Formulation symmetries in circle packing

Alberto Costa
Leo Liberti
Pierre Hansen

Abstract. The performance of Branch-and-Bound algorithms is severely impaired by the presence of symmetric optima in a given problem. We propose here a method to automatically find MINLP formulation symmetries. We show an application of our method to the "circle packing in a square" problem, in order to get a reformulation that should cut away symmetric optima.

1. Introduction

It is well known that problems involving a high degree of symmetry are particularly difficult to solve with Branch-and-Bound (BB) techniques. Intuitively, since optimal solutions are to be found at leaf nodes of the BB tree, the presence of many optima causes fewer prunings, longer branches, and hence a higher number of nodes to explore. One possibility for breaking symmetries, proposed in [2], is to reformulate the problem by adjoining *symmetry-breaking constraints* (SBC) to the original formulation, yielding a reformulation of the *narrowing* type [1]. The main theoretical contribution of this paper is the determination of the group structure of circle packing problem.

2. Automatic symmetry detection

In this section we discuss a method for computing Mathematical Program (MP) symmetries automatically; conceptually, it is the same as in [2] but the formal presentation is different. We consider a Mixed-Integer Nonlinear Program (MINLP) P:

$$\min\{f(x) \mid g(x) \leq 0 \wedge x \in \mathcal{X}\}, \tag{2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $x \in \mathbb{R}^n$, and $\mathcal{X} \subseteq \mathbb{R}^n$ is a set which might include variable ranges $x^L \leq x \leq x^U$ as well as integrality constraints on a subset of variables $\{x_i \mid i \in Z\}$ for some $Z \subseteq \{1, \ldots, n\}$. Let $\mathcal{G}(P)$ be the set of global optima of P and $\mathcal{F}(P)$ be its feasible region. The group $G^*_P = \text{stab}(\mathcal{G}(P), S_n)$ is called the *solution group* of P (where S_n is the symmetric group of order n). The solution group is the largest subgroup of S_n which maps every global optimum into another global optimum. Since G^*_P depends on $\mathcal{G}(P)$ it cannot, in general, be found before the solution process. We therefore try to find subgroups of G^*_P. In particular, we consider the subgroup of G^*_P consisting of all variable permutations which "fix the formulation" of P. For $\pi \in S_n$ and $\sigma \in S_m$ we define $\sigma P \pi$ to be the following MINLP:

$$\min\{f(\pi x) \mid \sigma g(\pi x) \leq 0 \wedge \pi x \in \mathcal{X}\}, \tag{2.2}$$

where σ acts on $g = (g_1, \ldots, g_m)$ by $\sigma g = (g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(m)})$. Consider the group $\bar{G}_P = \{\pi \in S_n \mid \exists \sigma \in S_m \ (\sigma P \pi) = P\}$. Whenever P is a Mixed-Integer Linear Program (MILP), $\bar{G}_P$ is called the *LP relaxation group* [3]. For general MINLPs, determining whether $\forall x \in \text{dom}(f)\ f(\pi x) = f(x)$ and $\forall x \in \text{dom}(g)\ \sigma g(\pi x) = g(x)$ is an undecidable problem.

We therefore introduce the following restriction: f, g_i $(i \leq m)$ must be strings of the formal language $\mathscr{L}$ on the alphabet $\mathscr{A}$ given by the operators in $\{+, -, \times, \div, \uparrow, \log, \exp, (,)\}$ (where $a \uparrow b = a^b$), the variable symbols in $\{x_1, \ldots, x_n\}$ and the constant symbols in $\mathbb{R}$. This restriction allow us to define the *formulation group* $G_P = \{\pi \in S_n \mid \exists \sigma \in S_m \ (\sigma P \pi \cong P)\}$ of P (where the symbol $\cong$ indicates that the formulations are equal respetct to this restriction). It is easy to show that $G_P \leq \bar{G}_P \leq G^*_P$. For MILPs, $G_P = \bar{G}_P$ [2].

E-mail:costa,liberti@lix.polytechnique.fr (corresponding author: `liberti`).
E-mail:pierre.hansen@gerad.ca. Secondary affiliation at *LIX, École Polytechnique*.
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Once G_P is known, we aim to find a reformulation Q of P which ensures that at least one symmetric optimum of P is in $\mathcal{G}(Q)$. Such reformulations are known as *narrowings* [1]. A set of constraints $h(x) \leq 0$ are SBCs with respect to $\pi \in G_P$ if there is $y \in \mathcal{G}(P)$ such that $h(\pi y) \leq 0$. Adjoining SBCs to P yields a narrowing Q of P [2].

3. Circle Packing in a square

We consider the following problem.

> Circle Packing in a Square (CPS). Given $N \in \mathbb{N}$ and $L \in \mathbb{Q}_+$, can N non-overlapping circles of unit radius be arranged in a square of side $2L$?

We formulate the CPS as the following nonconvex NLP:

$$\max\{\alpha \mid \forall i < j \leq N \ \|x_i - x_j\|^2 \geq 4\alpha \wedge x \in [1-L, L-1]^{2N}\} \tag{3.1}$$

For any given $N, L > 1$, if a global optimum (x^*, α^*) of (3.1) has $\alpha^* \geq 1$ then the CPS instance is a YES one. Using the theory above, we are able to prove this

Theorem
The formulation group of the CPS is isomorphic to $S_2 \times S_N$.

This result allow us to add some SBCs to the original formulation. Some preliminary tests show that solving the reformulated circle packing problem is more easy than solve the original one, in term of time and nodes generated by the Spatial Branch and Bound.

Bibliography

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Alberto Costa
LIX, École Polytechnique
91128 Palaiseau, France
costa@lix.polytechnique.fr

Leo Liberti
LIX, École Polytechnique
91128 Palaiseau, France
liberti@lix.polytechnique.fr

Pierre Hansen
GERAD, HEC, 3000 chemin de la
Côte-S.te-Catherine, H3T 2A7 Montreal, Canada
pierre.hansen@gerad.ca