

Computing with Polynomial Ordinary Differential Equations

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Abstract

In 1941, Claude Shannon introduced the General Purpose Analog Computer (GPAC) as a mathematical model of Differential Analysers, that is to say as a model of continuous-time analog (mechanical, and later on electronic) machines of that time.

Following Shannon's arguments, functions generated by the GPAC must satisfy a polynomial differential algebraic equation (DAE). As it is known that some computable functions like Euler's $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ or Riemann's Zeta function $\zeta(x) = \sum_{k=0}^\infty \frac{1}{k^x}$ do not satisfy any polynomial DAE, this argument has often been used to demonstrate in the past that the GPAC is less powerful than digital computation.

It was proved in (Bournez, Campagnolo, Graça, and Hainry, 2007), that if a more modern notion of computation is considered, i.e. in particular if computability is not restricted to real-time generation of functions, the GPAC is actually equivalent to Turing machines.

Our purpose is first to discuss the robustness of the notion of computation involved in (Bournez et al., 2007), by establishing that many natural variants of the notion of computation from this paper lead to the same computability result.

Second, to go from these computability results towards considerations about (time) complexity: we explore several natural variants for measuring time/space complexity of a computation.

Quite surprisingly, whereas defining a robust time complexity for general continuous time systems is a well known open problem, we prove that all variants are actually equivalent even at the complexity level. As a consequence, it seems that a robust and well defined notion of time complexity exists for the GPAC, or equivalently for computations by polynomial ordinary differential equations.

Another side effect of our proof is also that we show in some way that polynomial

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ordinary differential equations can actually be used as a kind of programming model, and that there is a rather nice and robust notion of ordinary differential equation (ODE) programming.

Keywords: Analog Computation, Continuous-Time Computations, General Purpose Analog Computer, Real Computations

1. Introduction

Claude Shannon introduced in (Shannon, 1941) the General Purpose Analog Computer (GPAC) as a model for Differential Analysers (Bush, 1931), which are mechanical (and later on electronic) continuous time analog machines, on which he worked as an operator. The model was later refined in (Pour-El, 1974), (Graça and Costa, 2003). It was originally presented by Shannon as a model based on circuits. Basically, a GPAC is any circuit (loops are allowed³) that can be built from the 4 basic units of Figure 1, which implement constants, addition, multiplication and integration, all of them working over analog real quantities (that were corresponding to angles in the mechanical Differential Analysers, and later on to voltages in the electronic versions). Note that the set of allowed constants will generally be restricted, for example to rational numbers, to avoid pathological issues. Given such a circuit, the function which gives the value of every wire (or a subset of the wires) over time is said to be *generated* by the circuit. In Definition 11, we consider an extension of this notion.

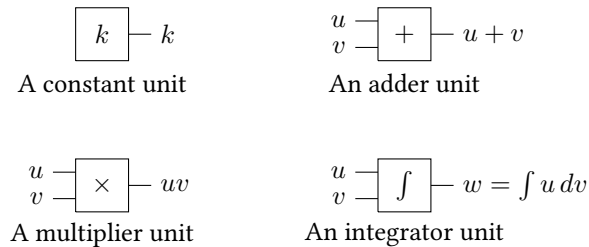


Figure 1: Circuit presentation of the GPAC: a circuit built from basic units. Presentation of the 4 types of units: constant, adder, multiplier, and integrator.

An important aspect of this model is that despite the apparent simplicity of its basic blocks, sophisticated functions can easily be generated. Figure 2 illustrates how the sine function can be generated using two integrators, with suitable initial states. Incidentally, the sine function is also the solution of a very simple ordinary differential equation. Shannon itself realized that functions generated by a GPAC are nothing more than solutions of a special class of polynomial differential equations. In particular it can be shown that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is generated by Shannon's model

³There are some syntactic restrictions to avoid ill-defined circuits.

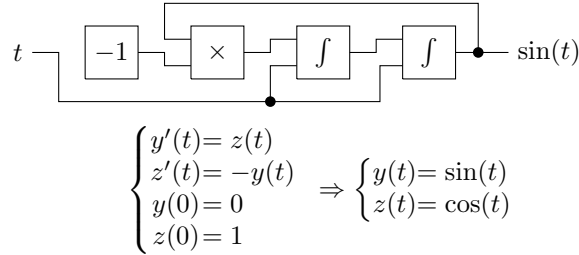


Figure 2: Example of a GPAC circuit computing the sine and cosine.

(Shannon, 1941), (Graça and Costa, 2003) if and only if it is a (component of the) solution of a polynomial initial value problem (PIVP) of the form:

$$\begin{cases} y'(t) = p(y(t)) \\ y(t_0) = y_0 \end{cases}, \quad t \in \mathbb{R} \quad (1)$$

where p is a vector of polynomials and $y(t)$ is vector. In other words, $f(t) = y_1(t)$, and $y'_i(t) = p_i(y(t))$ where p_i is a multivariate polynomial.

Intuitively, the link between a GPAC and a PIVP is the following: the idea is just to introduce a variable for each output of a basic unit, and write the corresponding ordinary differential equation (ODE), and observe that it can be written as an ODE with a polynomial right hand side.

While many of the usual real functions are known to be generated by a GPAC, a notable exception is Euler's Gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ function or Riemann's Zeta function $\zeta(x) = \sum_{k=0}^\infty \frac{1}{k^x}$ (Shannon, 1941), (Pour-El and Richards, 1989), which are known not to satisfy any polynomial DAE, i.e. they are not solutions of a system of the form (1). If we have in mind that these functions are known to be computable under the computable analysis framework (Pour-El and Richards, 1989), (Weihrauch, 2000) the previous result has long been interpreted as evidence that the GPAC is a somewhat weaker model than computable analysis.

In 2007, it was proved that this is more an artifact of the notion of real-time generation considered by Shannon than a true consideration about the computational power of the model. Indeed, Shannon assumes the GPAC computes in "real time" - a very restrictive form of computation: at time t the output of the machine must be $\Gamma(t)$. If we change this notion of computability to the kind of "converging computation" used in recursive analysis, or in modern computability theory, then the Γ function becomes computable (Graça, 2004), and more generally all functions over a bounded domain, computable in the sense of computable analysis, are actually GPAC computable (and conversely) (Bournez et al., 2007). The idea used in (Graça, 2004), (Bournez et al., 2007) to compute a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is to define a polynomial initial-value problem (PIVP) (1) such that the argument x of f is provided to the PIVP via the initial condition, and the system has a component which converges to $f(x)$. Moreover, the convergence rate of the component to $f(x)$ is known and we know exactly how much time we have to wait to get a desired accuracy when computing

$f(x)$. More precisely, the following was proved:

Definition 1 (GPAC computable function). $f : \mathbb{R} \rightarrow \mathbb{R}$ is called *GPAC-computable* if there are polynomials p and q with computable coefficients such that for any $x \in \mathbb{R}$, there exists (a unique) $y : I \rightarrow \mathbb{R}^d$ satisfying for all $t \in \mathbb{R}_+$:

- $y(0) = q(x)$ and $y'(t) = p(y(t))$ ▶ y satisfies a PIVP
- if $t \geq 1$ then $|y_1(t) - f(x)| \leq e^{-t}$ ▶ y_1 converges to $f(x)$

Proposition 2 ((Bournez et al., 2007)). *Let a and b be some computable reals. A function $f : [a, b] \rightarrow \mathbb{R}$ is computable⁴ if and only if it is GPAC-computable.*

In this paper our purpose is twofold: first explore natural variations on the notion of computability presented in Definition 1 and, second, go towards complexity theory and not only computability theory, by introducing some natural ways to measure complexity.

It is important to understand that talking about time complexity for continuous-time systems is known to be a non-trivial issue. Indeed, defining a robust (time) complexity notion for continuous time systems is a well known open problem (Bournez and Campagnolo, 2008) with no generic solution provided at this day. In short, the difficulty is that the naive idea of using the time variable of the ODE as a measure of “time complexity” is problematic, since time can be arbitrarily contracted in a continuous system due to the “Zeno phenomena” (e.g. by using functions like arctan which contract the whole real line into a bounded set). It follows that all computable languages can then be computed by a continuous system in time $O(1)$ (see e.g. (Ruohonen, 1993), (Ruohonen, 1994), (Moore, 1996), (Bournez, 1997), (Bournez, 1999), (Alur and Dill, 1990), (Calude and Pavlov, 2002), (Davies, 2001), (Copeland, 1998), (Copeland, 2002)).

Two first natural quantities will be considered: first, the time variable of the ordinary differential equation, that we will sometimes call *time*, and a bound on the norm of the involved variables, that we will sometimes call *space*.

As a reparameterization of the time variable of an ordinary differential equation leads to a new ordinary differential equation with the same solution curve, but which is traveled along time at a different speed, a natural idea is to try to consider quantities that are kept invariant by reparameterization. A natural choice for such quantity is the length of the curve. We recall that the length of a curve $y \in C^1(I, \mathbb{R}^n)$ defined over some interval $I = [a, b]$ is given by $\text{len}_y(a, b) = \int_I \|y'(t)\| dt$. Definition 1 leads then naturally to consider the following natural two variants of computability of functions over \mathbb{R}^n given below.

Given $x \in \mathbb{R}^n$, we write x_i for the i^{th} component of x and $x_{i..j}$ for the vector $(x_i, x_{i+1}, \dots, x_j)$. \mathbb{R}_P denotes the set of polynomial-time computable reals (Weihrauch, 2000). $\mathbb{K}[\mathbb{R}^n]$ denotes polynomial functions with n variables and with coefficients in \mathbb{K} , where variables live in \mathbb{R}^n and $\mathbb{R}_+ = [0, +\infty[$. In this document, $f : \subseteq X \rightarrow Y$

⁴In the classical sense, i.e. in the sense of computable analysis.

denotes a partial function, *i.e.* $f : Z \rightarrow Y$ where $X \subseteq Z$. We also take $\sup_{\delta} f(t) = \sup_{u \in [t, t-\delta] \cap \mathbb{R}_+} f(t)$. The intuition is that in Definition 1 we can reparameterize the time variable, but this will happen at the cost of space. Hence, if we want to know how many resources are needed to compute $f(x)$ with some accuracy μ , we should measure not only the time but also the space needed to obtain this accuracy. This is done in Definition 5, while Definition 4 is a variant which, instead of using of measuring accuracy against time and space, does this by measuring accuracy against the length of the solution curve needed to achieve that accuracy. Figures 3 and 4 illustrate those definitions.

Remark 3 (The space \mathbb{K} of the coefficients). In this paper, the coefficients of all considered polynomials will belong to \mathbb{K} . Formally, \mathbb{K} needs to be a *generable field*, as introduced in (Bournez, Graça, and Pouly, 2016). However, without a significant loss of generality, the reader can consider that

$$\mathbb{K} = \mathbb{R}_P$$

which is the set of polynomial time computable real numbers. All the reader needs to know about \mathbb{K} is that it is a field and it is stable by generable functions (introduced in Section 2), meaning that if $\alpha \in \mathbb{K}$ and f is generable then $f(\alpha) \in \mathbb{K}$. It is shown in (Bournez et al., 2016) that there exists a small generable field \mathbb{R}_G lying somewhere between \mathbb{Q} and \mathbb{R}_P , with probable strict inequality on both sides.

We now get to our first and main notion of computable function:

Definition 4 (Analog Length Computability). Let $n, m \in \mathbb{N}$, $f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\Omega : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$. We say that f is Ω -length-computable if and only if there exist $d \in \mathbb{N}$, and $p \in \mathbb{K}^d[\mathbb{R}^d]$, $q \in \mathbb{K}^d[\mathbb{R}^n]$ such that for any $x \in \text{dom } f$, there exists (a unique) $y : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ satisfying for all $t \in \mathbb{R}_+$:

- $y(0) = q(x)$ and $y'(t) = p(y(t))$ ► y satisfies a PIVP
- for any $\mu \in \mathbb{R}_+$, if $\text{len}_y(0, t) \geq \Omega(\|x\|, \mu)$ then $\|y_{1..m}(t) - f(x)\| \leq e^{-\mu}$
► $y_{1..m}$ converges to $f(x)$
- $\|y'(t)\| \geq 1$ ► technical condition: the length grows at least linearly with time⁵

We denote by $\text{ALC}(\Omega)$ the set of Ω -length-computable functions, and by ALP the set of (poly)-length-computable functions, and more generally by ALC the length-computable functions (for some Ω).

Definition 5 (Analog Time-Space computability). Let $n, m \in \mathbb{N}$, $f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\Upsilon, \Omega : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$. We say that f is (Υ, Ω) -time-space-computable if and only if there exist $d \in \mathbb{N}$, and $p \in \mathbb{K}^d[\mathbb{R}^d]$, $q \in \mathbb{K}^d[\mathbb{R}^n]$ such that for any $x \in \text{dom } f$, there exists (a unique) $y : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ satisfying for all $t \in \mathbb{R}_+$:

⁵This is a technical condition required for the proof. This can be weakened, for example to $\|p(y(t))\| \geq \frac{1}{\text{poly}(t)}$. The technical issue is that if the speed of the system becomes extremely small, it might take an exponential time to reach a polynomial length, and we want to avoid such “unnatural” cases.

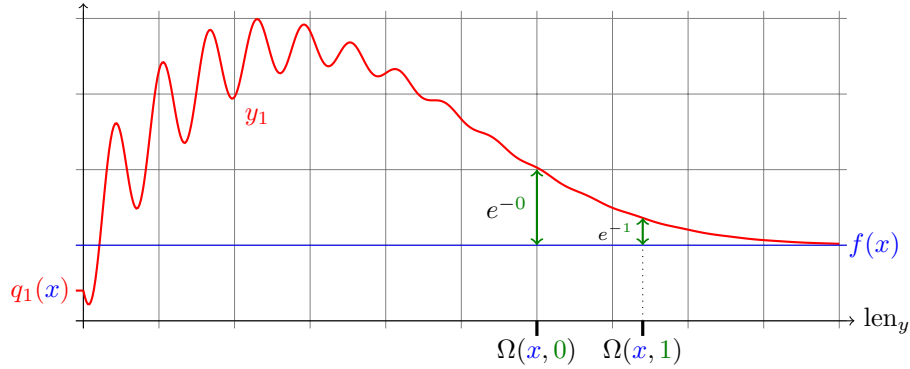


Figure 3: $\text{ALC}(\Omega)$: on input x , starting from initial condition $q(x)$, the PIVP $y' = p(y)$ ensures that $y_1(t)$ gives $f(x)$ with accuracy better than $e^{-\mu}$ as soon as the length of y (from 0 to t) is greater than $\Omega(\|x\|, \mu)$. Note that we did not plot the other variables y_2, \dots, y_d and the horizontal axis measures the length of y (instead of the time t).

- $y(0) = q(x)$ and $y'(t) = p(y(t))$ ▶ y satisfies a PIVP
- for all $\mu \in \mathbb{R}_+$, if $t \geq \Omega(\|x\|, \mu)$ then $\|y_{1..m}(t) - f(x)\| \leq e^{-\mu}$ ▶ $y_{1..m}$ converges to $f(x)$
- $\|y(t)\| \leq \Upsilon(\|x\|, t)$, for all $t \geq 0$ ▶ $y(t)$ is bounded

We denote by $\text{ATSC}(\Upsilon, \Omega)$ the set of (Υ, Ω) -time-space-computable functions, by ATSP the set of (poly, poly)-time-space-computable functions, and by ATSC the set of time-space-computable functions.

Indeed, Proposition 2 can be reformulated as:

Proposition 6. *Let a and b be some computable reals. A function $f : [a, b] \rightarrow \mathbb{R}$ is computable⁶ if and only if it is length-computable if and only if it is time-space-computable.*

More surprisingly, we prove that it turns out that both classes are the same, even at the complexity level.

Theorem 7. $\text{ALP} = \text{ATSP}$.

This turns out surprisingly to also be equivalent with many variants, both at the computability and complexity level.

For example, the error could also be given as input, via an initial condition. The intuition behind the following definition is that the initial condition also depends on the accuracy μ . Hence, instead of what happens in Definition 5, we are not guaranteed that a component converges to $f(x)$, only that it stays in a $e^{-\mu}$ -vicinity of $f(x)$ after some time, and that the space used is bounded.

⁶In the classical sense, i.e. in the sense of computable analysis.

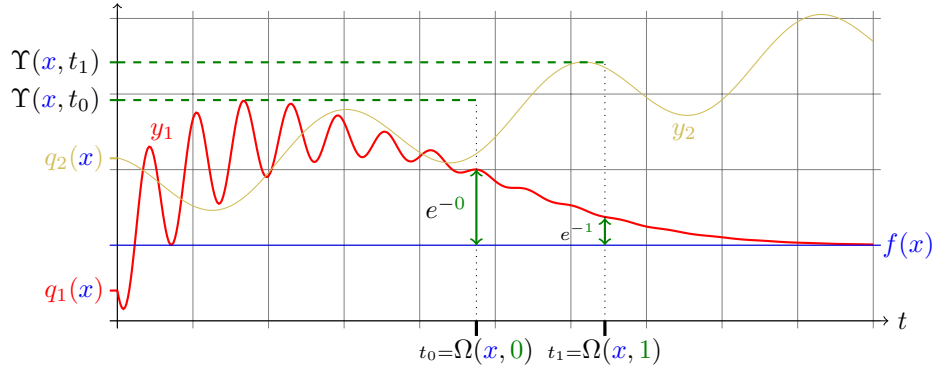


Figure 4: ATSC(Υ, Ω): on input x , starting from initial condition $q(x)$, the PIVP $y' = p(y)$ ensures that $y_1(t)$ gives $f(x)$ with accuracy better than $e^{-\mu}$ as soon as the time t is greater than $\Omega(\|x\|, \mu)$. At the same time, all variables y_j are bounded by $\Upsilon(\|x\|, t)$. Note that variables y_2, \dots, y_d need not converge to anything.

Definition 8 (Analog weak computability). Let $n, m \in \mathbb{N}$, $f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\Omega : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ and $\Upsilon : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$. We say that f is (Υ, Ω) -weakly-computable if and only if there exist $d \in \mathbb{N}$, $p \in \mathbb{K}^d[\mathbb{R}^d]$, $q \in \mathbb{K}^d[\mathbb{R}^{n+1}]$ such that for any $x \in \text{dom } f$ and $\mu \in \mathbb{R}_+$, there exists (a unique) $y : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ satisfying for all $t \in \mathbb{R}_+$:

- $y(0) = q(x, \mu)$ and $y'(t) = p(y(t))$ ► y satisfies a PIVP
- if $t \geq \Omega(\|x\|, \mu)$ then $\|y_{1..m}(t) - f(x)\| \leq e^{-\mu}$ ► $y_{1..m}$ approximates $f(x)$
- $\|y(t)\| \leq \Upsilon(\|x\|, \mu, t)$ ► $y(t)$ is bounded

We denote by $\text{AW}(\Upsilon, \Omega)$ the set of (Υ, Ω) -weakly-computable functions, by AWP the set of (poly, poly)-weakly-computable functions, and by AWC the set of weakly-computable functions.

Or we could consider a notion of online-computation, the intuition behind it being that if some external input $x(t)$ approaches a value \bar{x} sufficiently close, then by waiting enough time, and assuming that the external input stays near the value \bar{x} during that time interval, we will get an approximation of $f(\bar{x})$ with some desired accuracy. This process is illustrated in Figure 5. By constantly changing the external input $x(t)$ and “locking it” during some time near some value, we are able to compute approximations of $f(x)$ for several arguments in a single “run” of the GPAC.

Definition 9 (Online computability). Let $n, m \in \mathbb{N}$, $f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\Upsilon, \Omega, \Lambda : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$. We say that f is $(\Upsilon, \Omega, \Lambda)$ -online-computable if and only if there exist $\delta \geq 0$, $d \in \mathbb{N}$ and $p \in \mathbb{K}^d[\mathbb{R}^d \times \mathbb{R}^n]$ and $y_0 \in \mathbb{K}^d$ such that for any $x \in C^0(\mathbb{R}_+, \mathbb{R}^n)$, there exists (a unique) $y : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ satisfying for all $t \in \mathbb{R}_+$:

- $y(0) = y_0$ and $y'(t) = p(y(t), x(t))$
- $\|y(t)\| \leq \Upsilon(\sup_{\delta} \|x\| (t), t)$

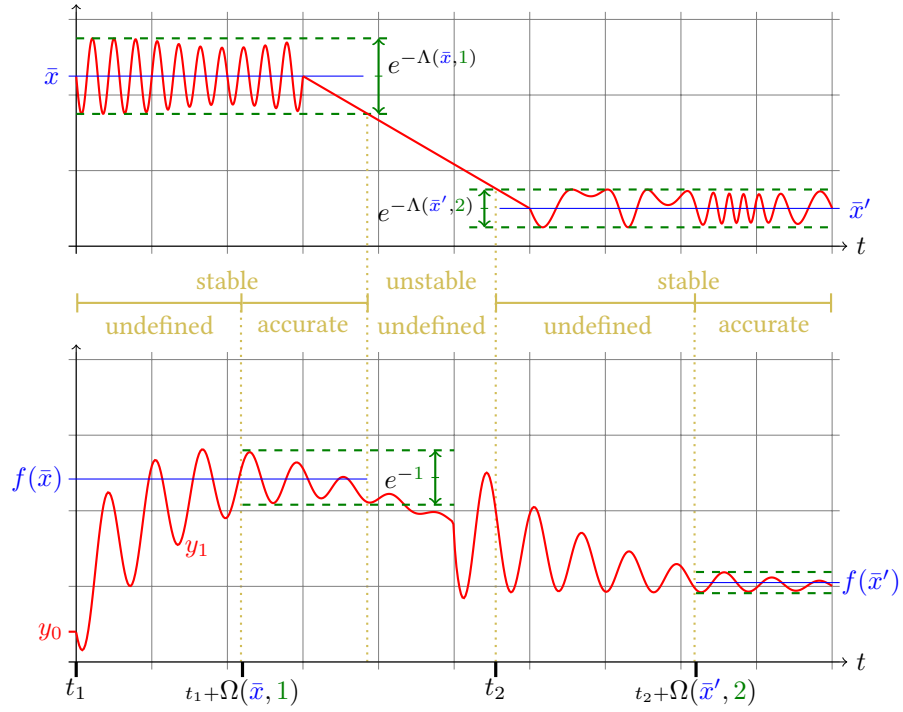


Figure 5: $\text{AOC}(\Upsilon, \Omega, \Lambda)$: starting from the (constant) initial condition y_0 , the PIVP $y'(t) = p(y(t), x(t))$ has two possible behaviors depending on the input signal $x(t)$. If $x(t)$ is unstable, the behaviour of the PIVP $y'(t) = p(y(t), x(t))$ is undefined. If $x(t)$ is stable around \bar{x} with error at most $e^{-\Lambda(\|\bar{x}\|, \mu)}$ then $y(t)$ is initially undefined, but after a delay of at most $\Omega(\|\bar{x}\|, \mu)$, $y_1(t)$ gives $f(\bar{x})$ with accuracy better than $e^{-\mu}$. In all cases, all variables $y_j(t)$ are bounded by a function (Υ) of the time t and the supremum of $\|x(u)\|$ during a small time interval $u \in [t - \delta, t]$.

- for any $I = [a, b] \subseteq \mathbb{R}_+$, if there exist $\bar{x} \in \text{dom } f$ and $\bar{\mu} \geq 0$ such that for all $t \in I$, $\|x(t) - \bar{x}\| \leq e^{-\Lambda(\|\bar{x}\|, \bar{\mu})}$ then $\|y_{1..m}(u) - f(\bar{x})\| \leq e^{-\bar{\mu}}$ whenever $a + \Omega(\|\bar{x}\|, \bar{\mu}) \leq u \leq b$.

We denote by $\text{AOC}(\Upsilon, \Omega, \Lambda)$ the set of $(\Upsilon, \Omega, \Lambda)$ -online-computable, by AOP the set of (poly, poly, poly)-online-computable functions and by AOC the set of online-computable functions.

Theorem 10. *All notions of computations are equivalent, both at the computability level:*

$$\text{ALC} = \text{ATSC} = \text{AWC} = \text{AOC}$$

and at the complexity level:

$$\text{ALP} = \text{ATSP} = \text{AWP} = \text{AOP}$$

The rest of the current paper is devoted to prove these equivalences between definitions. In Section 2 we recall some results established by (Shannon, 1941), and generalize several of them to multivariate functions. The proof of the previous Theorem 10 then follows but is however rather involved, and requires the introduction of other equivalent intermediate classes. We show several inclusions between these classes which will guarantee the result of Theorem 10. First we show that $\text{ATSP} \subseteq \text{AWP}$, which follows from the fact that it is possible to rescale the system using the length of the curve as a new variable to make sure it does not grow faster than a polynomial (Section 3). The other direction ($\text{AWP} \subseteq \text{ATSP}$) is really harder: the first step is to transform a computation into a computation that tolerates small perturbations of the dynamics ($\text{AWP} \subseteq \text{ARP}$, Section 5). The second problem is to avoid that the system explodes for inputs not in the domain of the function ($\text{ARP} \subseteq \text{ASP}$, Section 6). As a third step, we allow the system to have its inputs (input and precision) changed during the computation, but we require that the system has a maximum delay to react to these changes ($\text{ASP} \subseteq \text{AXP}$, Section 7). Finally, as a fourth step, we add a mechanism that feeds the system with the input and some precision. By continuously increasing the precision with time, we ensure that the system will converge when the input is stable. The result of these 4 steps is a lemma yielding a nice notion of online-computation ($\text{AXP} \subseteq \text{AOP}$, Section 8). Equality $\text{ATSP} = \text{AWP} = \text{AOP}$ follows because time and length are related for polynomially bounded systems.

A side effect of the closure properties of these classes, and of our proofs, is that programming with (polynomial length) ODE becomes a pleasant exercise, once the logic is understood. For example, simulating the assignment $y := g_\infty$ corresponds to the dynamics of $y(0) = y_0$, $y'(t) = \text{reach}(y(t), g(t)) + E(t)$, for a fixed function reach , tolerating bounded error $E(t)$ on dynamics, and g fluctuating around g_∞ . Other example: from a ATSP system computing f , just adding the corresponding AOP -equations for g , yields a PIVP computing $g \circ f$, by feeding the output of the system computing f to the (online) input of g .

2. The PIVP Class

This sections recalls some known results about the class of functions generated by polynomial initial value problems. We omitted the proofs but this section contains

all the necessary definitions and theorems needed to make this paper self-contained. Other lemmas related to the PIVP class are introduced in the paper when needed to avoid a long list of lemmas. A much more complete and detailed analysis of this class, with all the proofs, can be found in (Bournez et al., 2016) but we give a short overview below.

Terminology is important here: the functions of this class are called *generable*, and should not be confused with the notion of computable function introduced earlier. Informally, the main results on this class are the following:

- this class is stable by arithmetic operations and composition;
- this class contains many useful functions such as trigonometric functions;
- if $y' = f(y)$ where f in this class, then y is also in this class.

The general idea is that working directly with polynomial differential equations is a perilous exercise but it becomes easier if we can use more than polynomials. For example, assume that the above results are true, and consider the following differential equation:

$$y(0) = 1, y'(t) = \sin(y(t)).$$

It can be seen that \sin is generable so it follows that y is generable. Another example is the following differential equation:

$$y(0) = 1, y'(t) = \tanh(y(t)^2).$$

It can be seen again that \tanh is generable, and polynomials are also generable so $x \mapsto \tanh(x^2)$ is generable, thus y is generable. Hopefully these two examples will convince the reader that this class gives us a lot of flexibility when writing differential systems.

Another important aspect of this class is the growth of the functions. Without restrictions, it is very easy to build fast-growing functions, such as towers of exponentials. In this work, we crucially need to bound the growth of functions to limit the power of our systems. A necessary condition for this is that we should only write differential equations of the form $y' = f(y)$ where f is generable and bounded by a polynomial. Of course this condition is trivially satisfied by polynomials but is also verified by many other functions such as \sin or \tanh .

The following concept can be attributed to (Shannon, 1941): a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be a PIVP function if there exists a system of the form (1) with $f(t) = y_1(t)$ for all t , where y_1 denotes the first component of the vector y defined in \mathbb{R}^d . We need in our proof to extend this concept to talk about (i) multivariable functions and (ii) the growth of these functions. This leads to the following:

Definition 11 (Generable function (Bournez et al., 2016)). Let $d, e \in \mathbb{N}$, I be an open and connected subset of \mathbb{R}^d , $\text{sp} : \mathbb{R} \rightarrow \mathbb{R}_+$ and $f : I \rightarrow \mathbb{R}^e$. We say that $f \in \text{GVAL}[\text{sp}]$ if and only if there exist $n \geq e$, $p \in M_{n,d}(\mathbb{K})$ [\mathbb{R}^n], $x_0 \in \mathbb{K}^d$, $y_0 \in \mathbb{K}^n$ and $y : I \rightarrow \mathbb{R}^n$ satisfying for all $x \in I$:

- $y(x_0) = y_0$ and $J_y(x) = p(y(x))$ (i.e. $\partial_j y_i(x) = p_{ij}(y(x))$)
► y satisfies a differential equation

- $f(x) = y_{1..e}(x)$ ▶ f is a component of y
- $\|y(x)\| \leq \text{sp}(\|x\|)$ ▶ y is bounded by sp

Definition 12 (Polynomially bounded generable function). The class of generable functions with polynomially bounded value is called GPVAL:

$$f \in \text{GPVAL} \Leftrightarrow \text{there exists a polynomial } \text{sp} \text{ such that } f \in \text{GVAL}[\text{sp}]$$

The following closure properties can be seen as extensions of the results from (Graça, Buescu, and Campagnolo, 2009) to multivariate functions:

Lemma 13 (Arithmetic on generable functions (Bournez et al., 2016)). *Let $d, e, n, m \in \mathbb{N}$, $\text{sp}, \overline{\text{sp}} : \mathbb{R} \rightarrow \mathbb{R}_+$, $f : \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^n \in \text{GVAL}[\text{sp}]$ and $g : \subseteq \mathbb{R}^e \rightarrow \mathbb{R}^m \in \text{GVAL}[\overline{\text{sp}}]$. Then:*

- $f + g, f - g \in \text{GVAL}[\text{sp} + \overline{\text{sp}}]$ over $\text{dom } f \cap \text{dom } g$ if $d = e$ and $n = m$
- $f g \in \text{GVAL}[\max(\text{sp}, \overline{\text{sp}}, \text{sp} \overline{\text{sp}})]$ if $d = e$ and $n = m$
- $f \circ g \in \text{GVAL}[\max(\overline{\text{sp}}, \text{sp} \circ \overline{\text{sp}})]$ if $m = d$ and $g(\text{dom } g) \subseteq \text{dom } f$

Our key result is that the solution to an ODE whose right-hand side is generable, and possibly depends on an external and C^1 control, may be rewritten as a GPAC. A corollary of this result is that the solution of a generable ODE is generable.

Proposition 14 (Generable ODE rewriting (Bournez et al., 2016)). *Let $d, n \in \mathbb{N}$, $I \subseteq \mathbb{R}^n$, $X \subseteq \mathbb{R}^d$, $\text{sp} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $(f : I \times X \rightarrow \mathbb{R}^n) \in \text{GVAL}[\text{sp}]$. Define $\overline{\text{sp}} = \max(\text{id}, \text{sp})$. Then there exist $m \in \mathbb{N}$, $(g : I \times X \rightarrow \mathbb{R}^m) \in \text{GVAL}[\overline{\text{sp}}]$ and $p \in \mathbb{K}^m[\mathbb{R}^m \times \mathbb{R}^d]$ such that for any interval J , $t_0 \in \mathbb{K} \cap J$, $y_0 \in \mathbb{K}^n \cap J$, $y \in C^1(J, I)$ and $x \in C^1(J, X)$, if y satisfies:*

$$\begin{cases} y(t_0) = y_0 \\ y'(t) = f(y(t), x(t)) \end{cases}$$

then there exists $z \in C^1(J, \mathbb{R}^m)$ such that:

$$\begin{cases} z(t_0) = g(y_0, x(t_0)) \\ z'(t) = p(z(t), x'(t)) \end{cases} \quad \begin{cases} y(t) = z_{1..d}(t) \\ \|z(t)\| \leq \overline{\text{sp}}(\|y(t), x(t)\|) \end{cases}$$

A simplified version of this lemma shows that generable functions are closed under ODE solving.

Corollary 15 (Closure under ODE of generable functions (Bournez et al., 2016)).

Let $d \in \mathbb{N}$, $J \subseteq \mathbb{R}$ an interval, $\text{sp}, \overline{\text{sp}} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $f : \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^d$ in $\text{GVAL}[\text{sp}]$, $t_0 \in \mathbb{K} \cap J$ and $y_0 \in \mathbb{K}^d \cap \text{dom } f$. Assume there exists $y : J \rightarrow \text{dom } f$ satisfying for all $t \in J$:

$$\begin{cases} y(t_0) = y_0 \\ y'(t) = f(y(t)) \end{cases} \quad \|y(t)\| \leq \overline{\text{sp}}(t)$$

Then $y \in \text{GVAL}[\max(\overline{\text{sp}}, \text{sp} \circ \overline{\text{sp}})]$ and is unique.

It follows that many polynomially bounded usual analytic⁷ functions are in the class GPVAL.

We will also need the following results, which tell us how the solution of a GPAC varies if there is a slight change in the parameters defining it. In the next theorem Σp denotes the sum of the absolute values of the coefficients of the polynomial p .

Theorem 16 (Parameter dependency (Bournez et al., 2016)). *Let $I = [a, b]$, $p \in \mathbb{R}^n[\mathbb{R}^{n+d}]$, $k = \deg(p)$, $e \in C^0(I, \mathbb{R}^d)$, $x, \delta \in C^0(I, \mathbb{R}^n)$ and $y_0, z_0 \in \mathbb{R}^d$. Assume that $y, z : I \rightarrow \mathbb{R}^d$ satisfy:*

$$\begin{cases} y(a) = y_0 \\ y'(t) = p(y(t), x(t)) \end{cases} \quad \begin{cases} z(a) = z_0 \\ z'(t) = e(t) + p(z(t), x(t) + \delta(t)) \end{cases} \quad t \in I$$

Assume that there exists $\varepsilon > 0$ such that for all $t \in I$,

$$\begin{aligned} \mu(t) := & \left(\|z_0 - y_0\| + \int_a^t \|e(u)\| + k\Sigma p M^{k-1}(u) \|\delta(u)\| du \right) \exp \left(k\Sigma p \int_a^t M^{k-1}(u) du \right) \\ & < \varepsilon \quad (2) \end{aligned}$$

where $M(t) = \varepsilon + \|y(t)\| + \|x(t)\| + \|\delta(t)\|$. Then for all $t \in I$,

$$\|z(t) - y(t)\| \leq \mu(t)$$

Lemma 17 (Modulus of continuity (Bournez et al., 2016)). *Let $\text{sp} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $f \in \text{GVAL}[\text{sp}]$. There exists $q \in \mathbb{K}[\mathbb{R}]$ such that for any $x_1, x_2 \in \text{dom } f$, if $[x_1, x_2] \subseteq \text{dom } f$ then*

$$\|f(x_1) - f(x_2)\| \leq \|x_1 - x_2\| q(\text{sp}(\max(\|x_1\|, \|x_2\|))).$$

In particular, if $f \in \text{GPVAL}$ then there exists $q \in \mathbb{K}[\mathbb{R}]$ such that if $[x_1, x_2] \subseteq \text{dom } f$ then

$$\|f(x_1) - f(x_2)\| \leq \|x_1 - x_2\| q(\max(\|x_1\|, \|x_2\|)).$$

After these statements, we can go to the proof of Theorem 10. This is done by proving various implications.

3. Proof that ALP is ATSP

The purpose of the current section is to show the following.

Theorem 18. ATSP = ALP.

⁷Functions from GPVAL are necessarily analytic, as solutions of an analytic ODE are analytic.

3.1. Some remarks

We start by a remark:

Lemma 19 (Norm function, (Bournez et al., 2016)). *There is a family of functions $\text{norm}_{\infty, \delta} \in \text{GPVAL}$ such that, for any $x \in \mathbb{R}^n$ and $\delta \in]0, 1]$, we have:*

$$\|x\| \leq \text{norm}_{\infty, \delta}(x) \leq \|x\| + \delta.$$

3.2. The proof

In one direction the proof is simple because if the system uses polynomial time and space then there is a relationship between time and length and we only need to add one variable to the system to make sure that the technical condition holds. The other direction is more involved because we need to rescale the system using the length of the curve to make sure it does not grow faster than a polynomial, which is ensured by the technical condition.

Let $f \in \text{ATSC}(\Upsilon, \Omega)$ where Υ and Ω are polynomials, which we assume to be increasing functions. Apply Definition 5 to get d, p, q , let $k = \deg(p)$ and define:

$$\Omega^*(\alpha, \mu) = \Omega(\alpha, \mu) \left(1 + \Sigma p \max(1, \Upsilon(\|x\|, \Omega(\alpha, \mu)))^k\right)$$

Let $x \in \text{dom } f$ and consider the following system:

$$\begin{cases} y(0) = q(x) \\ z(0) = 0 \end{cases} \quad \begin{cases} y'(t) = p(y(t)) \\ z'(t) = 1 \end{cases}$$

Note that $z(t) = t$ (this variable is there only to ensure that the length of z grows at least linearly). Let $t, \mu \in \mathbb{R}_+$ and assume that $\text{len}_z(0, t) \geq \Omega^*(\|x\|, \mu)$. We will show that $t \geq \Omega(\|x\|, \mu)$ by contradiction. Assume the contrary and let $u \in [0, t]$. By definition:

$$\begin{aligned} \|y(u), z(u)\| &\leq 1 + \|y(u)\| \leq 1 + \Upsilon(\|x\|, t) \\ &< 1 + \Upsilon(\|x\|, \Omega(\|x\|, \mu)) \end{aligned}$$

and thus

$$\begin{aligned} \|y'(u), z'(u)\| &= \|1, p(y(u))\| \\ &< 1 + \Sigma p(1 + \Upsilon(\|x\|, \Omega(\|x\|, \mu)))^k. \end{aligned}$$

Consequently:

$$\text{len}_{y,z}(0, t) < t \sup_{u \in [0, t]} \|y'(u), z'(u)\| \leq \Omega^*(\|x\|, \mu)$$

which is absurd. Since $t \geq \Omega(\|x\|, \mu)$, by definition we get that

$$\|y_{1..m}(t) - f(x)\| \leq e^{-\mu}.$$

Finally, $\|y'(t), z'(t)\| \geq \|z'(u)\| \geq 1$ for all $t \in \mathbb{R}_+$. This shows that $f \in \text{ALC}(\Omega^*)$ where Ω^* is a polynomial.

Let $f \in \text{ALC}(\Omega)$ where Ω is a polynomial, which we assume to be an increasing function.

Apply Definition 4 to get Ω, d, p, q . Also assume that the polynomial Ω is an increasing function. Let $k = \deg(p)$. Apply Lemma 19 to get that $g(x) = \text{norm}_{\infty,1}(p(x))$ belongs to GPVAL. Apply Definition 11 to get the corresponding m, r, x_0 and z_0 . Let $x \in \text{dom } f$. For the analysis, it will be useful to consider the following systems:

$$\begin{cases} y(0) = q(x) \\ z(x_0) = z_0 \end{cases} \quad \begin{cases} y'(t) = p(y(t)) \\ J_z(x) = r(z(x)) \end{cases}$$

Note that by definition $z_1(x) = g(x)$. Define $\psi(t) = g(y(t))$ and $\hat{\psi}(u) = \int_0^u \psi(t) dt$. Now define the following system:

$$\begin{cases} \hat{y}(0) = q(x) \\ \hat{z}(0) = z(q(x)) \\ \hat{w}(0) = \frac{1}{g(q(x))} \end{cases} \quad \begin{cases} \hat{y}'(u) = \hat{w}(u)p(\hat{y}(u)) \\ \hat{z}'(u) = \hat{w}(u)r(\hat{z}(u))p(\hat{y}(u)) \\ \hat{w}'(u) = -\hat{w}(u)^3 r_1(\hat{z}(u))p(\hat{y}(u)) \end{cases}$$

where by r_1 we mean the first row of r . We will check that $\hat{y}(u) = y(\hat{\psi}^{-1}(u))$, $\hat{z}(u) = z(\hat{y}(u))$ and $\hat{w}(u) = (\hat{\psi}^{-1})'(u)$. We will use the fact that for any $h \in C^1$, $(h^{-1})' = \frac{1}{h' \circ h^{-1}}$. Also note that $\hat{\psi}' = \psi$.

- $\hat{y}(0) = y(\hat{\psi}^{-1}(0)) = y(0) = q(x)$
- $\hat{y}'(u) = (\hat{\psi}^{-1})'(u)y'(\hat{\psi}^{-1}(u)) = \hat{w}(u)p(y(\hat{\psi}^{-1}(u))) = \hat{w}(u)p(\hat{y}(u))$
- $\hat{z}(0) = z(\hat{y}(0)) = z(q(x))$
- $\hat{z}'(u) = J_z(\hat{y}(u))\hat{y}'(u) = \hat{w}(u)r(z(\hat{y}(u)))p(\hat{y}(u)) = \hat{w}(u)r(\hat{z}(u))p(\hat{y}(u))$
- $\hat{w}(0) = \frac{1}{\hat{\psi}'(\hat{\psi}^{-1}(0))} = \frac{1}{\psi(0)} = \frac{1}{g(q(x))}$
- $\hat{w}'(u) = \frac{-(\hat{\psi}^{-1})'(u)\hat{\psi}''(\hat{\psi}^{-1}(u))}{(\hat{\psi}'(\hat{\psi}^{-1}(u)))^2} = -\hat{w}(u)^3 \psi'(\hat{\psi}^{-1}(u)) = \nabla g(y(\hat{\psi}^{-1}(u))) \cdot y'(\hat{\psi}^{-1})$ and since $\nabla g(x) = r_1(z(x))^T$ (transpose of the first row of the Jacobian matrix of z because $g = z_1$) then

$$\hat{w}'(u) = -\hat{w}(u)^3 r_1(z(y(\hat{\psi}^{-1}(u))))^T \cdot p(y(\hat{\psi}^{-1}(u))) = -\hat{w}(u)^3 r_1(\hat{z}(u))p(\hat{y}(u))$$

We now claim that this system computes f quickly and has a polynomial bound. First note that by Lemma 19:

$$\|y'(t)\| \leq g(y(t)) \leq \|y'(t)\| + 1$$

and thus

$$\text{len}_y(0, t) \leq \hat{\psi}(t) \leq \text{len}_y(0, t) + t.$$

Thus

$$\begin{aligned}
\text{len}_{\hat{y}}(0, u) &= \int_0^u \|\hat{y}'(\xi)\| d\xi = \int_0^{\hat{\psi}^{-1}(u)} \|\hat{w}(\hat{\psi}(t))p(\hat{y}(\hat{\psi}(t)))\| \hat{\psi}'(t) dt \\
&= \int_0^{\hat{\psi}^{-1}(u)} \|(\hat{\psi}^{-1})'(\hat{\psi}(t))\hat{\psi}'(t)p(y(t))\| dt \\
&= \int_0^{\hat{\psi}^{-1}(u)} \|p(y(t))\| dt = \text{len}_y(0, \hat{\psi}^{-1}(u)) \leq \hat{\psi}(\hat{\psi}^{-1}(u)) \leq u.
\end{aligned}$$

It follows that

$$\|\hat{y}(u)\| \leq \|\hat{y}(0)\| + u \leq \|q(x)\| + u \leq \text{poly}(\|x\|, u).$$

Similarly:

$$\|\hat{z}(u)\| = \|z(\hat{y}(u))\| \leq \text{poly}(\|x\|, u)$$

because $z \in \text{GPVAL}$ and thus is polynomially bounded. Finally,

$$\|\hat{w}\| = \frac{1}{\psi(\hat{\psi}^{-1}(u))} = \frac{1}{g(\hat{y}(u))} \leq \frac{1}{\|y'(\hat{\psi}^{-1}(u))\|} \leq 1$$

because by hypothesis, $\|y'(t)\| \geq 1$ for all $t \in \mathbb{R}_+$. This shows that indeed $\|(\hat{y}, \hat{z}, \hat{w})(u)\|$ is polynomially bounded in $\|x\|$ and u . Now let $\mu \in \mathbb{R}_+$ and $t \geq 1 + \Omega(\|x\|, \mu)$ then

$$\begin{aligned}
\text{len}_{\hat{y}}(0, t) &= \text{len}_y(0, \hat{\psi}^{-1}(t)) \\
&\geq \hat{\psi}(\hat{\psi}^{-1}(t)) - \hat{\psi}^{-1}(t) \\
&\geq t - \hat{\psi}^{-1}(t) \\
&\geq 1 + \Omega(\|x\|, \mu) - \frac{1}{\psi(\hat{\psi}^{-1}(t))} \\
&\geq \Omega(\|x\|, \mu)
\end{aligned}$$

because, as we already saw, $\|\psi(\hat{\psi}^{-1}(t))\| \geq 1$. Thus by definition:

$$\|\hat{y}_{1..m}(t) - f(x)\| \leq e^{-\mu}$$

because $\hat{y}(t) = y(\hat{\psi}^{-1}(t))$. This shows that $f \in \text{ATSP}$.

4. Proof that ALP implies AWP

The purpose of the current section is to state the following.

Theorem 20. $\text{ATSP} = \text{AWP}$.

PROOF. The inclusion $\text{ATSP} \subseteq \text{AWP}$ is immediate from Definitions 5 and 8. The other inclusion will follow from the results of the other sections.

5. Proof that AWP implies ARP

The purpose of the current section is to state the following:

Theorem 21. $\text{AWP} \subseteq \text{ARP}$.

i.e. that it possible to transform a computation into a computation that tolerates small perturbations of the dynamics, where:

Definition 22 (Analog robust computability). Let $n, m \in \mathbb{N}$, $f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\Theta, \Omega : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ and $\Upsilon : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$. We say that f is $(\Upsilon, \Omega, \Theta)$ -robustly-computable if and only if there exist $d \in \mathbb{N}$, and $(h : \mathbb{R}^d \rightarrow \mathbb{R}^d), (g : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^d) \in \text{GPVAL}$ such that for any $x \in \text{dom } f, \mu \in \mathbb{R}_+, e_0 \in \mathbb{R}^d$ and $e \in C^0(\mathbb{R}_+, \mathbb{R}^d)$ satisfying

$$\|e_0\| + \int_0^\infty \|e(t)\| dt \leq e^{-\Theta(\|x\|, \mu)},$$

there exists (a unique) $y : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ satisfying for all $t \in \mathbb{R}_+$:

- $y(0) = g(x, \mu) + e_0$ and $y'(t) = h(y(t)) + e(t)$ \blacktriangleright y satisfies a generable IVP
- if $t \geq \Omega(\|x\|, \mu)$ then $\|y_{1..m}(t) - f(x)\| \leq e^{-\mu}$ \blacktriangleright $y_{1..m}$ approximates $f(x)$
- $\|y(t)\| \leq \Upsilon(\|x\|, \mu, t)$ \blacktriangleright $y(t)$ is bounded

We denote by $\text{ARC}(\Upsilon, \Omega, \Theta)$ the set of $(\Upsilon, \Omega, \Theta)$ -robustly-computable functions, and by ARP the set of (poly, poly, poly)-robustly-computable functions.

Intuitively, this definition says that even if the initial condition and the ODE defining the PIVP are (slightly) perturbed or have (small) errors in Definition 5, the PIVP is still capable of computing an approximation of $f(x)$.

Actually, we prove in this section that $\text{AWP} \subseteq \text{ARP}$. Then the equality will follow from results of other sections.

5.1. Some remarks

Remark 23 (Domain of definition of g and h). There is a subtle but important detail in this definition: we more or less replaced the polynomials p and q by generable functions g and h . It could have been tempting to take this opportunity to restrict the domain of definition of g to $\text{dom } f \times \mathbb{R}_+$ and that of h to a subset of \mathbb{R}^d where the dynamics takes place. We kept the entire euclidean space for good reasons. First it makes the definition simpler. Second, it makes the notion stronger and more useful. This last point is important because we are going to use robust computability (and the next notion of strong computability) in cases where we have less or no control over the errors and thus over the trajectory of the system. On the downside, this requires to check that g and h are indeed defined over the entire space !

The examples below show how to build robustly-computable functions. In the first example, we only need to define Θ so that it works, whereas in the second case, careful design of the system is needed for it to be robust.

Example 24 (Polynomials are robustly-computable). In order to make polynomials robustly-computable, we will play with the choice of Θ and see that this is enough to make the system robust. Let $q \in \mathbb{K}[\mathbb{R}^d]$ be a multivariate polynomial: we will show that $q \in \text{ARP}$. Let $x \in \mathbb{R}^d$, $\mu \in \mathbb{R}_+$, $e_0 \in \mathbb{R}$ and $e \in C^0(\mathbb{R}_+, \mathbb{R})$. Assume that $|e_0| + \int_0^\infty |e(t)|dt \leq e^{-\mu}$ and consider the following system for $t \in \mathbb{R}_+$:

$$y(0) = q(x) + e_0 \quad y'(t) = e(t)$$

We claim that this system satisfies Definition 22:

- The system is of the form $y(0) = \text{poly}(x) + e_0$ and $y'(t) = \text{poly}(y(t)) + e(t)$ where the polynomials have coefficients in \mathbb{K} .
- For any $t \geq 0$, we have:

$$\|y(t) - q(x)\| \leq |e_0| + \int_0^t |e(u)|du \leq |e_0| + \int_0^\infty |e(u)|du \leq e^{-\mu}$$

so we can take $\Omega(\alpha, \mu) = 0$.

- For any $t \in \mathbb{R}_+$, we have:

$$\|y(t)\| \leq \|q(x)\| + |e_0| + \int_0^t |e(u)|du \leq \|q(x)\| + 1 \leq \text{poly}(\|x\|)$$

so we can take Υ to be any polynomial such that $\Upsilon(\|x\|, \mu) \geq \|p(x)\| + 1$.

This shows that $q \in \text{ARC}(\Upsilon, \Omega, \Theta)$ where $\Theta(\alpha, \mu) = \mu$.

In the previous example, we saw that we could modify the associated system of some computable functions to make them robustly-computable. It appears that this is not a coincidence but a general fact. To understand how the proof works, one must first understand the problem. Let us consider a computable function $f : \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$ in $\text{AW}(\Upsilon, \Theta)$ and the associated system for $x \in \text{dom } f$ and $\mu \in \mathbb{R}_+$:

$$y(0) = q(x, \mu) \quad y'(t) = p(y(t))$$

This system converges to $f(x)$ very quickly: $\|y_1(t) - f(x)\| \leq e^{-\mu}$ when $t \geq \Omega(\|x\|, \mu)$ and $y(t)$ is bounded: $\|y(t)\| \leq \Upsilon(\|x\|, \mu, t)$. Let us introduce some errors in the system by taking $e_0 \in \mathbb{R}^d$ and $e \in C^0(\mathbb{R}_+, \mathbb{R}^d)$ such that $\|e_0\| + \int_0^\infty \|e(u)\| du \leq e^{-\Theta(\|x\|, \mu)}$ for some unspecified Θ and consider the perturbed system:

$$z(0) = q(x, \mu) + e_0 \quad z'(t) = p(z(t)) + e(t)$$

The relationship between this system and the previous one is given by Theorem 16 and can be informally written as:

$$\|z(t) - y(t)\| \leq \left(\|e_0\| + \int_0^t \|e(u)\| du \right) e^{\int_0^t k \Sigma p \|y(u)\|^{k-1} du} \quad (3)$$

$$\begin{aligned}
&\leq \left(\|e_0\| + \int_0^\infty \|e(u)\| du \right) e^{\int_0^t k\Sigma p \Upsilon(\|x\|, \mu, u)^{k-1} du} \\
&\hspace{15em} \text{using the bound of } y(t) \\
&\leq e^{k\Sigma p t \Upsilon(\|x\|, \mu, t)^{k-1} - \Theta(\|x\|, \mu)} \hspace{5em} \text{assuming that } \Upsilon \text{ is increasing}
\end{aligned}$$

One observes that this bound grows to infinity whatever we choose for Θ because of the dependency in t . On the other hand, we do not need to simulate y for arbitrary large t : as soon as $t \geq \Theta(\|x\|, \mu)$ we can stop the system and get a good enough result. Unfortunately, one does not simply stop a differential system, however we can slow it down. To this end, introduce $\psi(t) = (1 + \Theta(\|x\|, \mu)) \tanh(t)$ and $w(t) = z(\psi(t))$. If we show that w satisfies a differential system, then we are almost done. Indeed $\psi(t) \leq 1 + \Theta(\|x\|, \mu)$ for all $t \in \mathbb{R}_+$ and if $t \geq 1 + \Theta(\|x\|, \mu)$ then $\psi(t) \geq \Theta(\|x\|, \mu)$, so the system “kind of stops” between $\Theta(\|x\|, \mu)$ and $\Theta(\|x\|, \mu) + 1$. Furthermore, if $t \geq 1 + \Theta(\|x\|, \mu)$ then:

$$\begin{aligned}
\|w_1(t) - f(x)\| &\leq \|z(\psi(t)) - y(\psi(t))\| + \|y_1(\psi(t)) - f(x)\| \\
&\hspace{15em} \text{use the triangle inequality} \\
&\leq e^{k\Sigma p \psi(t) \Upsilon(\|x\|, \mu, \psi(t))^{k-1} - \Theta(\|x\|, \mu)} + e^{-\mu} \hspace{5em} \text{using (3)} \\
&\leq e^{k\Sigma p (1 + \Theta(\|x\|, \mu)) \Upsilon(\|x\|, \mu, 1 + \Theta(\|x\|, \mu))^{k-1} - \Theta(\|x\|, \mu)} + e^{-\mu} \\
&\hspace{15em} \text{using the bound on } \psi \\
&\leq 2e^{-\mu} \hspace{15em} \text{for a suitable choice of } \Theta
\end{aligned}$$

We are left with showing that $w(t) = z(\psi(t))$ can be generated by a generable IVP with perturbations. In the case of no perturbations, this is very easy because $w'(t) = \psi'(t)z'(t) = x(1 - \tanh(t))p(z(t))$ which is generable. The following lemma extends this idea to the case of perturbations.

Lemma 25 (PIVP Slow-Stop). *Let $d \in \mathbb{N}$, $y_0 \in \mathbb{R}^d$, $T, \theta \in \mathbb{R}_+$, $(e_{0,y}, e_{0,A}) \in \mathbb{R}^{d+1}$, $(e_y, e_A) \in C^0(\mathbb{R}_+, \mathbb{R}^{d+1})$ and $p \in \mathbb{K}^d[\mathbb{R}^d]$. Assume that $\|e_0\| + \int_0^\infty \|e(t)\| dt \leq e^{-\theta}$ and consider the following system:*

$$\begin{cases} y(0) = y_0 + e_{0,y} & \left\{ \begin{array}{l} y'(t) = \frac{1 + \tanh(A(t))}{2} p(y(t)) + e_y(t) \\ A'(t) = -1 + e_A(t) \end{array} \right. \\ A(0) = T + 2 + e_{0,A} \end{cases}$$

Then there exist an increasing function $\psi \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ and $z : \psi(\mathbb{R}_+) \rightarrow \mathbb{R}^d$ such that:

$$\psi(0) = 0 \quad z(0) = y_0 + e_{0,y} \quad z'(t) = p(z(t)) + (\psi^{-1})'(t) e_y(\psi^{-1}(t))$$

and $y(t) = z(\psi(t))$. Furthermore $\psi(T + 1) \geq T$ and $\psi(t) \leq T + 4$ for all $t \in \mathbb{R}_+$. Furthermore, $|A(t)| \leq T + 3$ for all $t \in \mathbb{R}_+$.

PROOF. Let $f(t) = \frac{1 + \tanh(A(t))}{2}$ and note that $0 < f(t) < 1$ for all $t \in \mathbb{R}_+$. Check that we can integrate A explicitly:

$$A(t) = T + 2 - t + e_{0,A} + \int_0^t e_A(u) du.$$

If we take $\psi(t) = \int_0^t f(u)du$ then ψ is an increasing function because $f > 0$, so it is a diffeomorphism from \mathbb{R}_+ onto $\psi(\mathbb{R}_+)$. Note that $\psi(t) \leq t$ for all $t \in \mathbb{R}_+$. Let $t \geq T + 3$, then

$$\begin{aligned} A(t) &\leq T + 2 - t + |e_{0,A}| + \int_0^t |e_A(u)|du \\ &\leq T + 2 + e^{-\theta} - t \leq T + 3 - t \leq 0 \end{aligned}$$

because $\theta \geq 0$. Apply Lemma 30 to get that $\tanh(A(t)) \leq -1 + e^{T+3-t}$ and thus

$$f(t) \leq \frac{1}{2}e^{T+3-t} \text{ for } t \geq T + 3.$$

Integrating this inequality shows that

$$\begin{aligned} \psi(t) &\leq \psi(T + 3) + \frac{1}{2} \int_{T+3}^t e^{T+3-u} du \\ &\leq T + 3 + \frac{1}{2}(1 - e^{T+3-t}) \leq T + 4. \end{aligned}$$

This shows that $\psi(t) \leq T + 4$ for all $t \in \mathbb{R}_+$.

Let $t \leq T + 1$, then by the same reasoning:

$$A(t) \geq T + 2 - t - e^{-\theta} \geq T + 1 - t \geq 0$$

thus $\tanh(A(t)) \geq 1 - e^{t-T-1}$ and $f(t) \geq 1 - \frac{1}{2}e^{t-T-1}$. Thus:

$$\psi(T + 1) \geq \int_0^{T+1} 1 + \frac{1}{2}e^{u-T-1} du = T + 1 + \frac{1}{2}(1 - e^{-1-T}) \geq T.$$

Finally, apply Lemma 31 to get that $y(t) = z(\psi(t))$ where z satisfies for $t \in \psi(\mathbb{R}_+)$:

$$z(0) = y(0) \quad z'(t) = p(z(t)) + (\psi^{-1})'(t)e_y(\psi^{-1}(t))$$

5.2. The proof

The proof of the implication AWP implies ARP of Theorem 21 is then the following.

Let Υ^*, Ω^* be polynomials such that $f \in \text{AW}(\Upsilon^*, \Omega^*)$. Without loss of generality, we assume they are increasing functions on both arguments. Apply Definition 8 to get $d \in \mathbb{N}$, $p \in \mathbb{K}^d[\mathbb{R}^d]$, $q \in \mathbb{K}^d[\mathbb{R}^{n+1}]$ and let $k = \deg(p)$. Define:

$$\begin{aligned} T(\alpha, \mu) &= \Omega^*(\alpha, \mu + \ln 2) \\ \Theta(\alpha, \mu) &= k \Sigma p(T(\alpha + 1, \mu) + 4)(\Upsilon^*(\alpha, \mu, T(\alpha + 1, \mu) + 4) + 1)^{k-1} + \mu + \ln 2 \\ \Omega(\alpha, \mu) &= T(\alpha + 1, \mu) + 1 \end{aligned}$$

Let $x \in \text{dom } f$, $(e_{0,y}, e_{0,A}) \in \mathbb{R}^{d+1}$, $(e_y, e_A) \in C^0(\mathbb{R}_+, \mathbb{R}^{d+1})$ and $\mu \in \mathbb{R}_+$ such that

$$\|e_0\| + \int_0^\infty \|e(t)\| dt \leq e^{-\Theta(\|x\|, \mu)}.$$

Apply Lemma 25 and consider the following systems (where ψ is given by the lemma):

$$\begin{cases} y(0) = q(x, \mu) + e_{0,y} \\ A(0) = T(\text{norm}_{\infty,1}(x), \mu) + 2 + e_{0,A} \end{cases} \quad \begin{cases} y'(t) = \frac{1+\tanh(A(t))}{2} p(y(t)) + e_y(t) \\ A'(t) = -1 + e_A(t) \end{cases}$$

$$\begin{cases} z(0) = q(x, \mu) + e_{0,y} \\ z'(t) = p(z(t)) + (\psi^{-1})'(t) e_y(\psi^{-1}(t)) \end{cases} \quad \begin{cases} w(0) = q(x, \mu) \\ w'(t) = p(w(t)) \end{cases}$$

By definition of p and q , if $t \geq \Omega^*(\|x\|, \mu)$ then $\|w_{1..m}(t) - f(x)\| \leq e^{-\mu}$. Furthermore, $\|w(t)\| \leq \Upsilon^*(\|x\|, \mu, t)$ for all $t \in \mathbb{R}_+$. Define $T^* = T(\text{norm}_{\infty,1}(x), \mu)$. Apply Lemma 19 to get that

$$\|x\| \leq \text{norm}_{\infty,1}(x) \leq \|x\| + 1$$

and thus

$$T(\|x\|, \mu) \leq T^* \leq T(\|x\| + 1, \mu).$$

By construction, $\psi(t) \leq T^* + 4$ for all $t \in \mathbb{R}_+$. Let $t \in \mathbb{R}_+$, apply Theorem 16 by checking that:

$$\begin{aligned} & \left(\|e_{0,y}\| + \int_0^{\psi(t)} \|(\psi^{-1})'(u) e_y(\psi^{-1}(u))\| du \right) e^{k\Sigma p \int_0^{\psi(t)} (\|w(u)\| + 1)^{k-1} du} \\ & \leq \left(\|e_{0,y}\| + \int_0^t \|e_y(u)\| du \right) e^{k\Sigma p \int_0^{\psi(t)} (\Upsilon^*(\|x\|, \mu, u) + 1)^{k-1} du} \\ & \leq e^{k\Sigma p \psi(t) (\Upsilon^*(\|x\|, \mu, \psi(t)) + 1)^{k-1} - \Theta(\|x\|, \mu)} \quad \text{by a change of variable} \\ & \leq e^{k\Sigma p (T(\|x\| + 1, \mu) + 4) (\Upsilon^*(\|x\|, \mu, T(\|x\| + 1, \mu) + 4) + 1)^{k-1} - \Theta(\|x\|, \mu)} \quad \text{by hypothesis on the error} \\ & \leq e^{-\mu - \ln 2} \leq 1 \quad \text{because } \psi \text{ is bounded} \\ & \quad \text{by definition of } \Theta \end{aligned}$$

Thus

$$\|z(\psi(t)) - w(\psi(t))\| \leq e^{-\mu - \ln 2} \text{ for all } t \in \mathbb{R}_+.$$

Furthermore, if $t \geq \Omega(\|x\|, \mu)$ then

$$\psi(t) \geq \psi(T(\|x\| + 1, \mu) + 1) \geq \psi(T^* + 1) \geq T^*.$$

By construction $\psi(T^*) \geq T^*$ so

$$\psi(t) \geq T^* \geq T(\|x\|, \mu) = \Omega^*(\|x\|, \mu + \ln 2)$$

thus

$$\|z(\psi(t)) - f(x)\| \leq e^{-\mu - \ln 2}.$$

Consequently, we have

$$\|y(t) - f(x)\| \leq \|z(\psi(t)) - w(\psi(t))\| + \|w(\psi(t)) - f(x)\|$$

$$\leq 2e^{-\mu - \ln 2} \leq e^{-\mu}.$$

Let $t \in \mathbb{R}_+$, then

$$\begin{aligned} \|y(t)\| &= \|z(\psi(t))\| \leq \|w(\psi(t))\| + e^{-\mu} \\ &\leq \Upsilon^*(\|x\|, \mu, \psi(t)) + 1 \leq \Upsilon^*(\|x\|, \mu, T(\|x\| + 1, \mu) + 4) + 1 \\ &\leq \Upsilon^*(\|x\|, \mu, \Omega^*(\|x\| + 1, \mu + \ln 2) + 4) + 1 \end{aligned}$$

which is polynomially bounded in $\|x\|$ and μ . Furthermore

$$|A(t)| \leq T^* + 4 \leq \Omega^*(\|x\| + 1, \mu + \ln 2) + 4$$

which are both polynomially bounded in $\|x\|$, μ .

Finally, $(y, A)(0) = g(x, \mu) + e_0$ and $(y, A)'(t) = h(y(t), A(t)) + e(t)$ where g and h belong to GPVAL because $\tanh, \text{norm}_{\infty, 1} \in \text{GPVAL}$.

Remark 26 (Polynomial versus generable). The proof of Theorem 21 also works if q is generable (i.e. $q \in \text{GPVAL}$) instead of polynomial in Definition 5 or Definition 8.

6. Proof that ARP implies ASP

This section is devoted to prove the following result: it is always possible to avoid that the system in Definition 22. explodes for inputs not in the domain of the function, or for perturbations of the dynamics which are too big. This motivates the following result and Definition 28.

Theorem 27 (Robust \subseteq strong). ARP = ASP.

where

Definition 28 (Analog strong computability). Let $n, m \in \mathbb{N}$, $f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\Theta, \Omega : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ and $\Upsilon : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$. We say that f is $(\Upsilon, \Omega, \Theta)$ -strongly-computable if and only if there exist $d \in \mathbb{N}$, and $(h : \mathbb{R}^d \rightarrow \mathbb{R}^d), (g : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^d) \in \text{GPVAL}$ such that for any $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}_+$, $e_0 \in \mathbb{R}^d$ and $e \in C^0(\mathbb{R}_+, \mathbb{R}^d)$, there is (a unique) $y : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ satisfying for all $t \in \mathbb{R}_+$ and $\hat{e}(t) = \|e_0\| + \int_0^t \|e(u)\| du$:

- $y(0) = g(x, \mu) + e_0$ and $y'(t) = h(y(t)) + e(t)$ \blacktriangleright y satisfies a generable IVP
- if $x \in \text{dom } f$, $t \geq \Omega(\|x\|, \mu)$ and $\hat{e}(t) \leq e^{-\Theta(\|x\|, \mu)}$ then $\|y_{1..m}(t) - f(x)\| \leq e^{-\mu}$
- $\|y(t)\| \leq \Upsilon(\|x\|, \mu, \hat{e}(t), t)$ \blacktriangleright $y(t)$ is bounded

We denote by $\text{AS}(\Upsilon, \Omega, \Theta)$ the set of $(\Upsilon, \Omega, \Theta)$ -strongly-computable functions, and by ASP the set of (poly, poly, poly)-strongly-computable functions.

Actually, we prove in this section that $\text{ARP} \subseteq \text{ASP}$. Equality follows from results in other sections.

6.1. Some remarks

The following Lemma can be proved by providing explicitly such a function:

Lemma 29 (Max function, (Bournez et al., 2016)). *There is a family of functions $\text{mx}_\delta \in \text{GPVAL}$ such that: For any $x, y \in \mathbb{R}$ and $\delta \in]0, 1]$ we have:*

$$\max(x, y) \leq \text{mx}_\delta(x, y) \leq \max(x, y) + \delta$$

For any $x \in \mathbb{R}^n$ and $\delta \in]0, 1]$ we have:

$$\max(x_1, \dots, x_n) \leq \text{mx}_\delta(x) \leq \max(x_1, \dots, x_n) + \delta$$

The following lemmas can also be established:

Lemma 30 (Bounds on tanh, (Bournez et al., 2016)). $1 - \text{sgn}(t) \tanh(t) \leq e^{-|t|}$ for all $t \in \mathbb{R}$.

Lemma 31 (Perturbed time-scaling). *Let $d \in \mathbb{N}$, $x_0 \in \mathbb{R}^d$, $p \in \mathbb{R}^d[\mathbb{R}^d]$, $e \in C^0(\mathbb{R}_+, \mathbb{R}^d)$ and $\phi \in C^0(\mathbb{R}_+, \mathbb{R}_+)$. Let $\psi(t) = \int_0^t \phi(u) du$. Assume that ψ is an increasing function and that $y, z : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ satisfy for all $t \in \mathbb{R}_+$:*

$$\begin{cases} y(0) = x_0 \\ y'(t) = p(y(t)) + (\psi^{-1})'(t)e(\psi^{-1}(t)) \end{cases} \quad \begin{cases} z(0) = x_0 \\ z'(t) = \phi(t)p(z(t)) + e(t) \end{cases}$$

Then $z(t) = y(\psi(t))$ for all $t \in \mathbb{R}_+$. In particular,

$$\int_0^{\psi(t)} \|(\psi^{-1})'(u)e(\psi^{-1}(u))\| du = \int_0^t \|e(u)\| du$$

and

$$\sup_{u \in [0, \psi(t)]} \|(\psi^{-1})'(u)e(\psi^{-1}(u))\| = \sup_{u \in [0, t]} \frac{\|e(u)\|}{\phi(u)}.$$

PROOF. Use that $\phi = \psi'$, $\psi' \cdot (\psi^{-1})' \circ \psi = 1$ and that $\psi' \geq 0$.

On a more technical side, we will need to “apply” Definition 22 over finite intervals and we need the following lemma to do so.

Lemma 32 (Finite time robustness). *Let $f \in \text{ARC}(\Upsilon, \Omega, \Theta)$, $I = [0, T]$, $x \in \text{dom } f$, $\mu \in \mathbb{R}_+$, $e_0 \in \mathbb{R}^d$ and $e \in C^0(I, \mathbb{R}^d)$ such that*

$$\|e_0\| + \int_I \|e(t)\| dt < e^{-\Theta(\|x\|, \mu)}.$$

Assume that $y : I \rightarrow \mathbb{R}^d$ satisfies for all $t \in I$:

$$y(0) = g(x, \mu) + e_0 \quad y'(t) = h(y(t)) + e(t)$$

where g, h come from Definition 22 applied to f . Then for all $t \in I$:

- $\|y(t)\| \leq \Upsilon(\|x\|, \mu, t)$
- if $t \geq \Omega(\|x\|, \mu)$ then $\|y_{1..m} - f(x)\| \leq e^{-\mu}$

PROOF. The trick is simply to extend e so that it is defined over \mathbb{R}_+ and such that:

$$\|e_0\| + \int_0^\infty \|e(u)\| du \leq e^{-\Theta(\|x\|, \mu)}$$

This is always possible because the truncated integral is strictly smaller than the bound. Formally, define for $t \in \mathbb{R}_+$:

$$\bar{e}(t) = \begin{cases} e(t) & \text{if } t \leq T \\ e(T)e^{\frac{\varepsilon(T)}{\varepsilon}(T-t)} & \text{otherwise} \end{cases}$$

$$\text{where } \varepsilon = e^{-\Theta(\|x\|, \mu)} - \|e_0\| - \int_I \|e(t)\| dt > 0$$

One easily checks that $\bar{e} \in C^0(\mathbb{R}_+, \mathbb{R}^d)$ and that:

$$\begin{aligned} \|e_0\| + \int_0^\infty \|\bar{e}(t)\| dt &= \|e_0\| + \int_0^T \|e(t)\| dt + \int_T^\infty e(T)e^{\frac{\varepsilon(T)}{\varepsilon}(T-t)} dt \\ &= e^{-\Theta(\|x\|, \mu)} - \varepsilon + \left[-\varepsilon e(T)e^{\frac{\varepsilon(T)}{\varepsilon}(T-t)} \right]_T^\infty \\ &= e^{-\Theta(\|x\|, \mu)} \end{aligned}$$

Assume that $z : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ satisfies for $t \in \mathbb{R}_+$:

$$z(0) = g(x, \mu) \quad z'(t) = g(z(t)) + \bar{e}(t)$$

Then z satisfies Definition 22 so $\|z\|(t) \leq \Upsilon(\|x\|, \mu)$ and if $t \geq \Omega(\|x\|, \mu)$ then $\|z_{1..m}(t) - f(x)\| \leq e^{-\mu}$. Conclude by noting that $z(t) = y(t)$ for all $t \in [0, T]$ since $e(t) = \bar{e}(t)$.

6.2. The proof

The proof of Theorem 27 is then the following.

PROOF. Let Ω, Θ, Υ be polynomials and $(f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m) \in \text{ARC}(\Upsilon, \Omega, \Theta)$. Without loss of generality, we assume that Ω, Θ, Υ are increasing functions of their arguments. Apply Definition 22 to get d, h and g . Let $x \in \mathbb{R}^n, \mu \in \mathbb{R}_+, (e_{0,y}, e_{0,\ell}) \in \mathbb{R}^{d+1}$ and $(e_y, e_\ell) \in C^0(\mathbb{R}_+, \mathbb{R}^{d+1})$. Define $\hat{e}(t) = \|e_0\| + \int_0^t \|e(u)\| du$, and consider the following system for $t \in \mathbb{R}_+$:

$$\begin{cases} y(0) = g(x, \mu) + e_{0,y} \\ y'(t) = \psi(t)h(y(t)) + e_y(t) \\ \ell(0) = \text{mx}_1(\text{norm}_{\infty,1}(x), \mu) + 1 + e_{0,\ell} \\ \ell'(t) = 1 + e_\ell(t) \end{cases}$$

$$\psi(t) = \frac{1 + \tanh(\Delta(t))}{2} \quad \Delta(t) = \Upsilon(\ell(t), \ell(t), \ell(t)) + 1 - \text{norm}_{\infty,1}(y(t))$$

We will first show that the system remains polynomially bounded. Apply Lemma 29 and Lemma 19 to get that:

$$\begin{aligned} \|\ell(0)\| &\leq \max(\|x\| + 1, \mu) + 1 + \|e_{0,\ell}\| \\ &\leq \text{poly}(\|x\|, \mu) + \|e_{0,\ell}\| \end{aligned}$$

Consequently:

$$\begin{aligned} \|\ell(t)\| &\leq \|\ell(0)\| + \int_0^t 1 + \|e_\ell(u)\| \, du \\ &\leq \text{poly}(\|x\|, \mu) + t + \|e_{0,\ell}\| + \int_0^t \|e_\ell(u)\| \, du \\ &\leq \text{poly}(\|x\|, \mu) + t + \hat{e}(t) \\ &\leq \text{poly}(\|x\|, \mu, t, \hat{e}(t)) \end{aligned} \tag{4}$$

Since $g, h \in \text{GPVAL}$, there exist two polynomials sp and $\overline{\text{sp}}$ such that $\|g(x)\| \leq \text{sp}(\|x\|)$ and $\|h(x)\| \leq \overline{\text{sp}}(\|x\|)$ for all $x \in \mathbb{R}^d$ and without loss of generality, we assume that sp and $\overline{\text{sp}}$ are increasing functions. Let $t \in \mathbb{R}_+$, there are two possibilities:

- If $\Delta(t) \geq 0$ then $\text{norm}_{\infty,1}(y(t)) \leq 1 + \Upsilon(\ell(t), \ell(t), \ell(t))$ so apply Lemma 19 and use (4) to conclude that $\|y(t)\| \leq \text{poly}(\|x\|, \mu, t, \hat{e}(t))$ and thus:

$$\begin{aligned} \|\psi(t)h(y(t))\| &\leq \overline{\text{sp}}(\|y(t)\|) && \text{use that } \tanh < 1 \\ &\leq \text{poly}(\|x\|, \mu, t, \hat{e}(t)) \end{aligned} \tag{5}$$

- If $\Delta(t) < 0$ then apply Lemma 30 to get that $\psi(t) \leq \frac{1}{2}e^{\Delta(t)} \leq e^{\Delta(t)}$. Apply Lemma 19 to get that $\Delta(t) \leq \Upsilon(\ell(t), \ell(t), \ell(t)) + 1 - \|y(t)\|$ and thus $\|y(t)\| \leq \Upsilon(\ell(t), \ell(t), \ell(t)) + 1 - \Delta(t)$ and thus:

$$\begin{aligned} \|\psi(t)h(y(t))\| &\leq e^{\Delta(t)} \overline{\text{sp}}(\|y(t)\|) && \text{use the bound on } \psi \\ &\leq e^{\Delta(t)} \overline{\text{sp}}(\Upsilon(\ell(t), \ell(t), \ell(t)) + 1 - \Delta(t)) \\ &&& \text{use the bound on } \|y(t)\| \\ &\leq \text{poly}(\ell(t)) e^{\Delta(t)} \text{poly}(-\Delta(t)) && \text{use that } \Upsilon \text{ is polynomial} \\ &\leq \text{poly}(\ell(t)) \\ &&& \text{use that } e^{-x} \text{poly}(x) = \mathcal{O}(1) \text{ for } x \geq 0 \text{ and fixed poly} \\ &\leq \text{poly}(\|x\|, \mu, t, \hat{e}(t)) \end{aligned} \tag{6}$$

Putting (5) and (6) together, we get that:

$$\begin{aligned} \|y(t)\| &\leq \|g(x, \mu)\| + \|e_{0,y}\| + \int_0^t \|\psi(u)h(y(u))\| + \|e_y(u)\| \, du \\ &\leq \text{sp}(\|x, \mu\|) + \int_0^t \text{poly}(\|x\|, \mu, u, \hat{e}(u)) \, du + \hat{e}(t) \end{aligned}$$

$$\leq \text{poly}(\|x\|, \mu, t, \hat{e}(t))$$

We will now analyze the behavior of the system when the error is bounded. Define $\Theta^*(\alpha, \mu) = \Theta(\alpha, \mu) + 1$. Define $\hat{\psi}(t) = \int_0^t \psi(u) du$ and note that it is a diffeomorphism since $\psi > 0$. Apply Lemma 31 to get that $y(t) = z(\hat{\psi}(t))$ for all $t \in \mathbb{R}_+$, where z satisfies for $\xi \in \hat{\psi}(\mathbb{R}_+)$:

$$z(0) = g(x, \mu) + e_{0,y} \quad z'(\xi) = h(z(\xi)) + \tilde{e}(\xi)$$

$$\text{where } \int_0^{\hat{\psi}(t)} \|\tilde{e}(\xi)\| d\xi = \int_0^t \|e_y(u)\| du$$

Assume that $x \in \text{dom } f$ and let $T \in \mathbb{R}_+$ such that $\hat{e}(T) \leq e^{-\Theta^*(\|x\|, \mu)}$. Then $\hat{e}(T) < e^{-\Theta(\|x\|, \mu)}$ and for all $t \in [0, T]$:

$$\begin{aligned} \|e_{0,y}\| + \int_0^{\hat{\psi}(t)} \|\tilde{e}\| (u) du &= \|e_{0,y}\| + \int_0^t \|e_y(u)\| du \\ &\leq \hat{e}(t) \leq e^{-\Theta(\|x\|, \mu)} \end{aligned}$$

Apply Lemma 32 to get for all $u \in [0, \hat{\psi}(T)]$:

$$\|z(u)\| \leq \Upsilon(\|x\|, \mu, u) \quad (7)$$

$$\text{if } u \geq \Omega(\|x\|, \mu) \text{ then } \|z_{1..m}(u) - f(x)\| \leq e^{-\mu} \quad (8)$$

Apply Lemmas 29 and 19 to get for all $t \in [0, T]$:

$$\begin{aligned} \ell(t) &\geq \text{mx}_1(\text{norm}_{\infty,1}(\|x\|, \mu)) + 1 - \|e_{0,\ell}\| + t - \int_0^t \|e_\ell(u)\| du \\ &\geq \max(\|x\|, \mu) + 1 + t - \hat{e}(t) \\ &\geq \max(\|x\|, \mu, t) \quad \text{using that } \hat{e}(t) \leq 1 \end{aligned}$$

Consequently, using Lemma 19, for all $t \in [0, T]$:

$$\begin{aligned} \Delta(t) &\geq \Upsilon(\ell(t), \ell(t), \ell(t)) - \|y(t)\| \\ &\geq \Upsilon(\|x\|, \mu, t) - \|y(t)\| \quad \text{using that } \ell(t) \geq \max(\|x\|, \mu, t) \\ &= \Upsilon(\|x\|, \mu, t) - \|z(\hat{\psi}(t))\| \quad \text{using that } y(t) = z(\hat{\psi}(t)) \\ &\geq 0 \quad \text{because } \hat{\psi}(t) \in [0, \hat{\psi}(T)] \end{aligned}$$

Consequently for all $t \in [0, T]$:

$$\hat{\psi}(t) = \int_0^t \psi(u) du = \int_0^t \frac{1 + \tanh(\Delta(u))}{2} du \geq \frac{t}{2}$$

Define $\Omega^*(\alpha, \mu) = 2\Omega(\alpha, \mu)$. Assume that $T \geq \Omega^*(\|x\|, \mu)$ then $\hat{\psi}(T) \geq \Omega(\|x\|, \mu)$ and thus $\|y_{1..m}(T) - f(x)\| = \|z(\hat{\psi}(T)) - f(x)\| \leq e^{-\mu}$.

Finally, $(y, \ell)(0) = g^*(x, \mu) + e_0$ where $g^* \in \text{GPVAL}$. Similarly $(y, \ell)'(t) = h^*((y, \ell)(t)) + e(t)$ where $h^* \in \text{GPVAL}$. Note again that both h^* and g^* are defined over the entire space. This concludes the proof that $f \in \text{AS}(\Omega^*, \text{poly}, \Theta^*)$.

7. Proof that ASP implies AXP

This section is devoted to prove the following: in Definition 28 we defined a class with a high degree of robustness to perturbations and related it to previous classes. However, the value $f(x)$ the system computes still depends on the initial condition (i.e. x is provided via the initial condition). Here we want robustness to errors like in Definition 28, but we also want to dynamically change the argument x during a computation, as done in Definition 9. Since these are two exigent requirements, we named this computability form as “extreme”. Here $\mathbb{1}_X$ denotes the function defined by $\mathbb{1}_X(x) = 1$ if $x \in X$ and $\mathbb{1}_X(x) = 0$ otherwise.

Theorem 33 (Strong \subseteq extreme, ASP \subseteq AXP). *$f \in \text{ASP}$ iff there exist polynomials $\Upsilon, \Lambda, \Theta$ and a constant polynomial⁸ Ω such that $f \in \text{AXC}(\Upsilon, \Omega, \Lambda, \Theta)$.*

where

Definition 34 (Extreme computability). Let $n, m \in \mathbb{N}$, $f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\Upsilon : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ and $\Omega, \Lambda, \Theta : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$. We say that f is $(\Upsilon, \Omega, \Lambda, \Theta)$ -extremely-computable if and only if there exist $\delta \geq 0$, $d \in \mathbb{N}$ and $(g : \mathbb{R}^d \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^d) \in \text{GPVAL}$ such that for any $x \in C^0(\mathbb{R}_+, \mathbb{R}^n)$, $\mu \in C^0(\mathbb{R}_+, \mathbb{R}_+)$, $y_0 \in \mathbb{R}^d$, $e \in C^0(\mathbb{R}_+, \mathbb{R}^d)$ there exists (a unique) $y : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ satisfying for all $t \in \mathbb{R}_+$:

- $y(0) = y_0$ and $y'(t) = g(t, y(t), x(t), \mu(t)) + e(t)$
- $\|y(t)\| \leq \Upsilon \left(\sup_\delta \|x\| (t), \sup_\delta \mu(t), \|y_0\| \mathbb{1}_{[1, \delta]}(t) + \int_{\max(0, t-\delta)}^t \|e(u)\| du \right)$
- For any $I = [a, b]$, if there exist $\bar{x} \in \text{dom } f$ and $\check{\mu}, \hat{\mu} \geq 0$ such that for all $t \in I$:

$$\mu(t) \in [\check{\mu}, \hat{\mu}] \text{ and } \|x(t) - \bar{x}\| \leq e^{-\Lambda(\|\bar{x}\|, \hat{\mu})} \text{ and } \int_a^b \|e(u)\| du \leq e^{-\Theta(\|\bar{x}\|, \hat{\mu})}$$

then

$$\|y_{1..m}(u) - f(\bar{x})\| \leq e^{-\hat{\mu}} \text{ whenever } a + \Omega(\|\bar{x}\|, \hat{\mu}) \leq u \leq b.$$

We denote by $\text{AXC}(\Upsilon, \Omega, \Lambda, \Theta)$ the set of $(\Upsilon, \Omega, \Lambda, \Theta)$ -extremely-computable functions and by AXP the set of (poly, poly, poly, poly)-extremely-computable functions.

Actually we prove the implication from left to right. The equivalence will follow from other sections.

⁸ $\Omega(x) = c$ for all x for some constant c .

7.1. Some remarks

A very common pattern in signal processing is known as “sample and hold”, where we have a variable signal and we would like to apply some process to it. Unfortunately, the processor often assumes (almost) constant input and does not work in real time (analog-to-digital converters are a typical example). In this case, we cannot feed the signal directly to the processor so we need some black box that samples the signal to capture its value, and holds this value long enough for the processor to compute its output. This process is usually used in a τ -periodic fashion: the box samples for time δ and holds for time $\tau - \delta$.

Lemma 35 (Sample and hold). *There is a family of functions $\text{sample}_{I,\tau}(t, \mu, x, g) \in \text{GPVAL}$, where $t \in \mathbb{R}, \mu, \tau \in \mathbb{R}_+, x, g \in \mathbb{R}, I = [a, b] \subsetneq [0, \tau]$, with the following property: let $\tau \in \mathbb{R}_+, I = [a, b] \subsetneq [0, \tau], y : \mathbb{R}_+ \rightarrow \mathbb{R}, y_0 \in \mathbb{R}, x, e \in C^0(\mathbb{R}_+, \mathbb{R})$ and $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing function. Suppose that for all $t \in \mathbb{R}_+$:*

$$y(0) = y_0 \quad y'(t) = \text{sample}_{I,\tau}(t, \mu(t), y(t), x(t)) + e(t)$$

Then:

$$|y(t)| \leq 2 + \int_{\max(0, t-\tau-|I|)}^t |e(u)| du + \max \left(|y(0)| \mathbb{1}_{[0,b]}(t), \sup_{\tau+|I|} |x(t)| \right)$$

Furthermore:

- if $t \notin I \pmod{\tau}$ then $|y'(t)| \leq e^{-\mu(t)} + |e(t)|$
- for $n \in \mathbb{N}$, if there exist $\bar{x} \in \mathbb{R}$ and $\nu, \nu' \in \mathbb{R}_+$ such that $|\bar{x} - x(t)| \leq e^{-\nu}$ and $\mu(t) \geq \nu'$ for all $t \in n\tau + I$ then

$$|y(n\tau + b) - \bar{x}| \leq \int_{n\tau+I} |e(u)| du + e^{-\nu} + e^{-\nu'}$$

- for $n \in \mathbb{N}$, if there exist $\tilde{x}, \hat{x} \in \mathbb{R}$ and $\nu \in \mathbb{R}_+$ such that $x(t) \in [\tilde{x}, \hat{x}]$ and $\mu(t) \geq \nu$ for all $t \in n\tau + I$ then

$$y(n\tau + b) \in [\tilde{x} - \varepsilon, \hat{x} + \varepsilon]$$

where $\varepsilon = 2e^{-\nu} + \int_{n\tau+I} |e(u)| du$.

- for any $J = [c, d] \subseteq \mathbb{R}_+$, if there exist $\nu, \nu' \in \mathbb{R}_+$ and $\bar{x} \in \mathbb{R}$ such that $\mu(t) \geq \nu'$ for all $t \in J$ and $|x(t) - \bar{x}| \leq e^{-\nu}$ for all $t \in J \cap (n\tau + I)$ for some $n \in \mathbb{N}$, then

$$|y(t) - \bar{x}| \leq e^{-\nu} + e^{-\nu'} + \int_{t-\tau-|I|}^t |e(u)| du$$

for all $t \in [c + \tau + |I|, d]$.

- if there exists $\Omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for any $J = [a, b]$ and $\bar{x} \in \mathbb{R}$ such that for all $\nu \in \mathbb{R}_+$, $n \in \mathbb{N}$ and $t \in (n\tau + I) \cap [a + \Omega(\nu), b]$ we have $|\bar{x} - x(t)| \leq e^{-\nu}$, then

$$|y(t) - \bar{x}| \leq e^{-\nu}$$

for all $t \in [a + \Omega^*(\nu), b]$ where

$$\Omega^*(\nu) = \max(\Omega(\nu + \ln 3), \mu^{-1}(\nu + \ln 3)) + \tau + |I|.$$

PROOF. Let $n \in \mathbb{N}$. Apply Lemma 36, Lemma 44 and Remark 43 to get that:

- For all $t \in I_n = [n\tau + a, n\tau + b]$:

$$y'(t) = \phi(t) \text{reach}(\tilde{\mu}(t), y(t), x(t)) + e(t)$$

where $\int_{I_n} \phi \geq 1$. Since $|x(t) - 0| \leq \sup_{u \in I_n} |x(u)|$ and

$$\int_{I_n} \phi \tilde{\mu} = \int_{I_n} \phi \frac{1 + \mu}{|I|} \geq 1$$

then

$$\begin{aligned} |y(n\tau + b) - 0| &\leq \sup_{I_n} |x(u)| + \int_{I_n} |e(u)| du + e^{-1} \\ &\leq 1 + \sup_{u \in I_n} |x(u)| + \int_{I_n} |e(u)| du. \end{aligned}$$

- For all $t \in [n\tau + b, (n+1)\tau + a]$:

$$|y'(t)| \leq |e(t)| + e^{-\hat{\mu}(t)} \leq |e(t)| + e^{-\ln(\tau - |I|)}$$

thus

$$\begin{aligned} |y(t) - 0| &\leq \int_{n\tau + b}^t |e(u)| du + (\tau - |I|) e^{-\ln(\tau - |I|)} \\ &\quad + 1 + \sup_{u \in I_n} |x(u)| + \int_{I_n} |e(u)| du \\ &\leq 2 + \sup_{u \in I_n} |x(u)| + \int_{n\tau + a}^t |e(u)| du. \end{aligned}$$

- For all $t \in I_{n+1}$:

$$y'(t) = \text{reach}(\phi(t)\tilde{\mu}(t), y(t), x(t))$$

where $\int_{I_n} \phi \geq 1$. Since $|x(t) - 0| \leq \sup_{u \in I_{n+1}} |x(u)|$ then

$$|y(t) - 0| \leq \max \left(\sup_{u \in [(n+1)\tau + a, t]} |x(u)|, |y((n+1)\tau + a) - 0| \right)$$

$$\begin{aligned}
& + \int_{(n+1)\tau+a}^t |e| \\
& \leq 2 + \sup_{u \in [n\tau+a, t]} |x(u)| + \int_{n\tau+a}^t |e(u)| du.
\end{aligned}$$

Note that this analysis is a bit subtle: the first point *does not* give a bound on $|y(t)|$ over I_n , it only gives a bound on $|y(n\tau+b)|$. On the contrary the two other points give bounds on $|y(t)|$ over $[n\tau+b, (n+1)\tau+b]$ which cover the whole period so by correctly putting everything together, we get that for all $|y(t)| \leq 2 + \sup_{u \in [t, t-\tau-|I|] \cap \mathbb{R}_+} |x(u)| + \int_{t-\tau-|I|}^t |e(u)| du$ for all $t \geq b$. The case of the initial segment is similar in aspect but uses the other result from Lemma 44:

- For all $t \in [0, a]$:

$$|y'(t)| \leq |e(t)| + e^{-\hat{\mu}(t)} \leq |e(t)| + e^{-\ln(\tau-|I|)}$$

thus

$$|y(t)| \leq \int_0^t |e(u)| du + ae^{-\ln(\tau-|I|)} + |y_0| \leq \int_0^t |e(u)| du + 1 + |y_0|.$$

- For all $t \in [a, b]$:

$$y'(t) = \text{reach}(\phi(t)\check{\mu}(t), y(t), x(t)) + e(t)$$

where $\int_{I_n} \phi \geq 1$. Since $|x(t) - 0| \leq \sup_{u \in [a, t]} |x(u)|$ then

$$\begin{aligned}
|y(t) - 0| & \leq \max\left(\sup_{u \in [a, t]} |x(u)|, |y(a) - 0|\right) + \int_a^t |e(u)| du \\
& \leq 1 + \int_0^t |e(u)| du + \max(|y_0|, \sup_{u \in [a, t]} |x(u)|).
\end{aligned}$$

Finally, we get that for all $t \in \mathbb{R}_+$:

$$|y(t)| \leq 2 + \int_{t-\tau-|I|}^t |e(u)| du + \max\left(|y(0)|\mathbb{1}_{[0, b]}(t), \sup_{\tau+|I|} |x|(t)\right)$$

The first extra statement is a trivial consequence of Lemma 36 and the fact that $\check{\mu}(t) \geq \mu(t)$.

The second extra statement has mostly been proved already and uses Lemma 36 and Lemma 44 again. Let $n \in \mathbb{N}$, assume there exist $\bar{x} \in \mathbb{R}$ and $\nu \in \mathbb{R}_+$ such as described. For all $t \in I_n = [n\tau+a, n\tau+b]$ we have

$$y'(t) = \phi(t) \text{reach}(\check{\mu}(t), y(t), x(t)) + e(t)$$

where $\int_{I_n} \phi \geq 1$. Since $|x(t) - \bar{x}| \leq e^{-\nu}$ and $\int_{I_n} \phi \check{\mu} = \int_{I_n} \phi \frac{1+\mu}{|I|} \geq \nu'$ then

$$|y(n\tau + b) - \bar{x}| \leq e^{-\nu} + \int_{I_n} |e(u)| du + e^{-\nu'}.$$

The third statement is a consequence of the previous one: since $n\tau + I$ is a compact set and x is a continuous function, it admits a maximum over $n\tau + I$. Apply the previous statement to $\frac{\bar{x} + \sup_{n\tau + I} x}{2} \geq \bar{x}$ to conclude.

The last extra statement requires more work. Let $\nu \geq 0$ and $n \in \mathbb{N}$ such that $n\tau + a \geq \Omega(\nu)$. Apply Lemma 36, Remark 43 and Lemma 44 to get that:

- For all $t \in I_n$:

$$y'(t) = \phi(t) \text{ reach}(\check{\mu}(t), y(t), x(t))$$

where $\int_{I_n} \phi \geq 1$. Since $t \geq n\tau + a \geq \Omega(\nu)$ and $t \in I_n$ then $|x(t) - \bar{x}| \leq e^{-\nu}$. And since

$$\int_{I_n} \phi \check{\mu} = \int_{I_n} \phi \frac{1+\mu}{|I|} \geq 1 + \mu(n\tau + a)$$

then

$$|y(n\tau + b) - \bar{x}| \leq e^{-\nu} + e^{-\mu(n\tau + a)}.$$

- For all $t \in [n\tau + b, (n+1)\tau + a]$:

$$|y'(t)| \leq e^{-\hat{\mu}(t)} \leq e^{-\hat{\mu}(n\tau + a)}$$

thus

$$\begin{aligned} |y(t) - \bar{x}| &\leq (\tau - |I|)e^{-\hat{\mu}(n\tau + a)} + e^{-\nu} + e^{-\mu(n\tau + a)} \\ &\leq e^{-\nu} + 2e^{-\mu(n\tau + a)}. \end{aligned}$$

- For all $t \in I_{n+1}$:

$$y'(t) = \phi(t) \text{ reach}(\check{\mu}(t), y(t), x(t))$$

where $\int_{I_n} \phi \geq 1$. Since $t \geq n\tau + a \geq \Omega(\nu)$ and $t \in I_n$ then $|x(t) - \bar{x}| \leq e^{-\nu}$. Thus

$$|y(t) - \bar{x}| \leq \max(e^{-\nu}, |y((n+1)\tau + a) - \bar{x}|) \leq e^{-\nu} + 2e^{-\mu(n\tau + a)}.$$

Finally, we get that

$$|y(t) - \bar{x}| \leq e^{-\nu} + 2e^{-\mu(n\tau + a)}$$

for all $t \in [n\tau + b, (n+1)\tau + b]$.

Define

$$\Omega^*(\nu) = \max(\Omega(\nu + \ln 3), \mu^{-1}(\nu + \ln 3)) + \tau + |I|.$$

Let $\nu \geq 0$ and $t \geq \Omega^*(\nu)$. Let $n \in \mathbb{N}$ such that $t \in [n\tau + b, (n+1)\tau + b]$. Then

$$\begin{aligned} n\tau + a &= (n+1)\tau + b - \tau - |I| \\ &\geq t - \tau - |I| \end{aligned}$$

$$\begin{aligned} &\geq \Omega^*(\nu) - \tau - |I| \\ &\geq \Omega(\nu + \ln 3). \end{aligned}$$

By the previous reasoning, we get that $|y(t) - \bar{x}| \leq e^{-\nu} + 2e^{-\mu(n\tau+a)}$. And since

$$\begin{aligned} n\tau + a &\geq \Omega^*(\nu) - \tau - |I| \\ &\geq \mu^{-1}(\nu + \ln 3) \end{aligned}$$

then $\mu(n\tau + a) \geq \nu + \ln 3$. Thus $|y(t) - \bar{x}| \leq 3e^{-\nu} \leq e^{-\nu}$.

Lemma 36 (“periodic low-integral-low”, (Bournez et al., 2016)). *There is a family of functions $\text{plil}_{I,\tau} \in \text{GPVAL}$ where $\mu, \tau \in \mathbb{R}_+$, $I = [a, b] \subsetneq [0, \tau]$ and $x \in \mathbb{R}$ with the following property: there exist a constant K and ϕ such that $\text{plil}_{I,\tau}(t, \mu, x) = \phi(t, \mu, x)x$ and:*

- $\text{plil}_{I,\tau}(\cdot, \mu, x)$ is τ -periodic
- for all $t \notin I$, $|\text{plil}_{I,\tau}(t, \mu, x)| < e^{-\mu}$
- for any $\alpha : I \rightarrow \mathbb{R}_+$, $\beta : I \rightarrow \mathbb{R}$:

$$1 \leq \int_a^b \phi(t, \alpha(t), \beta(t)) dt \leq K$$

7.2. The proof

We then get to the proof of Theorem 33

PROOF. Let $(f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m) \in \text{AS}(\Upsilon, \Omega, \Theta)$ where Υ, Ω, Θ are polynomials which we assume, without loss of generality, to be increasing functions of their inputs. Apply Definition 28 to get d, h and g .

Let $e = 1 + d + m$, $x \in C^0(\mathbb{R}_+, \mathbb{R}^n)$, $\mu \in C^0(\mathbb{R}_+, \mathbb{R}_+)$, $(\nu_0, y_0, z_0) \in \mathbb{R}^e$, $(e_\nu, e_y, e_z) \in C^0(\mathbb{R}_+, \mathbb{R}^e)$ and consider the following system:

$$\begin{cases} \nu(0) = \nu_0 \\ y(0) = y_0 \\ z(0) = z_0 \end{cases} \quad \begin{cases} \nu'(t) = \text{sample}_{[0,1],4}(t, \mu^*(t), \nu(t), \mu(t) + \ln \Delta + 7) + e_\nu(t) \\ y'(t) = \text{sample}_{[1,2],4}(t, \mu^*(t), y(t), g(x(t), \nu(t))) \\ \quad + \text{plil}_{[2,3],4}(t, \mu^*(t), A(t)h(y(t))) + e_y(t) \\ z'(t) = \text{sample}_{[3,4],4}(t, \mu^*(t), z(t), y_{1..m}(t)) + e_z(t) \end{cases}$$

where

$$\begin{aligned} \Delta &= 5 & \Delta' &= \ln \Delta + 10 \\ \mu^*(t) &= \mathfrak{U}^*(1 + \text{norm}_{\infty,1}(x(t)), \nu(t) + 4) \\ A(t) &= 1 + \Omega(1 + \text{norm}_{\infty,1}(x(t)), \nu(t)) \\ \Lambda^*(\alpha, \mu) &= \Theta^*(\alpha, \mu) = \mathfrak{U}^*(\alpha, \mu + \Delta') \\ \mathfrak{U}^*(\alpha, \mu) &= \mu + \ln \Delta + \Theta(\alpha, \mu) + \ln q(\alpha + \mu) \end{aligned}$$

Let $I = [a, b]$ and assume there exist $\bar{x} \in \text{dom } f$ and $\check{\mu}, \hat{\mu} \in \mathbb{R}_+$ such that for all $t \in I$, $\mu(t) \in [\check{\mu}, \hat{\mu}]$, $\|x(t) - \bar{x}\| \leq e^{-\Lambda^*(\|\bar{x}\|, \hat{\mu})}$ and $\int_a^b \|e(u)\| du \leq e^{-\Theta^*(\|\bar{x}\|, \hat{\mu})}$. Apply Theorem 17 to g to get $q \in \mathbb{K}[\mathbb{R}]$, without loss of generality we can assume that q is an increasing function and $q \geq 1$. We will use Lemma 19 to get that $\text{norm}_{\infty,1}(x(t)) + 1 \geq \|\bar{x}\|$ because $\|x(t) - \bar{x}\| \leq 1$. Also note that $\mu^*, \Theta^*, \Lambda^*$ are increasing functions of their arguments. Let $n \in \mathbb{N}$ such that $[4n, 4n + 4] \subseteq I$ and $t \in [4n, 4n + 4]$. We will first analyse the variable ν , note that the analysis is extremely rough to simplify the proof.

- **if $t \in [4n, 4n + 1]$ then $\mu^*(t) \geq 0$ so apply Lemma 35 to get that**

$$\nu(4n + 1) \in [\check{\mu} + \ln \Delta + 7 - \varepsilon, \hat{\mu} + \ln \Delta + 7 + \varepsilon]$$

where

$$\varepsilon \leq 2e^{-0} + \int_{4n}^{4n+1} |e_\nu(u)| du \leq 3$$

because $\int_a^b \|e(t)\| dt \leq 1$. Define $\bar{\nu} = \nu(4n + 1)$, then

$$\bar{\nu} \in [\check{\mu} + \ln \Delta + 4, \hat{\mu} + \underbrace{\ln \Delta + 10}_{=\Delta'}].$$

- **if $t \in [4n + 1, 4n + 4]$ then $\mu^*(t) \geq 0$ so apply Lemma 35 to get that**

$$|\nu'(t)| \leq e^{-0} + \int_{4n+1}^t |e_\nu(u)| du$$

and thus

$$|\nu(t) - \bar{\nu}| \leq (t - 4n - 1) + \int_{4n+1}^t \|e(u)\| du \leq 4$$

because $\int_a^b \|e(t)\| dt \leq 1$. In other words

$$\nu(t) \in [\bar{\nu} - 4, \bar{\nu} + 4].$$

Furthermore for $t \in [4n + 1, 4n + 4]$ we have:

$$\mu^*(t) \geq \Theta^*(1 + \text{norm}_{\infty,1}(x(t)), \nu(t) + 4) \geq \mathcal{U}^*(\|\bar{x}\|, \bar{\nu})$$

It will also be useful to note that:

$$\begin{aligned} \Lambda^*(\|\bar{x}\|, \hat{\mu}) &= \Theta^*(\|\bar{x}\|, \hat{\mu}) \geq \mathcal{U}^*(\|\bar{x}\|, \hat{\mu} + \Delta') \\ &\geq \mathcal{U}^*(\|\bar{x}\|, \bar{\nu}) \end{aligned}$$

We can now analyse y using this property:

- **if $t \in [4n + 1, 4n + 2]$ then**

$$|\nu'(t)| \leq e^{-\mu^*(t)} + |e_\nu(t)|$$

thus

$$|\nu(t) - \bar{\nu}| \leq e^{-\mathcal{U}^*(\|\bar{x}\|, \bar{\nu})} + \int_{4n+1}^{4n+2} |e_\nu(u)| du.$$

Furthermore

$$\sup_{[4n+1, 4n+2]} \|x\| \leq \|\bar{x}\| + 1,$$

thus:

$$\begin{aligned} & \|g(\bar{x}, \bar{\nu}) - g(x(t), \nu(t))\| \\ & \leq \max(|\nu(t) - \bar{\nu}|, \|x(t) - \bar{x}\|) q(\max(\|\bar{x}\|, |\bar{\nu}|)) \\ & \leq \max\left(e^{-\Theta^*(\|\bar{x}\|, \hat{\mu})} + e^{-\mathcal{U}^*(\|\bar{x}\|, \bar{\nu})}, e^{-\Lambda^*(\|\bar{x}\|, \hat{\mu})}\right) q(\|\bar{x}\| + \bar{\nu}) \\ & \leq 2e^{-\Theta(\|\bar{x}\|, \bar{\nu}) - \ln \Delta} \end{aligned}$$

Also note that

$$\left\| y'(t) - \text{sample}_{[1,2],4}(t, \mu^*(t), y(t), g(x(t), \nu(t))) \right\| \leq e^{-\mu^*(t)}$$

by Lemma 36. So we can apply Lemma 35 to get that

$$\begin{aligned} \|y(4n+2) - g(\bar{x}, \bar{\nu})\| & \leq 2e^{-\Theta(\|\bar{x}\|, \bar{\nu}) - \ln \Delta} + e^{-\mathcal{U}^*(\|\bar{x}\|, \bar{\nu})} \\ & \quad + \int_{4n+1}^{4n+2} \|e(u)\| du \\ & \leq 4e^{-\Theta(\|\bar{x}\|, \bar{\nu}) - \ln \Delta}. \end{aligned}$$

- **if** $t \in [4n+2, 4n+3]$ **then** apply Lemmas 35 and 36 to get ϕ such that $\int_{4n+2}^{4n+3} \phi(u) du \geq 1$ and

$$\|y'(t) - \phi(t)A(t)h(y(t))\| \leq e^{-\mu^*(t)} + \|e_y(t)\|.$$

Define $\psi(t) = \int_{4n+2}^t \phi(u)A(u)du$ then $\psi(4n+3) \geq \Omega(\|\bar{x}\|, \bar{\nu})$ since $A(u) \geq \Omega(\|\bar{x}\|, \bar{\nu})$ for $u \in [4n+2, 4n+3]$. Apply Lemma 31 over $[4n+2, 4n+3]$ to get that $y(t) = w(\psi(t))$ where w satisfies

$$w(0) = y(4n+2), \quad w'(\xi) = h(w(\xi)) + \tilde{e}(\xi)$$

where $\tilde{e} \in C^0(\mathbb{R}_+, \mathbb{R}^d)$ satisfies

$$\int_0^{\psi(t)} \|\tilde{e}(\xi)\| d\xi = \int_{4n+2}^t \|e_y(u)\| du \leq e^{-\Theta^*(\|\bar{x}\|, \hat{\mu})} \leq e^{-\Theta(\|\bar{x}\|, \bar{\nu}) - \ln \Delta}.$$

Hence $\|w(0) - g(\bar{x}, \bar{\nu})\| \leq 4e^{-\Theta(\|\bar{x}\|, \bar{\nu}) - \ln \Delta}$ from the result above. In other words:

$$w(0) = g(\bar{x}, \bar{\nu}) + \tilde{e}_0, \quad w'(t) = g(w(t)) + \tilde{e}(t)$$

where

$$\|\tilde{e}_0\| + \int_0^{\psi(t)} \|e(u)\| du \leq 5e^{-\Theta(\|\bar{x}\|, \bar{\nu}) - \ln \Delta} \leq e^{-\Theta(\|\bar{x}\|, \bar{\nu})}$$

because $\Delta \geq 5$. Apply Definition 28 to get that

$$\|w_{1..m}(\psi(4n+3)) - f(\bar{x})\| \leq e^{-\bar{\nu}}$$

since $\psi(4n+3) \geq \Omega(\|\bar{x}\|, \bar{\nu})$.

- **if** $t \in [4n+3, 4n+4]$ **then** $\|y'(t)\| \leq e^{-\mu^*(t)} + \|e_y(t)\|$ **thus**

$$\begin{aligned} \|y(t) - y(4n+3)\| &\leq e^{-\mathcal{U}^*(\|\bar{x}\|, \bar{\nu})} + \int_{4n+3}^t \|e_y(u)\| du \\ &\leq 2e^{-\bar{\nu}} \end{aligned}$$

so $\|y_{1..m}(t) - f(\bar{x})\| \leq 3e^{-\bar{\nu}}$.

Note that the above reasoning is also true for the last segment $[4n, b] \subseteq I$ in which case the result only applies up to time b of course. In other words, the results apply as long as $t \in [4n, 4+4] \cap I$ and $4n \geq a$. From this we conclude that if $t \in [a+4, b] \cap [4n+3, 4n+3]$ for some $n \in \mathbb{N}$ then $\|y_{1..m}(t) - f(\bar{x})\| \leq 3e^{-\bar{\nu}}$. Apply Lemma 35 to get, using that $\bar{\nu} \geq \bar{\mu} + \ln \Delta$ and $\Delta \geq 5$, that for all $t \in [a+5, b]$:

$$\begin{aligned} \|z(t) - f(\bar{x})\| &\leq 3e^{-\bar{\nu}} + e^{-\mathcal{U}^*(\|\bar{x}\|, \bar{\nu})} + \int_{t-5}^t \|e(u)\| du \leq 5e^{-\bar{\nu}} \\ &\leq e^{-\bar{\mu}} \end{aligned}$$

To complete the proof, we must also analyse the norm of the system. As a shorthand, we introduce the following notation:

$$\text{int}_\delta^+ \alpha(t) = \int_{\max(0, t-\delta)}^t \alpha(u) du$$

Apply Lemma 35 to get that:

$$\begin{aligned} |\nu(t)| &\leq 2 + \int_{\max(0, t-5)}^t |e_\nu(u)| du + \max(|\nu_0| \mathbb{1}_{[0,4]}(t), \sup_5 |\mu + \ln \Delta + 7|(t)) \\ &\leq \text{poly}(|\nu_0| \mathbb{1}_{[0,5]}(t) + \text{int}_5^+ |e_\nu|(t), \sup_5 \mu(t)) \end{aligned}$$

The analysis of y is a bit more painful, as it uses both results about the sampling function and the strongly-robust system we are simulating. Let $n \in \mathbb{N}$, and $t \in [4n, 4n+4]$:

- **if** $t \in [4n, 4n+1]$ **then** apply Lemmas 35 and 36 to get, using that $\mu(t) \geq 0$, that $\|y'(t)\| \leq 2 + \|e(t)\|$ and thus $\|y(t) - y(4n)\| \leq 2 + \int_{4n}^t \|e(u)\| du$.

- **if** $t \in [4n + 1, 4n + 2]$ **then** using the result on ν , we have

$$\begin{aligned} \|g(x(t), \nu(t))\| &\leq \sup_{[4n+1, t]} \text{poly}(\|x\|, \nu) \\ &\leq \text{poly}(|\nu_0| \mathbb{1}_{[0,5]}(t) + \text{int}_6^+ \|e\| (t), \text{sup}_6 \mu(t), \text{sup}_1 \|x\| (t)). \end{aligned} \quad (9)$$

Apply Lemmas 35 and 36 to get, using that $\mu(t) \geq 0$ and the result on ν , that:

$$\begin{aligned} \|y(4n + 2)\| &\leq \sup_{[4n+1, 4n+2]} \|g(x, \nu)\| + 2 + \int_{4n+1}^{4n+2} \|e(u)\| du \\ &\leq \text{poly}(|\nu_0| \mathbb{1}_{[0,5]}(4n + 2) + \text{int}_6^+ \|e\| (4n + 2), \text{sup}_6 \mu(4n + 2), \\ &\quad \text{sup}_1 \|x\| (4n + 2)) \end{aligned} \quad (10)$$

and also that:

$$\begin{aligned} \|y(t)\| &\leq \max \left(\sup_{[4n+1, t]} \|g(x, \nu)\| + 2, \|y(4n + 1)\| \right) + \int_{4n+1}^t \|e(u)\| du \\ &\leq \text{poly}(|\nu_0| \mathbb{1}_{[0,5]}(t) + \text{int}_6^+ \|e\| (t), \text{sup}_6 \mu(t), \text{sup}_1 \|x\| (t), \|y(4n)\|) \end{aligned}$$

- **if** $t \in [4n + 2, 4n + 3]$ **then** apply Lemma 35, Lemmas 36, 31 and 28 to get that

$$\|y(t)\| \leq \Upsilon(0, 0, \hat{e}(\hat{A}(t)), \hat{A}(t))$$

where $\hat{A}(t) = \int_{4n+2}^t A(u) du$ and

$$\hat{e}(\hat{A}(t)) = \|y(4n + 2) - g(0, 0)\| + \int_{4n+2}^t 1 + \|e(u)\| du.$$

Since Ω is a polynomial, and using the result on ν , we get that:

$$\begin{aligned} \hat{A}(t) &\leq \sup_{[4n+2, t]} \text{poly}(\|x\|, |\nu|) \\ &\leq \text{poly}(|\nu_0| \mathbb{1}_{[0,5]}(t) + \text{int}_6^+ \|e\|, \text{sup}_6 \mu(t), \text{sup}_1 \|x\| (t)) \end{aligned}$$

and using that $4n + 2 \leq t \leq 4n + 3$:

$$\begin{aligned} \|y(4n + 2) - g(0, 0)\| &\leq \|y(4n + 2)\| + \|g(0, 0)\| \\ &\leq \text{poly}(|\nu_0| \mathbb{1}_{[0,5]}(t) + \text{int}_6^+ \|e\|, \text{sup}_7 \mu(t), \text{sup}_2 \|x\| (t)) \end{aligned}$$

And since Υ is a polynomial, we conclude that:

$$\|y(t)\| \leq \text{poly}(|\nu_0| \mathbb{1}_{[0,5]}(t) + \text{int}_6^+ \|e\| (t), \text{sup}_7 \mu(t), \text{sup}_2 \|x\| (t))$$

- **if** $t \in [4n + 3, 4n + 4]$ **then** apply Lemmas 35 and 36 to get, using that $\mu(t) \geq 0$, that $\|y'(t)\| \leq 2 + \|e(t)\|$ and thus

$$\|y(t) - y(4n + 3)\| \leq 2 + \int_{4n+3}^t \|e(u)\| du.$$

From this analysis we can conclude that for all $t \in [0, 2]$:

$$\begin{aligned} \|y(t)\| &\leq \text{poly}(|\nu_0| \mathbb{1}_{[0,5]}(t) + \text{int}_6^+ \|e\|(t), \text{sup}_6 \mu(t), \text{sup}_1 \|x\|(t), \|y(0)\|) \\ &\leq \text{poly}(|\nu_0| + \text{int}_6^+ \|e\|(t), \text{sup}_6 \mu(t), \text{sup}_1 \|x\|(t), \|y_0\|) \end{aligned}$$

and for all $n \in \mathbb{N}$ and $t \in [4n + 2, 4n + 6]$:

$$\|y(t)\| \leq \text{poly}(|\nu_0| \mathbb{1}_{[0,5]}(t) + \text{int}_9^+ \|e\|(t), \text{sup}_9 \mu(t), \text{sup}_4 \|x\|(t))$$

Putting everything together, we get for all $t \in \mathbb{R}_+$:

$$\|y(t)\| \leq \text{poly}(\|y_0, \nu_0\| \mathbb{1}_{[0,5]}(t) + \text{int}_9^+ \|e\|(t), \text{sup}_9 \mu(t), \text{sup}_4 \|x\|(t))$$

Finally apply Lemma 35 to get a similar bound on z and thus on the entire system.

8. Proof that AXP implies AOP

We can prove

Theorem 37 (Extreme \subseteq online). AXP = AOP

Actually, we prove in this section that AXP \subseteq AOP. Equality will follow from other sections.

8.1. Some remarks

We start by the following lemmas:

Lemma 38 (AXP time rescaling). *If $f \in \text{AXP}$ then there exist polynomials $\Upsilon, \Lambda, \Theta$ and a constant polynomial⁹ Ω such that $f \in \text{AXC}(\Upsilon, \Omega, \Lambda, \Theta)$.*

PROOF. We go for the shortest proof: we will show that AXP \subseteq AWP and use Theorem 21 then Theorem 27 followed by Theorem 33 which proves exactly our statement.

The proof that AXP \subseteq AWP is next to trivial since because we are given an extreme system and some input and precision, we can simply store the input and precision into some variables and feed them into the (extreme) system. We make the system autonomous by using a variable to store the time.

Let $(f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m) \in \text{AXC}(\Upsilon, \Omega, \Lambda, \Theta)$, apply Definition 34 to get δ, d and g . Let $x \in \text{dom } f$ and $\mu \in \mathbb{R}_+$, and consider the following system:

$$\begin{cases} x(0) = x \\ \mu(0) = \mu \\ \tau(0) = 0 \\ y(0) = 0 \end{cases} \quad \begin{cases} x'(t) = 0 \\ \mu'(t) = 0 \\ \tau'(t) = 1 \\ y'(t) = g(t, y(t), x(t), \mu(t)) \end{cases}$$

⁹ $\Omega(x) = c$ for all x for some constant c .

Clearly the system is of the form $z(0) = h(x, \mu)$ and $z'(t) = H(z(t))$ where h and H belong to GPVAL (and are defined over the entire space). Apply the definition to get that:

$$\|y(t)\| \leq \Upsilon(\|x\|, \mu, 0)$$

And thus the entire system is bounded by a polynomial in $\|x\|$, μ and t . Furthermore, if $t \geq \Omega(\|x\|, \mu)$ then $\|y_{1..m}(t) - f(x)\| \leq e^{-\mu}$. To conclude the proof, we need to rewrite the system as a PIVP using Theorem 14.

8.2. Reaching a value

The notion of extreme computability might seem so strong at first that one can wonder if anything is really computable in this sense. In this section, we will introduce a very useful pattern which we call “reaching a value”. This can be seen as a proof that all constant functions or generable functions are extremely-computable, and this pattern will be used as a basic block to build more complicated extremely-computable functions. As an introductory example, consider the system:

$$y'(t) = \alpha - y(t)$$

This system can be shown to converge to α whatever the initial value is. In this section we extend this system in several non-trivial ways. In particular, we want to ensure a certain rate of convergence in all situations and we want to make this system robust to perturbations. In other words, we want to analyse:

$$y'(t) = \alpha(t) - y(t) + e(t)$$

where $e(t)$ is a perturbation and $\alpha(t) \approx \alpha$.

Definition 39 (Reach ODE). Let $T > 0$, $I = [0, T]$, $g, E : I \rightarrow \mathbb{R}$, $\phi : I \rightarrow \mathbb{R}_+^*$. Define (11) as the following differential equation for $t \in I$,

$$\begin{cases} y'(t) = \phi(t)X_3(g(t) - y(t)) + E(t) \\ y(0) = y_0 \end{cases} \quad \text{where } X_3(u) = u + u^3 \quad (11)$$

Lemma 40 (Reach ODE: integral error). Let $T > 0$, $I = [0, T]$, $g, E \in C^0(I, \mathbb{R})$, $\phi \in C^0(I, \mathbb{R}_+^*)$. Assume that there exist $\eta > 0$ and $\bar{g} \in \mathbb{R}$ such that for all $t \in I$ we have $|g(t) - \bar{g}| \leq \eta$. Then the solution y to (11) exists over I and satisfies:

$$|y(T) - \bar{g}| \leq \eta + \int_0^T |E(t)| dt + \frac{1}{\sqrt{\exp(2 \int_0^T \phi(u) du) - 1}}$$

Furthermore, for any $t \in I$:

$$|y(t) - \bar{g}| \leq \max(\eta, |y(0) - \bar{g}|) + \int_0^t |E(u)| du$$

PROOF. Write $f(t, x) = E(t) + \phi(t)X_3(g(t) - x)$, then $y'(t) = f(t, y(t))$. Define $I(t) = \int_0^t |E(u)|du$ and consider:

$$\begin{aligned} f_+(t, x) &= |E(t)| + \phi(t)X_3(\bar{g} + \eta - (x - I(t))) \\ f_-(t, x) &= -|E(t)| + \phi(t)X_3(\bar{g} - \eta - (x + I(t))) \end{aligned}$$

Since X_3 and I are increasing functions, it is easily seen that

$$f_-(t, x) \leq f(t, x) \leq f_+(t, x).$$

By a classical result of differential inequalities, we get that

$$y_-(t) \leq y(t) \leq y_+(t)$$

where $y_-(0) = y_+(0) = y(0)$ and $y'_\pm(t) = f_\pm(t, y_\pm(t))$. Now realize that:

$$\begin{aligned} y'_+(t) - I'(t) &= \phi(t)X_3(\bar{g} + \eta - (y_+(t) - I(t))) \\ y'_-(t) + I'(t) &= \phi(t)X_3(\bar{g} - \eta - (y_-(t) + I(t))) \end{aligned}$$

which are two instances of the following differential equation:

$$x(0) = x_0 \quad x'(t) = \phi(t)X_3(x_\infty - x(t))$$

Since ϕ and X_3 are continuous, this equation has a unique solution by the Cauchy-Lipschitz theorem and one can check that the following is a solution:

$$x(t) = x_\infty + \frac{x_0 - x_\infty}{\underbrace{\sqrt{(e^{2 \int_0^t \phi(u)du} - 1)(1 + (x_0 - x_\infty)^2) + 1}}_{:=\alpha(x_0, x_\infty, t)}}$$

Furthermore, one can check that for any $a, b \in \mathbb{R}$ and any $t > 0$:

- $|\alpha(a, b, t)| \leq \frac{1}{\sqrt{e^{2 \int_0^t \phi(u)du} - 1}}$
- $\min(0, a - b) \leq \alpha(a, b, t) \leq \max(0, a - b)$

It follows that:

$$\begin{aligned} \bar{g} - \eta - I(t) + \alpha(y(0), \bar{g} - \eta, t) &\leq y(t) \leq \bar{g} + \eta + I(t) + \alpha(y(0), \bar{g} + \eta, t) \\ -\eta - I(t) + \alpha(y(0), \bar{g} - \eta, t) &\leq y(t) - \bar{g} \leq \eta + I(t) + \alpha(y(0), \bar{g} + \eta, t) \end{aligned}$$

Using the first inequality on α we get that:

$$-\eta - I(t) - \frac{1}{\sqrt{e^{2 \int_0^t \phi(u)du} - 1}} \leq y(t) - \bar{g} \leq \eta + I(t) + \frac{1}{\sqrt{e^{2 \int_0^t \phi(u)du} - 1}}$$

Which proves the first result. And using the second inequality we get that:

$$-\eta - I(t) + \min(0, y(0) - (\bar{g} - \eta)) \leq y(t) - \bar{g} \leq \eta + I(t) + \max(0, y(0) - (\bar{g} + \eta))$$

This proves the second result by case analysis.

Sometimes though, the previous lemma lacks some precision. In particular when ϕ is never close to 0, where the intuition tells us that we should be able to replace $\int_0^t |E(u)|du$ with some bound that does not depend on t . The next lemma focuses on this case exclusively.

Lemma 41 (Reach ODE: worst error). *Let $T > 0$, $I = [0, T]$, $g, E : I \rightarrow \mathbb{R}$, $\phi : I \rightarrow \mathbb{R}_+^*$. Assume that there exist $\eta, \phi_{min}, E_{max} > 0$ and $\bar{g} \in \mathbb{R}$ such that*

- For all $t \in I$, $|g(t) - \bar{g}| \leq \eta$.
- For all $t \in I$, $|E(t)| \leq E_{max}$
- For all $t \in I$, $\phi(t) \geq \phi_{min}$

Then the solution y to (11) exists over I and satisfies for all $t \in I$:

$$|y(t) - \bar{g}| \leq \eta + \frac{E_{max}}{\phi_{min}} + \frac{1}{\sqrt{\exp(2 \int_0^t \phi(u)du) - 1}}$$

PROOF. Define $\psi(t) = \int_0^t \phi(u)du$ for $t \in I$. Since $\phi(t) \geq \phi_{min} > 0$ then ψ is an increasing function and admits an inverse ψ^{-1} . Define for all $\xi \in [0, \psi(T)]$:

$$z_\infty(\xi) = g(\psi^{-1}(\xi)) \quad \text{and} \quad z(\xi) = y(\psi^{-1}(\xi)).$$

One sees that z satisfies

$$z'(\xi) = \underbrace{X_3(z_\infty(\xi) - z(\xi)) + \frac{E(\psi^{-1}(\xi))}{\phi(\psi^{-1}(\xi))}}_{:=f(\xi, z(\xi))}$$

for $\xi \in [0, \psi(T)]$ and $z(0) = y(0)$. Furthermore, for all such ξ :

$$|z_\infty(\xi) - \bar{g}| \leq \eta \quad \text{and} \quad \left| \frac{E(\psi^{-1}(\xi))}{\phi(\psi^{-1}(\xi))} \right| \leq \frac{E_{max}}{\phi_{min}}.$$

Define $\alpha = \frac{E_{max}}{\phi_{min}}$,

$$f_+(x) = X_3(\bar{g} + \eta - x) + \alpha \quad \text{and} \quad f_-(x) = X_3(\bar{g} - \eta - x) - \alpha.$$

One can check that $f_-(x) \leq f(\xi, x) \leq f_+(x)$ for any ξ and x . Consider the solutions z_- and z_+ to $z'_- = f_-(z_-)$ and $z'_+ = f_+(z_+)$ where $z_-(0) = z_+(0) = z(0) = y(0)$. By a classical result of differential inequalities, we get that $z_-(\xi) \leq z(\xi) \leq z_+(\xi)$. By shifting the solutions, both are instances of a system of the form:

$$x(0) = x_0 \quad x'(t) = -X_3(x(t)) + \varepsilon$$

Since $x \mapsto -X_3(x) + \varepsilon$ is an increasing function, there exists a unique x_∞ such that $\varepsilon = X_3(x_\infty)$. Define $f(x) = -X_3(x) + \varepsilon$ and $f^*(x) = X_3(x_\infty - x)$. One checks that $f^*(x) - f(x) = 3x_\infty(x^2 - x_\infty^2)$, thus $f^*(x) \leq f(x)$ if $x \leq x_\infty$ and $f(x) \leq f^*(x)$

if $x_\infty \leq x$. Notice that $f(x_\infty) = 0$, so by a classical result of differential equations, $x(t) - x_\infty$ must have a constant sign for the entire life of the solution (i.e. $x(t)$ cannot “cross” x_∞). Consider the solutions x_- and x_+ to $x_- = f^*(x_-)$ and $x_+ = f^*(x_+)$ where $x_-(0) = \min(x_\infty, x_0)$ and $x_+(0) = \max(x_\infty, x_0)$. Then the previous remark and a standard result guarantees that $x_-(t) \leq x(t) \leq x_+(t)$. By the existence-uniqueness theorem for ODEs, the equations $x'_\pm = f^*(x_\pm)$ have a unique solution and one can check that the following are solutions:

$$x_\pm(t) = x_\infty + \frac{x_\pm(0) - x_\infty}{\sqrt{(e^{2t} - 1)(1 + (x_\pm(0) - x_\infty)^2) - 1}}$$

We immediately deduce that

$$|x_\pm(t) - x_\infty| \leq \frac{1}{\sqrt{e^{2t} - 1}}$$

and so

$$|x(t) - x_\infty| \leq \frac{1}{\sqrt{e^{2t} - 1}}.$$

Let δ_∞ be such that $X_3(\delta_\infty) = \alpha$. Unrolling the definitions, we get that

$$|z_\pm(\xi) - \bar{g} \mp \delta_\infty \mp \eta| \leq \frac{1}{\sqrt{e^{2\xi} - 1}}.$$

So

$$|z(\xi) - \bar{g}| \leq \eta + \delta_\infty + \frac{1}{\sqrt{e^{2\xi} - 1}}.$$

And finally, since $y(t) = z(\psi(t))$, we get that

$$|y(t) - \bar{g}| \leq \eta + \delta_\infty + \frac{1}{\sqrt{e^{2 \int_0^t \phi(u) du} - 1}}.$$

To conclude, it suffices to note that if $X_3(\delta_\infty) = \alpha$ then $\delta_\infty \leq \alpha$ since $X_3(x) \geq x$ for all x .

Definition 42 (Reach function). For any $\phi \geq 0$ and $y, g \in \mathbb{R}$, define

$$\text{reach}(\phi, y, g) = 2\phi X_3(g - y) \quad \text{where } X_3(x) = x + x^3$$

Remark 43. It is useful to note that for any $\phi, \psi \in \mathbb{R}_+$ and $y, g \in \mathbb{R}$,

$$\phi \text{ reach}(\psi, y, g) = \text{reach}(\phi\psi, y, g)$$

Lemma 44 (Reach). *There exists a function $\text{reach} \in \text{GPVAL}$ with the following property: given some arbitrary $I = [a, b]$, $\phi \in C^0(I, \mathbb{R}_+)$, $g, E \in C^0(I, \mathbb{R})$, $y_0, g_\infty \in \mathbb{R}$ and $\eta > 0$ such that for all $t \in I$, $|g(t) - g_\infty| \leq \eta$, let $y : I \rightarrow \mathbb{R}$ be the solution of*

$$\begin{cases} y(0) = y_0 \\ y'(t) = \text{reach}(\phi(t), y(t), g(t)) + E(t) \end{cases}$$

Then for any $t \in I$,

$$|y(t) - g_\infty| \leq \eta + \int_a^t |E(u)| du + \exp\left(-\int_a^t \phi(u) du\right) \quad \text{whenever } \int_a^t \phi(u) du \geq 1$$

And for any $t \in I$,

$$|y(t) - g_\infty| \leq \max(\eta, |y(0) - g_\infty|) + \int_0^t |E(u)| du$$

PROOF. Apply Lemma 40 and notice that if $\int_a^t \phi(u) du \geq 1$, then:

$$\begin{aligned} \sqrt{\exp\left(\int_a^t 4\phi(u) du\right) - 1} &\geq \sqrt{(\exp\left(2\int_a^t \phi(u) du\right) + 1)(\exp\left(2\int_a^t \phi(u) du\right) - 1)} \\ &\geq \exp\left(\int_a^t \phi(u) du\right) \sqrt{e^2 - 1} \geq \exp\left(\int_a^t \phi(u) du\right) \end{aligned}$$

8.3. The proof

We then get to the proof of $\text{AWP} \subseteq \text{AOP}$.

PROOF. Apart from the issue of the input, the system is quite intuitive: we constantly feed the extreme system with the (smoothed) input and some precision. By increasing the precision with time, we ensure that the system will converge when the input is stable. However there is a small catch: over a time interval I , if we change the precision within a range $[\check{\mu}, \hat{\mu}]$ then we must provide the extreme system with precision based on $\hat{\mu}$ in order to get precision $\check{\mu}$. Since the extreme system takes time $\Omega(\|x\|, \hat{\mu})$ to compute, we need to make arrangements so that the requested precision doesn't change too much over periods of this duration to make things simpler. We will use to our advantage that Ω can always be assumed to be a constant.

Let $(f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m) \in \text{AXC}(\Upsilon, \Omega, \Lambda, \Theta)$ where $\Upsilon, \Omega, \Lambda$ and Θ are polynomials, which we can assume to be increasing functions of their arguments. Apply Lemma 38 to get $\omega > 0$ such that for all $\alpha \in \mathbb{R}^n, \mu \in \mathbb{R}_+$:

$$\Omega(\alpha, \mu) = \omega$$

Apply Definition 34 to get δ, d and g . Define:

$$\tau = \omega + 2 \quad \delta' = \max(\delta, \tau + 1)$$

Let $x \in C^0(\mathbb{R}_+, \mathbb{R}^n)$ and consider the following systems:

$$\begin{cases} x^*(0) = 0 \\ y(0) = 0 \\ z(0) = 0 \end{cases} \quad \begin{cases} x^{*'}(t) = \text{reach}(\phi(t), x^*(t), x(t)) \\ y'(t) = g(t, y(t), x^*(t), \mu(t)) \\ z'(t) = \text{sample}_{[\omega+1, \omega+2], \tau}(t, \mu(t), z(t), y_{1..m}(t)) \end{cases}$$

where

$$\phi(t) = \ln 2 + \mu(t) + \Lambda^*(2 + x_1(t)^2 + \dots + x_n(t)^2, \mu(t)) \quad \mu(t) = \frac{t}{\tau}$$

Let $t \geq 1$, since $\phi \geq 1$ then Lemma 44 gives:

$$\|x^*(t)\| \leq \sup_1 \|x\| (t) + e^{-\int_{t-1}^t \phi(u) du} \leq \sup_1 \|x\| (t) + 1$$

Also for $t \in [0, 1]$ we get that:

$$\|x^*(t)\| \leq \sup_{[0,t]} \|x\|$$

This proves that $\|x^*(t)\| \leq \sup_1 \|x\| (t) + 1$ for all $t \in \mathbb{R}_+$. From this we deduce that:

$$\begin{aligned} \|y(t)\| &\leq \Upsilon(\sup_\delta \|x^*\| (t), \sup_\delta \mu(t), 0) \\ &\leq \text{poly}(\sup_\delta \|x\| (t), t) \end{aligned}$$

Apply Lemma 35 to get that:

$$\begin{aligned} \|z(t)\| &\leq 2 + \sup_{\tau+1} \|y\| (t) \\ &\leq \text{poly}(\sup_{\delta'} \|x\| (t), t) \end{aligned}$$

Let $I = [a, b]$ and assume there exist $\bar{x} \in \text{dom } f$ and $\bar{\mu}$ such that for all $t \in I$, $\|x(t) - \bar{x}\| \leq e^{-\Lambda(\|\bar{x}\|, \bar{\mu})}$. Note that

$$2 + \sum_{i=1}^n x_i(t)^2 \geq 1 + \|x(t)\| \geq \|\bar{x}\|$$

for all $t \in I$. Let $n \in \mathbb{N}$ such that $n \geq \bar{\mu} + \ln 2$ and $[n\tau, (n+1)\tau] \subseteq I$. Note that $\mu(t) \in [n, n+1]$ for all $t \in I_n$. Apply Lemma 44, using that $\phi \geq 1$, to get that for all $t \in [n\tau + 1, (n+1)\tau]$:

$$\begin{aligned} \|x^*(t) - \bar{x}\| &\leq e^{-\Lambda^*(\|\bar{x}\|, n)} + e^{-\int_{n\tau}^t \phi(u) du} \leq 2e^{-\Lambda^*(\|\bar{x}\|, n)} \\ &\leq e^{-\Lambda(\|\bar{x}\|, \bar{\mu} + \ln 2)} \end{aligned}$$

Using the definition of extreme computability, we get that:

$$\|y_{1..m} - f(\bar{x})\| \leq e^{-\bar{\mu} + \ln 2}$$

for all $t \in [n\tau + 1 + \omega, (n+1)\tau] = [n\tau + \omega + 1, n\tau + \omega + 2]$.

Define $J = [a + (1 + \bar{\mu} + \ln 2)\tau, b] \subseteq I$. Assume that $t \in J \cap [n\tau + 1, (n+1)\tau]$ for some $n \in \mathbb{N}$, then we must have $(n+1)\tau \geq (1 + \bar{\mu} + \ln 2)\tau$ and thus $n \geq \bar{\mu} + \ln 2$ so we can apply the above reasoning to get that

$$\|y_{1..m}(t) - f(x)\| \leq e^{-\bar{\mu} + \ln 2}.$$

Furthermore, we also have

$$\mu(t) \geq \frac{(1 + \bar{\mu} + \ln 2)\tau}{\tau} \geq \bar{\mu} + \ln 2$$

for all $t \in J$. Apply Lemma 35 to conclude that for any $t \in [a + \tau + \bar{\mu} + \ln 2 + \tau + 1, b]$, we have

$$\|z(t) - f(x)\| \leq 2e^{-\bar{\mu} + \ln 2} \leq e^{-\bar{\mu}}.$$

To conclude the proof, we need to rewrite the system as a PIVP using Lemma 14. Note that this works because we only rewrite the variable y , and doing so we require that x^* be a C^1 function (which is the case) and the new initial variable will depend on $x^*(0) = 0$ which is constant.

9. Proof that AOP implies ATSP

The purpose of the current section is to show one last inclusion which, in conjunction with all the inclusions of the previous sections, closes the circle of inclusions and shows Theorem 10.

Theorem 45. $\text{AOP} \subseteq \text{ATSP}$.

PROOF. The proof is trivial: given x , we store it in a variable and run the online system. Since the input has no error, we can directly apply the definition to get that the online system converges.

Let $(f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m) \in \text{AOC}(\Upsilon, \Omega, \Lambda)$. Apply Definition 9 to get δ, d, p and y_0 . Let $x \in \text{dom } f$ and consider the following system:

$$\begin{cases} x(0) = x \\ y(0) = y_0 \end{cases} \quad \begin{cases} x'(t) = 0 \\ y'(t) = p(y(t), x(t)) \end{cases}$$

We immediately get that:

$$\|y(t)\| \leq \Upsilon(\sup_{\delta} \|x\| (t), t) \leq \Upsilon(\|x\|, t)$$

Let $\mu \in \mathbb{R}_+$ and let $t \geq \Omega(\|x\|, \mu)$, then apply Definition 9 to $I = [0, t]$ to get that

$$\|y_{1..m}(t) - f(x)\| \leq e^{-\mu}$$

since $\|x(t) - x\| = 0$.

10. Conclusion

As a conclusion, we proved actually even a stronger statement than Theorem 10, namely:

Theorem 46. *All notions of computations are equivalent, both at the computability level:*

$$\text{ALC} = \text{ATSC} = \text{AWC} = \text{AOC}$$

and at the complexity level:

$$\text{ALP} = \text{ATSP} = \text{AWP} = \text{AOP}$$

References

- Alur, R., Dill, D. L., 1990. Automata for modeling real-time systems. In: Paterson, M. (Ed.), Automata, Languages and Programming, 17th International Colloquium, ICALP90, Warwick University, England, July 16-20, 1990, Proceedings. Vol. 443 of Lecture Notes in Computer Science. Springer, pp. 322–335.
- Bournez, O., 1997. Some bounds on the computational power of piecewise constant derivative systems (extended abstract). In: ICALP. pp. 143–153.
- Bournez, O., 1999. Achilles and the Tortoise climbing up the hyper-arithmetical hierarchy. *Theoret. Comput. Sci.* 210 (1), 21–71.
- Bournez, O., Campagnolo, M. L., 2008. New Computational Paradigms. Changing Conceptions of What is Computable. Springer-Verlag, New York, Ch. A Survey on Continuous Time Computations, pp. 383–423.
- Bournez, O., Campagnolo, M. L., Graça, D. S., Hainry, E., June 2007. Polynomial differential equations compute all real computable functions on computable compact intervals. *Journal of Complexity* 23 (3), 317–335.
- Bournez, O., Graça, D., Pouly, A., Jan. 2016. On the Functions Generated by the General Purpose Analog Computer. ArXiv e-prints, submitted to Information and Computations.
URL <http://arxiv.org/abs/1602.00546>
- Bush, V., 1931. The differential analyzer. A new machine for solving differential equations. *J. Franklin Inst.* 212, 447–488.
- Calude, C. S., Pavlov, B., Apr. 2002. Coins, quantum measurements, and Turing’s barrier. *Quantum Information Processing* 1 (1-2), 107–127.
- Copeland, B. J., 1998. Even Turing machines can compute uncomputable functions. In: Calude, C., Casti, J., Dinneen, M. (Eds.), *Unconventional Models of Computations*. Springer-Verlag.
- Copeland, B. J., 2002. Accelerating Turing machines. *Minds and Machines* 12, 281–301.
- Davies, E. B., 2001. Building infinite machines. *The British Journal for the Philosophy of Science* 52, 671–682.
- Graça, D. S., 2004. Some recent developments on Shannon’s General Purpose Analog Computer. *Math. Log. Quart.* 50 (4-5), 473–485.
- Graça, D. S., Buescu, J., Campagnolo, M. L., 2009. Computational bounds on polynomial differential equations. *Appl. Math. Comput.* 215 (4), 1375–1385.
- Graça, D. S., Costa, J. F., 2003. Analog computers and recursive functions over the reals. *Journal of Complexity* 19 (5), 644–664.

- Moore, C., 5 Aug. 1996. Recursion theory on the reals and continuous-time computation. *Theoretical Computer Science* 162 (1), 23–44.
- Pour-El, M. B., 1974. Abstract computability and its relations to the general purpose analog computer. *Trans. Amer. Math. Soc.* 199, 1–28.
- Pour-El, M. B., Richards, J. I., 1989. *Computability in Analysis and Physics*. Springer.
- Ruohonen, K., 1993. Undecidability of event detection for ODEs. *Journal of Information Processing and Cybernetics* 29, 101–113.
- Ruohonen, K., 1994. Event detection for ODEs and nonrecursive hierarchies. In: *Proceedings of the Colloquium in Honor of Arto Salomaa. Results and Trends in Theoretical Computer Science (Graz, Austria, June 10-11, 1994)*. Vol. 812 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, pp. 358–371.
- Shannon, C. E., 1941. Mathematical theory of the differential analyser. *Journal of Mathematics and Physics MIT* 20, 337–354.
- Weihrauch, K., 2000. *Computable Analysis: an Introduction*. Springer.