A focused framework for emulating modal proof systems

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Abstract

Several deductive formalisms (e.g., sequent, nested sequent, labeled sequent, hypersequent calculi) have been used in the literature for the treatment of modal logics, and some connections between these formalisms are already known. Here we propose a general framework, which is based on a focused version of the labeled sequent calculus by Negri, augmented with some parametric devices allowing to restrict the set of proofs. By properly defining such restrictions and by choosing an appropriate polarization of formulas, one can obtain different, concrete proof systems for the modal logic K and for its extensions by means of geometric axioms. In particular, we show how to use the expressiveness of the labeled approach and the control mechanisms of focusing in order to emulate in our framework the behavior of a range of existing formalisms and proof systems for modal logic.

Keywords: Modal logic, sequent calculi, labeled proof systems, focusing.

1 Introduction

Modal proof theory is a notoriously difficult subject and several proposals for it have been given in the literature (a general account is in [7]). Such proposals range over a set of different proof formalisms (e.g., sequent, nested sequent, labeled sequent, hypersequent calculi), each of them presenting its own features and drawbacks. For instance, proof systems based on ordinary sequents present a good behavior in terms of proof search, but they are typically designed for a specific modal logic and lack modularity when one tries to capture modal logics with specific frame conditions. Moreover, cut-elimination for an important modal logic like S5 is problematic. For this reason, more sophisticated formalisms have been adapted or introduced, e.g., several hypersequent cut-free formulations have been given for S5, while nested and labeled sequent have been used for giving modular presentations of large classes of modal logics.

Here we propose a general framework for emulating and comparing existing modal proof systems as well as for generating new ones. Such a framework is based on a focused version of the labeled sequent calculus by Negri [20] and is an extension of the one presented in [19]. Being based on the idea of encoding within syntax elements coming from the (Kripke-style) semantics of modal logics, labeled systems [9] are extremely expressive and rules in such a setting can be seen as corresponding to "atomic" inference steps. Inference rules in other settings, like ordinary sequent or hypersequent calculi, tend instead to group together more steps into a single inference rule. Several results concerning correspondences and connections between the different formalisms are known [8,12,15]. We use such results as the basis for defining a translation from each formalism considered into our labeled framework. By adding elements of polarization to such a translation and by properly defining a few other parameters of the general framework, we are able to exploit the control mechanisms provided by focusing in order to reproduce proofs of the original calculi with a high degree of adequacy.

We proceed as follows. After providing background notions concerning modal logic and focusing (Section 2), we present the general framework LMF^X (Section 3) and prove some results about the emulation of existing modal proof systems (Section 4). In this paper, we restrict our attention to the emulation of ordinary and nested sequent systems. We remark, however, that the framework has been designed with the goal of capturing more modal calculi in a wider range of formalisms, as we discuss in the concluding remarks (Section 5), where we also sum up our contributions and propose some directions for future work.

2 Background notions

2.1 Modal logic

The language of *(propositional) modal formulas* consists of a functionally complete set of classical connectives (here we will present the syntax by means of a minimal one, but other connectives, defined as usual, will be used in the rest of the paper), a *modal operator* $\Box$ (here we will also use explicitly its dual $\Diamond$) and a denumerable set $\mathcal{P}$ of *propositional symbols*, according to the following grammar:

$$A ::= P \mid \bot \mid A \supset A \mid \Box A \mid \Diamond A\,,$$

where $P \in \mathcal{P}$. In the following, we will use Γ, Δ, Σ to refer to multisets of modal formulas. The semantics is usually defined by means of *Kripke frames*, i.e., pairs $\mathcal{F} = (W, R)$ where W is a non empty set of *worlds* and R is a binary relation on W. A *Kripke model* is a triple $\mathcal{M} = (W, R, V)$ where (W, R) is a Kripke frame and $V : W \to 2^{\mathcal{P}}$ is a function that assigns to each world in W a (possibly empty) set of propositional symbols.

Truth of a modal formula at a point w in a Kripke structure $\mathcal{M} = (W, R, V)$ is the smallest relation $\models$ satisfying:

$$\begin{array}{rll}
\mathcal{M}, w \models P & \text{iff} & p \in V(w) \\
\mathcal{M}, w \models A \supset B & \text{iff} & \mathcal{M}, w \models A \text{ implies } \mathcal{M}, w \models B \\
\mathcal{M}, w \models \Box A & \text{iff} & \mathcal{M}, w' \models A \text{ for all } w' \text{ s.t. } wRw' \\
\mathcal{M}, w \models \Diamond A & \text{iff} & \text{there exists } w' \text{ s.t. } wRw' \text{ and } \mathcal{M}, w' \models A.
\end{array}$$

By extension, we write $\mathcal{M} \models A$ when $\mathcal{M}, w \models A$ for all $w \in W$ and we write $\models A$ when $\mathcal{M} \models A$ for every Kripke structure $\mathcal{M}$. The former definition characterizes the basic modal logic K. Several further modal logics can be

Axiom	**Condition**	**First-Order Formula**
T: $\Box A \supset A$	Reflexivity	$\forall x.R(x, x)$
4: $\Box A \supset \Box\Box A$	Transitivity	$\forall x, y, z.(R(x, y) \wedge R(y, z)) \supset R(x, z)$
5: $\Box A \supset \Box\Diamond A$	Euclideaness	$\forall x, y, z.(R(x, y) \wedge R(x, z)) \supset R(y, z)$
B: $A \supset \Box\Diamond A$	Symmetry	$\forall x, y.R(x, y) \supset R(y, x)$
D: $\Box A \supset \Diamond A$	Seriality	$\forall x \exists y.R(x, y)$

Table 1
Axioms and corresponding first-order conditions on the accessibility relation R.

defined as extensions of K by simply restricting the class of frames we consider. Many of the restrictions we are interested in are definable as formulas of first-order logic where the binary predicate $R(x, y)$ refers to the corresponding accessibility relation. Table 1 summarizes some of the most common frame logics, describing the corresponding frame property, together with the modal axiom capturing it [22]. We will refer to the logic satisfying a set of axioms $\{F_1, \ldots, F_n\}$ as $K\{F_1, \ldots, F_n\}$.

2.2 Focusing in first-order classical logic

In this paper, we will not use Gentzen's LK sequent calculus [10] directly but rather variants of a *focused* version of it called *LKF* [17] (displayed in Figure 1).

Asynchronous introduction rules

$$\frac{}{\vdash \Theta \Uparrow t^-, \Delta}\ t^-{}_F \qquad \frac{\vdash \Theta \Uparrow A, \Delta \quad \vdash \Theta \Uparrow B, \Delta}{\vdash \Theta \Uparrow A \wedge^- B, \Delta}\ \wedge^-_F$$

$$\frac{\vdash \Theta \Uparrow \Delta}{\vdash \Theta \Uparrow f^-, \Delta}\ f^-{}_F \qquad \frac{\vdash \Theta \Uparrow A, B, \Delta}{\vdash \Theta \Uparrow A \vee^- B, \Delta}\ \vee^-_F \qquad \frac{\vdash \Theta \Uparrow [y/x]B, \Delta}{\vdash \Theta \Uparrow \forall x.B, \Delta}\ \forall_F\dagger$$

Synchronous introduction rules

$$\frac{}{\vdash \Theta \Downarrow t^+}\ t^+{}_F \qquad \frac{\vdash \Theta \Downarrow B_1, \Delta_1 \quad \vdash \Theta \Downarrow B_2, \Delta_2}{\vdash \Theta \Downarrow B_1 \wedge^+ B_2, \Delta_1, \Delta_2}\ \wedge^+_F$$

$$\frac{\vdash \Theta \Downarrow B_i, \Delta}{\vdash \Theta \Downarrow B_1 \vee^+ B_2, \Delta}\ \vee^+_F, i \in \{1, 2\} \qquad \frac{\vdash \Theta \Downarrow [t/x]B, \Delta}{\vdash \Theta \Downarrow \exists x.B, \Delta}\ \exists_F$$

Identity rules

$$\frac{}{\vdash \neg P_a, \Theta \Downarrow P_a}\ init_F \qquad \frac{\vdash \Theta \Uparrow B \quad \vdash \Theta \Uparrow \neg B}{\vdash \Theta \Uparrow \cdot}\ cut_F$$

Structural rules

$$\frac{\vdash \Theta, C \Uparrow \Delta}{\vdash \Theta \Uparrow C, \Delta}\ store_F \qquad \frac{\vdash \Theta \Uparrow \Delta}{\vdash \Theta \Downarrow \Delta}\ release_F \qquad \frac{\vdash \Theta \Downarrow \Delta}{\vdash \Theta \Uparrow \cdot}\ decide_F$$

Fig. 1. The *LKF* focused proof system.

In Figure 1, P is a positive formula; N a negative formula; P_a a positive literal; C a positive formula or negative literal; and $\neg B$ is the negation normal form of the negation of B. The eigenvariables proviso † is the usual one: y is not free in Θ, in Δ, nor in $\forall x.B$. Finally, in the $release_F$ rule, Δ must contain only negative formulas and in the $decide_F$ rule, Δ must be a non-empty multiset of positive formulas all of which occur in Θ.

This proof system and the one presented in the next section (Figure 2) modifies Gentzen's sequent calculus with the following features.

Polarized formula *LKF* is a proof system of *polarized formula* built using atomic formulas, the usual first-order quantifiers $\forall$ and $\exists$, and polarized versions of the logical connectives and constants t^-, t^+, f^-, f^+, $\vee^-$, $\vee^+$, $\wedge^-$, and $\wedge^+$. The positive and negative versions of connectives and constants have identical truth conditions but different inference rules. All polarized formulas are either positive or negative: if a formula's top-level connective is t^+, f^+, $\vee^+$, $\wedge^+$, or $\exists$, then that formula is positive. Dually, if a formula's top-level connective is t^-, f^-, $\vee^-$, $\wedge^-$, or $\forall$, then it is negative. In this way, every polarized formula is classified except for literals: to polarize them, we are allowed to fix the polarity of atomic formulas in any way we see fit. We may ask that all atomic formulas are positive, that they are all negative, or we can mix polarity assignments. In any case, if A is a positive atomic formula, then it is a positive formula and $\neg A$ is a negative formula: conversely, if A is a negative atomic formula, then it is a negative formula and $\neg A$ is a positive formula.

Two sequent judgments *LKF* rules involve two kinds of sequents: $\vdash \Theta \Uparrow \Delta$ and $\vdash \Theta \Downarrow \Delta$, where Θ is a multiset of polarized formulas and Δ is a *list* of polarized formulas. Formula occurrences in Δ in $\Downarrow$ sequents are called the *foci* of that sequent. The original versions of focused proof systems used in [17,19] differ from the version used here in that sequents here of the form $\vdash \Theta \Downarrow \Delta$ are allowed to have more than one focus. While showing soundness and completeness of this *multifocusing* proof system is nearly trivial, *multifocusing* in this sense has been used to analyze parallelism in classical proof systems [3,5].

Two phases of inference rules All the "asynchronous" inference rules of *LKF* have $\Uparrow$-sequents in their premises and conclusion while all the "synchronous" inference rules have $\Downarrow$-sequents in their premises and conclusion. The only rules that mix these sequents are the $release_F$ and $decide_F$ rules. A maximal sequence of asynchronous or synchronous inferences form *phases* with interfaces between phases given by instances of the $release_F$ and $decide_F$ rules. These phases form, in fact, macro-level (synthetic) inference rules constructed from collections of the smaller rules of Gentzen's original sequent calculus.

Delays We shall find it important to break a sequence of negative or positive connectives by inserting *delays*: if B is a polarized formula then we define $\partial^-(B)$ to be (always negative) $B \wedge^- t^-$ and $\partial^+(B)$ to be (always positive) $B \wedge^+ t^+$. From such a definition, the following rules can be derived:

$$\frac{\vdash \Theta \Uparrow B, \Delta}{\vdash \Theta \Uparrow \partial^-(B), \Delta}\ \partial_F^- \qquad\qquad \frac{\vdash \Theta \Downarrow B, \Delta}{\vdash \Theta \Downarrow \partial^+(B), \Delta}\ \partial_F^+$$

To illustrate the use of delays, note that the sequent $\vdash \Theta \Downarrow \exists x \exists y.B(x,y)$ must be the result of applying (at least) two $\exists$-introduction rules. In contrast, the sequent

$\vdash \Theta \Downarrow \exists x \partial^-(\exists y.B(x,y))$ must be the conclusion of only one $\exists$-introduction rule: a separate instantiating of $\exists y$ can take place elsewhere in the proof.

The completeness of *LKF* can be stated as follows [17]. We say that $\hat{B}$ is a *polarization* of the (unpolarized) B if it results from placing superscripts $+$ and $-$ on the propositional connectives, assigning atomic formulas any mix of positive or negative polarization, and inserting any number of delays. Completeness is now the statement that if B is an (unpolarized) classical logic theorem and $\hat{B}$ is *any* polarization of B, then $\vdash \cdot \Uparrow \hat{B}$ is provable in *LKF*. Clearly, the choice of polarization does not affect provability but it can have a big impact on the structure of proofs. A polarized formula B is a *bipolar formula* if B is a positive formula and no positive subformula occurrence of B is in the scope of a negative connective in B. A *bipole* is a pair of a negative phase below a positive phase within *LKF*: thus, bipoles are macro inference rules in which the conclusion and the premises are $\Uparrow$-sequents with no formulas to the right of the up-arrow.

3 A focused labeled framework for modal logic

Let $\mathcal{L}$ be a set of variables that we will call *labels.* Then *LMF* formulas are either *labeled formulas* of the form $x\sigma : A$ or *relational atoms* of the form xRy, where x, y are labels, σ is a (possibly empty) sequence of labels and A is a polarized modal formula. In the following, we will use φ, ψ to denote labeled formulas and Θ, Ω to denote multisets of labeled formulas. Furthermore, we will sometimes write $x\sigma : \Gamma$ to denote the multiset of labeled formulas $\{x\sigma : A \mid A \in \Gamma\}$. φ is a $\Diamond$*-formula* ($\Box$*-formula*) if the main connective of A is $\Diamond$ ($\Box$). Let $\varphi \equiv x\sigma : A$ be a labeled formula; we say that x is the *present* of φ and σ is the *future* of φ. Intuitively, the present of a formula has the same role of labels in labeled modal proof systems (e.g., [20]), i.e., it indicates in which world of the corresponding Kripke structure such a formula holds. The future of a formula is used to constrain the behavior of the rule $\Diamond_F$. We note that for the emulation of the calculi presented in this paper, a future consisting of a single label is enough. We prefer, however, to present the framework in this more general version that allows for capturing also other behaviors.

An *LMF sequent* has the form $\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \Omega$ or $\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow \Omega$, where the *relational set (of the sequent)* $\mathcal{G}$ is a set of relational atoms, the *present (of the sequent)* $\mathcal{H}$ is a non-empty multiset of pairs $(x, \mathcal{F})$, where x is a label and $\mathcal{F}$ is a set of labels, and Θ and Ω are multisets of labeled formulas. Intuitively, the present $\mathcal{H}$ of a sequent is used to specify on which worlds we are currently working on and $\mathcal{F}$ to specify which worlds, amongst the reachable ones, are admissible. E.g., if we are in the position of applying a $decide_F$ (by proceeding bottom-up), then a pair $(x, \mathcal{F})$ contained in $\mathcal{H}$ says that: (i) we can (multi)focus on labeled formulas whose main connective is either classical or a $\Box$, labeled with x; and (ii) we can "move" to a y reachable from x (by multifocusing on $\Diamond$-formulas) if y is not in the set $\mathcal{F}$ of *forbidden futures* for x. We say that an *LMF* sequent is a *synchronized sequent* if it has the form $\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot$.

The general framework LMF^X is presented in Figure 2. The parameter X is a subset of $\{T, 4, 5, B, D\}$, which specifies which modal logic we are considering.

The system $LMF^{\emptyset}$, which we will just denote LMF in the following, is a system for the logic K and is obtained by including only the first four classes of rules (i.e., no relational rules). Any other system LMF^X is obtained by adding to LMF the set of relational rules $\{C_F \mid C \in X\}$.

In addition to being modular with respect to the relational properties considered, we can (and in the following will) obtain different proof systems by specializing the rule $decide_F$. In particular, specializations of $decide_F$ will be defined by playing with the following parameters:

- restrictions on the class of formulas on which multifocusing can be applied;
- restriction of the multiset $\mathcal{H}'$;
- restrictions on the definition of the future σ of formulas in Ω.

Theorem 3.1 *The system LMF^X is sound and complete with respect to the logic KX, for any polarization of formulas.*

Proof. The system LMF^X is a multifocused version of the system presented in [19], augmented with some devices for controlling the application of rules. Soundness follows from the fact that such devices can only introduce restrictions to the application of rules. Completeness is a direct consequence of completeness of the previous system, since in the liberal version presented in this section all new devices (including multifocusing) can just be ignored, or used in a trivial way, so that each proof in the previous system is also a valid proof in LMF^X.$\Box$

4 Emulation of other modal proof systems

In order to emulate proofs in the system OS^X by means of the focused framework LMF^X, we need to: (i) define a proper polarization of modal formulas; and (ii) put some restrictions on the general framework LMF^X, namely by using a specialization of the rule $decide_F$.

We present here a polarization that will be used in the rest of this section. When translating a modal formula into a polarized one, we are often in a situation where we are interested in putting a delay in front of the formula only in the case when it is negative and not a literal. For that purpose, we define A^{∂^+}, where A is a modal formula in negation normal form, to be A if A is a literal or a positive formula and $\partial^+(A)$ otherwise. We extend such a notion to a multiset Γ of formulas by defining $\Gamma^{\partial^+} = \{A^{\partial^+} | A \in \Gamma\}$. Then we define the translation $\lfloor\cdot\rfloor$ from modal formulas in negation normal form into polarized modal formulas as follows:

$$\begin{array}{rcl@{\qquad}rcl}
\lfloor P \rfloor & = & P & \lfloor A \wedge B \rfloor & = & \lfloor A \rfloor^{\partial^+} \wedge^- \lfloor B \rfloor^{\partial^+} \\
\lfloor \neg P \rfloor & = & \neg P & \lfloor A \vee B \rfloor & = & \lfloor A \rfloor^{\partial^+} \vee^- \lfloor B \rfloor^{\partial^+} \\
\lfloor \Box A \rfloor & = & \Box(\lfloor A \rfloor^{\partial^+}) & \lfloor \Diamond A \rfloor & = & \Diamond(\partial^-(\lfloor A \rfloor^{\partial^+}))
\end{array}$$

In the following, we will sometimes use the natural extension of this notion to multisets of modal formulas, i.e., $\lfloor \Gamma \rfloor = \{\lfloor A \rfloor \mid A \in \Gamma\}$. We say that a synchronized sequent $\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot$ is in *OS form* if for all $x : A \in \Theta$, $A = \lfloor B \rfloor$ for some modal formula B.

Asynchronous introduction rules

$$\frac{}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : t^-, \Omega}\ t^-{}_F \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : f^-, \Omega}\ f^-{}_F$$

$$\frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : A, \Omega \quad \mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : B, \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : A \wedge^- B, \Omega}\ \wedge^-_F \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : A, x : B, \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : A \vee^- B, \Omega}\ \vee^-_F$$

$$\frac{\mathcal{G} \cup \{xRy\} \vdash_{\mathcal{H}} \Theta \Uparrow y : B, \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : \Box B, \Omega}\ \Box_F$$

Synchronous introduction rules

$$\frac{}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : t^+}\ t^+{}_F \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : B_1, \Omega_1 \quad \mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : B_2, \Omega_2}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : B_1 \wedge^+ B_2, \Omega_1, \Omega_2}\ \wedge^+_F$$

$$\frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : B_i, \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow x\sigma : B_1 \vee^+ B_2, \Omega}\ \vee^+_F, i \in \{1,2\} \qquad \frac{\mathcal{G} \cup \{xRy\} \vdash_{\mathcal{H}} \Theta \Downarrow y\sigma : B, \Omega}{\mathcal{G} \cup \{xRy\} \vdash_{\mathcal{H}} \Theta \Downarrow xy\sigma : \Diamond B, \Omega}\ \Diamond_F$$

Identity rules

$$\frac{}{\mathcal{G} \vdash_{\mathcal{H}} x : \neg P_a, \Theta \Downarrow x : P_a}\ init_F \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : B \quad \mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : \neg B}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}\ cut_F$$

Structural rules

$$\frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta, x : C \Uparrow \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow x : C, \Omega}\ store_F \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \Omega^P}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Downarrow \Omega}\ release_F \qquad \frac{\mathcal{G} \vdash_{\mathcal{H}'} \Theta \Downarrow \Omega}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}\ decide_F$$

Relational rules

$$\frac{\mathcal{G} \cup \{yRy\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}\ T_F \qquad \frac{\mathcal{G} \cup \{yRz\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}{\mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}\ D_F \qquad \frac{\mathcal{G} \cup \{xRy, yRx\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}{\mathcal{G} \cup \{xRy\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}\ B_F$$

$$\frac{\mathcal{G} \cup \{xRy, yRz, xRz\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}{\mathcal{G} \cup \{xRy, yRz\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}\ 4_F \qquad \frac{\mathcal{G} \cup \{xRy, xRz, yRz\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}{\mathcal{G} \cup \{xRy, xRz\} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot}\ 5_F$$

Here, P_a is a positive literal, C is a positive formula or a negative literal and $\neg B$ is the negation normal form of the negation of B.
In $\Box_F$, y is different from x and does not occur in $\mathcal{G}$ nor in Θ.
In $decide_F$, if $x\sigma : A \in \Omega$ then $x : A \in \Theta$. Moreover, Ω contains only formulas of the form: (i) $x\sigma : A$, where A is not a $\Diamond$-formula and $(x, \mathcal{F}) \in \mathcal{H}$ for some $\mathcal{F}$; or (ii) $zy\sigma : A$ where A is a $\Diamond$-formula, $xRy, zRy \in \mathcal{G}$, $(x, \mathcal{F}) \in \mathcal{H}$ for some $\mathcal{F}$ and $y \notin \mathcal{F}$.
In $release_F$, Ω contains no positive formulas and $\Omega^P = \{x : A \mid x\sigma : A \in \Omega\}$.
In D_F, z is different from y and does not occur in $\mathcal{G}$ and Θ.

Fig. 2. *LMF*^{X}: a focused labeled framework for modal logic.

Identity and structural rules

$$\frac{}{\vdash \Gamma, P, \neg P}\ init \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta}\ cut \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ contr$$

Classical connectives rules

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\ \wedge \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\ \vee$$

$\Box$-rules

$$\frac{\vdash \Gamma, A}{\vdash \Diamond\Gamma, \Box A, \Delta}\ \Box_K \qquad \frac{\vdash \Diamond\Gamma, \Gamma', A}{\vdash \Diamond\Gamma, \Box A, \Delta}\ \Box_{K4} \qquad \frac{\vdash \Diamond\Gamma, \Gamma', \Box\Sigma, A}{\vdash \Diamond\Gamma, \Box\Sigma, \Box A, \Delta}\ \Box_{K45}$$

$\Diamond$-rules

$$\frac{\vdash \Diamond A, A, \Sigma}{\vdash \Diamond A, \Sigma}\ \Diamond_T \qquad \frac{\vdash \Gamma}{\vdash \Diamond\Gamma, \Delta}\ \Diamond_D$$

In $\Box_{K4}$, Δ does not contain any formula whose main connective is $\Diamond$.
In $\Box_{K45}$, Δ does not contain any formula whose main connective is $\Diamond$ or $\Box$.
$\Gamma' \subseteq \Gamma$. $\neg A$ is the negation normal form of the negation of A.

Fig. 3. OS^X: a family of ordinary sequent proof systems for modal logic.

4.1 Ordinary sequent calculi

Several "ordinary" sequent systems have been proposed in the literature for different modal logics (a general account is, e.g., in [13,21]). In our treatment, we will use a formalization of a class of modal sequent systems, presented in Figure 3, which is adapted mainly from the presentations in [7,23]. It can be seen as a family of proof systems, where the system of a specific logic is obtained by adding to the base classical system (consisting of *identity*, *structural* and *classical connective* rules) one of the $\Box$-rules and any (possibly empty) set of $\Diamond$-rules. As the name of the rule suggests, the rule $\Box_K$ alone gives a system for the logic K. We replace it with $\Box_{K4}$ or $\Box_{K45}$ in case we want to capture logics characterized by transitive or both transitive and euclidean frames, respectively. The rules $\Diamond_T$ and $\Diamond_D$ can be further added, modularly, in order to get systems for those logics enjoying reflexivity and seriality, respectively. For instance, by adding $\Box_{K4}$ and $\Diamond_T$ to the base system, we get a system for the logic $S4$, while by adding $\Box_{K45}$ and $\Diamond_T$, we get a system for $S5$. Formulas are assumed to be in negation normal form.

Furthermore, we specialize the rule $decide_F$ as follows:

$$\frac{\mathcal{G} \vdash_{\mathcal{H}'} \Theta \Downarrow \Omega}{\mathcal{G} \vdash_{\{(x,\mathcal{F})\}} \Theta \Uparrow \cdot}\ decide_{OS}$$

where (in addition to the general conditions of Figure 2) we have that either:

(i) there exists y s.t.:
- $xRy \in \mathcal{G}$;

- if $x \neq y$, then $\mathcal{H}' = \{(y, \mathcal{F} \cup \{x\})\}$ and Ω is a multiset of formulas of the form $zy : \Diamond A$, s.t. $zRy \in \mathcal{G}$, $z \in \mathcal{F}$;
- if $x = y$ then $\mathcal{H}' = \{(x, \mathcal{F})\}$ and $\Omega = \{xx : \Diamond A\}$ for some A;

(ii) $\Omega = \{x : A\}$ for some A and $\mathcal{H}' = \{(x, \mathcal{F})\}$.

Roughly speaking, the specialization with respect to the general framework consists in: (i) restricting the use of multifocusing to $\Diamond$-formulas; (ii) forcing such $\Diamond$-formulas to be labeled with the same future; (iii) when moving to a new label, adding the current label to the set of forbidden futures.

We call LMF^X_{OS} the system obtained from LMF^X by replacing the rule $decide_F$ with the rule $decide_{OS}$. It is easy to notice that, given the polarization above and the rule $decide_{OS}$, we can in fact restrict LMF^X_{OS} to deal with sequents whose present is always a singleton and such that the future of each labeled formula has length at most 1. In the rest of this section, for simplicity, we will write sequents using the following notation: $\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Updownarrow \Omega$.

In order to compare OS and LMF sequents, we define an interpretation $\mathcal{I}^X_{OS}(\cdot)$ of synchronized sequents as multisets of modal formulas, where the X denotes the fact that the interpretation is also parametric in the logic considered; namely, we use a different interpretation in the case of those logics whose frames enjoy transitivity.

$$\mathcal{I}^X_{OS}(\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot) = \begin{cases} \{A \mid x : \lfloor A \rfloor^{\partial^+} \in \Theta\} \cup \{\Box B \mid y : \partial^+(\lfloor B \rfloor) \in \Theta, xRy \in \mathcal{G}^*, y \notin \mathcal{F}\}, & \text{if } 4 \notin X \\ \{A \mid x : \lfloor A \rfloor^{\partial^+} \in \Theta\} \cup \{\Box B \mid y : \partial^+(\lfloor B \rfloor) \in \Theta, xRy \in \mathcal{G}^*, y \notin \mathcal{F}\} \cup & \\ \{\Diamond C \mid z : \lfloor \Diamond C \rfloor \in \Theta, zRx \in \mathcal{G}^*, z \in \mathcal{F}\}, & \text{otherwise} \end{cases}$$

where $\mathcal{G}^*$ denotes the closure of $\mathcal{G}$ with respect to those properties amongst reflexivity, transitivity and euclideaness contained in X.

We notice that in an LMF^X_{OS} derivation (reading from the end-sequent upwards), when we decide on a formula, we keep a copy of it in the context, i.e., we implicitly apply a contraction. For this reason, we have that an LMF^X_{OS} derivation tends to keep some information that is lost in the corresponding OS^X derivation (again, reading bottom-up). We define a notion of extension of a sequent that will help compare the two systems. Given a synchronized sequent $S \equiv \mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot$ and an OS sequent $\vdash \Gamma$, we say that S *extends* $\vdash \Gamma$ if there exists $S' \equiv \mathcal{G} \vdash_{\mathcal{H}} \Theta' \Uparrow \cdot$ such that $\Theta \supseteq \Theta'$ and $\mathcal{I}^X_{OS}(S') = \Gamma$.

Lemma 4.1 *Let* $\dfrac{S_1}{S}\ r$ $\left(\dfrac{S_1 \quad S_2}{S}\ r\right)$ *be an application of a non-structural rule in* OS^X*. Then for any synchronized sequent* S' *that is in OS form and extends* S*, there exists a derivation* $\begin{matrix} S'_1 \\ \vdots \\ S' \end{matrix}$ $\left(\begin{matrix} S'_1 \quad S'_2 \\ \vdots \\ S' \end{matrix}\right)$ *in* LMF^X_{OS}*, such that* S'_1 *is in OS form and extends* S_1 *(*S'_1*,* S'_2 *are in OS form and extend* S_1*,* S_2*, respectively). Furthermore, if* $\overline{S}\ r$ *is a rule application in* OS^X*, then for any synchronized sequent* S' *in OS form extending* S*, there exists a proof of* S' *in* LMF^X_{OS}*.*

Proof. The proof proceeds by considering all the non-structural rules of OS^X. The case of initial and classical connectives rules is trivial and we omit it. We consider in detail one case; further cases are given in Appendix A. Let us consider an application of the rule $\dfrac{\vdash \Gamma, A}{\vdash \Diamond\Gamma, \Box A, \Delta}\ \Box_K$, where Δ does not contain any formula whose main connective is $\Diamond$. Now assume that $S' \equiv \mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot$ is in OS form and extends $\vdash \Diamond\Gamma, \Box A, \Delta$. Notice that we are in the case when 4 does not occur in X. It follows that $x : \lfloor\Diamond\Gamma\rfloor \in \Theta$. We have two cases: either (a) $x : \partial^+(\lfloor\Box A\rfloor) \in \Theta$ or (b) $y : \lfloor A\rfloor^{\partial^+} \in \Theta$ and $xRy \in \mathcal{G}$. Then the LMF^X derivation corresponding to this rule application consists in the following steps (reading the derivation bottom-up):

(i) decide on $x : \partial^+(\lfloor\Box A\rfloor)$, ending up by adding xRy to $\mathcal{G}$ (note that this step is only required if we are in case (a));

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G} \cup \{xRy\} \vdash_{x,\mathcal{F}} \Theta, y : \lfloor A\rfloor^{\partial^+} \Uparrow \cdot}{\mathcal{G} \cup \{xRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow y : \lfloor A\rfloor^{\partial^+}}\ store_F}{\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow x : \Box\lfloor A\rfloor^{\partial^+}}\ \Box_F}{\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Downarrow x : \partial^+(\Box\lfloor A\rfloor^{\partial^+})}\ \partial^+_F, release_F}{\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot}\ decide_{OS}$$

(ii) multi-decide on $x : \lfloor\Diamond\Gamma\rfloor$ choosing y as the future.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G} \cup \{xRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta, y : \lfloor A\rfloor^{\partial^+}, y : \lfloor\Gamma\rfloor^{\partial^+} \Uparrow \cdot}{\mathcal{G} \cup \{xRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta, y : \lfloor A\rfloor^{\partial^+} \Uparrow y : \partial^-(\lfloor\Gamma\rfloor^{\partial^+})}\ \partial^-_F, store_F}{\mathcal{G} \cup \{xRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta, y : \lfloor A\rfloor^{\partial^+} \Downarrow y : \partial^-(\lfloor\Gamma\rfloor^{\partial^+})}\ release_F}{\mathcal{G} \cup \{xRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta, y : \lfloor A\rfloor^{\partial^+} \Downarrow xy : \Diamond\partial^-(\lfloor\Gamma\rfloor^{\partial^+})}\ \Diamond_F}{\mathcal{G} \cup \{xRy\} \vdash_{x,\mathcal{F}} \Theta, y : \lfloor A\rfloor^{\partial^+} \Uparrow \cdot}\ decide_{OS}$$

□

Theorem 4.2 *Let Π be an OS^X derivation of a sequent $S \equiv \vdash A$ from the sequents $S_1, \ldots, S_n$ and let $S' \equiv \emptyset \vdash_{\{x,\emptyset\}} x : (\lfloor A\rfloor)^{\partial^+} \Uparrow \cdot$ for some x. Then there exists an LMF^X_{OS} derivation Π' of S' from $S'_1, \ldots, S'_n$, where $S'_1, \ldots, S'_n$ extend $S_1, \ldots, S_n$, respectively. Moreover, Π' is such that each rule application in Π, deriving a sequent $\hat{S}$, corresponds to a sequence s of bipoles in Π' such that s ends with a synchronized sequent $\hat{S}'$ extending $\hat{S}$.*

Proof. For simplicity, we assume that in Π the rule *contr* is only applied to a given formula immediately below a rule that introduces an occurrence of such a formula. We proceed bottom-up by starting from the root of Π and build Π' by repeatedly applying Lemma 4.1. At each step, we get as leaves sequents that are extensions of the ones in Π, so that Lemma 4.1 can be applied again. □

We say that a synchronized sequent $S \equiv \mathcal{G} \vdash_{\mathcal{H}} \Theta \Uparrow \cdot$ is a *contraction of an OS sequent* $\vdash \Gamma$ if S is in OS form, Γ contains $\mathcal{I}^X_{OS}(S)$ and for each formula A in Γ there is at least one occurrence of A in $\mathcal{I}^X_{OS}(S)$.

Lemma 4.3 *Let $S' \equiv \mathcal{G} \vdash_{\{x,\mathcal{F}\}} \Theta \Uparrow \cdot$ be a synchronized sequent in OS form. For each derivation of the form* $\begin{array}{c} S'_1 \\ \vdots \\ S' \end{array} \left(\begin{array}{c} S'_1 \quad S'_2 \\ \vdots \\ S' \end{array} \right)$ *in LMF^X_{OS} that is a bipole, there exists an OS sequent S such that: (i) S' is a contraction of S; and (ii) if $\mathcal{I}^X_{OS}(S'_1) \neq \mathcal{I}^X_{OS}(S')$ $(\mathcal{I}^X_{OS}(S'_1) \neq \mathcal{I}^X_{OS}(S')$ and $\mathcal{I}^X_{OS}(S'_2) \neq \mathcal{I}^X_{OS}(S'))$, then there exists a rule application $\dfrac{S_1}{S}$ $\left(\dfrac{S_1 \quad S_2}{S}\right)$ in OS^X such that $\mathcal{I}^X_{OS}(S'_1) = S_1$ $(\mathcal{I}^X_{OS}(S'_1) = S_1$ and $\mathcal{I}^X_{OS}(S'_2) = S_2)$. Furthermore, for each proof of S' that is a bipole, there exist: (i) an OS sequent S such that S' is a contraction of S; and (ii) a rule application $\overline{S}$ in OS^X.*

Proof. In Appendix A. $\Box$

Theorem 4.4 *Let Π' be a proof of a sequent $S' \equiv \emptyset \vdash_{\{x,\emptyset\}} x : (\lfloor A \rfloor)^{\partial^+} \Uparrow \cdot$ for some x. Then there exists a proof Π of a sequent S in OS^X, where S' is a contraction of S, such that each bipole in Π' corresponds to one rule application in Π,* plus possible applications of *contr*.

Proof. We proceed top-down starting from the leaves of Π' and build Π by repeatedly applying Lemma 4.3. At each step, we get as the conclusion of an OS^X rule application a sequent S^* such that the one obtained in the corresponding step of Π' is a contraction of S^*. By applying *contr*, we can transform the OS^X derivation built so far in order to remove possible undesired multiple occurrences of a formula. $\Box$

Theorem 4.5 *The system LMF^X_{OS} is sound and complete for the logic KX.*

Proof. Soundness is obvious, since LMF^X_{OS} is just a restriction of LMF^X. Completeness follows from Theorem 4.2 and completeness of the system OS^X. $\Box$

4.1.1 A different formulation for ordinary sequent systems

The system LMF^X_{OS} is designed with the aim of emulating the behavior of OS^X as much as possible in a rule-by-rule way. It is also possible to give a different polarization (obtained by using delays less intensively) such that a bipole in the focused system corresponds to a larger, but well identified, block of an OS^X derivation. In fact, we can read an OS^X derivation (from the root upwards) as composed of blocks, where, as observed, e.g., in [16], we first apply all the classical reasoning (intuitively, on a given world) and then we apply a modal rule (thus moving on a different world). We can define $\lfloor \cdot \rfloor_{OS'}$ as follows:

$$
\begin{array}{rclrcl}
\lfloor P \rfloor_{OS'} & = & P & \lfloor A \wedge B \rfloor_{OS'} & = & \lfloor A \rfloor_{OS'} \wedge^- \lfloor B \rfloor_{OS'} \\
\lfloor \neg P \rfloor_{OS'} & = & \neg P & \lfloor A \vee B \rfloor_{OS'} & = & \lfloor A \rfloor_{OS'} \vee^- \lfloor B \rfloor_{OS'} \\
\lfloor \Box A \rfloor_{OS'} & = & \Box(\lfloor A \rfloor_{OS'}{}^{\partial^+}) & \lfloor \Diamond A \rfloor_{OS'} & = & \Diamond(\partial^-(\lfloor A \rfloor_{OS'}{}^{\partial^+}))
\end{array}
$$

In this setting, each new bipole is started by choosing a successor of the current world and by multifocusing on a set of $\Diamond$-formulas, i.e., we can define the following rule:

$$
\frac{\mathcal{G} \vdash_{\mathcal{H}'} \Theta \Downarrow \Omega}{\mathcal{G} \vdash_{\{(x,\mathcal{F})\}} \Theta \Uparrow \cdot} \; decide_{OS'}
$$

where (in addition to the general conditions of Figure 2) we require that Ω is a multiset of $\Diamond$-formulas and there exists y such that:

- $xRy \in \mathcal{G}$;
- if $x \neq y$, then $\mathcal{H}' = \{(y, \mathcal{F} \cup \{x\})\}$ and Ω is a multiset of formulas of the form $zy : \Diamond A$, s.t. $zRy \in \mathcal{G}$, $z \in \mathcal{F}$;
- if $x = y$ then $\mathcal{H}' = \{(x, \mathcal{F})\}$ and $\Omega = \{xx : \Diamond A\}$ for some A.

4.2 Nested sequent calculi

Nested sequents (first introduced by Kashima [14], and then independently rediscovered by Poggiolesi [21], as *tree-hypersequents*, and by Brünnler [2]) are an extension of ordinary sequents to a structure of tree, where each node represents the scope of a modal $\Box$. We write a nested sequent as a multiset of formulas and *boxed sequents*, according to the following grammar, where A can be any modal formula in negative normal form: $\Gamma ::= \emptyset \mid A, \Gamma \mid [\Gamma], \Gamma$

In a nested sequent calculus, a rule can be applied at any depth in this tree structure, that is, inside a certain nested sequent context. A *context* written as $\Gamma\{\ \}\cdots\{\ \}$ is a nested sequent with a number of holes occurring in place of formulas (and never inside a formula). Given a context $\Gamma\{\ \}\cdots\{\ \}$ with n holes and n nested sequents $\Delta_1, \ldots, \Delta_n$, we write $\Gamma\{\Delta_1\}\cdots\{\Delta_n\}$ to denote the nested sequent where the i-th hole in the context has been replaced by Δ_i, with the understanding that if $\Delta_i = \emptyset$ then the hole is simply removed.

We are going to consider the nested sequent system NS^X (on Figure 4) introduced by Brünnler in [2]. The first two categories of rules constitute a complete system for the modal logic K. It can then be extended modularly by a subset $\mathsf{X}^{\Diamond}$ of the $\Diamond$-rules to give a complete system for any logic built from *45-closed* [1] set of axioms X among D, T, B, 4 and 5.

We want to specify the general framework LMF^X in order to emulate the proofs produced by NS^X. We can use here the same polarization $\lfloor\cdot\rfloor$ as in the case of ordinary sequent systems and specialize the rule $decide_F$ as follows:

$$\frac{\mathcal{G} \vdash_{\{(x,\emptyset) \mid x \in \mathcal{L}\}} \Theta \Downarrow x : A}{\mathcal{G} \vdash_{\{(x,\emptyset) \mid x \in \mathcal{L}\}} \Theta \Uparrow \cdot}\ decide_{NS}$$

where, as defined in Section 3, $\mathcal{L}$ denotes the set of all labels.

We can also use LMF^X in order to emulate the behavior of focused nested sequent calculi, like the one in [4]. Such a system can be captured by defining a polarization that does not apply delays intensively, like the one given in Section 4.1.1 for an ordinary sequent focused system.

[1] X is said to be 45-closed: if whenever 4 is derivable in $K + X$, $4 \in X$, and whenever 5 is derivable in $K + X$, $5 \in X$. This condition is not restrictive as any logic obtained from a combination of axioms among D, T, B, 4 and 5 always has an equivalent 45-closed axiomatisation.

Identity and structural rules

$$\frac{}{\Gamma\{P, \neg P\}}\ init \qquad \frac{\Gamma\{A\} \quad \Gamma\{\neg A\}}{\Gamma\{\emptyset\}}\ cut$$

Connectives rules

$$\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}\ \wedge \qquad \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}\ \vee \qquad \frac{\Gamma\{A\}}{\Gamma\{\Box A\}}\ \Box \qquad \frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}\ \mathsf{k}^{\Diamond}$$

$\Diamond$-rules

$$\frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}}\ \mathsf{t}^{\Diamond} \qquad \frac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}}\ \mathsf{d}^{\Diamond} \qquad \frac{\Gamma\{A, [\Diamond A, \Delta]\}}{\Gamma\{[\Diamond A, \Delta]\}}\ \mathsf{b}^{\Diamond}$$

$$\frac{\Gamma\{\Diamond A, [\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}\ \mathsf{4}^{\Diamond} \qquad \frac{\Gamma\{\Diamond A\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathsf{5}^{\Diamond}$$

In $\mathsf{5}^{\Diamond}$, the first hole in $\Gamma\{\ \}\{\ \}$ can not occur at the root of the sequent tree.

Fig. 4. NS^X: a family of nested sequent proof systems for modal logic.

5 Concluding remarks

We have presented LMF^X as a general framework, based on a focused version of a labeled system, for emulating the behavior of several known modal proof systems from different proof formalisms. We have considered, in particular, ordinary sequent systems and nested sequent systems. The case of ordinary sequents is interesting, because such calculi are proven to be optimal from the point of view of the efficiency of proof search. By decorating the sequents used in our framework with information (the *present* of the sequent) that specifies which world we are currently working on and which worlds are not reachable anymore, we are able to reproduce the mechanism that constrains (and improves) proof search in such calculi. We remark that a similar result is obtained in [16], by using a different technique. By analysing the case of ordinary sequents, we conclude that the behavior of modal rules in such a setting can be seen as corresponding to the application of two bipoles in our (1-sided) focused framework: the first bipole concerns a formula whose main connective is a $\Box$, while the second corresponds to a phase in which we multifocus on formulas with $\Diamond$ as the main connective. In the case of logics extending K, in addition to such bipoles, the application of relational rules can also be required. The case of nested sequents, on the other hand, allows for illustrating the use of sequents decorated with a present which is a set of worlds and not just a singleton.

We believe that our framework is general enough to capture modal proof systems defined in other formalisms, such as prefixed tableaux systems [6,11], 2-sequents [18] and their generalization to linear nested sequents [15]. In particular, we are currently working on formalizing a parametrization of LMF^X that allows for capturing the modal hypersequent systems of, e.g., [1]. The basic idea consists in using a present which is a multiset, in representing external structural rules as operations on such a present and in seeing modal communication rules as a combination of relational and modal rules in our setting.

We showed that LMF^X, when properly instantiated, can emulate several modal proof systems with high precision: individual modal inference rules correspond to certain chains of bipoles in the encoded LMF^X system *and vice versa*. Thus implementations of the LMF^X proof system can be seen as providing a theorem prover and a proof checker for the emulated proof systems. Although the LMF^X proof system imposes a lot of structure on the search for proofs, several important details are free to be implemented in differing ways. For example, one is free to implement the closure of the underlying world structure $\mathcal{G}^*$ via saturated bottom-up or top-down proof search.

While in this work we focused on the emulation of existing calculi, we believe that LMF^X can also be seen as a tool for developing new, original (focused) proof systems for modal logics, obtained by properly fine-tuning the parametrical aspects of the framework. This will also be object of future research.

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Appendix

A Proofs

A.1 Proof of Lemma 4.1

Other cases: (1) Let us consider an application of the rule $\Box_{K4}$:

$$\frac{\vdash \Diamond\Gamma, \Gamma, A}{\vdash \Diamond\Gamma, \Box A, \Delta}\ \Box_{K4}$$

where Δ does not contain any formula whose main connective is $\Diamond$. Assume that $S' \equiv \mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot$ is in OS form and extends $\vdash \Diamond\Gamma, \Box A, \Delta$. As in (i), we can have two cases: either (a) $x : \partial^+(\lfloor \Box A \rfloor) \in \Theta$ or (b) $y : \lfloor A \rfloor^{\partial^+} \in \Theta$ and $xRy \in \mathcal{G}^*$. Moreover, for each $B \in \Gamma$, one of the following two cases holds: either (c) $x : \lfloor \Diamond B \rfloor \in \Theta$ or (d) $z : \lfloor \Diamond B \rfloor \in \Theta$ and $zRx \in \mathcal{G}^*$ for some z. After possible applications of relational rules that lead to a sequent whose relational set contains xRy (if we are in case (b)) and zRx (if we are in case (d)), the

LMF^X derivation corresponding to this rule application consists in the following bipoles (reading the derivation bottom-up):

(i) decide on $x : \partial^+(\lfloor\Box A\rfloor)$, ending up by adding xRy to $\mathcal{G}$ (note that this step is only required if we are in case (a));

$$\dfrac{\dfrac{\dfrac{\mathcal{G} \cup \{xRy\} \vdash_{x,\mathcal{F}} \Theta, y : \lfloor A\rfloor^{\partial^+} \Uparrow \cdot}{\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow x : \Box\lfloor A\rfloor^{\partial^+}}\ \Box_F, store_F}{\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Downarrow x : \partial^+(\Box\lfloor A\rfloor^{\partial^+})}\ \partial^+_F, release_F}{\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot}\ decide_{OS}$$

(ii) for those $B \in \Gamma$ such that case (d) holds, we apply the rule 4_F to zRx and xRy (ending up by adding zRy to the relation set);

$$\dfrac{\mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot}{\mathcal{G} \cup \{zRx, xRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot}\ 4_F$$

(iii) multi-decide on all the $w : \lfloor\Diamond B\rfloor$ such that wRy is in the relation set and $B \in \Gamma$, choosing y as the future.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta, y : \lfloor B\rfloor^{\partial^+}, \Omega'' \Uparrow \cdot}{\mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta \Uparrow y : \partial^-(\lfloor B\rfloor^{\partial^+}), \Omega'}\ \partial^-_F, store_F}{\mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta \Downarrow y : \partial^-(\lfloor B\rfloor^{\partial^+}), \Omega'}\ release_F}{\mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta \Downarrow wy : \Diamond\partial^-(\lfloor B\rfloor^{\partial^+}), \Omega}\ \Diamond_F}{\mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot}\ decide_{OS}$$

(2) Let us consider an application of the rule $\Box_{K45}$:

$$\dfrac{\vdash \Diamond\Gamma, \Gamma, \Box\Sigma, A}{\vdash \Diamond\Gamma, \Box\Sigma, \Box A, \Delta}\ \Box_{K45}$$

where Δ does not contain any formula whose main connective is $\Diamond$ or $\Box$. Assume that $S' \equiv \mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot$ is in OS form and extends $\vdash \Diamond\Gamma, \Box\Sigma, \Box A, \Delta$. We focus on the treatment of the formulas in Σ, which is the difference with respect to case (ii). Let $B \in \Sigma$. By hypothesis, either (a) $x : \partial^+(\lfloor\Box B\rfloor) \in \Theta$ or $(b) y : \lfloor B\rfloor^{\partial^+} \in \Theta$ and $xRy \in \mathcal{G}^*$. If we are in case (a), then an application of $\Box_F$ followed by an application of 5_F will eventually lead to a synchronized sequent S'_1 such that $\Box B \in \mathcal{I}^X_{OS}(S'_1)$. If we are in case (b), then an application of 5_F, plus possible relational rules to get xRy in the relational set, will suffice.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G}\cup\{xRy,xRu,yRu\}\vdash_{x,\mathcal{F}}\Theta,u:\lfloor B\rfloor^{\partial^+}\Uparrow\cdot}{\mathcal{G}\cup\{xRy,xRu\}\vdash_{x,\mathcal{F}}\Theta,u:\lfloor B\rfloor^{\partial^+}\Uparrow\cdot}\ 5_F}{\mathcal{G}\cup\{xRy,xRu\}\vdash_{x,\mathcal{F}}\Theta\Uparrow x:\Box\lfloor B\rfloor^{\partial^+}}\ \Box_F, store_F}{\mathcal{G}\cup\{xRy\}\vdash_{x,\mathcal{F}}\Theta\Downarrow x:\partial^+(\Box\lfloor B\rfloor^{\partial^+})}\ \partial^+_F, release_F}{\mathcal{G}\cup\{xRy\}\vdash_{x,\mathcal{F}}\Theta\Uparrow\cdot}\ decide_{OS}
$$

(3) Let us consider an application of the rule $\Diamond_T$:

$$
\dfrac{\vdash \Diamond A, A, \Sigma}{\vdash \Diamond A, \Sigma}\ \Diamond_T
$$

and assume that $S' \equiv \mathcal{G}\vdash_{x,\mathcal{F}}\Theta\Uparrow\cdot$ is in OS form and extends $\vdash \Diamond A, \Sigma$. We have that either (a) $x : \lfloor\Diamond A\rfloor \in \Theta$ or (b) we are in a case where X contains 4 and $z : \lfloor\Diamond A\rfloor \in \Theta$ and $zRx \in \mathcal{G}^*$. After possible applications of relational rules that lead to a sequent whose relational set contains zRx (if we are in case (b)), the LMF^X derivation corresponding to this rule application consists in the following bipoles (reading the derivation bottom-up):

(i) if we are in case (a), apply the rule T_F in order to add xRx to $\mathcal{G}$; then decide on $x : \lfloor\Diamond A\rfloor$;

$$
\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G}\cup\{xRx\}\vdash_{x,\mathcal{F}}\Theta,x:\lfloor A\rfloor^{\partial^+}\Uparrow\cdot}{\mathcal{G}\cup\{xRx\}\vdash_{x,\mathcal{F}}\Theta\Downarrow x:\partial^-(\lfloor A\rfloor^{\partial^+})}\ release_F,\partial^-_F,store_F}{\mathcal{G}\cup\{xRx\}\vdash_{x,\mathcal{F}}\Theta\Downarrow xx:\Diamond\partial^-(\lfloor A\rfloor^{\partial^+})}\ \Diamond_F}{\mathcal{G}\cup\{xRx\}\vdash_{x,\mathcal{F}}\Theta\Uparrow\cdot}\ decide_{OS}}{\mathcal{G}\vdash_{x,\mathcal{F}}\Theta\Uparrow\cdot}\ T_F
$$

(ii) if we are in case (b), then decide on $z : \lfloor\Diamond A\rfloor$ and choose x as the future.

$$
\dfrac{\dfrac{\dfrac{\mathcal{G}\cup\{zRx\}\vdash_{x,\mathcal{F}}\Theta,x:\lfloor A\rfloor^{\partial^+}\Uparrow\cdot}{\mathcal{G}\cup\{zRx\}\vdash_{x,\mathcal{F}}\Theta\Downarrow x:\partial^-(\lfloor A\rfloor^{\partial^+})}\ release_F,\partial^-_F,store_F}{\mathcal{G}\cup\{zRx\}\vdash_{x,\mathcal{F}}\Theta\Downarrow zx:\Diamond\partial^-(\lfloor A\rfloor^{\partial^+})}\ \Diamond_F}{\mathcal{G}\cup\{zRx\}\vdash_{x,\mathcal{F}}\Theta\Uparrow\cdot}\ decide_{OS}
$$

(4) Let us consider an application of the rule $\Diamond_D$:

$$
\dfrac{\vdash \Gamma}{\vdash \Diamond\Gamma, \Delta}\ \Diamond_D
$$

where Δ does not contain any formula whose main connective is $\Diamond$. Assume that $S' \equiv \mathcal{G}\vdash_{x,\mathcal{F}}\Theta\Uparrow\cdot$ is in OS form and extends $\vdash \Diamond\Gamma, \Delta$. For each $B \in \Gamma$, one of the following two cases holds: either (a) $x : \lfloor\Diamond B\rfloor \in \Theta$ or (b) $z : \lfloor\Diamond B\rfloor \in \Theta$

and $zRx \in \mathcal{G}^*$ (note that this is only possible if X contains 4). After possible applications of relational rules that lead to a sequent whose relational set contains zRx (if we are in case (b)), the LMF^X derivation corresponding to this rule application consists in the following bipoles (reading the derivation bottom-up):

(i) apply the rule D_F in order to add xRy to $\mathcal{G}$ for some "fresh" y;

$$\frac{\mathcal{G} \cup \{xRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot}{\mathcal{G} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot}\ D_F$$

(ii) for those $B \in \Gamma$ such that case c holds, apply the rule 4_F to zRx and xRy (ending up by adding zRy to the relation set);

$$\frac{\mathcal{G} \cup \{zRx, xRy, zRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot}{\mathcal{G} \cup \{zRx, xRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot}\ 4_F$$

(iii) multi-decide on all the $w : \lfloor \Diamond B \rfloor$ such that wRy is in the relation set and $B \in \Gamma$, choosing y as the future.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G} \cup \{wRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta, y : \lfloor B \rfloor^{\partial^+}, \Omega'' \Uparrow \cdot}{\mathcal{G} \cup \{wRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta \Uparrow y : \partial^-(\lfloor B \rfloor^{\partial^+}), \Omega'}\ \partial^+_F, store_F}{\mathcal{G} \cup \{wRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta \Downarrow y : \partial^-(\lfloor B \rfloor^{\partial^+}), \Omega'}\ release_F}{\mathcal{G} \cup \{wRy\} \vdash_{y,\mathcal{F}\cup\{x\}} \Theta \Downarrow wy : \Diamond\partial^-(\lfloor B \rfloor^{\partial^+}), \Omega}\ \Diamond_F}{\mathcal{G} \cup \{wRy\} \vdash_{x,\mathcal{F}} \Theta \Uparrow \cdot}\ decide_{OS}$$

A.2 Proof of Lemma 4.3

We can distinguish cases according to the main connective of the formula(s) on which we decide. The case of classical connectives is trivial, since we have that there is an exact correspondence between a bipole in LMF^X_{OS} and a rule application in OS^X. The case of a formula with $\Box$ as the main connective is also simple, because we have that $\mathcal{I}^X_{OS}(S') = \mathcal{I}^X_{OS}(S'_1)$. Relational rules do not change interpretation of the sequent either. If we consider a decide on a multiset of formulas, whose main operator is $\Diamond$, we have that, by inspecting the cases arising from condition (i) in the definition of the rule $decide_{OS}$, one can see that this corresponds to an application of $\Box_K$, $\Box_{K4}$, $\Box_{K45}$, $\Diamond_T$ or $\Diamond_D$ according to the logic considered and the label chosen as the next one.