Complete Herbrand Interpretations giving a Symmetric Characterization of Failure and Success *

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Abstract

The treatment of success and failure in the semantics of positive logic programs presents some asymmetric aspects. The paper presents a solution to this problem by first giving a symmetric characterization of the set of successful non-ground atoms and of the set of the finitely failed ones.

We extend the usual notions of Herbrand Universe, Base and Interpretations to complete ones, thus allowing the representation of infinite computations. In this extended setting the immediate consequence operator is downward continuous, and therefore its greatest fixpoint is reachable in ω steps. This result allows to symmetrically characterize the above introduced sets also from the point of view of the fixpoint semantics. In fact, the set of successful non-ground atoms is obtained as the least fixpoint of this transformation, while the set of the finitely failed ones is given by the complement of its greatest fixpoint.

Key words and phrases: logic programming, finite failure, downward continuity, greatest fixpoint.

1 Introduction.

Some recent papers [FLMP89a],[FLMP89b] have developed a new approach to the declarative semantics of logic programs which allows to capture various operational properties relatively to successful computations.

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This effort was first applied to positive logic programs and should be applied to general logic programs. A first step in this direction is to look at the problem of characterizing finitely failed atoms and goals. In the literature, the set of finitely failed atoms is well characterized (operationally and declaratively), only in the ground case. In this paper we study the set of (possibly non ground) finitely failed atoms, within the framework developed in [FLMP89a].

The first asymmetry we encounter is that the set of non ground finitely failed atoms is not the counterpart of the atomic consequences set shown in [FLMP89a] to be equivalent to the set of atoms having a successful computation with an empty computed answer substitution. Thus we introduce the notion of non-ground success set consisting of those atoms which have a successful computation (with some computed answer substitution).

Another asymmetry shows up when we consider the fixpoint semantics (i.e., the fixpoints of the immediate consequence operator T on Herbrand interpretations; the powers of T are defined in section 2.1). As pointed out in [JLM86] the theory of logic programs is asymmetric in the following sense. A ground atom A belongs to the success set iff $A \in T\uparrow_{\omega} = \mathit{lfp}(T)$ holds. On the other hand, a ground atom A belongs to the finite failure set iff $A \notin T\downarrow_{\omega}$, which, in general, is different from $\mathit{gfp}(T)$. $\mathit{gfp}(T) = T\downarrow_{\omega}$ holds for *canonical* programs only, and not all programs are canonical. Jaffar and Stuckey [JS86] studied canonical programs and proved that every logic program is equivalent to a canonical program. Blair and Brown in [BB90] choose a specific pre-interpretation for which the corresponding transformation T is such that $\mathit{gfp}(T) = T\downarrow_{\omega}$.

In this paper, we tackle the same problem considered by [BB90] and we show that it is possible to give a greatest fixpoint characterization of the set of non-ground finitely failed atoms.

Let us discuss which kind of structures we need. It is well known (theorem of Knaster-Tarski) that the greatest fixpoint of a monotonic transformation T on a complete lattice always exists. Moreover, if T is downward continuous then the greatest fixpoint can be reached in ω steps starting from the top of the lattice. In general, downward continuity requires certain additional topological properties on the elements of the lattice. One sufficient condition is the closedness (in the topological sense) of those elements. This requires the underlying structure to be a complete domain. In the case of the Herbrand interpretations, we have therefore to perform a completion of the Herbrand universe (i.e. we have to embed the Herbrand universe in a complete domain). In the literature there are examples of this kind of construction. See, for instance, the metric method of [Nai84], and the approach based on a partial ordering (completion by ideals) of [Gol88]. The metric topologies have less *discriminating power* than the CPO-based topologies, in the sense that the former identify more elements. For instance, the sequences $\{p(f^n(x), f^n(x))\}_{n\geq 0}$ and $\{p(f^n(x), f^{n+1}(x))\}_{n\geq 0}$ converge to the same metric limit $p(f^{\omega}, f^{\omega})$, whilst they have different *lubs* when considered as chains of ideals. The same applies to the sequences $\{p(f^n(a))\}_{n\geq 0}$ and $\{p(f^n(x))\}_{n\geq 0}$ (to be more precise, $\{p(f^n(a))\}_{n\geq 0}$ does not correspond to any chain and has no *lub*). To solve our problem, we need to discriminate the first two sequences. Intuitively, this is because we want to model the difference between the program

$$\{p(f(x), f(x)) \leftarrow p(x, x)\}$$

and the program

$$\{p(f(x), f(f((x))) \leftarrow p(x, f(x))\}$$

since the goal $\leftarrow p(x, x)$ finitely fails in the second but not in the first. On the other hand, we must introduce limits for sequences like $\{p(f^n(a))\}_{n\geq 0}$. More precisely, we must force interpretations to be closed (i.e. to contain the limits) with respect to these sequences. Intuitively, this is because the immediate consequence operator associated to a program like

$$\begin{aligned} \{\ & q(a) \leftarrow p(x) \\ & p(f(x)) \leftarrow p(x) \quad \} \end{aligned} \tag{1}$$

is not downward continuous on the decreasing chain of interpretations

$$\{p(f^k(a)) \mid k \geq n\}_{n\geq 0},$$

since its intersection is empty (see [AvE82]). To obtain downward continuity it is sufficient to add the limit element. The chain becomes

$$\{\{p(f^k(a)) \mid k \geq n\} \cup \{p(f^\omega)\}\}_{n\geq 0}$$

which has a non empty intersection.

To get this "intermediate" discrimination power, we suggest a method based on the combination of both approaches. More precisely we define a metric topology on top of the space obtained as completion by ideals. This allows to obtain downward continuity for every program. In the construction proposed by Golson [Gol88], downward continuity holds for programs which do not contain local variables only. Note that program (1) has local variables.

The completion of the Herbrand universe does not affect the classical results as far as the least fixpoint of T is concerned (see [Fit86] for an analysis of the properties of T on compact topological spaces), therefore also in this extended framework the success set is characterized by $T \uparrow_\omega = lfp(T)$. So, we obtain the symmetrical characterization of success and failure we were aiming at.

In section 2 some preliminary results are presented which describe our notion of Herbrand structures with variables and related properties. Section 3 is devoted to a characterization of the non-ground success set and the non-ground finite failure set in terms of the least upper bound and the greatest lower bound of two specific chains of interpretations. In sections 4 and 5, we define a complete Herbrand universe with the above described technique (our explicit notion of domain for interpretations is another difference with respect to the approach of [Gol88]) and we extend the transformation T_v on the new lattice of interpretations defined in [FLMP89a]. We show that the new transformation is continuous (in the metric sense), and that the chains and decreasing chains are Cauchy sequences. This allows to prove continuity and downward continuity with respect to the lattice structure. Finally, we prove that the greatest fixpoint of this new transformation corresponds to (the complement of) the set of non-ground finitely failed atoms.

2 Preliminaries.

In this section we recall the terminology and the basic results in the semantics of logic programs. For more details we refer [Llo87], [Apt90].

Let the language $\mathbf{L}$ consist of: a finite set *Con* of *data constructors* $a, b, c \ldots f, g, h \ldots$, a finite set *Pred* of *predicate symbols* $p, q, r \ldots$, a denumerable set *Var* of *variable symbols* $x, y, z \ldots$. The sets *Con* and *Pred* are partitioned with respect to the arity. Con_n and $Pred_n$ denote function and predicate symbols of arity n.

Let *Term* be the set of terms $t, u \ldots$ built on *Con* and *Var*. The Herbrand Universe $\mathbf{U}$ is the set of all the ground terms. A substitution is a mapping $\vartheta : Var \rightarrow Term$ such that the set $dom(\vartheta) = \{x | \vartheta(x) \neq x\}$ (*domain* of ϑ) is finite. ϵ denotes the empty substitution. The composition $\vartheta\sigma$ of the substitutions ϑ and σ is defined as the functional composition. The pre-ordering $\leq$ on substitutions is such that $\vartheta \leq \sigma$ iff there exists ϑ' such that $\vartheta\vartheta' = \sigma$.

The *atoms* $A, B \ldots$ are objects of the form $p(t_1, \ldots, t_n)$. The Herbrand Base $\mathbf{B}$ is the set of all ground atoms. The application of the substitution ϑ to the atom A is denoted by $A\vartheta$. We define $A \leq A'$ (A is more general than A') iff there exists ϑ such that $A\vartheta = A'$. The relation $\leq$ is a preorder. Let $\approx$ be the associated equivalence relation (*renaming*).

A *definite clause* is a formula $H \leftarrow B_1, \ldots, B_n$ ($n \geq 0$), where H and the B_i's are atoms, "$\leftarrow$" and "," denote logic implication and conjunction respectively, and all variables are universally quantified. H is the *head* of the clause and $B_1, \ldots, B_n$ is the *body*. If the body is empty the clause is a *unit clause*. A *program* is a finite set of definite clauses $W = \{C_1, \ldots, C_n\}$. A *goal statement* is a formula $\leftarrow A_1, \ldots, A_m$, where each A_i is an atom. The *derivation step* is defined by $\leftarrow A_1, \ldots, A_i, \ldots, A_m \overset{\vartheta}{\longmapsto} \leftarrow (A_1, \ldots, A_{i_1}, B_1, \ldots B_n, A_{i+1}, \ldots, A_m)\vartheta$ iff $\exists H \leftarrow B_1, \ldots B_n \in W$ (renamed apart) such that $\vartheta \in mgu(A_i = H)$. The renaming of the clauses is done in order to avoid variable clash with the goal. Given a particular *selection rule* (rule for the choice of the atom of the goal for the derivation step), an *SLD-derivation* $G \overset{\vartheta}{\longmapsto}^* G'$ is a sequence of derivation steps $G \overset{\vartheta_1}{\longmapsto} \ldots \overset{\vartheta_n}{\longmapsto} G'$ such that $\vartheta = \vartheta_1 \ldots \vartheta_n$. An SLD-refutation is obtained when G' is the empty goal $\Box$. An SLD-derivation is called *failed* if it is finite, maximal (i.e. it is not the prefix of a longer derivation) and it is not a refutation.

An SLD-tree is a tree such that

(i) the nodes are goals,

(ii) the children of a node are all the goals obtained by performing a resolution step with a clause of W whose head is unifiable with the selected atom.

An SLD-tree is *successful* if it contains the empty clause, and it is *finitely failed* if it is finite, maximal and not successful.

The following definitions are all relative to a given program W.

Let us first recall the basic operational characterizations of atoms which have a successful and a finitely failed computation. In the classical approach ([Llo87], [Apt90]) the following sets have been studied and characterized.
The (ground) success set:

$$SS = \{A \mid A \text{ is ground and } \leftarrow A \overset{\epsilon\ *}{\longmapsto} \Box\}$$

The (ground) finite failure set:

$$FF = \{A \mid A \text{ is ground and } \leftarrow A \text{ has a finitely failed SLD-tree}\}$$

In [FLMP89a] another set has been considered, the atomic consequences set, proved equivalent to the set of those atoms having a successful computation with an empty computed answer substitution:

$$ACS = \{A \mid\ W \models A\} = \{A \mid \leftarrow A \overset{\epsilon\ *}{\longmapsto} \Box\}$$

In this paper we want to characterize also the set of non-ground atoms which have a finitely failed computation. This is not the counterpart of the set ACS. In fact the atomic consequences are (implicitly) universally quantified, whereas in the case of finitely failed atoms it is the negation of their existential closure which is seen as a consequence of the program. Thus we will discuss about a larger set of successful atoms, which have a successful computation with whatever computed answer substitution (the atoms whose existential closure is a consequence of the program). The operational concepts that we characterize are the following.

- The non-ground success set:

$$NGSS = \{A \mid \exists\vartheta \leftarrow A \overset{\vartheta\ *}{\longmapsto} \Box\}$$

- The non-ground finite failure set:

$$NGFF = \{A \mid \leftarrow A \text{ has a finitely failed SLD-tree}\}$$

2.1 Interpretations and models.

We introduce here a definition of interpretations and models slightly different from (but equivalent to) the standard one. The idea is to extend the notion of domain so to include tuples in the domain itself [1]. This will help in the definition of an appropriate complete domain.

A *domain* is a pair (D, π), where D is a set and $\pi = \{\pi_n\}_{n\geq 0}$ is a partition of D, i.e.

- $\forall n\ [\pi_n \subseteq D]$,
- $\forall m, n\ [(m \neq n) \Rightarrow \pi_m \cap \pi_n = \emptyset]$,
- $\bigcup_{n\geq 0} \pi_n = D$.

Additionally we require

- $\pi_0 = \{\langle\rangle\}$.

An *interpretation* for the language $\mathbf{L}$ is given by a domain (D, π) and a function I such that

- $\forall f \in Con_n$, $I(f)$ is a function from π_n to π_1
- $\forall p \in Pred_n$, $I(p)$ is a function from π_n to $\{true, false\}$.

When the domain is clear from the context, we will denote an interpretation simply by I.

The correspondence with the standard notion is immediate. For instance, a standard Herbrand interpretation I (a subset of the Herbrand Base) can be viewed as an interpretation whose domain is $(\mathbf{U}^*, \pi)$, where

- $\mathbf{U}$ is the standard Herbrand domain,
- $\mathbf{U}^* = \bigcup_{n\geq 0} \mathbf{U}^n$, where $\mathbf{U}^n$ is the cartesian product $\mathbf{U} \times \ldots \times \mathbf{U}$ n times,
- $\pi_n = \mathbf{U}^n$

and

- $\forall f \in Con_n[I(f)\langle t_1, \ldots, t_n\rangle = \langle f(t_1, \ldots, t_n)\rangle]$
- $\forall p \in Pred_n$,

$$I(p)\langle t_1, \ldots, t_n\rangle = \begin{cases} \mathit{true} & \text{if } p(t_1, \ldots, t_n) \in I \\ \mathit{false} & \text{if } p(t_1, \ldots, t_n) \notin I \end{cases}$$

[1] This idea was suggested by H. Blair.

We will use a bar to denote generic tuples and sequences. For instance, $\bar{t}$ stands for a tuple of terms, $\bar{A}$ for a sequence of atoms, etc. The truth value of a formula of $\mathbf{L}$ in I is defined as usual. A model M of a set $\mathcal{F}$ of formulae in $\mathbf{L}$ ($M \models \mathcal{F}$) is an interpretation in which all elements of $\mathcal{F}$ are true. We write $\mathcal{F} \models \mathcal{A}$ to denote the validity of $\mathcal{A}$, namely that $\mathcal{A}$ is true in every model of $\mathcal{F}$.

Let us recall the following definitions of the powers of an operator T on a complete lattice D with top element $\top$ and bottom element $\bot$.

- $T\!\uparrow_0 = \bot$
- $T\!\uparrow_{n+1} = T(T\!\uparrow_n)$
- $T\!\uparrow_\omega = lub_{n\geq 0}(T\!\uparrow_n)$

and

- $T\!\downarrow_0 = \top$
- $T\!\downarrow_{n+1} = T(T\!\downarrow_n)$
- $T\!\downarrow_\omega = glb_{n\geq 0}(T\!\downarrow_n)$

2.2 Herbrand structures and the immediate consequence operator

The set of Herbrand interpretations, ordered by set inclusion, is a complete lattice; $\bot$ is $\emptyset$, $\top$ is $\mathbf{B}$, the lub is $\bigcup$, and the glb is $\bigcap$. On this structure, the immediate consequence operator T [Llo87], [Apt90], defined by:

$$
\begin{aligned}
T(I) = \{A \mid\ & \exists H \leftarrow \bar{B} \in W \\
& \exists\vartheta \text{ ground} \\
& \bar{B}\vartheta \in I \\
& A = H\vartheta\}
\end{aligned}
$$

is continuous.

By monotonicity of T, there exist the least and the greatest fixpoints of T, $\mathit{lfp}(T)$ and $\mathit{gfp}(T)$, and

$$T\!\uparrow_\omega \subseteq \mathit{lfp}(T) \subseteq \mathit{gfp}(T) \subseteq T\!\downarrow_\omega$$

hold. Moreover, by continuity of T, $\mathit{lfp}(T) = T\!\uparrow_\omega$, whilst the converse, $\mathit{gfp}(T) = T\!\downarrow_\omega$ in general does not hold. The program (1) in the introduction is a well known counter-example for it. Note that $T\!\downarrow_\omega = \{q(a)\}$, while $\mathit{gfp}(T) = T\!\downarrow_{\omega+1} = \emptyset$.

We recall the basic results:

- Model-theoretic and fixpoint characterization of the success set

$$
\begin{aligned}
SS &= \{A \mid A \text{ is ground and } W \models A\} \qquad (2)\\
&= M \text{ (the least Herbrand model)}\\
&= T\!\uparrow_\omega
\end{aligned}
$$

- Fixpoint characterization of the finite failure set

$$FF = \mathbf{B} \backslash T\!\downarrow_\omega \qquad (3)$$

2.3 Herbrand structures enriched with variables and the extended immediate consequence operator

The definitions and the results of previous sections have been extended, first by Clark [Cla79] and then by Falaschi et al. [FLMP89a], to interpretations and models whose domain is obtained by enriching the Herbrand universe with elements corresponding to the names of variables. To be more precise, in [FLMP89a] the actual structure is the quotient set with respect to variable renaming. We follow here the approach of [FLMP89a].

Let $\mathbf{U}_v$ be the set of all terms ("Herbrand universe with variables"). The domain we consider is $((\mathbf{U}_v{}^*)_{/\approx}, \pi_v)$, where $\pi_{v_n} = (\mathbf{U}_v{}^n)_{/\approx}$ [2]. For the sake of simplicity, we will denote the equivalence class of a tuple with the tuple itself. The "Herbrand base with variables" $\mathbf{B}_v$ is the set

$$\{p(\bar{t}) \mid \ p \in \mathit{Pred}_n \ \wedge \ \bar{t} \in \pi_{v_n}\}$$

We introduce the following operators on subsets I of $\mathbf{B}_v$

- $\mathit{Down}(I) = \{A \mid \exists A' \in I\ [A \leq A']\}$
- $\mathit{Up}(I) = \{A \mid \exists A' \in I\ [A' \leq A]\}$
- $\mathit{Ground}(I) = \{A \mid A \in I \ \wedge A \text{ is ground}\}$

The interpretations on this domain ("Herbrand interpretations with variables"), are those subsets I of $\mathbf{B}_v$ that are *upward closed*, i.e., $I = \mathit{Up}(I)$. The set $\mathcal{I}_v$ of interpretations, ordered by set inclusion, is a complete lattice. $\mathbf{B}_v$ is the supremum. On this structure, the extended immediate consequence operator T_v, defined by

$$\begin{aligned} T_v(I) = \{H\vartheta \mid \ & H \leftarrow \bar{B} \in W \\ & \bar{B}\vartheta \in I\} \end{aligned}$$

is continuous. Again, by monotonicity of T_v, there exist the least and the greatest fixpoints of T_v, $\mathit{lfp}(T_v)$ and $\mathit{gfp}(T_v)$, and

$$T_v \uparrow_\omega \subseteq \mathit{lfp}(T_v) \subseteq \mathit{gfp}(T_v) \subseteq T_v \downarrow_\omega$$

hold. Moreover, by continuity of T_v, $\mathit{lfp}(T_v) = T_v \uparrow_\omega$ holds, whilst the converse, $\mathit{gfp}(T_v) = T_v \downarrow_\omega$ in general does not hold. Again the program (1) provides a counter-example.

We recall the basic results, that extend the ones listed in (2), i.e. the model-theoretic and fixpoint characterization of the atomic consequence set ACS

$$\begin{aligned} ACS &= \{A \mid W \models A\} \\ &= M_v \text{ (the least Herbrand model with variables)} \\ &= T_v \uparrow_\omega \end{aligned} \tag{4}$$

Until now, there was no counterpart of the result in (3) characterizing the non-ground finite failure set. This can be considered one main contribution of this paper. As an example, the program (1) is such that $T_v \downarrow_\omega = \{q(a)\}$, and we see that the simple solution of defining *NGFF* as $\mathbf{B}_v \backslash T_v \downarrow_\omega$ does not work. In fact *NGFF* would contain atoms like $p(x)$ which are not finitely failed.

In the following we will show the relation between ACS and *NGSS*. This last notion represents all successful computations, and thus it is the notion which is really symmetric with respect to *NGFF*.

[2] Note that $(\mathbf{U}_v{}^*)_{/\approx}$ is not isomorphic to $(\mathbf{U}_{v/\approx})^*$. This is the first reason why we introduce in section 2.1 the new notion of interpretation. The other reason is the kind of completion we define in section 4.

3 Non-ground finite failure and success sets.

In this section we present a characterization of the non-ground success set and the non-ground finite failure set in terms of the least upper bound and the greatest lower bound of two chains of interpretations. The elements of the chains can be obtained by repeatedly applying the immediate consequence operator T starting from the empty set and the Herbrand base, and then by performing the downward closure. The same construction can be done by using T_v instead of T. We will characterize the two sets by using T_v, because the extension that we will propose in the following sections (transformation on complete Herbrand interpretations) is based on T_v. Nevertheless, in the technical proofs we will use both T_v and T, according to convenience. Let us first consider the relation between the non-ground success set and these transformations.

Theorem 3.1

$$NGSS = Down(ACS) = Down(lfp(T_v))$$

Proof From [Cla79], [FLMP89a] we know that

$$\begin{aligned} lfp(T_v) &= \{A \mid \leftarrow A \overset{\epsilon\ *}{\longmapsto} \Box\} \\ &= \{A \mid \ W \models A\} \end{aligned}$$

$\subseteq$) Let $A \in NGSS$. Let ϑ be a computed answer substitution for $\leftarrow A$. Then $W \models A\vartheta$ ([Cla79]). Hence $A\vartheta \in lfp(T_v)$ and therefore $A \in Down(lfp(T_v))$.

$\subseteq$) Let $A \in Down(lfp(T_v))$. Then there exists $A' \in lfp(T_v)$ such that $A \leq A'$. Hence $\leftarrow A'$ has a successful SLD-derivation with empty computed answer substitution, and therefore, by the lifting lemma ([Llo87]), $\leftarrow A$ has a successful SLD-derivation, i.e. $A \in NGSS$. □

Note that $NGSS$ is also equal to $Down(SS)$ and $Down(lfp(T))$. A model-theoretic characterization of $NGSS$ is $\{A \mid \ W \models \exists A\}$, i.e., the set of atoms whose existential quantification is a logical consequence of the program.

Proposition 3.2 *Down is continuous with respect to the lattice $(\mathcal{I}_v, \subseteq)$, i.e. for every chain of interpretations $\{I_n\}_{n\geq 0}$, $Down(\bigcup_{n\geq 0} I_n) = \bigcup_{n\geq 0} Down(I_n)$ holds.*

Proof

$\subseteq$) Let $A \in Down(\bigcup_{n\geq 0} I_n)$. Then there exists $n \geq 0, A' \in I_n$ such that $A \leq A'$. Therefore $A \in Down(I_n)$, hence $A \in \bigcup_{n\geq 0} Down(I_n)$.

$\supseteq$) Let $A \in \bigcup_{n\geq 0} Down(I_n)$. Then there exists $n \geq 0, A' \in I_n$ such that $A \leq A'$. Therefore $A \in Down(\bigcup_{n\geq 0} I_n)$. □

Corollary 3.3

$$NGSS = Down(lfp(T_v)) = Down(\bigcup_{n\geq 0} T_v\!\uparrow_n) = \bigcup_{n\geq 0} Down(T_v\!\uparrow_n)$$

To characterize $NGFF$, let us recall two results from [Apt90].

Lemma 3.4 (Apt [Apt90], lemma 5.7) *Let A be an atom and let ϑ be a substitution. If $\leftarrow A$ has a finitely failed SLD-tree of depth $\leq n$ then so has $\leftarrow A\vartheta$.*

Lemma 3.5 (Apt [Apt90], lemma 5.9) *Let A be a ground atom. If $\leftarrow A$ has a finitely failed SLD-tree of depth $\leq n$ then $A \notin T\!\downarrow_n$.*

The following lemma corresponds to a classical result.

Lemma 3.6 *Let A be an atom. If $\leftarrow A$ has an infinite fair SLD-tree then for every $n \geq 0$ there exists a (ground) substitution ϑ such that $A\vartheta \in T\downarrow_n$.*

Proof It is an immediate consequence of lemma 5.10 in [Apt90].

□

We present now our construction. We give first the definition of the elements of the chain that allow to characterize $NGFF$.

Definition 3.7

- *For every $n \geq 0$ define $I_n = Down(T\downarrow_n)$.*
- $I_\omega = \bigcap_{n\geq 0} I_n$.

Then we have the following result, which is the non-ground counterpart of theorem 5.6 in [Apt90].

Theorem 3.8 *The following three statements are equivalent*

1. $A \in NGFF$.
2. $A \notin I_\omega$.
3. *Every fair SLD-tree for $\leftarrow A$ is finitely failed.*

Proof

1⇒2) We prove a stronger property, corresponding to lemma 3.5 for non-ground atoms. Namely, we prove that if $\leftarrow A$ has a finitely failed SLD-tree of depth $\leq n$ then $A \notin Down(T\downarrow_n)$. Assume (by contradiction) that $A \in Down(T\downarrow_n)$. Then there exists a ground substitution ϑ such that $A\vartheta \in T\downarrow_n$. Thus, by lemma 3.5, $\leftarrow A\vartheta$ has no finitely failed SLD-trees of depth $\leq n$. By lemma 3.4, also $\leftarrow A$ has no finitely failed SLD-trees of depth $\leq n$ (contradiction).

2⇒3) Assume, by contradiction, that there exists a fair SLD-tree for $\leftarrow A$ which is not finitely failed. There are two cases.

i) The tree contains an empty clause. In this case, the computed answer substitution ϑ is a correct answer substitution, therefore for any ground substitution ϑ' we have $A\vartheta\vartheta' \in T\uparrow_\omega$. Since $T\uparrow_\omega \subseteq T\downarrow_\omega$ [Apt90], and

$$\begin{array}{rcl} T\downarrow_\omega & = & \bigcap_{n\geq 0} T\downarrow_n \\ & \subseteq & \bigcap_{n\geq 0} Down(T\downarrow_n) \\ & = & \bigcap_{n\geq 0} I_n \\ & = & I_\omega \end{array}$$

we have $A\vartheta\vartheta' \in I_\omega$. Since I_ω is downward closed, then $A \in I_\omega$ (contradiction).

ii) The tree is infinite. In this case, for every $n \geq 0$ there exists a (ground) substitution ϑ such that $A\vartheta \in T\downarrow_n$ (by lemma 3.6). Therefore for every $n \geq 0$ $A \in Down(T\downarrow_n)$. Hence $A \in I_\omega$ (contradiction).

3⇒1) Immediate. □

Let us note that I_ω can be obtained by using T_v instead of T.

Lemma 3.9 *For every n, $T\downarrow_n = Ground(T_v\downarrow_n)$ holds.*

Proof Immediate, by induction on n. □

Lemma 3.10 *If there exists at least one constant symbol, then for every interpretation I,*

$$Down(Ground(Up(I))) = Down(Up(I)).$$

Proof

$\subseteq$) Immediate, since $Ground(Up(I)) \subseteq Up(I)$.

$\supseteq$) Let $A \in Down(Up(I))$. Then $\exists A' \in Up(I)$ such that $A \leq A'$. Let ϑ be a ground substitution for A'. Then $A \leq A'\vartheta$ and $A'\vartheta \in Ground(Up(I))$. □

Proposition 3.11 *For every $n, Down(T\downarrow_n) = Down(T_v\downarrow_n)$ holds.*

Proof

$$\begin{aligned}
Down(T\downarrow_n) &= Down(Ground(T_v\downarrow_n)) && \text{(by lemma 3.9)}\\
&= Down(Ground(Up(T_v\downarrow_n))) && \text{(since } T_v\downarrow_n \text{ is upward closed)}\\
&= Down(Up(T_v\downarrow_n)) && \text{(by lemma 3.10)}\\
&= Down(T_v\downarrow_n). && \text{(since } T_v\downarrow_n \text{ is upward closed)}
\end{aligned}$$

□

From this proposition the relation between I_ω (and therefore *NGFF*) and T_v follows immediately.

Corollary 3.12 $NGFF = \mathbf{B}_v \backslash I_\omega = \mathbf{B}_v \backslash \bigcap_{n\geq 0} Down(T_v\downarrow_n)$

Example 3.13 *Let us give an example of this result using program (1).*

$$T_v\downarrow_1 = \{p(f^n(a)), p(f^n(x)), q(a) \mid n \geq 1\}$$

$$Down(T_v\downarrow_1) = T_v\downarrow_1 \bigcup \{p(x), q(x)\}$$

$$T_v\downarrow_k = \{p(f^n(a)), p(f^n(x)), q(a) \mid n \geq k\}$$

$$Down(T_v\downarrow_k) = T_v\downarrow_k \bigcup \{p(f^n(x)), q(x) \mid n < k\}$$

$$T_v\downarrow_\omega = \{q(a)\}$$

$$\bigcap_{n\geq 0} Down(T_v\downarrow_n) = \{p(f^n(x)), q(a), q(x) \mid n \geq 0\}$$

$$NGFF = \{p(f^n(a)), q(f^{n+1}(a)), q(f^{n+1}(x)) \mid n \geq 0\}$$

A first symmetry can be noted looking at corollaries 3.3 and 3.12. *NGSS* is characterized as the union of an infinite chain whose elements are obtainable using $T_v\uparrow_n$, while *NGFF* is the complement of a set characterized as the intersection of an infinite chain whose elements are obtainable using $T_v\downarrow_n$.

4 Complete Herbrand domain, interpretations and models.

In the following sections, we will show that it is possible to give a greatest fixpoint characterization of *NGFF*.

We define a complete Herbrand universe and we extend the transformation T_v on the new lattice of interpretations. Finally, we show that the new transformation is continuous (in the metric sense), and that the greatest fixpoint of this transformation corresponds to (the complement of) *NGFF*.

We first recall some basic definitions about metric spaces and partial orders. We refer to [Eng77] for more details.

4.1 Completion by ideals

Let $(P, \leq)$ be a preorder. A *directed set* in P is a subset D of P such that

$$\forall \alpha, \beta \in D \; \exists \gamma \in D \; [\alpha \leq \gamma \; \wedge \; \beta \leq \gamma].$$

An *ideal* S is a directed set which is *downward closed*, i.e. such that

$$\forall \alpha \in S \; [\beta \leq \alpha \Rightarrow \beta \in S].$$

The set of ideals of P, ordered by set inclusion, will be denoted by $(Id(P), \subseteq)$. It is well known that it is a Complete Partial Order (CPO) and that it contains a sub-CPO isomorphic to $(P_{/\approx}, \leq)$, where $\approx$ is the equivalence relation induced by $\leq$.

Every atom $p(t)$ is characterized by the ideal with $p(t)$ as the top element, i.e., the set $\{A \mid A \leq p(t)\}$. Infinite elements come also into the picture with set such as $\{p(f^n(x)) \mid n \geq 0\}$, which represents an infinite element $p(f^\omega)$.

4.2 Complete metric spaces

A metric space is a pair (S, d) where S is a non empty set and the distance function $d : S \times S \to [0, 1]$ satisfies the following properties

1. $\forall \alpha, \beta \in S \; [d(\alpha, \beta) = 0 \text{ iff } \alpha = \beta]$
2. $\forall \alpha, \beta \in S \; [d(\alpha, \beta) = d(\beta, \alpha)]$
3. $\forall \alpha, \beta, \gamma \in S \; [d(\alpha, \gamma) \leq d(\alpha, \beta) + d(\beta, \gamma)]$

A sequence $\{\alpha_n\}_{n \geq 0}$ of elements of S is a *Cauchy sequence* iff

$$\forall \delta > 0 \; \exists k \; \forall i, j \geq k \; [d(\alpha_i, \alpha_j) \leq \delta]$$

The sequence $\{\alpha_n\}_{n \geq 0}$ *converges* to a limit α (notation: $\lim_{n \to \infty} \alpha_n = \alpha$) iff

$$\forall \delta > 0 \; \exists k \; \forall i \geq k \; [d(\alpha_i, \alpha) \leq \delta]$$

Two Cauchy sequences $\{\alpha_n\}_{n \geq 0}$ and $\{\beta_n\}_{n \geq 0}$ are *equivalent* iff

$$\forall \delta > 0 \; \exists k \; \forall n \geq k \; [d(\alpha_n, \beta_n) < \delta].$$

Let (S, d) be a metric space. A subset $S' \subseteq S$ is *complete* (or *closed*) if each Cauchy sequence in S' converges to a limit in S', and it is *compact* if each sequence $\{\alpha_n\}_{n \geq 0}$ in S' contains a subsequence $\{\alpha_{n_j}\}_{j \geq 0}$ converging to a limit in S'. Given a metric space (S, d), there is a standard way to embed it into a complete metric space, that consists of adding the limits of all the Cauchy sequences. Formally, this is done by considering the quotient of the space of all the sequences, with respect to the above defined equivalence. The distance between two sequences is given by the limit of the distances on individuals.

Given the complete metric spaces (S,d), (S',d'), a mapping $\mathcal{F} : S \leftarrow S'$ is *continuous* iff for each Cauchy sequence $\{\alpha_n\}_{n\geq 0}$ in (S,d), $\{\mathcal{F}(\alpha_n)\}_{n\geq 0}$ is a Cauchy sequence in (S',d'), and

$$\lim_{n\to\infty} \mathcal{F}(\alpha_n) = \mathcal{F}(\lim_{n\to\infty}(\alpha_n))$$

Let (S,d) be a metric space and let $\mathcal{P}(S)$ denote the set of subsets of S. The *Hausdorff distance* $d_H : \mathcal{P}(S)\times\mathcal{P}(S) \to [0,1]$ is defined as follows

$$d_H(X,Y) = \begin{cases} max\{sup_{\alpha\in X} d_S(\alpha,Y), sup_{\beta\in Y} d_S(\beta,X)\} & \text{if } X,Y \neq \emptyset \\ 0 & \text{if } X = Y = \emptyset \\ 1 & \text{otherwise} \end{cases}$$

where

$$d_S(\alpha,Y) = inf_{\beta\in Y} d(\alpha,\beta)$$

If (S,d) is compact [complete], then the set of the compact [complete] subsets of S, with the Hausdorff distance, is complete. Note that, if (S,d) is compact, then every complete subset of S is compact.

4.3 The Herbrand interpretations topology

In this section we extend the usual notions of Herbrand universe, base, interpretation by means of some suitable topological structure.

Definition 4.1 (Complete Herbrand domain)
The Complete Herbrand Domain $(\mathbf{U}_c, \pi_c)$ *is defined by:*

$$\mathbf{U}_c = Id({\mathbf{U}_v}^*),$$

where $\leq$ *on* ${\mathbf{U}_v}^*$ *is the usual ordering on tuples of terms, and*

$$\pi_{c_n} = Id({\mathbf{U}_v}^n).$$

According to the results reported in section 4.1, $(\mathbf{U}_c, \subseteq)$ contains a sub-CPO isomorphic to the CPO $(({\mathbf{U}_v}^*)_{/\approx}, \subseteq)$, (the Herbrand universe with variables defined in section 2.3). In other words, there is an isomorphism $\mathcal{F}$ between $(({\mathbf{U}_v}^*)_{/\approx}, \subseteq)$ and a subset of $(\mathbf{U}_c, \subseteq)$ that preserves the ordering. This isomorphism also preserves the function π_v, namely $\pi_{c_n} = \mathcal{F}(\pi_{v_n})$. In fact

$$\begin{aligned} \pi_{c_n} &= Id({\mathbf{U}_v}^n) \text{ (by definition)} \\ &= \mathcal{F}(({\mathbf{U}_v}^n)_{/\approx}) \text{(tuples of different length are not comparable)} \\ &= \mathcal{F}(\pi_{v_n}) \end{aligned}$$

Let *Dsubst* be the set of *directed sets* of substitutions, with typical elements $\Theta, \Psi, \ldots$. We define the application of Θ to the tuple $\bar{t}$ by

$$\bar{t}\Theta = \{\bar{u} \mid \exists\vartheta\in\Theta\ [\bar{u} \leq \bar{t}\vartheta]\}$$

Lemma 4.2 (Representation lemma) *Any element α of $\mathbf{U}_c$ can be represented by a tuple of terms and an ideal of substitutions. More formally*

$$\forall \alpha \in \mathbf{U}_c \; \exists \bar{t} \in \mathbf{U}_v{}^* \; \exists \Theta \in Dsubst \; [\alpha = \bar{t}\Theta]$$

Proof See [Gol88].

Definition 4.3 (Complete Herbrand base) *The complete Herbrand base $\mathbf{B}_c$ is defined as*

$$\{p(\bar{t})\Theta \mid \; p \in Pred_n \; \wedge \; \bar{t}\Theta \in \pi_{c_n}\}$$

Because of the representation lemma, every element of $\mathbf{B}_c$ can be represented as $A\Theta$, where A is an appropriate atom and Θ is an appropriate directed set of substitutions.

We define now a distance d on $\mathbf{B}_c$. To this purpose, we consider the representation of atoms as trees, where the nodes are labelled by the constructor symbols. The depth of an atom A, i.e. the maximum number of nodes in a path root-leaf, will be denoted by $|A|$. The truncation at depth n of A, denoted $\lfloor A \rfloor_n$, is the tree obtained from A by replacing all the symbols at depth $n+1$ by fresh variables, and by eliminating the arcs and nodes at depth $> n+1$.

Example 4.4 *Let $A = p(a, f(a, b))$. Then $|A| = 3$ and $\lfloor A \rfloor_2 = p(a, f(x, y))$.*

Definition 4.5 *Given two objects $\alpha, \beta \in \mathbf{B}_c$, their distance d is defined as*

$$d(\alpha, \beta) = \begin{cases} 0 & \textit{if } \alpha = \beta \\ 2^{-min\{|A| \mid \; A \in (\alpha \setminus \beta) \cup (\beta \setminus \alpha)\}} & \textit{otherwise} \end{cases}$$

It is easy to see that the above definition of distance is correct. In fact, $d(\alpha, \alpha) = 0$ and $d(\alpha, \beta) = d(\beta, \alpha)$ are obvious. To show that $d(\alpha, \beta) \leq d(\alpha, \gamma) + d(\gamma, \beta)$, let's assume $d(\alpha, \beta) = 2^{-|A|}$ with, say, $A \in \alpha \setminus \beta$. If $A \notin \gamma$, then $d(\alpha, \gamma) \geq 2^{-|A|}$. If $A \in \gamma$, then $d(\gamma, \beta) \geq 2^{-|A|}$. Therefore $d(\alpha, \beta) \leq max(d(\alpha, \gamma), d(\gamma, \beta))$ holds (so, d is even an *ultrametric*). Note that the maximal distance between two objects is 2^{-2}, since all atoms have depth ≥ 2.

The following example shows that $(\mathbf{B}_c, d)$ is not isometric to the standard metric on the Herbrand universe, based on the (standard) distance on trees ($d'(A, B) = 2^{-min\{n \mid \; \lfloor A \rfloor_n \neq \lfloor B \rfloor_n\}}$), and used, for instance, by Lloyd ([Llo87]).

Example 4.6 *Consider the atoms $A = p(f^n(a), f^n(b))$, $B = p(f^n(a), f^n(a))$ and consider $\alpha = Down(\{A\})$, $\beta = Down(\{B\})$. We have that $d(\alpha, \beta) = 2^{-2}$, in fact $p(x, x) \in \beta \setminus \alpha$. The standard distance between A and B is $2^{-(n+2)}$, since the two atoms are equal till depth $n+1$.*

The reason why we use the metric d instead of the standard one is that the latter has not enough discriminating power, as explained in the introduction. It identifies the limits of sequences that cannot be identified, when we want to model finite failure. The following example shows the difference with d, in this respect.

Example 4.7
Let $\{A_n\}_{n \geq 0} = \{p(f^n(x), f^n(x))\}_{n \geq 0}$ and $\{B_n\}_{n \geq 0} = \{p(f^n(x), f^{n+1}(x))\}_{n \geq 0}$ be the sequences considered in the introduction. Consider the sequences $\{\alpha_n\}_{n \geq 0} = \{Down(\{A_n\})\}_{n \geq 0}$, $\{\beta_n\}_{n \geq 0} = \{Down(\{B_n\})\}_{n \geq 0}$.
We have that in the standard metric

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} B_n = p(f^\omega(x), f^\omega(x)),$$

whilst

$$\lim_{n\to\infty} \alpha_n = \bigcup_{n\geq 0} Down(\{p(f^n(x), f^n(x))\}) \neq$$

$$\lim_{n\to\infty} \beta_n = \bigcup_{n\geq 0} Down(\{p(f^n(x), f^{n+1}(x))\}).$$

(for instance, $p(x,x)$ belongs to the first and not to the second).

Proposition 4.8 $(\mathbf{B}_c, d)$ *is a complete metric space.*

Proof Let $\{\alpha_n\}_{n\geq 0}$ be a Cauchy sequence in $\mathbf{B}_c$. Then

$$\forall h > 0\ \exists n_h\ \forall m, n \geq n_h\ [d(\alpha_m, \alpha_n) < 2^{-h}]. \tag{5}$$

Without loss of generality, we can assume that n_h depends monotonically on h. For $h \geq 2$ define

$$\mathcal{A}_h = \{A \in \alpha_{n_h} \mid |A| \leq h\}.$$

In general, $\mathcal{A}_h \notin \mathbf{B}_c$, because it is not an ideal (it is not a directed set)[3], it is, however, downward closed. Moreover, by definition of d,

$$\forall n \geq n_h\ [\mathcal{A}_h \subseteq \alpha_n] \tag{6}$$

holds. Consider now

$$\alpha = \bigcup_{h\geq 1} \mathcal{A}_h,$$

we show that $\alpha \in \mathbf{B}_c$ and that $\lim_{n\to\infty} \alpha_n = \alpha$.

$\boldsymbol{\alpha \in \mathbf{B}_c}$**)** Since it is the union of downward closed sets, α is downward closed. It remains to show that it is a directed set. Let $A_1, A_2 \in \alpha$. Then $\exists h_1, h_2$ such that $A_1 \in \mathcal{A}_{h_1}$ and $A_2 \in \mathcal{A}_{h_2}$. Assume, without loss of generality, that $h_1 \leq h_2$. Then, by (6), we have $\forall n \geq n_{h_2}\ [A_1, A_2 \in \alpha_n]$. Therefore, since the α_n's are directed and downward closed sets

$$\forall n \geq n_{h_2}\ [A_3 = lub(A_1, A_2) \in \alpha_n]$$

holds. Let $h_3 = max\{|A_3|, h_2\}$. Then $A_3 \in \mathcal{A}_{h_3}$ and therefore $A_3 \in \alpha$.

$\boldsymbol{\lim_{n\to\infty} \alpha_n = \alpha}$**)** We show that $\forall h > 0\ \forall n \geq n_h$

1. $\forall A \in \alpha\ [|A| \leq h \Rightarrow A \in \alpha_n]$
2. $\forall A \in \alpha_n\ [|A| \leq h \Rightarrow A \in \alpha]$

1. Let $A \in \alpha$ and let h' be such that $A \in \mathcal{A}_{h'}$. We have two cases

   $\boldsymbol{h' < h}$**)** Immediate by (6), and by monotonicity of n_h ($n_{h'} < n_h$).

   $\boldsymbol{h' \geq h}$**)** Since $A \in \alpha_{n_{h'}}$, and $d(\alpha_{n_{h'}}, \alpha_{n_h}) < 2^{-h}$, we have $A \in \alpha_{n_h}$. Then, $A \in \mathcal{A}_h$ holds. The rest follows by (6).

2. Let $n > n_h$ and $A \in \alpha_n$ such that $|A| \leq h$. Since $d(\alpha_n, \alpha_{n_h}) < 2^{-h}$ then $A \in \alpha_{n_h}$ holds. Hence $A \in \mathcal{A}_h$, from which we get $A \in \alpha$. □

The metric we have defined has an even stronger property: the compactness.

Proposition 4.9 $(\mathbf{B}_c, d)$ *is a compact metric space.*

[3] Let $A = p(f(a), g(f(a)))$ and $\alpha_{n_3} = Down(\{A\})$. Then $p(x, g(x)), p(f(a), y) \in \mathcal{A}_3$, but their least upper bound (A), having depth 4, does not belong to $\mathcal{A}_3$.

Proof It is sufficient to show that every sequence $\{\alpha_n\}_{n\geq 0}$ admits a Cauchy subsequence. The convergency of it follows then from the completeness (proposition 4.8). We have two cases.

1. only finitely many α_n's contain atoms of arbitrary depth.

2. for any $h \geq 1$ there exists n such that α_n contains an atom of depth $> h$.

The first case is trivial: since the set $Atom_h$ of atoms of depth $\leq h$ is finite (apart from variable renaming), also the set of the (downward closed) subsets of $Atom_h$ is finite, and therefore there exists α such that $\alpha_n = \alpha$ for infinitely many n.

Let's now consider the second case. We show how to build a Cauchy subsequence $\{\alpha_{n_h}\}_{h\geq 2}$. Let $\{\alpha_n^1\}_{n\geq 0} = \{\alpha_n\}_{n\geq 0}$ For $h \geq 2$, we define the set of atoms $\mathcal{A}_h$, the sequence $\{\alpha_n^h\}_{n\geq 0}$, and the object α_{n_h} as follows.

- Consider the set of the (downward closed) subsets of $Atom_h$. Since it is finite, there is one of them, $\mathcal{A}_h$, such that for infinitely many n, $\mathcal{A}_h = \alpha_n^{h-1} \cap Atom_h$.

- Let $\{\alpha_n^h\}_{n\geq 0}$ be the subsequence of $\{\alpha_n^{h-1}\}_{n\geq 1}$ whose elements satisfy the above property ($\forall n\ [\mathcal{A}_h = \alpha_n^h \cap Atom_h]$).

- Define $\alpha_{n_h} = \alpha_0^h$.

It is easy to see that $\{\alpha_{n_h}\}_{h\geq 2}$ is a Cauchy sequence. In particular, we have by construction

$$\forall h \geq 2\ \forall h', h'' \geq h\ [d(\alpha_{n_{h'}}, \alpha_{n_{h''}}) < 2^{-h}].$$

In fact, $\alpha_{n_{h'}} \cap Atom_h = \alpha_{n_{h''}} \cap Atom_h = \mathcal{A}_h$. $\square$

We extend now the notion of interpretation on the complete Herbrand universe. Essentially, we still consider interpretations as subsets of the (complete) Herbrand base, provided some conditions that guarantee the metric completeness. This property is obtained by considering complete subsets only (since $(\mathbf{B}_c, d_H)$ is a complete metric space), see section 4.2.

Definition 4.10 (Metric space of complete Herbrand interpretations) *A complete Herbrand interpretation $I \in \mathcal{I}_c$ is any complete*[4] *subset of $\mathbf{B}_c$. We define the distance d_c on $\mathcal{I}_c$ as the Hausdorff distance based on $(\mathbf{B}_c, d)$.*

According to the results reported in section 4.2, $(\mathcal{I}_c, d_c)$ is a complete metric space. Moreover, the set $\mathcal{I}_c$, ordered by set inclusion, is also a complete lattice. The greatest lower bound operator (*glb*) is given by the set intersection $\cap$, in fact the intersection of complete sets is complete.

The least upper bound operator (*lub*) is given by the metric closure of the set union, i.e. for every set of interpretations $\mathcal{L}$, $lub(\mathcal{L}) = Cl(\bigcup_{I\in\mathcal{L}} I)$. The function $Cl : \mathcal{P}(\mathbf{B}_c) \rightarrow \mathcal{I}_c$ adds to a set all the limits of the Cauchy sequences. More formally, for a subset I of $\mathbf{B}_c$, $Cl(I)$ is the smallest complete set containing I. Of course $Cl(\bigcup_{I\in\mathcal{L}} I)$ is complete, and it is easy to see that is smaller than every upper bound of $\mathcal{L}$.

The top of $\mathcal{I}_c$ is $\mathbf{B}_c$, and the bottom is $\emptyset$.

There is a "natural" correspondence Ω between $\mathcal{I}_v$ and $\mathcal{I}_c$, that amounts to perform the completion. Given an $I \in \mathcal{I}_v$, this correspondence is defined as:

$$\Omega(I) = Cl(\{Down(\{A\}) \mid A \in I\})$$

[4]the completeness is the main difference with the approach of Golson [Gol88], and the reason why the transformation we define in section 5 has the property that the greatest fixpoint is obtainable in ω steps from the top, without the need of any additional restriction on the programs.

Of course, $\Omega(I)$ is complete, so this definition is correct.

Notice that Ω is a poset homomorphism, i.e., it preserves the set inclusion ordering. Namely

$$I \subseteq I' \Rightarrow \Omega(I) \subseteq \Omega(I')$$

The lattice of (upward closed) interpretations with variables, however, is not a sublattice of $(\mathcal{I}_c, \subseteq)$. In fact, the natural correspondence Ω does not preserve the *glb* operation. This is the point in which we gain in expressivity: the *glb* of a decreasing chain in $(\mathcal{I}_c, \subseteq)$ contains here more information than the corresponding chain of standard interpretations.

Example 4.11 *Let $\{I_n\}_{n\geq 0}$ be the decreasing chain of interpretations with variables defined by $I_n = Up(\{p(f^j(x)) \mid j \geq n\}) = \{p(f^j(t)) \mid t \in \mathbf{U}_v, j \geq n\}$. We have*

$$\Omega(I_n) = Cl(\{p(x)\Theta_{t,j} \mid t \in \mathbf{U}_v, j \geq n\}),$$

where

$$\Theta_{t,j} = \{\epsilon, \{x/f(x)\}, \ldots, \{x/f^j(t)\}\},$$

Note now that all Cauchy sequences in this set have as limit $p(x)\Theta$, with

$$\Theta = \{\epsilon, \{x/f(x)\}, \ldots, \{x/f^j(x)\}, \ldots\}.$$

Therefore

$$\Omega(I_n) = \{p(x)\Theta_{t,j} \mid t \in \mathbf{U}_v, j \geq n\} \cup \{p(x)\Theta\}$$

Hence

$$\bigcap_{n\geq 0} \Omega(I_n) = \{p(x)\Theta\},$$

whilst

$$\bigcap_{n\geq 0} I_n = \emptyset.$$

In this case, Golson [Gol88] finds the same result, since it is sufficient to force the upward closure (with respect to the set inclusion), as he does, to get the limit element. In fact:

$$p(x)\Theta \in Up_c(\{p(x)\Theta_{x,j}\}),$$

where, for $I \in \mathcal{I}_c$, $Up_c(I) = \{\alpha \mid \exists \alpha' \in I\ [\alpha' \subseteq \alpha]\}$.

However, we can show a similar example in which our approach differs from the one of Golson:

Example 4.12 *Let $\{I_n\}_{n\geq 0}$ be the decreasing chain of interpretations with variables defined by $I_n = \{p(f^j(a)) \mid j \geq n\}$. We have*

$$\Omega(I_n) = Cl(\{p(x)\Theta_j \mid j \geq n\}),$$

where

$$\Theta_j = \{\epsilon, \{x/f(x)\}, \ldots, \{x/f^j(a)\}\},$$

All Cauchy sequences converge to the limit $p(x)\Theta$, with Θ defined as in example 4.11. Hence

$$\Omega(I_n) = \{p(x)\Theta_j \mid j \geq n\} \cup \{p(x)\Theta\}$$

and

$$\bigcap_{n\geq 0} \Omega(I_n) = \{p(x)\Theta\},$$

In [Gol88] the metric limit is not required, and

$$p(x)\Theta \notin Up_c(\{p(x)\Theta_j\})$$

Hence the corresponding sequence would be $I'_n = \{p(x)\Theta_j \mid j \geq n\}$ so obtaining

$$\bigcap_{n\geq 0} I'_n = \emptyset,$$

Note that the objects I'_n's are not allowed as interpretations in our definition, since they are not complete.

Intuitively, the metric structure allows to add the limit of the sequence $Down(\{p(f^n(a))\}_{n\geq 0})$, in the same way as in the case $Down(\{p(f^n(x))\}_{n\geq 0})$, since these sequences are equivalent (in the metric sense).

However, our approach has still more discrimination power than in the pure metric approaches. This is shown in the following example, that is the continuation of example 4.7.

Example 4.13 *Consider the sequences*

$$\{p(f^n(x), f^n(x))\}_{n\geq 0}$$

and

$$\{p(f^{n+1}(x), f^n(x))\}_{n\geq 0}.$$

These sequences are equivalent in the standard metric and their limit is $p(f^\omega, f^\omega)$.

Consider now the corresponding sequences in $\mathcal{I}_c$

$$\{I_n\}_{n\geq 0} = \{Down(p(f^n(x), f^n(x)))\}_{n\geq 0}$$

and

$$\{I'_n\}_{n\geq 0} = \{Down(p(f^{n+1}(x), f^n(x)))\}_{n\geq 0}$$

in $(\mathbf{B}_c, d_H)$. *They are not equivalent, in fact*

$$\forall n\ [\{p(x, x), \ldots\} \in I_n \setminus I'_n]$$

therefore

$$\forall n\ [d_c(I_n, I'_n) = 2^{-2}]$$

5 Fixpoint characterization of non-ground success set and non-ground finite failure set.

This section presents the fixpoint characterization of *NGSS* and *NGFF*.

We extend the immediate consequence operator on $\mathcal{I}_c$ as follows

$$T_c(I) = \{H\Theta \mid \begin{array}{l} \exists H \leftarrow \bar{B} \in W \\ \bar{B}\Theta \in I\} \end{array}$$

The following results present the most important characteristics of this transformation.

Proposition 5.1 *T_c is continuous with respect to the metric d_c.*

Proof We need the following lemmata:

Lemma 5.2 *Let (S, d), (S', d') be complete metric spaces. Consider a mapping $\mathcal{F} : S \to S'$ which is* uniformly continuous*, i.e.*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall \alpha, \beta \in S\ [d(\alpha, \beta) < \delta \Rightarrow d'(\mathcal{F}(\alpha), \mathcal{F}(\beta)) < \varepsilon].$$

Then $\mathcal{F}$ is continuous.

Proof Standard. □

Lemma 5.3 *If $B_1 \in \alpha_1 \wedge \ldots \wedge B_n \in \alpha_n$, then $\exists \Theta\ [B_1\Theta = \alpha_1 \wedge \ldots \wedge B_n\Theta = \alpha_n]$.*

Proof We prove it for $n = 1$. The general case follows from the possibility of considering a "term" $f(B_1, \ldots, B_n) \in \alpha'$ where $\alpha' = \{f(A_1, \ldots, A_n) \mid A_1 \in \alpha_1, \ldots, A_n \in \alpha_n\}$. Consider $B \in \alpha$. Define $\Theta = \{\vartheta \mid B\vartheta \in \alpha\}$. Then:

- Θ is a directed set. In fact, if $\vartheta_1, \vartheta_2 \in \Theta$, then $B\vartheta_1, B\vartheta_2 \in \alpha$. Hence, since α is directed, there exists A such that $B\vartheta_1, B\vartheta_2 \leq A$. For the properties of unification, there exists ϑ_3 such that $\vartheta_1, \vartheta_2 \leq \vartheta_3$ and $B\vartheta_3 = A$.
- Θ is downward closed. In fact, if $\vartheta \in \Theta$ and $\vartheta' \leq \vartheta$, then $B\vartheta' \leq B\vartheta \in \alpha$. Therefore, $\vartheta' \in \Theta$.
- $B\Theta = \alpha$. In fact, obviously $B\Theta \subseteq \alpha$ holds. Moreover, if $A \in \alpha$, then since α is a directed set, there exists $A' \in \alpha$ such that $B, A \leq A'$. Let ϑ be such that $B\vartheta = A' \in B\Theta$. Since $B\Theta$ is downward closed, $A \in B\Theta$.

□

We prove now Proposition 5.2 by showing that T_c is uniformly continuous.

Let $h = max\{|A| \mid A \text{ occurs in } W\}$, let m be the maximum number of symbols that occur in the atoms of W, and let r be the maximum arity of the constructor symbols occurring in W. Given $\varepsilon > 0$, fix

$$\delta = min\{2^{-h}, 2^{-h}2^{-u(-log\ \varepsilon)}\} \tag{7}$$

where the function $u(x)$ is defined as

$$u(x) = (m + r^x)(h + x) \tag{8}$$

(the reasons of this choice will become clear later). Consider two interpretations I_1, I_2 such that $d_c(I_1, I_2) < \delta$. We show that $d_c(T_c(I_1), T_c(I_2)) < \varepsilon$.

Let $\alpha \in T_c(I_1)$. Then there exist Θ_1 and a clause $H \leftarrow B_1, \ldots, B_n$ in W such that $\alpha = H\Theta_1$ and $B_1\Theta_1, \ldots, B_n\Theta_1 \in I_1$. If $n = 0$, then $\alpha \in T_c(I_1)$ holds.

If $n \geq 1$, then choose $\beta_1 \ldots, \beta_n \in I_2$ such that $\forall k \in [1, n]\ [d(B_k\Theta_1, \beta_k) \leq d_c(I_1, I_2)]$. Since $d_c(I_1, I_2) < \delta \leq 2^{-h}$ and $|B_k| \leq h$, we have that $\forall k \in [1, n]\ [B_k \in \beta_k]$. By lemma 5.3, it follows that there exists Θ_2 such that $\forall k \in [1, n]\ [\beta_k = B_k\Theta_2]$. Thus we obtain $H\Theta_2 \in T_c(I_2)$.

Without loss of generality, we can assume that the substitutions in Θ_1 and Θ_2 affect only the variables of $B_1, \ldots, B_k$. Define $|\vartheta| = |p(\bar{x})\vartheta|$ and $d(\Theta_1, \Theta_2) = d(p(\bar{x})\Theta_1, p(\bar{x})\Theta_2)$, where $\bar{x}$ are all the variables occurring in $B_1, \ldots, B_k$. Observe that for each ϑ and for each k $|B_k\vartheta| \leq |B_k| + |\vartheta|$ holds, hence

$$d(\Theta_1, \Theta_2) \leq 2^h d(B_k\Theta_1, B_k\Theta_2),$$

from which it follows

$$d(\Theta_1, \Theta_2) \leq 2^h d_c(I_1, I_2). \tag{9}$$

Summarizing, we have shown that

$$\forall \alpha \in T_c(I_1)\ \exists H, \Theta_1, \Theta_2\ [\alpha = H\Theta_1\ \wedge\ H\Theta_2 \in T_c(I_2)\ \wedge\ d(\Theta_1, \Theta_2) \leq 2^h d_c(I_1, I_2).]$$

Analogous result can be shown for $\alpha \in T_c(I_2)$.

Let $d(H\Theta_1, H\Theta_2) = 2^{-|A|}$ with, say, $A \in H\Theta_1 \setminus H\Theta_2$. Consider a minimal $\vartheta \in \Theta_1 \setminus \Theta_2$ (the set of substitutions is well founded) such that $A \le H\vartheta$. Consider the most general unifier μ of A and H. We can assume, without loss of generality, that the variables of A and H are disjoint. Therefore μ can be written as $\mu_A \cup \mu_H$, where μ_A affects only the variables of A, and μ_H affects only the variables of H. Observe now that $\mu_H \le \vartheta$ holds by definition of most general unifier; furthermore $\vartheta \le \mu_H$, since $A \le H\mu_H$, $\mu_H \in \Theta_1$ (as $\mu_H \le \vartheta \in \Theta_1$ and Θ_1 is a directed set), and $\mu_H \notin \Theta_2$ (otherwise it would contradict the hypothesis $A \notin H\Theta_2$). Hence we conclude

$$\mu_H = \vartheta. \tag{10}$$

By induction on the number of replacement steps made by the unification algorithm [Apt90], it is easy to prove that

$$|\mu_H| \le (m + r^{|A|})(|A| + |H|). \tag{11}$$

(This property motivates the definition of $u(x)$ in (8).)

Let's collect the results:

$$\begin{array}{ll} & 2^{-u(|A|)} \\ \le & \quad ((11) \text{ and } (10)) \\ & 2^{-|\vartheta|} \\ \le & \quad (\text{minimality of } \vartheta \in \Theta_1 \setminus \Theta_2) \\ & d(\Theta_1, \Theta_2) \\ \le & \quad ((9)) \\ & 2^h d_c(I_1, I_2) \\ < & \quad (\text{initial hypothesis}) \\ & 2^h \delta \\ \le & \quad ((7)) \\ & 2^{-u(-log\ \varepsilon)}. \end{array}$$

Since $u(x)$ is a monotonic function, we conclude $|A| > -log\ \varepsilon$, i.e. $2^{-|A|} < \varepsilon$. Finally, observe that $d_c(T_c(I_1), T_c(I_2)) \le 2^{-|A|}$. □

Notice that we didn't show yet that the definition of T_c is correct, namely, for $I \in \mathcal{I}_c$, $T_c(I) \in \mathcal{I}_c$, i.e. $T_c(I)$ is complete. The proof of continuity given above, however, does not make use of it.

Proposition 5.4 *If $I \in \mathcal{I}_c$, then $T_c(I) \in \mathcal{I}_c$, i.e. $T_c(I)$ is complete.*

Proof Since the finite union of complete sets is complete, and $T_c(I)$ can be defined as $T_{c_1}(I) \cup \ldots \cup T_{c_k}(I)$, where T_{c_i} is the transformation associated to the i-th clause of the program, it is sufficient to prove the proposition for a program containing only one clause, say $H \leftarrow B_1, \ldots B_m$.

Let $\{\alpha_n\}_{n\ge0}$ be a Cauchy sequence in $T_c(I)$. We have $\alpha_n = H\Theta_n$ for appropriate Θ_n's such that $B_1\Theta_n, \ldots B_m\Theta_n \in I$. Consider the sequence $\{I_n\}_{n\ge0}$, where $I_n = \{B_1\Theta_n, \ldots B_m\Theta_n\}$. By compactness of $\mathbf{B}_c$, there exists a converging subsequence $\{I_{n_i}\}_{i\ge0}$. Let

$$B_1\Theta = \lim_{n\to\infty} B_1\Theta_n, \quad \ldots, \quad B_m\Theta = \lim_{n\to\infty} B_m\Theta_n$$

By continuity,

$$\lim_{i\to\infty}\{H\Theta_{n_i}\} = \lim_{i\to\infty} T_c(I_{n_i}) = T_c(\lim_{i\to\infty} I_{n_i}) = T_c(\{B_1\Theta,\dots B_m\Theta\}) = \{H\Theta\}$$

Hence $\lim_{i\to\infty} H\Theta_{n_i} = H\Theta$. Since any subsequence of a Cauchy sequence converges to the same limit, we have $\lim_{n\to\infty}\alpha_n = H\Theta$. Finally, since I is complete, $B_1\Theta,\dots B_m\Theta \in I$, and therefore $H\Theta \in T_c(I)$. □

We analyze now the properties of T_c with respect to the CPO structure of $(\mathcal{I}_c, \subseteq)$. We note that, by definition, T_c is monotonic. The next proposition shows that chains and decreasing chains in $(\mathcal{I}_c, \subseteq)$ have a strong property with respect to the metric structure: they are Cauchy sequences. Moreover the limits coincide in both topologies.

Proposition 5.5

1. *Each chain $\{I_n\}_{n\geq 0}$ is a Cauchy sequence, and $\lim_{n\to\infty} I_n = lub_{n\geq 0} I_n$ holds.*
2. *Each decreasing chain $\{I_n\}_{n\geq 0}$ is a Cauchy sequence, and $\lim_{n\to\infty} I_n = glb_{n\geq 0} I_n$ holds.*

Proof We recall that $Atom_h$ denotes the set of all atoms of depth smaller or equal to h.

1. By definition $lub_{n\geq 0} I_n = Cl(\bigcup_{n\geq 0} I_n)$. It is sufficient to show that

$$\forall h > 0\ \exists n_h\ \forall i \geq n_h\ \forall \alpha \in Cl(\bigcup_{n\geq 0} I_n)\ \exists \alpha' \in I_i\ [d(\alpha,\alpha') < 2^{-h}]$$

   For $\alpha \in Cl(\bigcup_{n\geq 0} I_n)$, define

$$[\alpha]_h = \{A \in \alpha \mid\ |A| \leq h\} = \alpha \cap Atom_h.$$

   Let n_α be the smallest index such that

$$[\alpha]_h \subseteq \bigcup_{\alpha' \in I_{n_\alpha}} \alpha'.$$

   Since $\bigcup_{\alpha \in Cl(\bigcup_{n\geq 0} I_n)} [\alpha]_h$ is finite, there exists a finite

$$n_h = max\{n_\alpha \mid\ \alpha \in Cl(\bigcup_{n\geq 0} I_n)\}$$

   that (since $\{I_n\}_{n\geq 0}$ is a chain) satisfies the required property.

2. By definition $glb_{n\geq 0} I_n = \bigcap_{n\geq 0} I_n$. It is sufficient to show that

$$\forall h > 0\ \exists n_h\ \forall i \geq n_h\ \forall \alpha \in I_i\ \exists \alpha' \in \bigcap_{n\geq 0} I_n\ [d(\alpha,\alpha') < 2^{-h}]$$

   For $A \in Atom_h$ define

$$n_A = \begin{cases} 0 & \text{if } \exists \alpha \in \bigcap_{n\geq 0} I_n\ [A \in \alpha] \\ k & \text{if } \exists k\ \forall n \geq k\ \forall \alpha \in I_n\ [A \notin \alpha] \end{cases}$$

   (since $\{I_n\}_{n\geq 0}$ is a decreasing chain there are only these two alternatives) then, since $Atom_h$ is finite, there exists a finite

$$n_h = max\{n_A \mid\ A \in Atom_h\}$$

   that satisfies the required property. □

Corollary 5.6

1. *Tc is continuous with respect to the CPO structure, i.e., for every chain $\{I_n\}_{n\geq 0}$, we have $lub_{n\geq 0}T_c(I_n) = T_c(lub_{n\geq 0}I_n)$.*
2. *Tc is downward continuous with respect to the CPO structure, i.e., for every decreasing chain $\{I_n\}_{n\geq 0}$, we have $glb_{n\geq 0}T_c(I_n) = T_c(glb_{n\geq 0}I_n)$.*

Proof Immediate by monotonicity of T_c, by proposition 5.5 and by metric continuity of T_c. □

Corollary 5.7

1. *$lfp(T_c) = T_c{\uparrow}_\omega$*
2. *$gfp(T_c) = T_c{\downarrow}_\omega$*

Proof Standard. □

The following result shows that T_c extends T_v.

Proposition 5.8 *For every n,*

1. *$T_c{\uparrow}_n = \Omega(T_v{\uparrow}_n)$,*
2. *$T_c{\downarrow}_n = \Omega(T_v{\downarrow}_n)$.*

Proof Observe that $\emptyset = \Omega(\emptyset)$ and $\mathbf{B}_c = \Omega(\mathbf{B}_v)$. Moreover, it is easy to see that $T_c(\Omega(I)) = \Omega(T_v(I))$. The rest follows by induction on n. □

The following definition allows to obtain usual interpretations from the complete ones.

Definition 5.9 *The function $\mho : \mathcal{I}_c \to \mathcal{P}(\mathbf{B}_v)$ is defined as*

$$\mho(I) = \bigcup_{\alpha\in I} \alpha$$

Lemma 5.10 *For every $I \in \mathcal{I}_v$, $Down(I) = \mho(\Omega(I))$ holds.*

Proof

$\subseteq$) Immediate, since $I \subseteq \mho(\Omega(I))$ and, for every $J \in \mathcal{I}_c$, $Down(\mho(J)) = \mho(J)$.

$\supseteq$) Let $A \in \mho(\Omega(I))$. Then there exists $\alpha \in \Omega(I)$ such that $A \in \alpha = \lim_{n\to\infty} Down(\{A_n\})$ and $\{A_n\}_{n\geq 0} \subseteq I$. By definition of Hausdorff distance, $\exists j\ \forall n \geq j\ [A \in Down(\{A_n\})]$. Hence $A \in Down(I)$. □

Lemma 5.11 *For every n,*

1. *$Down(T_v{\uparrow}_n) = \mho(T_c{\uparrow}_n)$,*
2. *$Down(T_v{\downarrow}_n) = \mho(T_c{\downarrow}_n)$.*

Proof We prove (2). Part (1) is similar.

$$\begin{aligned} Down(T_v{\downarrow}_n) &= \mho(\Omega(T_v{\downarrow}_n)) \text{ (by lemma 5.10)} \\ &= \mho(T_c{\downarrow}_n) \text{ (by proposition 5.8, part 2)} \end{aligned}$$

□

Proposition 5.12 *Let $I_n \in \mathcal{I}_c$ for every n:*

1. *for each chain $\{I_n\}_{n\geq 0}$, $\mho(lub_{n\geq 0} I_n) = \bigcup_{n\geq 0} \mho(I_n)$ holds,*
2. *for each decreasing chain $\{I_n\}_{n\geq 0}$, $\mho(glb_{n\geq 0} I_n) = \bigcap_{n\geq 0} \mho(I_n)$ holds.*

Proof

1. $\subseteq$) Let $A \in \mho(lub_{n\geq 0} I_n)$. Then there exists $\alpha = \lim_{n\to\infty} \alpha_n$ such that $A \in \alpha$, for a sequence $\{\alpha_n\}_{n\geq 0}$ in $I = \bigcup_{n\geq 0} I_n$. Then for some (infinitely many) α_n's we have $A \in \alpha_n$, and therefore

$$A \in \bigcup_{\alpha \in I} \alpha = \bigcup_{n\geq 0} \mho(I_n).$$

   $\supseteq$) Obvious.

2. $\subseteq$) Obvious.

   $\supseteq$) Let $A \in \bigcap_{n\geq 0} \mho(I_n)$. Then for every index n, there exists $\alpha_n \in I_n$ such that $A \in \alpha_n$. Consider the sequence $\{\alpha_n\}_{n\geq 0}$. Because of the compactness of $(\mathbf{B}_c, d)$, there exists a converging subsequence $\{\alpha_{n_j}\}_{j\geq 0}$. Let $\alpha = \lim_{j\to\infty} \alpha_{n_j}$. Then $A \in \alpha$. Moreover, since $\{I_n\}_{n\geq 0}$ is decreasing, and every I_n is complete, $\forall n\ [\alpha \in I_n]$ holds, and therefore $\alpha \in glb_{n\geq 0} I_n$.

□

We state, finally, the main results that relate this new construction to *NGSS* and *NGFF*.

Proposition 5.13

1. $\bigcup_{n\geq 0} Down(T_v\!\uparrow_n) = \mho(lub_{n\geq 0} T_c\!\uparrow_n)$
2. $\bigcap_{n\geq 0} Down(T_v\!\downarrow_n) = \mho(glb_{n\geq 0} T_c\!\downarrow_n)$

Proof We prove (2). Part (1) is analogous.

$$\begin{aligned}
\textstyle\bigcap_{n\geq 0} Down(T_v\!\downarrow_n) &= \textstyle\bigcap_{n\geq 0} \mho(T_v\!\downarrow_n) \text{ (by lemma 5.11, part 2)}\\
&= \mho(\textstyle\bigcap_{n\geq 0} T_v\!\downarrow_n) \text{ (by proposition 5.12, part 2)}
\end{aligned}$$

□

Theorem 5.14 *$NGSS = \mho(lfp(T_c))$.*

Proof

$$\begin{aligned}
NGSS &= Down(lfp(T_v)) \text{ (by theorem 3.1)}\\
&= \textstyle\bigcup_{n\geq 0} Down(T_v\!\uparrow_n) \text{ (by corollary 3.3)}\\
&= \mho(lub_{n\geq 0} T_c\!\uparrow_n) \text{ (by proposition 5.13, part 1)}\\
&= \mho(lfp(T_c)) \text{ (by corollary 5.7, part 1)}
\end{aligned}$$

□

Theorem 5.15 *$NGFF = \mathbf{B}_v \setminus \mho(gfp(T_c))$*

Proof

$$\begin{aligned}
NGFF &= \mathbf{B}_v \setminus \textstyle\bigcap_{n\geq 0} Down(T_v\!\downarrow_n) \text{ (by corollary 3.12)}\\
&= \mathbf{B}_v \setminus \mho(glb_{n\geq 0} T_c\!\downarrow_n) \text{ (by proposition 5.13, part 2)}\\
&= \mathbf{B}_v \setminus \mho(gfp(T_c)) \text{ (by corollary 5.7, part 2)}
\end{aligned}$$

□

6 Conclusions.

In this paper we have studied various properties of the non-ground finitely failed atoms and we have proved that *NGFF* is the complement of the greatest fixpoint of the immediate consequence operator extended to complete Herbrand interpretations.

This is the first step towards an extension to general goals and programs of the approach presented in [FLMP89b].

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