Information-hiding Protocols as Opaque Channels

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Based on joint work with
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Plan of the talk

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- Motivation

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- Statistical inference and Bayesian risk

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- Statistical inference and Bayesian risk
- Conclusion and future work

Information-hiding

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More precise definition @ www.freehaven.net/anonbib/cache/terminology.pdf

Privacy in Global/Pervasive Computing

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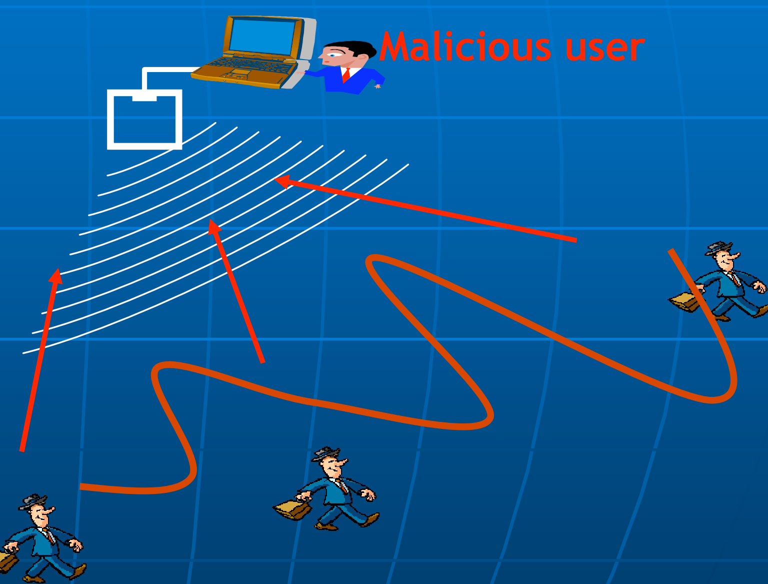
RFID tags may be everywhere...
Artificial leg
model #459382
Faxed hair
model #4456
(polyester)
Bakunin
letters
5 x $100
banknotes
Serial #:
597387,389473...
Lingerie
model Penelope Bras
(Size 4)
Coutesy by: Giuseppe Bianchi

… and at stake

Personal data Gathering

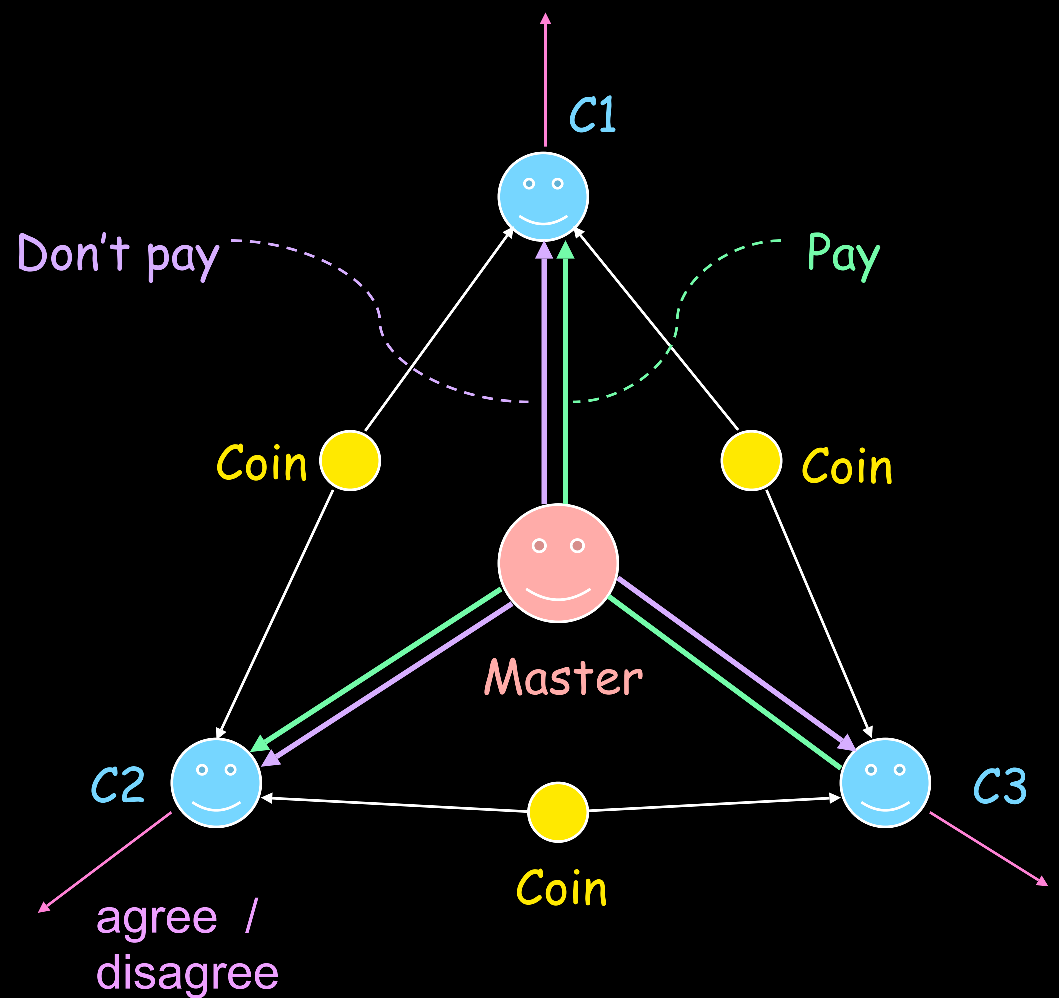
Tracking



Can't kill the RFID tag when dealing with tagged cash…

Courtesy by: Giuseppe Bianchi

Example: the dining cryptographers



$$
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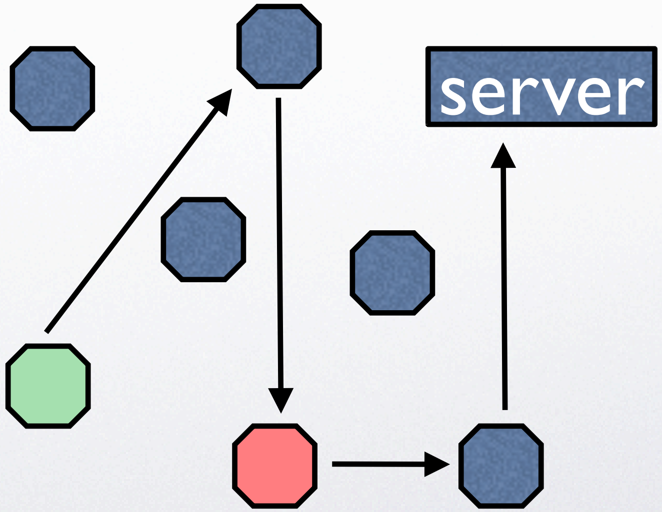
&\qquad \mid \ (\nu \vec{c})(\Pi_{i=0}^{2} Crypt_i \ \mid \ \Pi_{i=0}^{2} Coin_i)\ )
\end{aligned}
$$

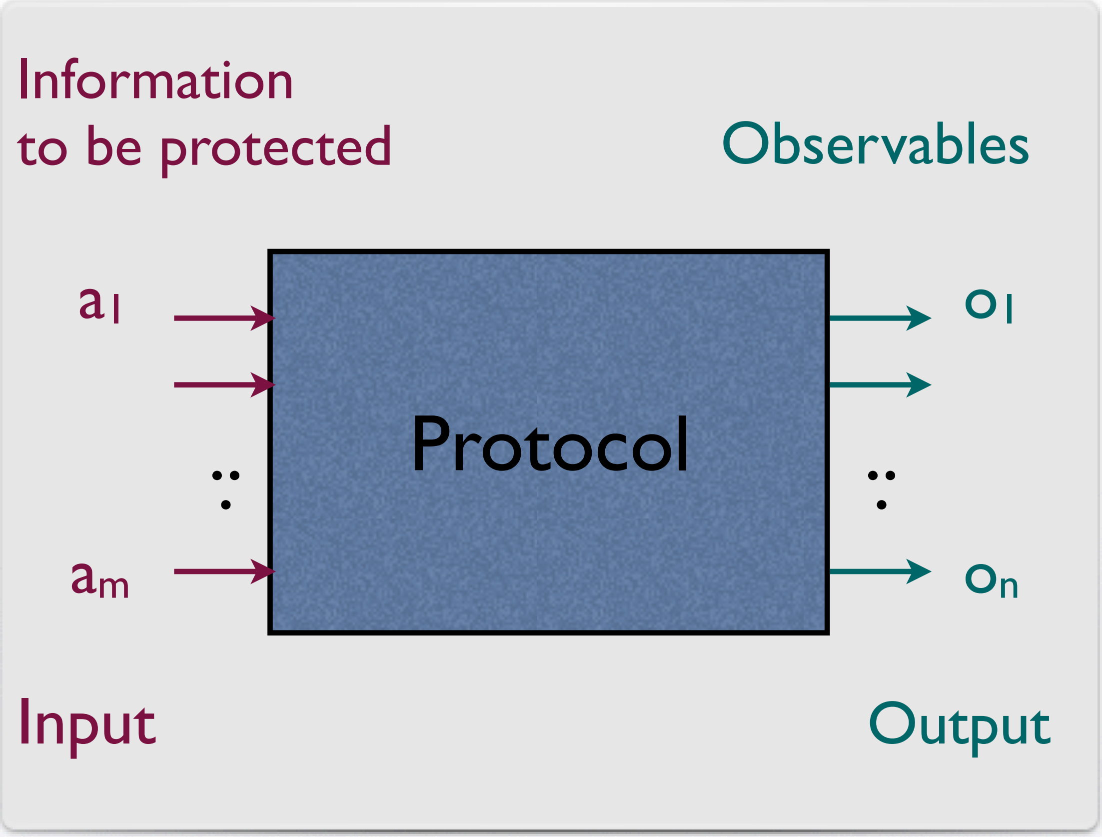
Crowds

- A crowd is a group of n nodes
- The initiator selects randomly a node (called forwarder) and forwards the request to it
- A forwarder:
  - With prob. $1-p_f$ selects randomly a new node and forwards the request to him
  - With prob. p_f sends the request to the server

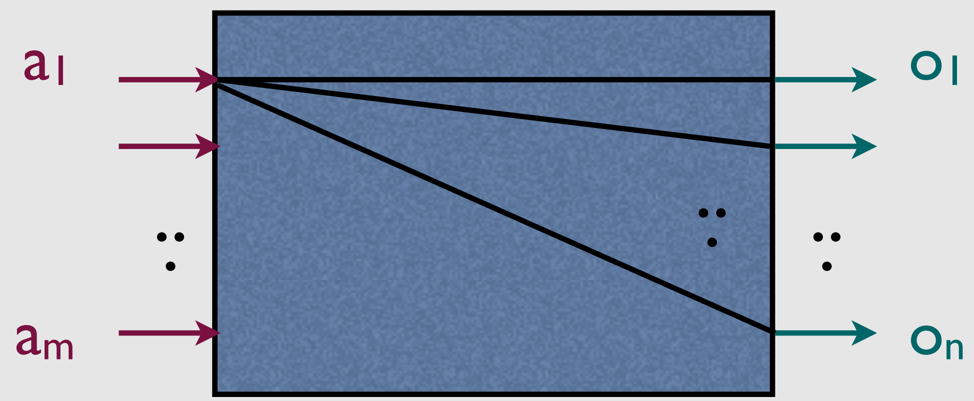


Common features of information-hiding protocols

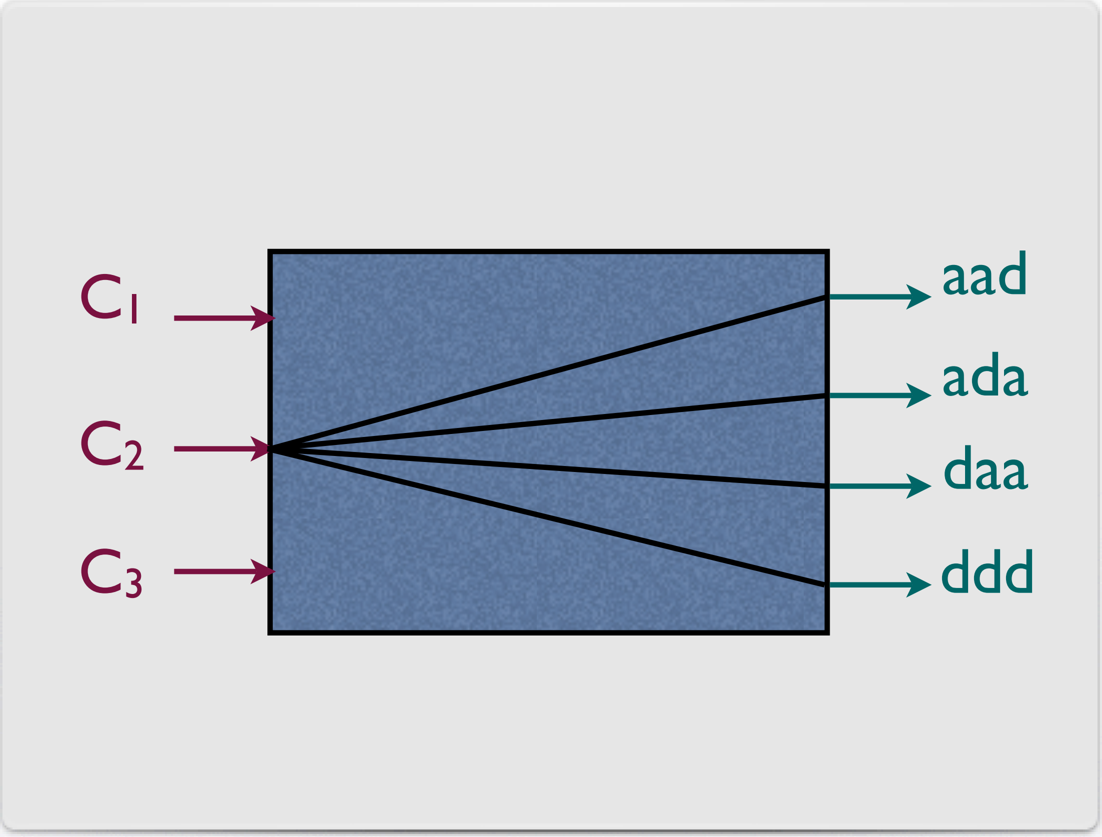
- There is information that we want to keep hidden
  - the user who pays in D.C.
  - the user who initiates the request in Crowds
- There is information that is revealed
  - agree/disagree in D.C.
  - the users who forward messages to a corrupted user in Crowds
- Protocols often use randomization to hide the link between anonymous and observable events
  - coin tossing in D.C.
  - random forwarding in Crowds to a corrupted user in Crowds



Protocols as channels



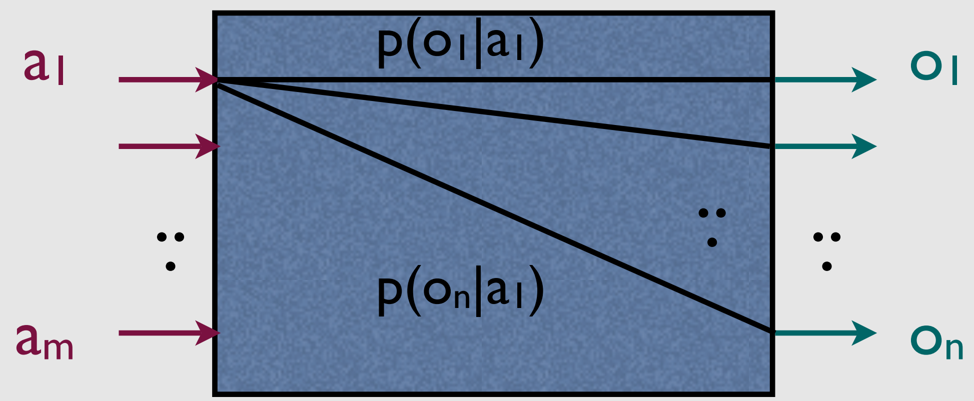
Protocols as noisy channels



The protocol of the dining cryptographers

Protocols as noisy channels

- We consider a probabilistic approach
  - Inputs: elements of a random variable A
  - Outputs: elements of a random variable O
  - For each input a_i, the probability that we obtain an observable o_j is given by $p(o_j \mid a_i)$
- We assume that the protocol receives exactly one input at each session
- We want to define the degree of protection independently from the input's distribution, i.e. the users



The conditional probabilities

Information-hiding protocols as opaque channels

	o_1	...	o_n
a_1	$p(o_1\|a_1)$	...	$p(o_n\|a_1)$
$\vdots$	$\vdots$		
a_m	$p(o_1\|a_m)$		$p(o_n\|a_m)$

The channel is completely characterized by the array of conditional probabilities

Preliminaries of Information Theory

- The entropy $H(A)$ measures the uncertainty about the anonymous events:

$$H(A) = -\sum_{a \in \mathcal{A}} p(a) \log p(a)$$

- The conditional entropy $H(A|O)$ measures the uncertainty about A after we know the value of O (after the execution of the protocol).
- The mutual information $I(A;O)$ measures how much uncertainty about A we lose by observing O:

$$I(A;O) = H(A) - H(A|O)$$

Opacity

- Necessity to give a quantitative measure of the degree of protection provided by a protocol
- We define Opacity as the converse of the Capacity of the channel:

$$C = \max_{p(a)} I(A; O)$$

- Note that this definition is independent from the distribution on the inputs, as desired

Relative privacy

- Some information about A may be revealed intentionally
- Example: elections

- We model the revealed information with a third random variable R

R = number of users who voted for c

Relative privacy

- We use the notion of conditional mutual information

$$I(A;O|R) = H(A|R) - H(A|R,O)$$

- And define the conditional capacity similarly

$$C_R = \max_{p(a)} I(A;O|R)$$

Partitions: a special case of relative privacy

- We say that R partitions $\mathcal{X}$ iff $p(r|x)$ is either 0 or 1 for every r, x
- Examples: elections, group anonymity

Theorem

If R partitions $\mathcal{A}$ and $\mathcal{O}$ then the transition matrix of the protocol is of the form

$$\begin{array}{c|cccc} & \mathcal{O}_1 & \mathcal{O}_2 & \dots & \mathcal{O}_l \\ \hline \mathcal{A}_1 & M_1 & 0 & \dots & 0 \\ \mathcal{A}_2 & 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_l & 0 & 0 & \dots & M_l \end{array}$$

and

$$C_R \leq d \quad \Leftrightarrow \quad C_i \leq d, \forall i \in 1..l$$

where C_i is the capacity of matrix M_i.

Relation with existing notions

Strong probabilistic anonymity

$p(a) = p(a|o) \quad \forall a, o$ — [Chaum, 88], aka "conditional anonymity" [Halpern and O'Neill, 03].

$p(o|a_i) = p(o|a_j) \quad \forall o, i, j$ — [Bhargava and Palamidessi, 05]

Proposition

An anonymity protocol satisfies strong probabilistic anonymity iff $C = 0$.

Example: Dining cryptographers

	100	010	001	111
a_1	$1/4$	$1/4$	$1/4$	$1/4$
a_2	$1/4$	$1/4$	$1/4$	$1/4$
a_3	$1/4$	$1/4$	$1/4$	$1/4$

How to compute the capacity of the channel associated to a protocol

- Express the protocol in your favorite formalism
- Establish the anonymous events (inputs) and the observable events (outputs)
- The matrix of the channel (i.e. the conditional probabilities) is completely determined by the protocol and can be computed either by hand or by model checking
- The capacity is completely determined by the matrix and can be approximated by using the Arimoto-Blahut algorithm. In some particular cases is given by a formula

Example: D.C. in the probabilistic asynchronous π-calculus

$$\begin{aligned}
Master &= \textstyle\sum_{i=0}^{2} \tau \,.\, \overline{m}_i \mathsf{p} \,.\, \overline{m}_{i\oplus 1} \mathsf{n} \,.\, \overline{m}_{i\oplus 2} \mathsf{n} \,.\, 0 \\
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Nondeterministic choice

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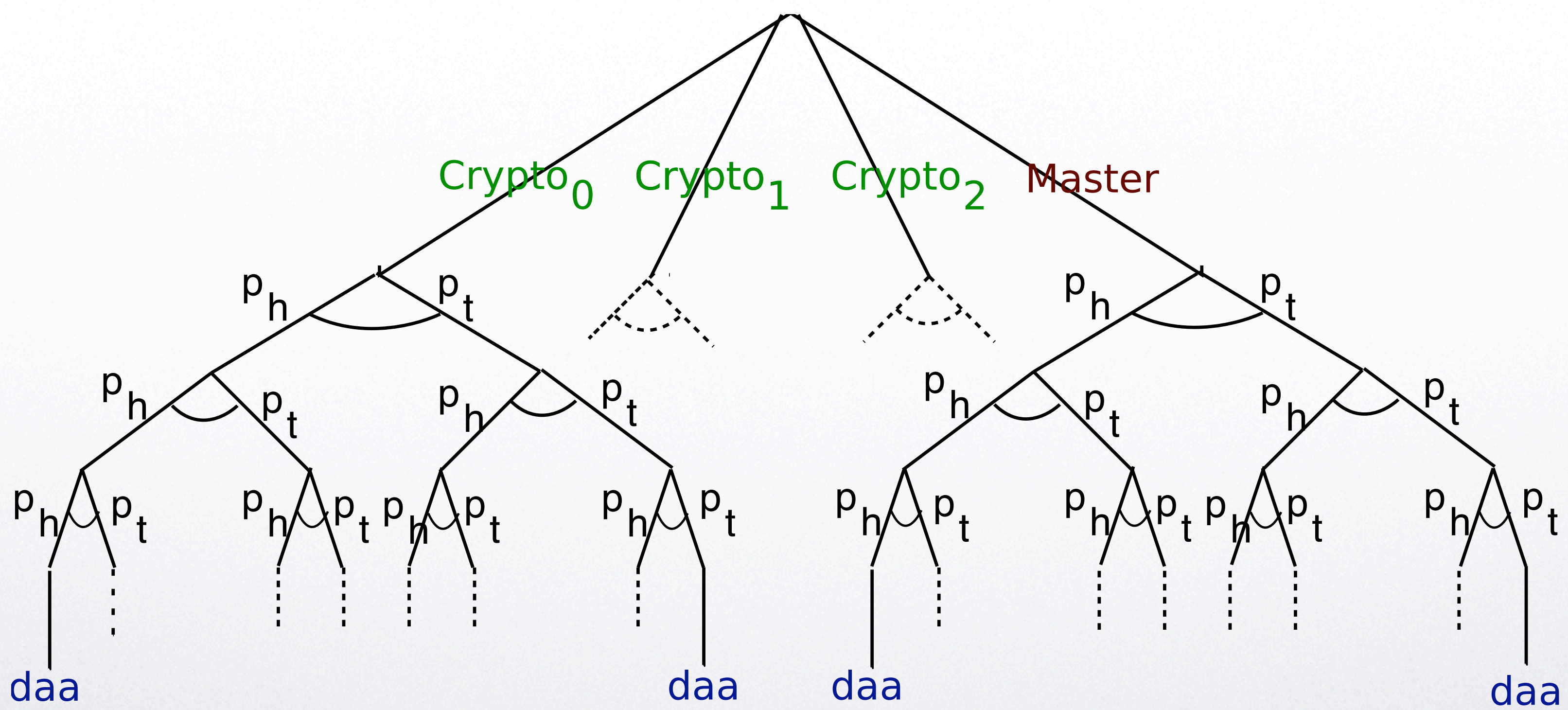
Probabilistic automaton associated to the probabilistic π program for the D.C.



Examples of channel matrices

- Dining cryptographers, while varying the probability p of the coins to give heads

  - p = 0.5

	daa	ada	aad	ddd	aaa	dda	dad	add
c_1	1/4	1/4	1/4	1/4	0	0	0	0
c_2	1/4	1/4	1/4	1/4	0	0	0	0
c_3	1/4	1/4	1/4	1/4	0	0	0	0
m	0	0	0	0	1/4	1/4	1/4	1/4

  - p = 0.7

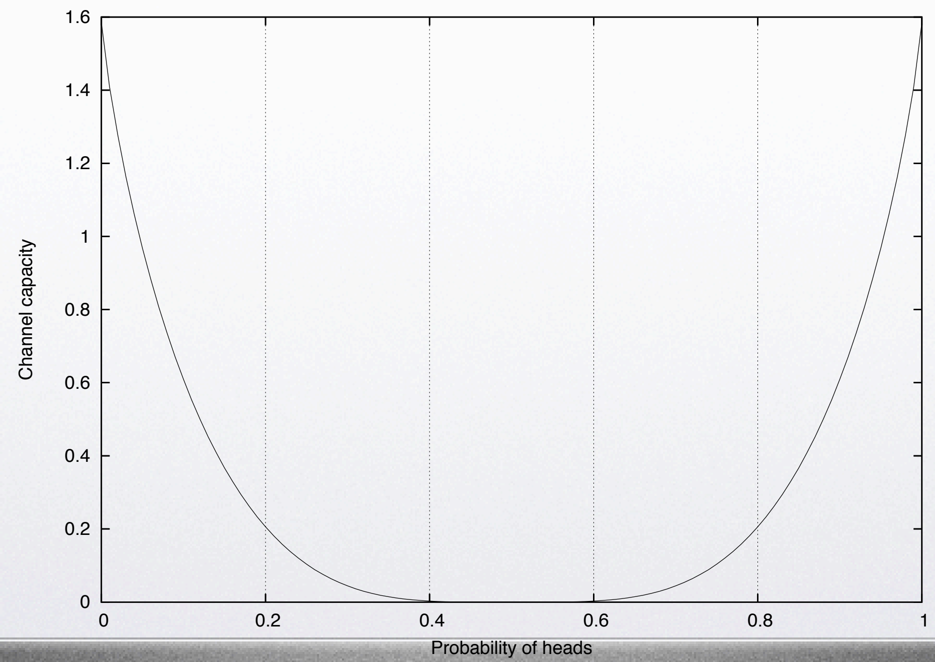
	daa	ada	aad	ddd	aaa	dda	dad	add
c_1	0.37	0.21	0.21	0.21	0	0	0	0
c_2	0.21	0.37	0.21	0.21	0	0	0	0
c_3	0.21	0.21	0.37	0.21	0	0	0	0
m	0	0	0	0	0.37	0.21	0.21	0.21

Computing the capacity from the matrix

- General case: using the Arimoto-Blahut algorithm
  - Approximates the capacity to a given precision
- In particular cases we can exploit the protocol's symmetries
  - Symmetric channel: all rows and all columns are permutations of each other
  - In a symmetric channel: $C = \log|\mathcal{O}| - H(\mathbf{r})$
  - Can be extended to weaker notions of symmetry

Test-case: dining cryptographers

- **Fair coins:** the protocol is strongly anonymous (C=0)
- **Totally biased coins:** the payer can be always identified (maximum capacity C = log 3)



Privacy and Statistical Inference

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  Ok, it seems 'reasonable'.
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- An uncontroversially natural notion is be the 'probability of error' of an adversary trying to infer the hidden information (input) from the observables (output)

Statistical inference

- $\boldsymbol{o} = o_1, o_2, \ldots, o_n$: a sequence of n observations
- f : the function used by the adversary to infer the input from a sequence of observations
- Error region of f for input a: $E_f(a) = \{\boldsymbol{o} \in \mathcal{O}^n \mid f(\boldsymbol{o}) \neq a\}$
- Probability of error for input a: $\eta(a) = \sum_{\boldsymbol{o} \in E_f(a)} p(\boldsymbol{o}|a)$
- Probability of error for f :

$$P_{f_n} = \sum_{a \in A} p(a)\eta(a)$$

MAP decision functions

- *MAP: Maximum Aposteriory Probability*
- Applicable when the input's distribution is known.
  Use Bayes theorem:

$$p(a \mid o) = (p(o \mid a)\, p(a)) / p(o)$$

- f is a MAP decision function if $f(o) = a$ implies

$$p(o \mid a)\, p(a) \;>=\; p(o \mid a')\, p(a') \qquad \text{for all } a, a' \text{ and } o$$

- **Proposition:** the MAP decision functions minimize the probability of error (which in this case is called Bayesian risk)

Independence from the input distribution

- Under certain conditions, for large sequences of observations the input distribution becomes negligible:
- **Proposition:** A MAP decision function f can be approximated by a function g such that $g(o) = a$ implies

$$p(o \mid a) > p(o \mid a') \qquad \text{for all } a, a' \text{ and } o$$

- "approximated" means that the more observations we make, the smaller is the difference in the error probability of f and g

Bayesian Risk and Information Theory

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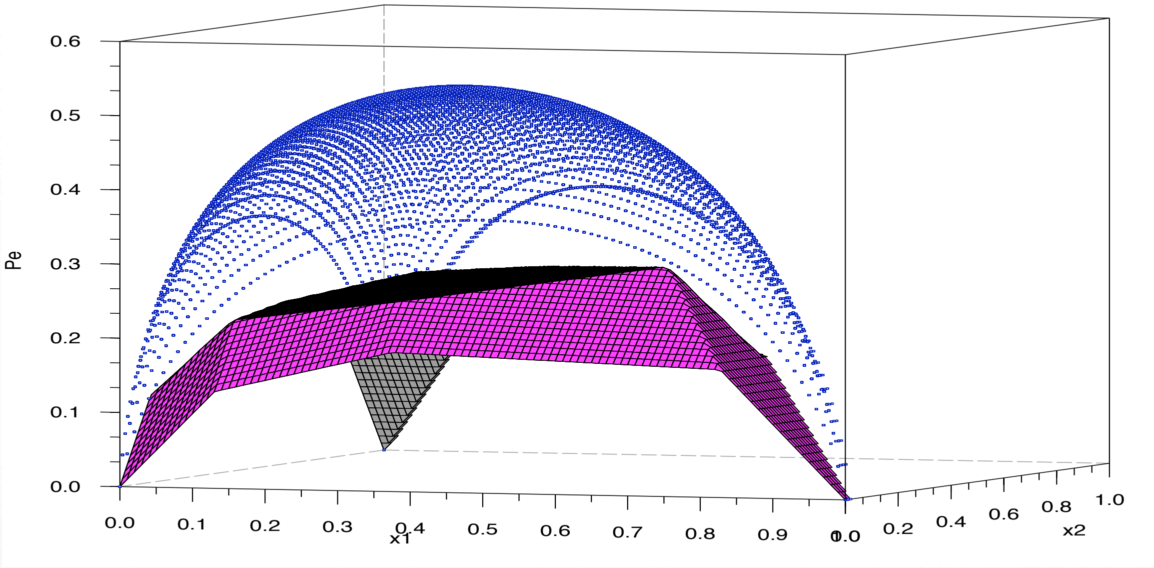
- Philosophical and practical motivations



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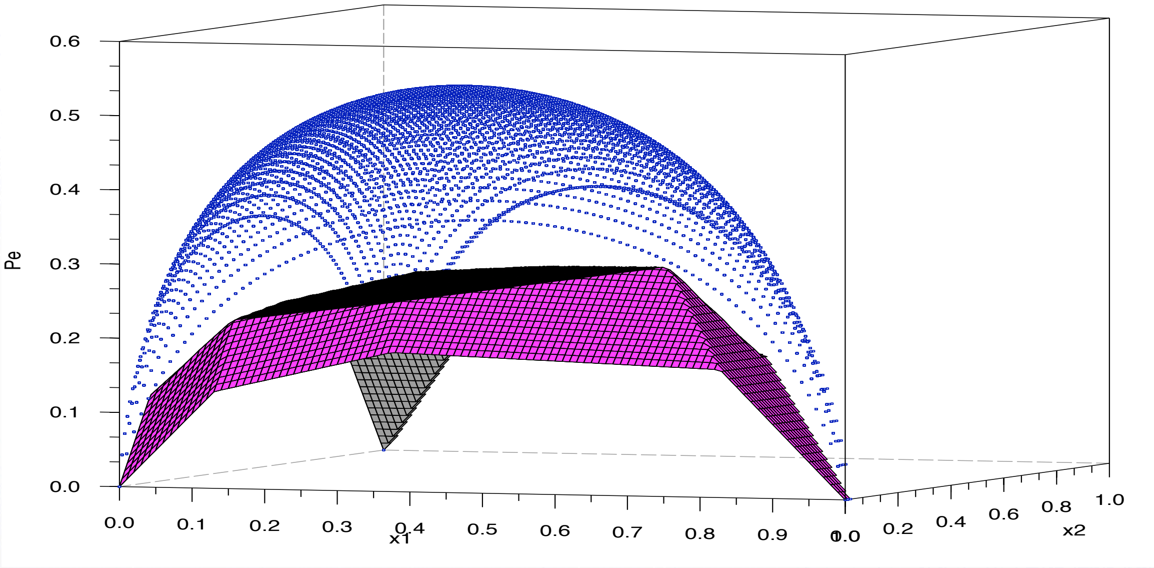

- Relation with Conditional Entropy $H(A|O)$

Bayesian Risk and Information Theory

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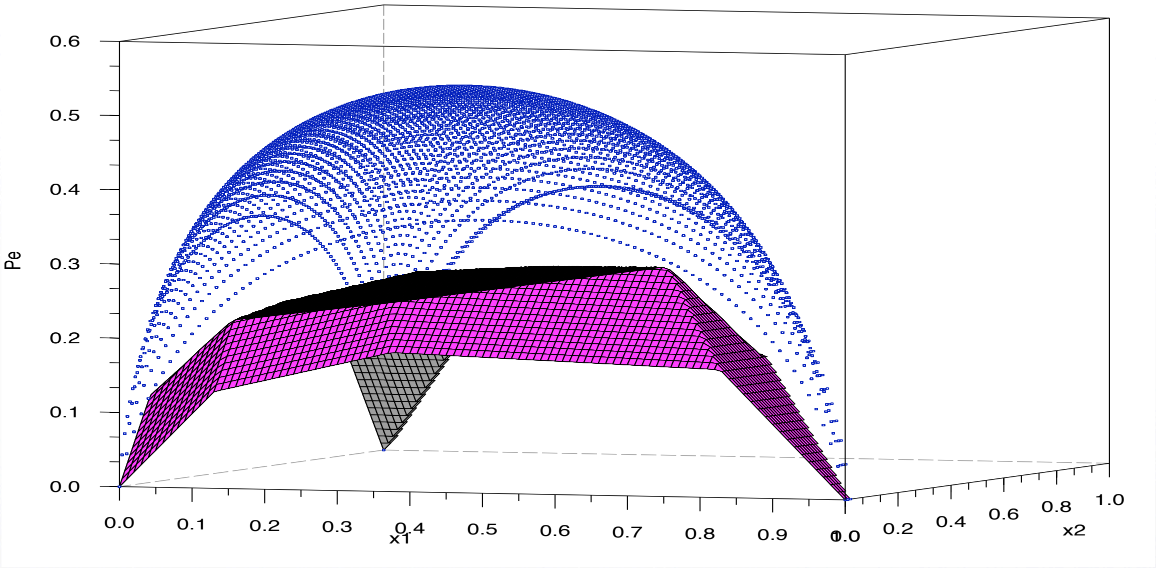

- Philosophical and practical motivations
- Relation with Conditional Entropy   H(A|O)
- Bounds by Rény ’66, Hellman-Raviv ’70, Santhi-Vardy ‘06

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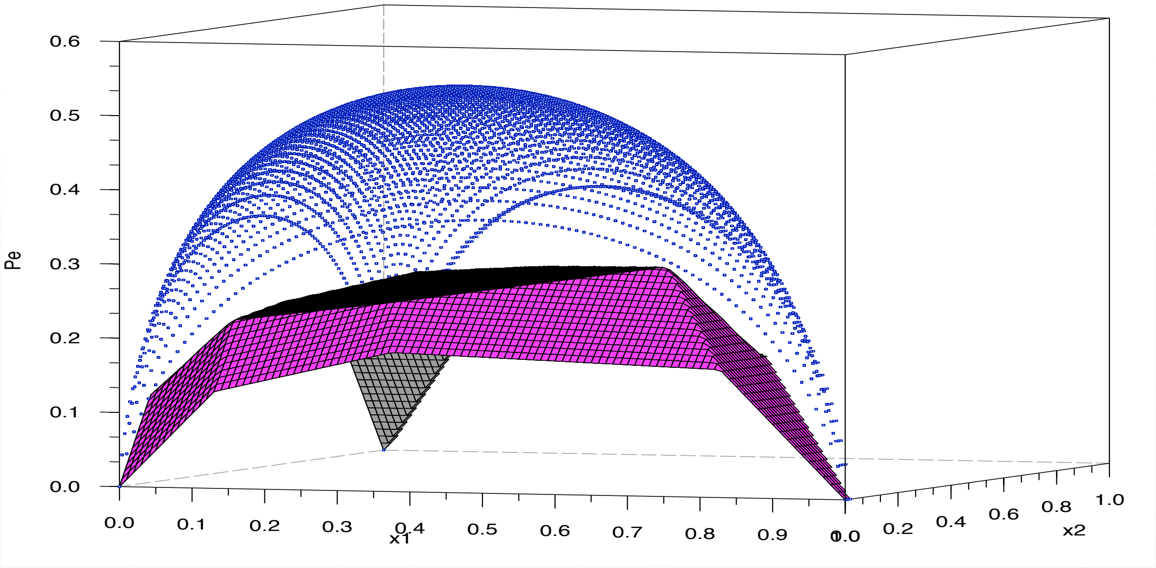

- Relation with Conditional Entropy   H(A|O)
- Bounds by Rény ’66, Hellman-Raviv ’70, Santhi-Vardy ‘06
- Tighter bound obtained by studying the ‘corner points’

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- $p(a|o)$ vs $H(A|O)$

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- $p(a|o)$ vs $H(A|O)$
- $p(a|o) / p(a)$ vs $H(A|O) - H(A)$?

Future work

- Explore more in depth the relation between the capability of inferring info about the input and the capacity, or other quantitative notions depending on the channel's matrix.
- Inference of the input distribution without the power of forcing the input to remain the same through the observations
- Characterizations of other (weaker) notions of privacy which are easy to model check, in the sense that they do not require to analyze the capacity as a function of the input distribution
- Develop a logic for efficient model checking

Thank you !