On Subexponentials, Synthetic Connectives, and Multi-Level Delimited Control

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Outline

- A computational phenomena that needs a proof theory treatment, a la Curry-Howard.
- Proof theory: subexponentials & synthetic connectives
- A fragment of subexponential linear logic called MC for *multi-colored* classical logic.
- Sequent calculus for MC
- Natural Deduction for MC: bounded $\lambda\mu$-terms

Computational Motivation

A *control operator* captures its continuation context: e.g.,

$$(+\ 2\ (+\ (\mathit{control}\ \ k.(+\ (k\ 1)\ (k\ 3)))\ 1))\ \longrightarrow\ 10$$

Here, k is bound to its calling context $\lambda x.(+\ 2\ (+\ x\ 1))$.

One wishes to *delimit* the scope of control operator: e.g.,

$$(+\ 2\ \sharp(+\ (\mathit{control}\ \ k.(+\ (k\ 1)\ (k\ 3)))\ 1))\ \longrightarrow\ 6$$

Here, k is bound to $\lambda x.(+\ x\ 1)$.

Multi-levels of delimitation are also desirable: e.g.,

$$(f\ \ \sharp_2(g\ \ \sharp_5(h\ \ \sharp_4\ \mathit{control}^3\ \ k.E[\mathit{control}^1\ \ k'.s])))$$

Here, $\mathit{control}^3 k.$ delimited by $\sharp_2$, captures h and g but not f.

Scope extrusion

Consider the formula $(C \supset A) \vee B$.

In intuitionistic logic, C has scope over A and not B.

In classical logic, the assumption C is allowed to *extrude* its scope over also B.

$$\begin{aligned}(C \supset A) \vee B &\equiv (\neg C \vee A) \vee B \\ &\equiv A \vee (\neg C \vee B) \equiv A \vee (C \supset B) \\ &\equiv \neg C \vee (A \vee B) \equiv C \supset (A \vee B)\end{aligned}$$

Gentzen's sequent calculus (1935) models the difference between intuitionistic and classical logic as one of *scope extrusion*.

Gentzen did not use this term: we borrow it from the π-calculus.

Sequents and binding structure

The *single-conclusion* sequent

$$x_1 \colon B_1, \ldots, x_n \colon B_n \vdash t \colon B_0$$

encodes the judgment that the term t is a properly typed functional program fragment of type B_0 given that the variables $x_1, \ldots, x_n$ have type $B_1, \ldots, B_n$, respectively.

Continuation variables are added as *multiple conclusions*.

$$x_1 \colon B_1, \ldots, x_n \colon B_n \vdash t \colon B_0, k_1 \colon C_1, \ldots, k_m \colon C_m$$

where C_i is the type expected by the continuation k_i.

Later: $t : (x_1 \colon B_1, \ldots, x_n \colon B_n \vdash B_0 \mid k_1 \colon C_1, \ldots, k_m \colon C_m)$.

Sequent rules and program structure

In *intuitionistic* (single-conclusion) proof systems, the left-hand side can be treated as *monotone*: once a variable appears in that context, it stays in that context as we move up in a proof.

In *classical* (multiple-conclusion) proof systems, both left-hand and right-hand sides can be treated as monotone.

If we can add restrictions to monotonicity, then we can have more expressive typing.

E.g., declare that certain lambda and continuation variables cannot appear within certain regions inside their scopes.

Sequent calculus and program execution

We do not wish to change evaluation models here.

Instead, we want richer types that can forbid variables from occurring in certain parts of their scope.

Basic β-reduction (cut-elimination) is untouched.

The paper addresses both the call-by-name and call-by-value variants of evaluation.

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We now consider the logic of types. Only at the end, do we return to the terms that they intend to qualify.

Intuitionistic logic and multiple conclusions

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ C \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ W \qquad \text{Right structural rules}$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B, \Delta}\ IL \qquad \frac{A, \Gamma \vdash B, \Delta}{\Gamma \vdash A \supset B, \Delta}\ CL \qquad \text{Right implication}$$

You either forget the formulas in Δ (intuitionistic logic) or keep all those formulas (classical logic).

Informally, we can hope for something intermediate:

$$\frac{A, \Gamma \vdash B, \Delta_1 \Delta_2}{\Gamma \vdash A \overset{1}{\supset} B, \Delta_1 \Delta_2} \qquad \frac{A, \Gamma \vdash B, \Delta_2}{\Gamma \vdash A \overset{2}{\supset} B, \Delta_1 \Delta_2} \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \overset{3}{\supset} B, \Delta_1 \Delta_2}$$

Introducing an implication at level 2 requires forgetting level 1 conclusions.

Two lessons from linear logic

1. Implications are built using *exponentials*: $A \supset B \equiv\, !A \multimap B$.
2. Exponentials are *not canonical*.

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Tensor ($\otimes$) is canonical since if we have two versions of $\otimes$, i.e.,

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'} \quad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'},$$

we can prove that $A \otimes B \vdash A \otimes B$ and $A \otimes B \vdash A \otimes B$.

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we can prove that $A \otimes B \vdash A \otimes B$ and $A \otimes B \vdash A \otimes B$.
All linear logic connectives are canonical except for the exponentials. The promotion rule is the culprit.

$$\frac{!\,\Gamma \vdash B, ?\,\Delta}{!\,\Gamma \vdash\ !\,B, ?\,\Delta}$$

This leaves no way to prove $!\,B \vdash\ !\,B$.

Subexponentials

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Consider having a collection of pairs of $!_i, ?_i$ where

- $i \in I$, a set of *indexes* and
- the set I has a *pre-order* $\preceq$.

The logical connectives $!_i, ?_i$ are called *subexponentials* and their promotion rule is the following:

$$\frac{!_{n_1} A_1, \ldots !_{n_k} A_k \vdash \quad B, ?_{m_1} C_1, \ldots ?_{m_j} C_k}{!_{n_1} A_1, \ldots !_{n_k} A_k \vdash \quad !_j B, ?_{m_1} C_1, \ldots ?_{m_j} C_k} \; j \leq n_1, \ldots n_k, m_1, \ldots m_j$$

In particular, if $i \preceq j$ then $!_j B \vdash !_i B$.
Promotion for $?_j$ takes place on the left.

Exponentials vs Subexponentials

The other inference rules for subexponentials are standard:

$$\frac{\Gamma \vdash ?_i A, ?_i A, \Delta}{\Gamma \vdash ?_i A, \Delta}\ CR \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?_i A, \Delta}\ WR \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?_i A, \Delta}\ DR$$

$$\frac{\Gamma, !_i A, !_i A \vdash \Delta}{\Gamma, !_i A \vdash \Delta}\ CL \qquad \frac{\Gamma \vdash \Delta}{\Gamma, !_i A \vdash \Delta}\ WL \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, !_i A \vdash \Delta}\ DL$$

Subexponentials do not need to permit weakening and contraction: in those cases, the *exponential laws* $!_i(A \,\&\, B) \equiv !_i A \otimes !_i B$ do not hold. Hence, the name.

In this paper, all *subexponentials* are, in fact, *exponentials*.

Subexponentials are not an extension of linear logic but a feature of linear logic.

Many versions of the implication $A \supset B$

$$!A \multimap B \qquad !A \multimap\, !B \qquad !?A \multimap\, ?B \qquad !A \multimap\, ?!B$$

All of the occurrences of ! and ? can now have indexes.

Implications should be transitive (i.e., cuts can be eliminated).

$$\dfrac{\dfrac{!A \qquad !A \multimap\, ?B}{?B} \qquad !B \multimap\, ?C}{?}\ \textit{oops}$$

Proving the sequent $!_1 A, !_2 B \vdash !_2 C$ requires weakened $!_1 A$.

If term $\lambda x \lambda y.t$ has type $!_1 A \multimap !_2 B \multimap !_2 C$, then x is not free in t.

This represents a form of resource control: not *how many times* it is used but *where* it can appear.

Designing a new implication

Recall that $A \multimap B$ can be built from the (multiplicative) disjunction and negation $A^\perp \parr B$.

We highlight two equivalences of linear logic.

- $?(?A \parr ?B) \equiv ?A \parr ?B$: contraction on a (negative) compound formula can be replaced by contraction on subformulas.
- $!(!A \multimap !B) \equiv !(!A \multimap B)$: some occurrences of ! are superfluous. Thus, $!(!A \multimap !(!B \multimap !C)) \equiv !(!A \multimap !B \multimap C)$.

With indexed exponentials, these equivalences are preserved under the following conditions:

- $?_{i'}(?_k A \parr ?_{j'} B) \equiv ?_k A \parr ?_{j'} B$ if and only if $i' \leq k, j'$
- $!_i(!_k A \multimap !_j B) \equiv !_i(!_k A \multimap B)$ if and only if $j \leq k, i$.

The $!_i?_k$ and $?_k!_i?_k$ modals

Let $i, k \in I$ be two indexes. We shall make use of the following two strings of exponentials (along with their abbreviations).

$$!_i?_k = \binom{i}{k} \quad \text{and} \quad ?_k!_i?_k = \begin{bmatrix} k \\ i \end{bmatrix}.$$

We have

$$\binom{i}{k} \equiv !_i?_k \equiv !_i?_k!_i?_k \equiv !_i \begin{bmatrix} k \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} k \\ i \end{bmatrix} \equiv ?_k \binom{i}{k}$$

If we write only equivalence classes of these modalities, then promotion and dereliction become inverse operations:

$$\dfrac{\dfrac{!?A}{?!?A}\ \textit{derelict}}{!?A}\ \textit{promote} \qquad \dfrac{\dfrac{\binom{i}{k}A}{\begin{bmatrix} k \\ i \end{bmatrix}A}\ \textit{derelict}}{\binom{i}{k}A}\ \textit{promote}$$

A new synthetic connective

Consider the following possible encoding of the implication $A \supset B$:

$$\binom{i}{i'}\left(\binom{k}{k'}A \multimap \binom{j}{j'}B\right) \quad \text{where } i' \leq k, j' \text{ and } j \leq i, k$$

This forms a proper *synthetic connective* in the sense that a sequent calculus for it can be given (next slide) and both cuts and (non-atomic) initials can be eliminated.

For example, $!_4?_0(!_5?_1A \multimap !_3?_2(!_8?_6B \multimap !_2?_2C))$ is a legal formula.

Focusing can be used to build synthetic connectives. Unfortunately, there are too many exponentials here to employ focusing to support this claim.

Sequents for MC

Sequents are of the form $\Gamma \vdash \mathcal{C} \mid \Delta$, where

- Γ denotes a set of formulas of the form $\binom{i}{i'}B$.
- $\mathcal{C}$ ranges over Γ but contains at most one formula.
- Δ denotes a set of formulas of the form $\left[{i' \atop i}\right]B$.

We introduce the following schematic variables, where $n \in I$.

- $\Gamma^{(n)}$ denotes a set of formulas $\binom{i}{i'}B$ such that $n \leq i$.
- $\mathcal{C}^{(n)}$ ranges over $\Gamma^{(n)}$ but contains at most one element.
- $\Delta^{(n)}$ denotes a set of formulas $\left[{i' \atop i}\right]B$ such that $n \leq i'$.

Sequent calculus proof rule for MC

$$\frac{\Gamma^{(n)} \vdash A \mid \Delta^{(n)} \qquad B, \Gamma^{(n)} \vdash \mathcal{C}^{(n)} \mid \Delta^{(n)}}{\binom{i}{i'}(A \multimap B), \Gamma', \Gamma^{(n)} \vdash \mathcal{C}^{(n)} \mid \Delta^{(n)}, \Delta'} \supset L, \;\; i' \leq n$$

$$\frac{A, \Gamma^{(n)} \vdash B \mid \Delta^{(n)}}{\Gamma', \Gamma^{(n)} \vdash \binom{i}{i'}(A \multimap B) \mid \Delta^{(n)}, \Delta'} \supset R, \;\; i \leq n$$

$$\frac{\Gamma^{(n)} \vdash \cdot \mid \left[{i \atop k}\right] A, \Delta^{(n)}}{\Gamma', \Gamma^{(n)} \vdash \binom{k}{i} A \mid \Delta^{(n)}, \Delta'} \; \mathit{Produce}, \;\; k \leq n$$

$$\frac{\Gamma \vdash \binom{j}{i} A \mid \Delta}{\Gamma \vdash \cdot \mid \Delta, \left[{i \atop j}\right] A} \; \mathit{Consume}$$

$$\frac{}{\binom{a}{b} q, \Gamma \vdash \binom{c}{d} q \mid \Delta} \; !DR/Id, \;\; c \leq a, \;\; b \leq d, \;\; q \text{ atomic}$$

Intuitionistic logic in MC

Intuitionistic logic occurs in MC when we use just, say, $\binom{2}{1}$ and $\begin{bmatrix}1\\2\end{bmatrix}$.

Classical logic occurs in MC when we use just, say, $\binom{1}{1}$ and $\begin{bmatrix}1\\1\end{bmatrix}$.

It is easy to see that Peirce's formula $((P \supset Q) \supset P) \supset P$, cannot be proved using only $\binom{i}{k}$ and $\begin{bmatrix}k\\i\end{bmatrix}$ with $i \not\leq k$.

For example, the MC formula

$$\binom{2}{1}\left(\binom{2}{1}\left(\binom{2}{1}\left(\binom{2}{1}P \multimap \binom{2}{1}Q\right) \multimap \binom{2}{1}P\right) \multimap \binom{2}{1}P\right)$$

is not provable.

Natural Deduction in MC

$$\frac{u : \Gamma \vdash \binom{i}{k} A \mid \Delta}{[d]u : \Gamma \vdash \cdot \mid d : \left[\begin{smallmatrix}k\\ i\end{smallmatrix}\right] A, \Delta}\ Name \qquad \frac{t : \Gamma \vdash \cdot \mid d : \left[\begin{smallmatrix}k\\ i\end{smallmatrix}\right] A, \Delta}{\mu^k d.t : \Gamma \vdash \binom{i}{k} A \mid \Delta}\ Unname$$

$$\frac{f : \Gamma^{(n)} \vdash \binom{i}{i'}\left(\binom{k}{k'} A \multimap \binom{j}{j'} B\right) \mid \Delta^{(n)} \qquad t : \Gamma^{(m)} \vdash \binom{k}{k'} A \mid \Delta^{(m)}}{(f \sharp_{n'} t) : \Gamma^{(n)}\Gamma^{(m)} \vdash \binom{j}{j'} B \mid \Delta^{(n)}\Delta^{(m)}}\ App$$

provided that $n' \leq min(n, j')$.

$$\frac{t : \quad x : \binom{k}{k'} A, \Gamma^{(n)} \vdash \binom{j}{j'} B \mid \Delta^{(n)}}{\lambda x.t : \Gamma'\Gamma^{(n)} \vdash \binom{i}{i'}\left(\binom{k}{k'} A \multimap \binom{j}{j'} B\right) \mid \Delta^{(n)}\Delta'}\ Abs,\ i \leq n$$

$$\frac{t : \Gamma^{(n)} \vdash \cdot \mid \left[\begin{smallmatrix}i\\ k\end{smallmatrix}\right] A, \Delta^{(n)}}{t : \Gamma'\Gamma^{(n)} \vdash \binom{k}{i} A \mid \Delta'\Delta^{(n)}}\ Produce\ (k \leq n) \qquad \frac{}{x :\ x : \binom{i}{j} C, \Gamma \vdash \binom{i}{j} C \mid \Delta}\ Id$$

$$\frac{t : \Gamma \vdash \binom{a}{b} q}{t : \Gamma \vdash \binom{c}{d} q}\ !DR\ (a \geq c,\ b \leq d,\ q \text{ atomic})$$

Bounded $\lambda\mu$-terms

$t = [d]t$	named term
$= \mu^k d.t$	binder
$= (t \sharp_n t)$	application and bounded reset indicator
$= \lambda x.t$	function abstraction
$= x$	bound variable

A proof of Peirce's formula

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dots \vdash \binom{j}{j'}\left(\binom{k}{k'}\left(\binom{a}{a'}P \multimap \binom{b}{b'}Q\right) \multimap \binom{a}{a'}P\right) \mid \begin{bmatrix} a' \\ a \end{bmatrix} P
        \qquad
        \dfrac{
          \dfrac{
            \dfrac{
              \binom{a}{a'}P \vdash \binom{a}{a'}P \mid \begin{bmatrix} b' \\ b \end{bmatrix} Q
            }{
              \binom{a}{a'}P \vdash \cdot \mid \begin{bmatrix} b' \\ b \end{bmatrix} Q, \begin{bmatrix} a' \\ a \end{bmatrix} P
            }
          }{
            \binom{a}{a'}P \vdash \binom{b}{b'}Q \mid \begin{bmatrix} a' \\ a \end{bmatrix} P
          }\; b \le a, a'
        }{
          \vdash \binom{k}{k'}\left(\binom{a}{a'}P \multimap \binom{b}{b'}Q\right) \mid \begin{bmatrix} a' \\ a \end{bmatrix} P
        }\; k \le a'
      }{
        \binom{j}{j'}\left(\binom{k}{k'}\left(\binom{a}{a'}P \multimap \binom{b}{b'}Q\right) \multimap \binom{a}{a'}P\right) \vdash \binom{a}{a'}P \mid \begin{bmatrix} a' \\ a \end{bmatrix} P
      }
    }{
      \binom{j}{j'}\left(\binom{k}{k'}\left(\binom{a}{a'}P \multimap \binom{b}{b'}Q\right) \multimap \binom{a}{a'}P\right) \vdash \cdot \mid \begin{bmatrix} a' \\ a \end{bmatrix} P
    }
  }{
    \binom{j}{j'}\left(\binom{k}{k'}\left(\binom{a}{a'}P \multimap \binom{b}{b'}Q\right) \multimap \binom{a}{a'}P\right) \vdash \binom{a}{a'}P \mid \cdot
  }\; a \le j
}{
  \vdash \binom{i}{i'}\left(\binom{j}{j'}\left(\binom{k}{k'}\left(\binom{a}{a'}P \multimap \binom{b}{b'}Q\right) \multimap \binom{a}{a'}P\right) \multimap \binom{a}{a'}P\right) \mid \cdot
}
$$

Needed restrictions: $k \le a'$ and $b \le a'$. The proof term is

$$\lambda x.\mu^{a'} d.[d](x \sharp_m (\lambda y.\mu^{b'} e.[d]y))$$

where $m \le min(j, a')$.

Conclusions

We have built an implication that has some controls on the resources in its environment.

These resources are bound variables, denoting formal parameters as well as continuations.

These controls allow mixing intuitionistic and classical principles.

Next we need to design functional programming primitives that can exploit some of the riches of the many parameters (i, i', j, k, etc).