	$\Pi x.A'$	$\Sigma x.B'$
$+L$	$-R$	$?A' \parr !B'$	$?A' \& !B'$	$?A' \otimes !B'$	$?A' \oplus !B'$	$\Pi x.?A'$	$\Sigma x.!B'$
$+R$	$-L$	$A' \parr B'$	$A' \& B'$	$A' \otimes B'$	$A' \oplus B'$	$\Pi x.A'$	$\Sigma x.B'$

There are some cases for the quantifiers that are missing from the above table: for $+L$ formula A and $-R$ formula B, $(\Pi x.B)' = \Pi x.!B$ and $(\Sigma x.A)' = \Sigma x.?A'$. The other cases can be inferred by duality.

A	B	$(A \wedge^+ B)'$	$(A \boxtimes B)'$	$(A \vee^+ B)'$	$(A \boxplus B)'$	$(\exists x.A)'$	$(\exists^l x.B)'$
+1	–1	N/A	N/A	N/A	$A' \oplus B'$	N/A	$\Sigma x.B'$
$+L$	$-R$	N/A	$A' \otimes !B'$	N/A	$A' \oplus !B'$	N/A	$\Sigma x.!B'$
$+R$	$-L$	$A' \otimes !B'$	$A' \otimes B'$	$A' \oplus !B'$	$A' \oplus B'$	$\Sigma x.A'$	$\Sigma x.B'$
$-L$	–1	$!A' \otimes !B'$	N/A	$!A' \otimes !B'$	$A' \oplus B'$	$\Sigma x.!A'$	$\Sigma x.B'$
–1	$-R$	$!A' \otimes !B'$	$A' \otimes !B'$	$!A' \oplus !B'$	$A' \oplus !B'$	$\Sigma x.!A'$	$\Sigma x.!B'$
$+L$	$+L$	N/A	N/A	N/A	$A' \oplus B'$	N/A	$\Sigma x.B'$
$+L$	$-L$	N/A	N/A	N/A	$A' \oplus B'$	N/A	$\Sigma x.B'$
$-R$	$+R$	$!A' \otimes B'$	$!A' \otimes B'$	$!A' \oplus B'$	$!A' \oplus B'$	$\Sigma x.!A'$	$\Sigma x.B'$

The translation of atoms and the constants 1_r, 0_r, 0_l, $\top_l$, $\top_r$ and $\bot_l$ are invariant.

Table 1. Translation of GIL to linear logic.

It is important to emphasize that all Right-polarity formulas A translate into forms A' such that $A' \equiv !A'$ regardless of whether A is positive or negative. Likewise, for all Left-polarity formulas B, $B' \equiv ?B'$. A major difference between the GIL interpretations and previous studies of polarization is the decoupling of $!$ and $?$ from their status as positive and negative operators respectively. A formula of the form $!(A \,\&\, B)$ is considered negative. In fact, the promotion rule is clearly *invertible* when it's applicable.

The GIL end-sequent $\vdash:\Uparrow^1 A_1, \ldots, A_n$ is translated into LLF as $\vdash:\Uparrow (A_1 \parr \ldots \parr A_n \parr \bot)'$. The "classical" end-sequent $\vdash:\Uparrow^\bullet A_1, \ldots, A_n$ is translated as $\vdash:\Uparrow (A_1 \vee^- \ldots \vee^- A_n \vee^- \bot_l)'$. The "intuitionistic" end-sequent $\vdash:\Uparrow^\circ \Gamma, A$ where A is a Right-formula, is translated in the same way as general GIL sequents *with the* $!$ *in font of* A' *removed if A is a -R formula.* Left formulas in Γ translate to forms that are equivalent to $?$-formulas, so the invertible promotion is applicable. This will allow an asynchronous right formula to be eagerly decomposed.

This translation of GIL preserves provability as well as proofs as long as the $\top$ rule is not involved. It does not preserve incomplete proofs as discussed in Section 5. However, as Girard notes in [5], proofs involving $\top$ have the same, vacuous semantic interpretation. Thus the translation allows GIL to inherit the semantics of linear logic in so far as complete proofs are concerned.

7 Cut Elimination, Contraction, and the Degree of Polarization

The origins of the "classical" polarities $+R$/$-L$ can be attributed to Girard's LC and LU systems. The polarities +1 and -1 originate from focusing in linear logic as defined by Andreoli. What is new in GIL is the introduction of the polarities $+L$/$-R$ and the amalgamation of all six polarities into a single system. The need for the new axis of polarization can be explained in terms of cut-elimination, especially when some *but not all* formulas are subject to contraction. Consider the reduction of cut above a contraction:

$$\dfrac{\vdash A, \Delta \qquad \dfrac{\vdash A^{\perp}, A^{\perp}, \Gamma}{\vdash A^{\perp}, \Gamma}\, C}{\vdash \Delta\Gamma}\, cut \quad \longmapsto \quad \dfrac{\dfrac{\vdash A, \Delta \qquad \dfrac{\vdash A, \Delta \quad \vdash A^{\perp}, A^{\perp}, \Gamma}{\vdash A^{\perp}, \Delta\Gamma}\, cut}{\vdash \Delta\Delta\Gamma}\, cut}{\vdash \Delta\Gamma}\, C^{*}$$

If the context Δ contain linear formulas not subject to contraction, then this reduction cannot be made. Intuitionistic logic has been described as "classical on the left, linear on the right" and we in fact seek to expand this hybrid characteristic of intuitionistic logic. To ensure the admissibility of cut in this setting, we can consider three approaches.

The first approach is the traditional intuitionistic restriction to a "single conclusion" (i.e., the cut formula is the only linear formula). If one is only interested in embedding intuitionistic logic, then clearly this approach is enough. But this property can hardly be kept if we wish to aggressively mix intuitionistic deduction with linear deduction. A second approach involves using the ! of linear logic. The !, however, is not compatible with focusing; indeed it obscures the synchronous/asynchronous duality. In the terminology of Girard, this problem can also be described as the loss of denotational associativity. The third approach is to replace ! (and thus ?) with polarity information. LU was the first system to make such a use of polarities. However, in LU the "negative" intuitionistic formulas, along with $+1$ and -1 formulas, were all classified as having "neutral" polarity; that is, linear. Although LU's translation tables also suggest that mixing connectives from different logics is possible, this aspect of LU was never fully explored. LU includes classical, intuitionistic and linear logics as fragments but its ability to mix formulas in a single sequent is, in fact, limited. In particular, its three admissible cuts do not include the following case:

$$\dfrac{;\Gamma \vdash ; A, \Delta \qquad ; A, \Gamma' \vdash ; \Delta'}{;\Gamma\Gamma' \vdash ; \Delta\Delta'}\, cut$$

where A is a negative intuitionistic formula and Δ is a non-empty context of formulas that are not subject to contraction. The problem is that the polarity of A is not distinguished from that of linear formulas and consequently the introductions rules for such formulas may take place in the presence of other linear formulas. In contrast, a cut between $!A$ and $?A^{\perp}$ is admissible in *any* context in linear logic. The permutation of such a cut above a contraction can be delayed until the promotion rule is applied. The polarization scheme of LU is not enough to completely replace the role of the exponential operators or the single-conclusion restriction. More polarities are needed.

By considering where structural rules are needed in a *focused* proof, a refined polarization scheme emerges. The "Right" polarities of GIL, $+R$ and $-R$, are equivalent (provability-wise) to !-formulas in linear logic, but unlike Andreoli's system, they are not both considered "positive". Focusing on $+R$ formulas requires an empty linear context. The $+L$ polarity enables the focalization of left-introduction rules in intuitionistic logic, which must take place in the presence of a *non-empty* linear context. The $+L$ polarity must, therefore, be distinguished from $+R$. The focusing of a left-side intuitionistic formula should be *preceded by a contraction.* Thus the $+L$ polarity should also be distinguished from +1, which indicates positive MALL formulas. The contraction at the border of the focusing phase entails that inference rules for the dual polarity -R require an empty linear context. Thus the -R polarity should likewise be distinguished from -1. The fine-grained sensitivity to cut-elimination found in GIL is possible because of the additional polarities and modes of focusing. Cut elimination will fail if, for example, we were to allow arbitrary transitions from the $\Uparrow^{\bullet}$ mode to the $\Uparrow^{1}$ mode (i.e., allow the inverse of $L\Uparrow^{1}$).

The consequence of the enriched polarization scheme is a general form of admissible cut that is in fact *independent of polarity restrictions.* We refer to it as the *end cut* as it applies to GIL end-sequents:

$$\frac{\vdash:\Uparrow^{1} A, \Theta \quad \vdash:\Uparrow^{1} A^{\perp}, \Theta'}{\vdash:\Uparrow^{1} \Theta\Theta'}\ Cut$$

This is a stronger form of cut than those found in LU (and LKU). During the permutation of cuts *starting with the end cut,* the following intermediate cuts are also needed:

$$\frac{\vdash \Gamma : \Delta, A \Uparrow^{n} \Theta \quad \vdash \Gamma' : \Delta' \Uparrow^{m} A^{\perp}, \Theta'}{\vdash \Gamma\Gamma' : \Delta\Delta' \Uparrow^{1} \Theta\Theta'}\ cut_1 \qquad \frac{\vdash \Gamma, A : \Delta \Uparrow^{n} \Theta \quad \vdash \Gamma' :\Uparrow^{m} A^{\perp}, \Upsilon}{\vdash \Gamma\Gamma' : \Delta \Uparrow^{n} \Theta\Upsilon}\ cut_2$$

The multiset Υ may contain only Left-formulas. In the cut_1 rule, $\Uparrow^{n}$ cannot be $\Uparrow^{\circ}$ and in cut_2, $\Uparrow^{m}$ cannot be $\Uparrow^{\circ}$. Rules similar to cut_1 and cut_2 are found in LU and LKU. They would be enough if one only considered classical, linear and intuitionistic logics as *independent* fragments. In the expanded setting of GIL, the usual consequences and applications of cut-elimination are only possible in general with the stronger Cut rule. This rule justifies GIL's status as *logic.*

The detailed proof of cut-elimination, which is omitted here, also takes advantage of the structure of focused proofs. Positive and negative introduction rules are generalized to allow us to concentrate on cut reduction only where it matters the most: at the borders marking polarity transitions.

Theorem 1. *The Cut, cut_1 and cut_2 rules are admissible in GIL.*

Initial elimination can be proved using the same technical devices as cut-elimination. This property tests if the connectives of our logic are *"small enough"*.

Theorem 2. *For all formulas A, the sequent $\vdash:\Uparrow^{1} A, A^{\perp}$ is provable in GIL.*

8 Conclusion

One of the strengths of intuitionistic logic is its ability to embed classical logic. The various double-negation translations suggest that intuitionistic logic already contains the characteristics of a *unified framework* for logical deduction.

Typically, a double-negation translation embeds the intuitionistic taboo $A \vee \neg A$ as the more harmless-looking $\neg(A \wedge \neg A)$. The left-hand side of intuitionistic sequents can be used for classical reasoning. The advent of linear logic, however, has allowed us to consider this intuitionistic concept in a new light. Intuitionistic logic can also be embedded inside linear logic, which, with its De Morgan style dualities, has a single sided sequent calculus. In this context, "double negating" a formula would in fact change little: an intuitionistic conjunction $p \wedge q$ on the "left" would appear as something like $?(p' \oplus q')$. Yet one should not dismiss double negation as a syntactic cover-up, for what is revealed by such analyses is that intuitionistic logic can be said to contain, at least, two different versions of disjunction, only one of which is subject to contraction.

The possibility, therefore, arises for the formulation of an intuitionistic-like system in which linear and classical components are also found. However, instead of elaborate embeddings, one chooses a particular logic by simply selecting the *appropriate versions* of the logical connectives. Furthermore, these connectives can mix with few restrictions. We have developed such a generalization of intuitionistic logic from a refined notion of polarization and focusing, one that is delicately sensitive to the preservation of cut elimination.

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