Symmetry breaking constraints for the circle packing in a square problem

Alberto Costa[1], Pierre Hansen[1,2], Leo Liberti[1]

[1] LIX, École Polytechnique
91128 Palaiseau, France
{costa,liberti}@lix.polytechnique.fr
[2] GERAD, HEC
3000 chemin de la Côte-S.te-Catherine, H3T 2A7 Montreal, Canada
pierre.hansen@gerad.ca

Keywords : *circle packing in a square, reformulation, narrowing, symmetry breaking constraints.*

1 Introduction

Optimization problems involving a high degree of symmetry are very difficult to solve using Branch-and-Bound (BB) algorithms, because the BB tree becomes large and the exploration to reach the leaves (where we find the optimal solutions) takes longer. In order to break these symmetries, we can adjoin some *symmetry breaking constraints* (SBCs) to the original formulation [1, 2], obtaining a *narrowing* reformulation [3]. We present three classes of SBCs for the circle packing in a square problem (CPS).

2 Circle packing in a square

Consider the following problem.

> CIRCLE PACKING IN A SQUARE (CPS). Given $N \in \mathbb{N}$ and $S \in \mathbb{Q}_+$, can N non-overlapping circles of unit radius be arranged in a square of side $2S$?

We formulate the CPS as the following nonconvex NLP:

$$\max\{\alpha \mid (x_i - x_j)^2 + (y_i - y_j)^2 \geq 4\alpha \forall i < j \leq N \wedge x_i, y_i \in [1-S, S-1] \forall i \in N\} \tag{1}$$

where x_i and y_i are the coordinates of the center of the circle i.

For any given $N, S > 1$, if a global optimum (x^*, y^*, α^*) of (1) has $\alpha^* \geq 1$ then the CPS instance is a YES one.

3 Formulation group for CPS

We can describe the symmetric structure for the CPS by means of the formulation group. A permutation π belongs to the formulation group of a problem if there exists a permutation σ which acts on the constraints such that after applying the permutation π on the variables and the permutation σ on the constraints we obtain the original problem.

The CPS presents a high degree of symmetry; it is proven that the formulation group of the CPS is isomorphic to $C_2 \times S_N$ [4]. In this case C_2 (the cyclic group of order 2) represents the permutation between x and y axes, while S_N (the symmetric group of order N) represents the permutations of the circles.

4 Symmetry Breaking Constraints

As stated earlier, we can break the symmetries of the problem by adjoining some SBCs. In this way we obtain a narrowing: we eliminate some of the symmetric global optima, but at least one is preserved.

In order to break the symmetries of CPS, we test three classes of constraints:

- *weak* constraints [4]: $x_1 \leq x_i, \forall i \leq N,\ x_1 \leq y_i, \forall i \leq N$;
- *strong* constraints [4]: $x_i \leq x_{i+1}, \forall i < N$;
- *mixed* constraints [5]: let $L = \lfloor S \rfloor$; starting from the *strong* constraints, replace $x_{iL} \leq x_{iL+1}$ with $y_{1+(i-1)L} \leq y_{1+iL}, \forall i \in \{1, 2, \ldots, \lceil \frac{N}{L} \rceil - 1\}$.

The tests performed in [4] show that the formulations obtained with the *weak* and the *strong* SBCs are better than the original CPS formulation; however, the formers are less efficient than the latters. Nevertheless, the best results are obtained with the *mixed* constraints, as reported in [5].

5 Conclusion and future work

Adjoining SBCs improves the results, because with these new formulations it is possible to solve more instances of CPS with the same time limit, although this approach does not scale so well to large ones. Although the upper bound does not decrease with these narrowings, we notice a welcome but unexpected behaviour associated to the formulation with the *mixed* SBCs: the optimal solution is found earlier on during the search. We are trying to use other SBCs which change the bounds on the variables: some preliminary experiments, performed both for the original formulation and for its SDP (Semi Definite Programming) relaxation, seem to show that this approach can allow to obtain best bounds on the objective function.

References

[1] L. Liberti, *Automatic generation of symmetry-breaking constraints*, in: B. Yang, D.-Z. Du and C. Wang, editors, *COCOA Proceedings*, LNCS **5165**:328–338, 2008.

[2] L. Liberti, *Reformulations in mathematical programming: Automatic symmetry detection and exploitation*, Mathematical Programming, to appear.

[3] L. Liberti, *Reformulations in mathematical programming: Definitions and systematics*, RAIRO-RO **43**:55–86, 2009.

[4] P. Hansen, A. Costa and L. Liberti, *Formulation symmetries in circle packing*, ISCO 2010 Proceedings, in M. Haouari, A.R. Mahjoub (Eds.), Electronic Notes in Discrete Mathematics **36**:1303–1310. Elsevier, 2010.

[5] A. Costa, P. Hansen and L. Liberti, *Static symmetry breaking in circle packing*, in U. Faigle, R. Schrader, D. Herrmann (Eds.), CTW 2010 Proceedings, 47–50, Köln, Germany, 2010.