Weak Relative Pseudo-Complements of Closure Operators

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Abstract

We define the notion of weak relative pseudo-complement on meet semi-lattices, and we show that it is strictly weaker than relative pseudo-complementation, but stronger than pseudo-complementation. Our main result is that if a complete lattice $\mathcal{L}$ is meet-continuous, then every closure operator on $\mathcal{L}$ admits weak relative pseudo-complements with respect to continuous closure operators on $\mathcal{L}$.

1 Introduction

Closure operators have been widely used in various fields of theoretical computer science, like semantics of declarative languages and program analysis. In particular, Cousot and Cousot proved in [3] that the lattice of abstract interpretations, or equivalently the lattice of the semantics derivable by abstraction, for an arbitrary program whose data-objects have a lattice structure $\mathcal{L}$, is isomorphic to the lattice $uco(\mathcal{L})$ of the upper closure operators on $\mathcal{L}$. One facility offered by this result is that the meet operation in $uco(\mathcal{L})$ can be applied to compose semantics, thus getting more concrete (precise) information about the program behavior. A natural question that arises in this setting is whether it is possible to define the inverse operation, namely an operation which, starting from two semantics ρ and η for the same program, with ρ more abstract than η, gives as result the most abstract semantics φ whose composition with ρ is exactly η. Such an operation would be useful to isolate properties of ρ which are not shared by η. In terms of closure operators this amounts to define an operation analogous to the *relative pseudo-complement*, the only difference is that we require φ to be the maximum of the set of closure operators whose meet with ρ is *equal to* η, instead than *smaller than or equal to* η. It is easy to see that if the relative pseudo-complement of ρ w.r.t. η exists, then it is the intended element. The converse is not true.

We define the notion described above as a partial operation (which we call *weak relative pseudo-complement*) on arbitrary meet semi-lattices. We investigate conditions on $\mathcal{L}$ under which pairs of elements in $uco(\mathcal{L})$ admit the weak relative pseudo-complement. In particular, we are interested in the case in which $\mathcal{L}$ is a complete lattice (in which case also $uco(\mathcal{L})$ is a complete lattice). Our main result is that the weak relative pseudo-complement of a closure operator ρ w.r.t. η, for η continuous, always exists when $\mathcal{L}$ is a meet-continuous complete lattice (i.e., when in $\mathcal{L}$ the meet preserves the join of chains). In [2], this result is applied to static program analysis.

Note that the pseudo-complement is a particular case of weak relative pseudo-complement, hence the meet-continuity of $\mathcal{L}$ also guarantees that $uco(\mathcal{L})$ is pseudo-complemented. On the other hand, the meet-continuity of $\mathcal{L}$ is much less restrictive than the condition under which $uco(\mathcal{L})$ is relatively pseudo-complemented. In fact, in a complete lattice, relative pseudo-complementation is equivalent to complete meet-distributivity ([1]), and complete meet-distributivity in $uco(\mathcal{L})$ is equivalent to distributivity ([6]). Now, for $uco(\mathcal{L})$ to be distributive, it is necessary and sufficient that $\mathcal{L}$ is a complete well-ordered chain ([4]). This condition is also necessary and sufficient for $uco(\mathcal{L})$ to be complemented ([7]).

2 Weak relative pseudo-complement

Before defining our notion of weak relative pseudo-complement, we first recall some basic notions and terminology. We refer to [1, 5] for all the notions recalled in the paper.

Let $\mathcal{L} = \langle L, \preceq, \wedge, \vee, \top, \bot \rangle$ be a complete lattice, where L is a set, $\preceq$ is the partial ordering relation, $\wedge$ is the meet, $\vee$ is the join, $\top$ is the top element and $\bot$ is the bottom element. The complete lattice $\mathcal{L}$ is *completely meet-distributive* if for each $x \in L$ and $Y \subseteq L$, $x \wedge \bigvee Y = \bigvee_{y \in Y}(x \wedge y)$ holds. Given $x, y \in L$, the *relative pseudo-complement* of x w.r.t. y, if it exists, is the (unique) element $x * y \in L$ such that $x \wedge x * y \preceq y$, and for each $z \in L$, if $x \wedge z \preceq y$ then $z \preceq x * y$. If $x * y$ exists for every $x, y \in L$, then $\mathcal{L}$ is *relatively pseudo-complemented*. It is well known that this property holds iff $\mathcal{L}$ is completely meet-distributive. The element $x * \bot$, if it exists, is called the *pseudo-complement* of x, and it is denoted by x^*. If x^* exists for every $x \in L$, then $\mathcal{L}$ is *pseudo-complemented*. Obviously, (relative) pseudo-complementation is definable for a mere meet semi-lattice.

Definition 2.1 Let $\mathcal{L} = \langle L, \preceq, \wedge \rangle$ be a meet semi-lattice, and let $x, y \in L$ with $y \preceq x$. The *weak relative pseudo-complement* of x w.r.t. y, if it exists, is the (unique) element $wr(x, y) \in L$ such that

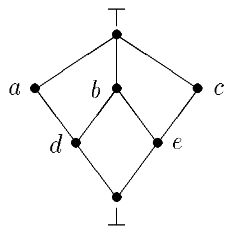
(i) $x \wedge wr(x, y) = y$,

(ii) for each $z \in L$, if $x \wedge z = y$ then $z \preceq wr(x, y)$.

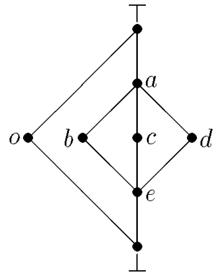
If $wr(x, y)$ exists for every $x, y \in L$ such that $y \preceq x$, then $\mathcal{L}$ is *weakly relatively pseudo-complemented*. □

The reason why we require $y \preceq x$, in the definition of $wr(x, y)$, is that for $y \not\preceq x$ Condition (i) would never be satisfied.

In the context of Definition 2.1, note that for $y \preceq x$ Condition (i) could equivalently be rewritten as $x \wedge wr(x, y) \preceq y$; the difference with the notion of relative pseudo-complement (within the case $y \preceq x$) is only that in Condition (ii) we require $x \wedge z = y$ instead of $x \wedge z \preceq y$. It is easy to see that for $y \preceq x$, if $x * y$ exists then $wr(x, y) = x * y$ (hence in that case, if $\mathcal{L}$ is a complete lattice then $wr(x, y)$ can be defined as the join of all the elements z satisfying the premise of Condition (ii)). The converse is not true. For instance the following lattice is weakly relatively pseudo-complemented, but $a * d$ does not exist (while $wr(a, d) = b$.)



Concerning the relation with the pseudo-complement, we have that if $wr(x, \bot)$ exists, then it coincides with x^*. However, a pseudo-complemented lattice may not be weakly relatively pseudo-complemented. For instance, the following lattice is pseudo-complemented, but $wr(b, e)$ does not exist.



3 Weak relative pseudo-complements of closure operators

In this section, we study the conditions for the existence of weak relative pseudo-complements of closure operators. An (*upper*) *closure operator* on $\mathcal{L}$ is a mapping $\rho : L \rightarrow L$ such that $x \preceq \rho(x)$, $\rho(\rho(x)) = \rho(x)$, and for every $x, y \in L$, if $x \preceq y$ then $\rho(x) \preceq \rho(y)$ (obviously, this notion can be given in general for posets). As usual, we say that ρ is *continuous* if for every chain $C \subseteq L$, $\rho(\bigvee C) = \bigvee_{x \in C} \rho(x)$ holds. A closure operator ρ is uniquely determined by the set of its fixpoints, which coincides with its image $\rho(L)$. A set $X \subseteq L$ is the set of fixpoints of some closure operator iff X contains $\top$ and it is meet-closed, i.e. for any $Y \subseteq X$, with $Y \neq \emptyset$, $\bigwedge Y \in X$ holds.

Let $uco(L)$ be the set of all closure operators on L. We will denote by $uco(\mathcal{L})$ the complete lattice $\langle uco(L), \sqsubseteq, \sqcap, \sqcup, \mho, \iota \rangle$ where $\rho \sqsubseteq \eta$ iff for each $x \in L$, $\rho(x) \preceq \eta(x)$, $(\sqcap_{i \in I} \rho_i)(x) = \bigwedge_{i \in I} \rho_i(x)$, $(\sqcup_{i \in I} \rho_i)(x) = x$ iff for each $i \in I$, $\rho_i(x) = x$, and $\mho(x) = \top$ and $\iota(x) = x$. Note that $\rho \sqsubseteq \eta$ iff $\eta(L) \subseteq \rho(L)$. Note also that the set of fixpoints of $\sqcap_{i \in I} \rho_i$ is the smallest meet-closed set which contains all the fixpoints of each ρ_i, and that the set of fixpoints of $\sqcup_{i \in I} \rho_i$ is the intersection of all the sets of fixpoints of the ρ_i's, i.e. $(\sqcup_{i \in I} \rho_i)(L) = \bigcap_{i \in I} \rho_i(L)$.

We recall that a complete lattice $\mathcal{L}$ is *meet-continuous* if for any chain $C \subseteq L$, and for each element $x \in L$, $x \wedge \bigvee C = \bigvee_{y \in C}(x \wedge y)$ holds. It is worth noting that meet-continuity is strictly weaker than complete meet-distributivity. For instance, any finite lattice is meet-continuous, but not necessarily completely meet-distributive (i.e. distributive). Also an infinite lattice can be meet-continuous without being completely meet-distributive: consider, for instance, the ordinal sum of an ω-chain and a finite non-distributive lattice.

The following theorem expresses the main result of this paper. We prove that if $\mathcal{L}$ is a meet-continuous complete lattice, then $uco(\mathcal{L})$ admits weak relative pseudo-complements w.r.t. continuous closure operators. In the proof, ordinal numbers will be denoted by greek letters $\alpha, \beta, \ldots$.

Theorem 3.1 *If $\mathcal{L}$ is a complete, meet-continuous lattice then for every $\rho, \eta \in uco(L)$ such that $\eta \sqsubseteq \rho$ and η is continuous, there exists $wr(\rho, \eta)$.*

Proof. For $\rho, \eta \in uco(L)$ such that $\eta \sqsubseteq \rho$ and η is continuous, let $P_{\rho\eta} = \{\varphi \in uco(L) : \rho \sqcap \varphi = \eta\}$, and define $\bar{\rho} = \sqcup P_{\rho\eta}$. Notice that $\bar{\rho}(L) = \bigcap \{\varphi(L) : \varphi \in P_{\rho\eta}\}$. We prove that $\bar{\rho} = wr(\rho, \eta)$.
Clearly $\varphi \sqsubseteq \bar{\rho}$ for every $\varphi \in P_{\rho\eta}$. Furthermore, $\eta \sqsubseteq \rho \sqcap \bar{\rho}$ holds, because $P_{\rho\eta}$ contains η. Hence, we have only to show that $\rho \sqcap \bar{\rho} \sqsubseteq \eta$, which amounts to prove that $\eta(L) \subseteq (\rho \sqcap \bar{\rho})(L)$.
Assume that there exists $\bar{x} \in \eta(L)$ such that $\bar{x} \notin (\rho \sqcap \bar{\rho})(L)$. We show that for each ordinal α we are able to construct a strictly increasing chain $\{x_\beta\}_{\beta \leq \alpha}$ in $\eta(L)$. Since the cardinality of this chain is $|\alpha|$, this leads to a contradiction when $|\alpha| > |\eta(L)|$.
More precisely, we prove by transfinite induction on α that it is possible to construct in $\eta(L)$ two sets $\{x_\beta\}_{\beta \leq \alpha}$, with $x_0 = \bar{x}$, and $\{y_\beta\}_{\beta < \alpha}$, such that:

(i) $\{x_\beta\}_{\beta \leq \alpha}$ is a strictly increasing chain in $\eta(L) \setminus (\rho \sqcap \bar{\rho})(L)$,

(ii) $\forall \beta < \alpha.\ y_\beta \in \rho(L)$,

(iii) $x_0 = x_\alpha \wedge (\bigwedge_{\beta<\alpha} y_\beta)$, and

(iv) $\forall \beta, \gamma.\ \beta \leq \gamma < \alpha \Rightarrow x_\beta \prec y_\gamma$.

Let us consider the three cases $\alpha = 0$, α successor ordinal, and α limit ordinal.

($\alpha = 0$) Define $x_0 = \bar{x}$. Then $x_0 \in \eta(L) \setminus (\rho \sqcap \bar{\rho})(L)$ by definition, thus (i) is satisfied. Conditions (ii), (iii) and (iv) are vacuously satisfied.

($\alpha + 1$) By the induction hypothesis (i), $x_\alpha \notin \bar{\rho}(L)$, hence there exists a closure operator $\varphi \in P_{\rho\eta}$ such that $x_\alpha \notin \varphi(L)$. Also, again by the induction hypothesis (i), $x_\alpha \notin \rho(L)$ and $x_\alpha \in \eta(L)$. By the latter and the fact that $\rho \sqcap \varphi = \eta$, we derive $x_\alpha \in (\rho \sqcap \varphi)(L)$. Therefore, there must exist $x \in \varphi(L) \setminus \rho(L)$ and $y \in \rho(L) \setminus \varphi(L)$ such that $x_\alpha = x \wedge y$. Define $x_{\alpha+1} = x$ and $y_\alpha = y$, and observe that (ii) and (iv) are satisfied.
We prove now (iii). By the induction hypothesis (iii), we have $x_0 = x_\alpha \wedge (\bigwedge_{\beta<\alpha} y_\beta)$, hence we derive $x_0 = x_{\alpha+1} \wedge y_\alpha \wedge (\bigwedge_{\beta<\alpha} y_\beta) = x_{\alpha+1} \wedge (\bigwedge_{\beta<\alpha+1} y_\beta)$.
Concerning (i), observe that $x_\alpha \preceq x_{\alpha+1}$, and $x_\alpha \neq x_{\alpha+1}$, since $x_\alpha \notin \varphi(L)$ while $x_{\alpha+1} \in \varphi(L)$. Furthermore, $x_{\alpha+1} \in \eta(L)$ since $\varphi(L) \subseteq \eta(L)$. Finally, note that the term $\bigwedge_{\beta<\alpha+1} y_\beta$ in (iii) is an element of $\rho(L)$, being $\rho(L)$ meet-closed. Since $\rho(L) \subseteq (\rho \sqcap \bar{\rho})(L)$, and $(\rho \sqcap \bar{\rho})(L)$ is meet-closed, we deduce that $x_{\alpha+1}$ cannot belong to $(\rho \sqcap \bar{\rho})(L)$, otherwise also x_0 would.

(α limit ordinal) Define $x_\alpha = \bigvee_{\beta<\alpha} x_\beta$. Conditions (ii) and (iv) are trivially satisfied by the corresponding induction hypothesis.
Consider now (iii). By the induction hypothesis (iii), we have that, for any $\beta < \alpha$, $x_0 = x_\beta \wedge (\bigwedge_{\gamma<\beta} y_\gamma)$ holds. Hence, we derive $x_0 = \bigvee_{\beta<\alpha}(x_\beta \wedge (\bigwedge_{\gamma<\beta} y_\gamma))$. By the induction hypothesis (iv), for $\beta \leq \gamma < \alpha$, $x_\beta \preceq y_\gamma$ holds. Therefore, we have $x_0 = \bigvee_{\beta<\alpha}(x_\beta \wedge (\bigwedge_{\beta\leq\gamma<\alpha} y_\gamma) \wedge (\bigwedge_{\gamma<\beta} y_\gamma)) = \bigvee_{\beta<\alpha}(x_\beta \wedge (\bigwedge_{\gamma<\alpha} y_\gamma))$, from which we get, by meet-continuity, $x_0 = (\bigvee_{\beta<\alpha} x_\beta) \wedge (\bigwedge_{\gamma<\alpha} y_\gamma) = x_\alpha \wedge (\bigwedge_{\gamma<\alpha} y_\gamma)$.
Finally, we show (i). Since all the elements of $\{x_\beta\}_{\beta<\alpha}$ are distinct, also x_α is strictly greater than x_β, for any $\beta < \alpha$. Furthermore, $x_\alpha \in \eta(L)$: in fact $\eta(x_\alpha) = \eta(\bigvee_{\beta<\alpha} x_\beta) =$ (by continuity of η) $= \bigvee_{\beta<\alpha} \eta(x_\beta) =$ (since, by the induction hypothesis (i), $x_\beta \in \eta(L)$, i.e. $\eta(x_\beta) = x_\beta$) $= \bigvee_{\beta<\alpha} x_\beta = x_\alpha$. The proof that $x_\alpha \notin (\rho \sqcap \bar{\rho})(L)$ is like in the case of α successor ordinal. ■

In the above theorem, the meet-continuity of $\mathcal{L}$ and the continuity of η play a fundamental role. The following example shows that if η is not continuous, then $wr(\rho, \eta)$ might not exist, even if $\mathcal{L}$ is meet-continuous.

Example 3.2 Consider the complete, meet-continuous lattice $\mathcal{L}$ obtained as direct product of the two elements chain $\{0, 1\}$ and the $(\omega + 1)$-chain $\mathbb{N}^\omega = \{0, 1, \ldots, \omega\}$. Consider the closure operators ρ and η such that $\rho(L) = \{\langle 1, x\rangle : x \in \mathbb{N}^\omega\}$, and $\eta(L) = L \setminus \{\langle 0, \omega\rangle\}$. Clearly, η is not continuous. For any $k \in \mathbb{N}$, consider the closure operator ρ_k such that $\rho_k(L) = \{\langle 0, n\rangle : n \in \mathbb{N},\ k \leq n\} \cup \{\langle 1, \omega\rangle\}$, and observe that for each k, $\rho \sqcap \rho_k = \eta$. Define $\hat{\rho} = \sqcup_{k\in\mathbb{N}}\rho_k$, and observe that $wr(\rho, \eta)$, if it exists, must be greater than or equal to $\hat{\rho}$. But $\hat{\rho}(L) = \{\top\}$, and therefore $\rho \sqcap \hat{\rho} = \rho$, which is strictly greater than η. □

Vice-versa, if $\mathcal{L}$ is not meet-continuous, then $wr(\rho, \eta)$ might not exist, even if η is continuous.

Example 3.3 Consider the lattice $\mathcal{L}$ defined as in the previous example, but with the element $\langle 0, \omega\rangle$ removed. In this case, $\mathcal{L}$ is not meet-continuous. The same lattice is used in [8, page 16] as an example of a distributive complete lattice which is not pseudo-complemented. To see that $uco(\mathcal{L})$ is not weakly relatively pseudo-complemented just define ρ, the ρ_k's, and $\hat{\rho}$ as in the previous example, and observe that for any k, we have $\rho \sqcap \rho_k = \iota$, but $\rho \sqcap \hat{\rho} = \rho$, namely $wr(\rho, \iota)$ does not exist. □

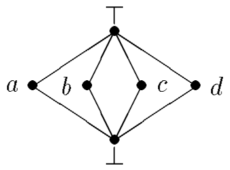
The following are easy consequences of Theorem 3.1.

Corollary 3.4 *If $\mathcal{L}$ is meet-continuous then $uco(\mathcal{L})$ is pseudo-complemented.*

Corollary 3.5 *If $\mathcal{L}$ does not have infinite ascending chains, then for any $\rho, \eta \in uco(L)$, with $\eta \sqsubseteq \rho$, there exists $wr(\rho, \eta)$.*

One might, at this point, raise the question whether $uco(\mathcal{L})$ satisfies the Stone identity (namely, for every $\rho \in uco(L)$, $\rho^* \sqcup \rho^{**} = \mho$), in case the conditions for it to be pseudo-complemented hold. Next example answers this question negatively.

Example 3.6 Let $\mathcal{L}$ be the following lattice:



Consider the closure operator ρ on $\mathcal{L}$ for which $\rho(L) = \{\top, a, b, \bot\}$. It is clear that $\rho^*(L) = \{\top, c, d, \bot\}$ and $\rho^{**} = \rho$. Thus, the Stone identity does not hold since $(\rho^* \sqcup \rho^{**})(L) = \{\top, \bot\}$. $\square$

We conclude by observing that the hypothesis of meet-continuity of $\mathcal{L}$ could be weaken. Given a complete lattice $\mathcal{L} = \langle L, \preceq, \wedge, \vee, \top, \bot \rangle$, and $\eta \in uco(L)$, the structure $\eta(\mathcal{L}) = \langle \eta(L), \preceq, \wedge \rangle$ is a complete meet-sublattice of $\mathcal{L}$. Furthermore, $\eta(\mathcal{L})$ is indeed a complete lattice, but, in general, not a complete sublattice of $\mathcal{L}$, because the join in $\eta(\mathcal{L})$ may be different from $\vee$. However, it is easy to show that if η is continuous, then the join of chains in $\eta(\mathcal{L})$ is the same as in $\mathcal{L}$. Thus, in the hypotheses of Theorem 3.1, one could replace the condition of meet-continuity of $\mathcal{L}$ (used in the limit ordinal case) with the condition of meet-continuity of $\eta(\mathcal{L})$. The present formulation of the theorem can then be retrieved as a consequence, by observing that if $\mathcal{L}$ is meet-continuous and $\eta \in uco(L)$ is continuous, then $\eta(\mathcal{L})$ is meet-continuous (an analogous result is proved in [5] for algebraic complete lattices).

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References

[1] G. Birkhoff. *Lattice Theory.* AMS Colloquium Publications vol. XXV, 3rd edition, 1967.

[2] A. Cortesi, G. Filé, R. Giacobazzi, C. Palamidessi, and F. Ranzato. Complementation in abstract interpretation. In A. Mycroft, ed., *Proc. of the 2nd Static Analysis Symposium*, Lecture Notes in Computer Science 983, pp. 100–117. Springer-Verlag, 1995.

[3] P. Cousot and R. Cousot. Systematic design of program analysis frameworks. In *Conference Record of the 6th ACM Symposium on Principles of Programming Languages*, pp. 269–282. ACM Press, 1979.

[4] P. Dwinger. On the closure operators of a complete lattice. *Indagationes Math.*, 16:560–563, 1954.

[5] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices.* Springer-Verlag, 1980.

[6] J. Morgado. On complete congruences of complete lattices. *Portugal. Math.*, 21(1):11–25, 1962.

[7] J. Morgado. Note on complemented closure operators of complete lattices. *Portugal. Math.*, 21(3):135–142, 1962.

[8] J. Varlet. Contribution à l'étude des treillis pseudo-complémentés et des treillis de Stone. *Mémoires de la Société Royale des Sciences de Liège*, 5eme série, 8(4), 71 pages, 1963.