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Recursive Analysis Characterized as a Class of Real Recursive Functions*

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Abstract. Recently, using a limit schema, we presented an analog and machine independent algebraic characterization of elementary functions over the real numbers in the sense of recursive analysis.

In a different and orthogonal work, we proposed a minimalization schema that allows to provide a class of real recursive functions that corresponds to extensions of computable functions over the integers.

Mixing the two approaches we prove that computable functions over the real numbers in the sense of recursive analysis can be characterized as the smallest class of functions that contains some basic functions, and closed by composition, linear integration, minimalization and limit schema.

1. Introduction

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Recursive analysis, also called computable analysis, has been introduced by Turing [35], Grzegorczyk [16], Lacombe [19]. It has shown to provide a very robust concept of computability, that enables to discuss most arguments of mathematical analysis from the computability point of view: see *e.g.* monograph [36].

In this framework, a function $f : \mathbb{R} \to \mathbb{R}$ over the reals is considered as computable, if there is some computable functional, or Type 2 machine, that maps any sequence quickly converging to some x to a sequence quickly converging to $f(x)$, for all x. That means that this notion of computability requests *a priori* to deal with functionals, or higher order Turing machines.

In a recent work [4, 6], extending some classes proposed by Campagnolo, Moore and Costa [10, 11, 12] by a suitable limit schema, we proved that a particular subclass of computable functions over the reals can be characterized algebraically in a machine independent way. Indeed, elementary functions in the sense of recursive analysis were characterized as the smallest class of functions that contains some basic functions, and closed by composition, linear integration, and a simple limit schema.

This results was obtained by using deeply a result from Campagnolo, Moore, and Costa [10, 11, 12] characterizing algebraically functions over the reals that extend elementar functions over the integers.

However, an algebraic and machine characterization of the whole class of computable functions over the real numbers was missing. If we were to follow the steps of the arguments of [4, 6], the point was first to be able to find a minimalization schema that could provide a result similar to the previous one for computable functions over the integers, and second to understand how and whether it could be arranged with the arguments of [4, 6], to provide such a characterization.

The first step was solved recently in paper [5]. This journal paper presents detailed proofs of the claims of [5], with several extensions. In particular, it characterizes also non-total functions. More importantly, it proves that this is indeed possible to do the second step: mix these constructions with the ones of [4, 6] to get a characterization of the whole class of computable functions over the reals. This is done by extending the constructions of [4], and in particular provides extensions of [4, 6] that allow to talk about functions defined on non-compact domains.

Indeed, computable functions over the reals are characterized in an algebraic and machine independent way as the smallest class of functions that contains some basic functions, and closed by composition, linear integration, minimalization and limit schema.

This result has several consequences. First, that proves that it is possible to define computability in the sense of recursive analysis in a machine independent way, avoiding to talk about higher order Turing machines, or functionals, nor less natural characterization such as [8].

Second, that proves that the study of mathematical concepts through recursive analysis can be investigated by talking in terms of these algebraic classes and operators, providing a rather natural continuous setting to deal with continuous problems, instead of needing to discuss continuous problems with discrete models.

Third, it provides strong connections with several analog models. Indeed, the classes from Campagnolo, Costa and Moore, are inspired from a class of functions over the reals, called real recursive functions, introduced by Moore in [21]. Real recursive functions have been shown (see [21] with corrections from Graça and Costa in [15]) to be strongly connected to functions computable by the General Purpose Analog Computer (GPAC) of Shannon [33]. GPAC is in turn an abstraction of some systems that really existed [34, 9, 7], or is an abstraction of easy realizable systems using today's electronic. Extensions of the GPAC have been discussed in [32] and [20].

Fourth, these results show that the provided class of functions does not exhibit super-Turing phenomena such as [21, 3, 2, 17, 14], and benefits from all the robustness results that have been established for computable functions in recursive analysis.

The paper is organized as follows. In Section 2, we recall some basic mathematical properties that we will use, as well as basic definitions from classical recursion theory. Section 3 recalls recursive analysis. Section 4 recalls some results established by Campagnolo, Costa and Moore. Section 5 introduced our proposed minimalization schema, and shows that adding this schema provides a class that corresponds to extensions of computable functions over the integers. Section 6 discuss some alternative to our minimalization schema. Section 7 presents our limit schema, and proves our main result.

2. Preliminaries

2.1. Mathematical preliminaries

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^{\geq 0}$ denote the set of integers, natural integers, the set of rational numbers, the set of real numbers, and the set of non-negative real numbers respectively. Given $x \in \mathbb{R}^n$, we write $\overrightarrow{x}$ to emphasize that x is a vector.

Lemma 2.1. (Bounding Lemma for Linear Differential Equations (see e.g. [1]))
For linear differential equation $\overrightarrow{x}' = A(t)\overrightarrow{x}$, if A is defined and continuous on interval $I = [a, b]$, where $a \leq 0 \leq b$, then, for all $\overrightarrow{x}_0$, the solution of $\overrightarrow{x}' = A(t)\overrightarrow{x}$ with initial condition $\overrightarrow{x}(0) = \overrightarrow{x}_0$ is defined and unique on I. Furthermore, we know that the solution satisfies

$$\|\overrightarrow{x}(t)\| \leq \|\overrightarrow{x}_0\| \exp(\sup_{\tau \in [0,t]} \|A(\tau)\| t).$$

Lemma 2.2. (Implicit Functions Theorem (see e.g. [30]))
Let $f : \mathcal{D} \times \mathcal{I} \subset \mathbb{R}^{k+1} \to \mathbb{R}$ where $\mathcal{D} \times \mathcal{I}$ is a product of closed intervals be a function of class[1] $\mathcal{C}^k$, for $k \geq 1$. Assume that for all $\overrightarrow{x} \in \mathcal{D}$, the equation $f(\overrightarrow{x}, y) = 0$ has exactly one solution y_0 and that this y_0 belongs to the interior of $\mathcal{I}$. Assume for all $\overrightarrow{x}$ that

$$\frac{\partial f}{\partial y}(\overrightarrow{x}, y_0) \neq 0$$

in the corresponding root y_0. Then function $g : \mathbb{R}^k \to \mathbb{R}$ that maps $\overrightarrow{x}$ to the corresponding root y_0 is defined over $\mathcal{D}$ and also of class $\mathcal{C}^k$.

[1]Recall that function $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$, $k, l \in \mathbb{N}$, is said to be of class $\mathcal{C}^r$ if it is r-times continuously differentiable on $\mathcal{D}$. It is said to be of class $\mathcal{C}^\infty$ if it is of class $\mathcal{C}^r$ for all r.

Lemma 2.3. (Basic fact (see e.g. [6]))
Let $F : \mathbb{R} \times \mathcal{V} \subset \mathbb{R}^{k+1} \to \mathbb{R}^l$ be a function of class $\mathcal{C}^1$, and $\beta(\overrightarrow{x}) : \mathcal{V} \to \mathbb{R}$, $K(\overrightarrow{x}) : \mathcal{V} \to \mathbb{R}$ be some continuous functions. Assume that for all t and $\overrightarrow{x}$,

$$\|\frac{\partial F}{\partial t}(t, \overrightarrow{x})\| \leq K(\overrightarrow{x}) \exp(-t\beta(\overrightarrow{x})).$$

Let $\mathcal{D}$ be the subset of the $\overrightarrow{x} \in \mathcal{V}$ with $\beta(\overrightarrow{x}) > 0$.
Then,

- for all $\overrightarrow{x} \in \mathcal{D}$, $F(t, \overrightarrow{x})$ has a limit $L(\overrightarrow{x})$ in $t = +\infty$.
- Function $L(\overrightarrow{x})$ is a continuous function.
- Furthermore
$$\|F(t, \overrightarrow{x}) - L(\overrightarrow{x})\| \leq \frac{K(\overrightarrow{x}) \exp(-t\beta(\overrightarrow{x}))}{\beta(\overrightarrow{x})}.$$

2.2. Classical Recursion Theory

Classical recursion theory deals with functions over integers. Most classes of classical recursion theory can be characterized as closures of a set of basic functions by a finite number of basic rules to build new functions [13, 31, 27, 28]: given a set $\mathcal{F}$ of functions and a set $\mathcal{O}$ of operators on functions (an operator is an operation that maps one or more functions to a new function), $[\mathcal{F}; \mathcal{O}]$ will denote the closure of $\mathcal{F}$ by $\mathcal{O}$ [13].

Proposition 2.1. (Classical settings: see e.g. [31, 27, 28])
Let f be a function from $\mathbb{N}^k$ to $\mathbb{N}$ for $k \in \mathbb{N}$. Function f is

- *elementar* iff it belongs to $\mathcal{E} = [0, S, U, +, \ominus; \text{COMP}, \text{BSUM}, \text{BPROD}]$;
- *primitive recursive* iff it belongs to $\mathcal{PR} = [0, S, U; \text{COMP}, \text{REC}]$;
- *recursive*[2] iff it belongs to $\mathcal{R}ec = [0, S, U; \text{COMP}, \text{REC}, \text{MU}]$.

A function $f : \mathbb{N}^k \to \mathbb{N}^l$ is elementar (resp: primitive recursive, recursive) iff its projections are elementar (resp: primitive recursive, recursive).

The basic functions $0, (U_i^m)_{i,m \in \mathbb{N}}, S, +, \ominus$ and the operators BSUM, BPROD, COMP, REC, MU are given by

1. 0 is the constant 0; $U_i^m : \mathbb{N}^m \to \mathbb{N}$, $U_i^m : (n_1, \dots, n_m) \mapsto n_i$; $S : \mathbb{N} \to \mathbb{N}, S : n \mapsto n + 1$; $+ : \mathbb{N}^2 \to \mathbb{N}$, $+ : (n_1, n_2) \mapsto n_1 + n_2$; $\ominus : \mathbb{N}^2 \to \mathbb{N}$, $\ominus : (n_1, n_2) \mapsto max(0, n_1 - n_2)$;
2. BSUM : bounded sum. Given f, $h = \text{BSUM}(f)$ is defined by $h : (\overrightarrow{x}, y) \mapsto \sum_{z<y} f(\overrightarrow{x}, z)$; BPROD : bounded product. Given a function f, the bounded product $h = \text{BPROD}(f)$ is defined by $h : (\overrightarrow{x}, y) \mapsto \prod_{z<y} f(\overrightarrow{x}, z)$;

[2] This class is often called *partial recursive* since it contains partial functions as opposed to the class of total recursive functions.

3. COMP : composition. Given $f_1, \ldots, f_p$ and g, $h = \text{COMP}(f_1, \ldots, f_p, g)$ is defined as the function verifying $h(\overrightarrow{x}) = g(f_1(\overrightarrow{x}), \ldots, f_p(\overrightarrow{x}))$;

4. REC : primitive recursion . Given f and g, $h = \text{REC}(f, g)$ is defined as the function verifying $h(\overrightarrow{x}, 0) = f(\overrightarrow{x})$ and $h(\overrightarrow{x}, n+1) = g(\overrightarrow{x}, n, h(\overrightarrow{x}, n))$;

5. MU : minimalization. Given a function f, function μf is defined on all $\overrightarrow{x}$ for which there is a y such that $\forall z \leq y$, $f(\overrightarrow{x}, z)$ is defined and $f(\overrightarrow{x}, y) = 0$. For such $\overrightarrow{x}$, the minimalization of f is $\mu f : \overrightarrow{x} \mapsto \inf\{y; f(\overrightarrow{x}, y) = 0\}$.

Observe that minimalization operator can actually be reinforced into a *unique* minimalization operator as follows:

Proposition 2.2. A function f from $\mathbb{N}^k$ to $\mathbb{N}^l$, for $k, l \in \mathbb{N}$, is recursive iff its projections belong to $[0, U, S; \text{COMP}, \text{REC}, \text{UMU}]$ where operator UMU is defined as follows:

1. UMU: unique minimalization. Given f, that satisfies that for all $\overrightarrow{x}$, there is at most one y with $f(\overrightarrow{x}, y)$ defined and equal to 0, the unique minimalization of f, denoted by $!\mu(f)(\overrightarrow{x})$, is defined on all $\overrightarrow{x}$ for which there is a (unique) y with $f(\overrightarrow{x}, y) = 0$. For such $\overrightarrow{x}$, $!\mu(f)(\overrightarrow{x})$ is defined as that unique y.

Proof:
The inclusion $[0, U, S; \text{COMP}, \text{REC}, \text{UMU}] \subset \mathcal{R}ec$ is easy: given f with $!\mu(f)$ defined, given $\overrightarrow{x}$, $!\mu(f)(\overrightarrow{x})$ can be computed by testing in parallel for all y whether $f(\overrightarrow{x}, y) = 0$ until one finds the correct y, if it exists.

Conversely, let ϕ be a function from $\mathcal{R}ec$. It is well known [18, 31] that ϕ can be written as $\phi = \chi \circ \mu(\psi)$ with χ and ψ in $\mathcal{E}$ and such that for all $\overrightarrow{x}$ on which ϕ is defined, there is at least a y with $\psi(\overrightarrow{x}, y) = 0$. Let σ be the elementar function defined by $\sigma(m, n) = \prod_{z<n} \psi(m, z)$. Given m, let us note $n_0 = \mu(\psi)(m)$. We have $\forall n \leq n_0$, $\sigma(m, n) \neq 0$ and $\forall n > n_0$, $\sigma(m, n) = 0$.Let $\kappa(m, n) = sgn(\max(1 \ominus \sigma(m, n), \sigma(m, n+1)))$.

We have clearly $\forall n < n_0$, $\kappa(m, n) = 1$, $\kappa(m, n_0) = 0$ and $\forall n > n_0$, $\kappa(m, n) = 1$, hence $\mu(\kappa) = !\mu(\kappa) = \mu(\psi)$. κ is an elementar function and we have $\phi = \chi \circ !\mu(\kappa)$, hence ϕ belongs to $[0, U, S; \text{COMP}, \text{REC}, \text{UMU}]$. □

We have $\mathcal{E} \subseteq \mathcal{PR} \subseteq \mathcal{R}ec$, and the inclusions are known to be strict [31, 27, 28]. If TIME(t) and SPACE(t) denote the classes of functions that are computable with time and space t, then, $\mathcal{E} = \text{TIME}(\mathcal{E})$, and $\mathcal{PR} = \text{TIME}(\mathcal{PR}) = \text{SPACE}(\mathcal{PR})$ [31, 27, 28]. Class $\mathcal{PR}$ corresponds to functions computable using *For-Next programs.* Class $\mathcal{E}$ corresponds to computable functions bounded by some iterate of the exponential function [31, 27, 28]. At most two nested For-Next loops are required for a function of class $\mathcal{E}$, whereas general functions from class $\mathcal{PR}$ may require an arbitrary high number of such nested loops.

In classical computability, more general objects than functions over the integers can be considered, in particular functionals, i.e. functions $\Phi : (\mathbb{N}^m)^{\mathbb{N}} \times \mathbb{N}^k \to \mathbb{N}^l$. A functional will be said to be *elementarily* (or *primitive recursively*, *recursively*) computable when it belongs to the corresponding[3] class.

3. Computable Analysis

The idea sustaining *computable analysis*, also called *recursive analysis*, is to define computable functions over real numbers by considering functionals over fast-converging sequences of rationals [35, 19, 16, 36].

Let $\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q}$ be the following representation[4] of rational numbers by integers:

$$\nu_{\mathbb{Q}}(\langle p, r, q \rangle) \mapsto \frac{p-r}{q+1},$$

where $\langle ., ., . \rangle : \mathbb{N}^3 \to \mathbb{N}$ is a computable bijection.

A sequence of integers $(x_i) \in \mathbb{N}^{\mathbb{N}}$ *represents a real number* x if $(\nu_{\mathbb{Q}}(x_i))$ converges quickly toward x (denoted by $(x_i) \rightsquigarrow x$) in the following sense : $\forall i, |\nu_{\mathbb{Q}}(x_i) - x| < \exp(-i)$. For a sequence of k-tuples $(\overrightarrow{x}_i) \in (\mathbb{N}^k)^{\mathbb{N}}$, we write $(\overrightarrow{x}_i) \rightsquigarrow \overrightarrow{x}$ when it holds componentwise.

Note that many sequences can represent the same real number and also that the chosen bound is arbitrary and could be replaced by another function converging fast toward 0. In particular, our notion of computability is equivalent to the one of [36], or [12].

Definition 3.1. (Recursive analysis [36])
A function $f : \mathbb{R}^k \to \mathbb{R}$ is said *computable* (or *real computable*) if there exists a recursive functional $\Phi : (\mathbb{N}^k)^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that for all $\overrightarrow{x} \in \mathbb{R}^k$, for all sequence $X = (\overrightarrow{x}_n) \in (\mathbb{N}^k)^{\mathbb{N}}$, we have $(\Phi(X, j))_j \rightsquigarrow f(\overrightarrow{x})$ whenever $X \rightsquigarrow \overrightarrow{x}$. A function $f : \mathbb{R}^k \to \mathbb{R}^l$, with $l > 1$, is said computable if all its projections are.

A function f will be said *elementary* whenever the corresponding functional Φ is elementarily computable. The class of computable (respectively elementary) functions over the reals will be denoted by $\mathcal{R}ec(\mathbb{R})$ (resp. $\mathcal{E}(\mathbb{R})$).

4. Real sub-recursive and sub-recursive functions

Campagnolo, Moore and Costa proposed in [10, 11, 12] to consider the following class, built in analogy with elementar functions over the integers.

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[3]Formally, a function f over the integers can be considered as functional $\overline{f} : (V, \overrightarrow{n}) \mapsto f(\overrightarrow{n})$. Similarly, an operator Op on functions $f_1, \dots, f_m$ over the integers can be extended to an operator over functionals by fixing first argument $\overline{Op}(F_1, \dots, F_m) : (V, \overrightarrow{n}) \mapsto Op(f_1(V, .), \dots, f_m(V, .))(\overrightarrow{n})$.
In that spirit, given some set $\mathcal{F}$ of basic functions $\mathbb{N}^k \to \mathbb{N}^l$ and a set $\mathcal{O}$ of operators on functions over the integers, we will still (abusively) denote by $[f_1, \dots, f_p; O_1, \dots, O_q]$ for the smallest class of functionals that contains basic functions $\overline{f_1}, \dots, \overline{f_p}$, plus the functional $Map : (V, n) \to V_n$, the nth element of sequence V, and which is closed by the operators $\overline{O_1}, \dots, \overline{O_q}$. For example, a functional will be said elementarily computable iff it belongs to $\mathcal{E} = [Map, \overline{0}, \overline{S}, \overline{U}, \overline{+}, \overline{\ominus}; \overline{\text{COMP}}, \overline{\text{BSUM}}, \overline{\text{BPROD}}]$.
[4]Many other natural representations of rational numbers can be chosen and provide the same class of computable functions: see [36].

Definition 4.1. ([12, 11])
Let us define $\mathcal{L}$ as being the class of functions $f : \mathbb{R}^k \to \mathbb{R}^l$, for some $k, l \in \mathbb{N}$, defined by $\mathcal{L} = [0, 1, -1, \pi, U, \theta_3; \text{COMP}, \text{LI}]$ where the basic functions 0, 1, -1, π, $(U_i^m)_{i,m\in\mathbb{N}}$, θ_3 and the schemata COMP and LI are the following:

1. $0, 1, -1, \pi$ are the corresponding constants; $U_i^m : \mathbb{R}^m \to \mathbb{R}$ are, as in the classical settings, projections: $U_i^m : (x_1, \dots, x_m) \mapsto x_i$;
2. $\theta_3 : \mathbb{R} \to \mathbb{R}$ is defined as $\theta_3 : x \mapsto x^3$ if $x \geq 0$, 0 otherwise;
3. COMP: composition is defined as in the classical settings: Given $f_1, f_2, \dots, f_p$ and g, $h = \text{COMP}(f_1, \dots, f_p; g)$ is defined by $h(\overrightarrow{x}) = g(f_1(\overrightarrow{x}), \dots, f_p(\overrightarrow{x}))$;
4. LI: linear integration. From g and h, $\text{LI}(g, h)$ is the *maximal* solution of the *linear* differential equation $\frac{\partial f}{\partial y}(\overrightarrow{x}, y) = h(\overrightarrow{x}, y) f(\overrightarrow{x}, y)$ with $f(\overrightarrow{x}, 0) = g(\overrightarrow{x})$.

   In this schema, if g goes to $\mathbb{R}^n$, $f = \text{LI}(g, h)$ goes to $\mathbb{R}^{n+1}$ and $h(\overrightarrow{x}, y)$ is a $(n+1) \times (n+1)$ matrix with elements in $\mathcal{L}$.

Class $\mathcal{L}$ includes common functions like +,sin,cos,$-$,$\times$,exp, or constants r for all $r \in \mathbb{Q}$ (see [12, 11]), but contains only total functions [11]:

Proposition 4.1. ([11])
All functions from $\mathcal{L}$ are continuous, defined everywhere, and of class $\mathcal{C}^2$.

Actually, observing the proofs from [12, 11], schema LI can be strengthened as follows:

Proposition 4.2. Class $\mathcal{L}$ is also the class of functions $f : \mathbb{R}^k \to \mathbb{R}^l$, for some $k, l \in \mathbb{N}$, defined by $\mathcal{L} = [0, 1, -1, \pi, U, \theta_3; \text{COMP}, \text{CLI}]$ where CLI is the following schema:

1. CLI: controlled linear integration. From g and h, and c, with h differentiable and norm[5] of first partial derivatives of h bounded by c, $\text{CLI}(g, h, c)$ is the maximal solution of the *linear* differential equation $\frac{\partial f}{\partial y}(\overrightarrow{x}, y) = h(\overrightarrow{x}, y) f(\overrightarrow{x}, y)$ with $f(\overrightarrow{x}, 0) = g(\overrightarrow{x})$.

   In this schema, if g goes to $\mathbb{R}^n$, $f = \text{CLI}(g, h, c)$ goes to $\mathbb{R}^{n+1}$ and $h(\overrightarrow{x}, y)$ is a $(n+1) \times (n+1)$ matrix with elements in $\mathcal{L}$.

Class $\mathcal{L}$ can be related to the class $\mathcal{E}$ of elementar functions over the integers. A *real extension* $\tilde{f}$ of a function $f : \mathbb{N}^k \to \mathbb{N}^l$ over the integers is a function $\tilde{f}$ from $\mathbb{R}^k$ to $\mathbb{R}^l$ whose restriction to $\mathbb{N}^k$ is f. Observe that a function $\tilde{f} : \mathbb{R}^k \to \mathbb{R}^l$ over the reals is an extension of a function over the integers iff its preserves integers: $\tilde{f}(\mathbb{N}^k) \subset \mathbb{N}^l$.

Definition 4.2. (Discrete Part)
Given a class $\mathcal{C}$ of real functions, we denote by $DP(\mathcal{C})$ the class of functions over the integers that have a real extension in $\mathcal{C}$.

Proposition 4.3. ([12, 11])
$\mathcal{E} = DP(\mathcal{L})$. I.e.:

[5] Say sup norm. All norms would provide equivalent results.

- If a function from $\mathcal{L}$ preserves integers, then its restriction to integers is elementar.
- Any elementar function over *the integers*, has a real extension that belongs to $\mathcal{L}$.

Actually, class $\mathcal{L}$ can also be partially related to the class $\mathcal{E}(\mathbb{R})$ of functions over the real numbers elementary in the sense of recursive analysis: any function from $\mathcal{L}$ is in $\mathcal{E}(\mathbb{R})$ [12, 11]. We proved in [4] that the inclusion is actually strict, but that adding a limit schema to class $\mathcal{L}$, allows us to capture the whole class $\mathcal{E}(\mathbb{R})$ for functions defined over a compact domain.

5. Real recursive and recursive functions

We are now going to extend the class $\mathcal{L}$ with a minimalization schema in order to get a class whose discrete part corresponds to recursive functions over the integers.

To do so, we need to introduce a zero-finding operator that permits to simulate the classical discrete minimalization schema over the integers. However, this operator needs to be stricter than a simple "return the smallest root" since this idea, investigated in [21], has shown to be the source of numerous problems, including ill-defined problems and super-Turing Zeno phenomena. These problems are discussed, and pointed in [12, 11, 23, 22, 21]. Papers [23, 25] do provide well-defined alternatives, replacing minimalizations by limit-takings. We propose here to keep to a minimalization schema, not as general as the one from [21]. Compared to the approach from Costa and Mycka, our schemata are also more restricted than theirs.

Our idea is to use the alternative UMU schema which is equivalent to schema MU for classical computability, but has real counterparts which turn out to preserve real computability. It means that in a discrete context, the search of a unique zero is sufficient to capture the whole class of discrete recursive functions, and moreover in this continuous context, computing a unique zero does not demonstrate the over-power of the standard minimalization operator.

Indeed, motivated by Proposition 2.2, by Lemma 2.2, and by results from recursive analysis about the computability of zeros (see e.g. [36] where theorems 6.3.5 and 6.3.8 state that the search of a unique zero is computable), we define our unique-zero-finding operator UMU as follows:

Definition 5.1. Given a differentiable function f from $(\mathcal{D} \times \mathcal{I}) \subset \mathbb{R}^{k+1}$ to $\mathbb{R}$ where $\mathcal{D} \times \mathcal{I}$ is a product of closed intervals, if for all $\overrightarrow{x} \in \mathcal{D}$, $y \mapsto f(\overrightarrow{x}, y)$ is a non-decreasing function with a unique root y_0 on $\mathcal{I}$ such that y_0 is in the interior of $\mathcal{I}$ and $\frac{\partial f}{\partial y}(\overrightarrow{x}, y_0) > 0$, then UMU$(f)$ is defined on $\mathcal{D}$ as follows:

$$\mathrm{UMU}(f) : \begin{cases} \mathcal{D} & \longrightarrow & \mathbb{R} \\ \overrightarrow{x} & \mapsto & y_0 \text{ such that } f(\overrightarrow{x}, y_0) = 0. \end{cases}$$

We also slightly modify CLI schema, by allowing not-necessarily maximal solutions of linear differential equations to be considered. By abuse of notation, CLI will denote this schema in what follows.

Definition 5.2. (CLI **schema)**
From g and h, and c, with h differentiable and the norm[6] of first partial derivatives of h bounded by c, $\mathrm{CLI}(g,h,c)$ is any solution *defined on a product of closed intervals* of the the linear differential equation $\frac{\partial f}{\partial y}(\overrightarrow{x},y) = h(\overrightarrow{x},y)f(\overrightarrow{x},y)$ with $f(\overrightarrow{x},0) = g(\overrightarrow{x})$.

In this schema, if g goes to $\mathbb{R}^n$, $f = \mathrm{CLI}(g,h,c)$ goes to $\mathbb{R}^{n+1}$ and $h(\overrightarrow{x},y)$ is a $(n+1)\times(n+1)$ matrix with elements in $\mathcal{L}$.

Definition 5.3. (Class $\mathcal{L}+!\mu$)
Let $\mathcal{L}+!\mu$ be the set of functions defined by

$$\mathcal{L}+!\mu = [0, 1, U, \theta_3; \mathrm{COMP}, \mathrm{CLI}, \mathrm{UMU}].$$

Remark 5.1. The previous schema CLI yields a function in the class from g,h and c in the class. The reason why the first partial derivative of h is required to be bounded by c in the schema is to ensure computability of f from computability of g, h and c. Notice that the bound is on the derivative of h, and not on h, and hence functions like exponential can still be defined. Notice, that when UMU operator is not present, it follows from Campagnolo, Moore and Costa [10, 11, 12], that there is always some such function c in the class, and hence that there is no need to require this bound.

Lemma 5.1. $\mathcal{L} \subset \mathcal{L}+!\mu$.

Proof:
(sketch) We only need to prove that constants -1 and π are in $\mathcal{L}+!\mu$. Indeed, -1 is the unique root of $x \mapsto x+1$, and $\pi = 4\arctan(1)$, where $\arctan(x)$ is the solution of linear differential equation $\arctan(0) = 0$ and $\arctan'(x) = \frac{1}{1+x^2}$, and $x \mapsto \frac{1}{1+x^2}$ can be obtained by applying UMU on $x, y \mapsto (1+x^2)y - 1$. □

Lemma 5.2. All functions from $\mathcal{L}+!\mu$ are of class $\mathcal{C}^2$ and defined on a product of closed intervals.

Proof:
By structural induction. Basic functions U, θ_3 are defined on $\mathbb{R}^k$ and of class $\mathcal{C}^2$. Now, the properties on the domain are preserved by the definition of composition, linear integration, and schema UMU. The $\mathcal{C}^2$ property is also preserved by Lemma 2.2 for schema UMU, and classical results about differential equations (see e.g. [1]) for schema CLI. □

It follows in particular, that there is no way to obtain functions such as $x \mapsto 1/x$ defined on $(0,+\infty)$. It can be shown that any restriction to a closed interval of this function is in class $\mathcal{L}+!\mu$.

Now, observe that operator UMU preserves real computability.

[6] Say sup norm. All norms would provide equivalent results.

Lemma 5.3. ([36])
Given $f : \mathcal{D} \times \mathcal{I} \longrightarrow \mathbb{R}$ real computable, if $\mathrm{UMU}(f)$ is defined, then $\mathrm{UMU}(f)$ is also real computable.

Proof:
We reprint here mostly a restatement of a (slight generalization of) Corollary 6.3.9 from [36], that we think helpful in order to understand next remark.

Write $\mathcal{I} = [a, b]$ with a and b possibly infinite, where $\mathcal{I}$ is as in the definition of schema UMU.

Given $\overrightarrow{x} \in \mathcal{D}$, let $y_0 \in \mathcal{I}$ be the unique y_0 with $f(\overrightarrow{x}, y_0) = 0$. Since $f(\overrightarrow{x}, .)$ is continuous, non-decreasing, and with a unique root, we have $f(\overrightarrow{x}, y) < 0$ for $y < y_0$, and $f(\overrightarrow{x}, y) > 0$ for $y > y_0$.

There exists $m \in \mathbb{N}$, such that $f(\overrightarrow{x}, \max(a, -m)) < 0$ and $f(\overrightarrow{x}, \min(m, b)) > 0$: one just need to take any integer m with $-m < y_0 < m$. Actually, such an m can be computed as follows.

$m = 1$
Repeat
  Compute $f_1 = f(\overrightarrow{x}, \min(b, m))$ and $f_2 = f(\overrightarrow{x}, \max(a, -m)))$ at precision $\pm 2^{-m}$
  $m = m + 1$
Until $(f_1 > 2^{-m}$ and $f_2 < -2^{-m})$
Return m

Indeed, given any integer $m_0 \in \mathbb{N}$ with $-m_0 < y_0 < m_0$, (for example $\lfloor |y_0| \rfloor + 1$), we have for all $m \geq m_0$, $f(\overrightarrow{x}, \min(m, b)) \geq f(\overrightarrow{x}, m_0) > 0$ and $f(\overrightarrow{x}, \max(a, -m)) \leq f(\overrightarrow{x}, -m_0) < 0$. Now, for m big enough (i.e. $m \geq m_0$, $2^{-m} \leq |f(\overrightarrow{x}, \max(a, -m_0))|$, and $2^{-m} \leq |f(\overrightarrow{x}, \min(b, m_0))|$) we have $f_1 > 2^{-m}$ and $f_2 < -2^{-m}$ and the process halts with an m such that$f(\overrightarrow{x}, \max(a, -m)) < 0$ and $f(\overrightarrow{x}, \min(m, b)) > 0$.

Computing y_0 then reduces to compute the unique root of function $f(\overrightarrow{x}, .)$ over a compact $[-m, m] \cap \mathcal{I}$. The fact that this is indeed computable can be seen as a consequence of the results in [36]. See [5] for a direct proof. □

Remark 5.2. The proof is non constructive in the following sense: it follows from constructions from [26] that there is no way to determine effectively from the code of f whether $\mathrm{UMU}(f)$ is defined. Now, when it is, $\mathrm{UMU}(f)$ is real computable from previous arguments.

Lemma 5.4. Given h, g and c real computable, then $f = \mathrm{CLI}(g, h, c)$ is also real computable.

Proof:
Observing carefully [12, 11], if given $\overrightarrow{x} \in \mathbb{R}^k$ and some $\overline{y} \in \mathbb{Q}$ one can bound effectively the norms of $h(\overrightarrow{x}, y)$, $f(\overrightarrow{x}, y)$, $\frac{\partial^2 f}{\partial y^2}(\overrightarrow{x}, y)$ for $|y| \leq \overline{y}$, then f will be real computable: use the constructions and bounds based on Euler's method to prove preservation of elementarity by linear integration in [12, 11], but replacing elementar bounds by computable bounds.

Now, from [36], it is known that one can bound effectively the norm of any real computable function on a closed domain, and so we only need to care about $f(\overrightarrow{x},y)$ and $\frac{\partial^2 f}{\partial y^2}(\overrightarrow{x},y)$. But the norm of $f(\overrightarrow{x},y)$ can be bounded effectively by Lemma 2.1 from bounds on the norms of $g(\overrightarrow{x})$ and $h(\overrightarrow{x},y)$ on the corresponding domain, which are computable by previous argument. Now,

$$\|\frac{\partial^2 f}{\partial y^2}(\overrightarrow{x},y)\| = \|(h^2(\overrightarrow{x},y) + \frac{\partial h}{\partial y}(\overrightarrow{x},y))f(\overrightarrow{x},y)\|,$$

hence is bounded by $(\|h^2(\overrightarrow{x},y)\| + \|c(\overrightarrow{x},y))\|) \times \|f(\overrightarrow{x},y)\|$. First factor can still be bounded effectively since $h^2(\overrightarrow{x},y)$ and $c(\overrightarrow{x},y)$ are particular real computable functions, and we just saw that second factor can be bounded effectively. □

From previous two Lemmas, the fact that basic functions are real computable and observing that composition is known to preserve real computability for functions defined over closed intervals (see [36]), we obtain:

Theorem 5.1. Every function belonging to $\mathcal{L}+!\mu$ is real computable.

Remark 5.3. As observed above, the proof is non-constructive. There is no way to obtain effectively a functional that would compute a function f from the code of f. However, when f belongs to $\mathcal{L}+!\mu$, there is one functional.

We now prove the converse direction. The following lemma is a weaker form of Lemma 7.3. It provides a "canonical extension" of all functions in $\mathcal{L}$. This terminology comes from [24] where a similar result is proved. The idea is to exhibit a function $\tilde{f}$ that matches f on each integer points, but is kept controlled with respect to x and y on each square.

Lemma 5.5. Given $f : \mathbb{R}^2 \to \mathbb{R}$ in $\mathcal{L}$, there exists $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ in $\mathcal{L}$ such that $\forall (m,n) \in \mathbb{Z}^2$, $\forall (x,y) \in \mathbb{R}^2$,

- $\tilde{f}(m,n) = f(m,n)$
- $\tilde{f}(m,y) \in [f(m,\lfloor y \rfloor), f(m,\lfloor y+1 \rfloor)]$ (or $[f(m,\lfloor y+1 \rfloor), f(m,\lfloor y \rfloor)]$).
- $\tilde{f}(x,n) \in [f(\lfloor x \rfloor,n), f(\lfloor x+1 \rfloor,n)]$ (or $[f(\lfloor x+1 \rfloor,n), f(\lfloor x \rfloor,n)]$).

Proof:
Let $\zeta = \frac{3\pi}{2}$. Let $\omega : x \mapsto \zeta\theta_3(\sin(2\pi x))$. $\forall i, \int_i^{i+1} \omega = 1$ and ω is equal to 0 on $[i+\frac{1}{2}, i+1]$ for $i \in \mathbb{Z}$. Let Ω its primitive equal to 0 at 0, and $int : x \mapsto \Omega(x-\frac{1}{2})$. Function int is a function similar to the integer part: $\forall i \in \mathbb{Z}$, $\forall x \in [i, i+\frac{1}{2}]$, $int(x) = i = \lfloor x \rfloor$. Figures 1 and 2 show graphical representations of ω and int.

Let $\Delta(i,y) = f(i,y+1) - f(i,y)$. Then for all $i \in \mathbb{Z}$, $y \in \mathbb{R}$, we have

$$\omega(y)\Delta(i,int(y)) = \begin{cases} 0 & \textit{whenever } y - \lfloor y \rfloor \geq 1/2 \\ \omega(y)\Delta(i,\lfloor y \rfloor) & \textit{otherwise.} \end{cases}$$

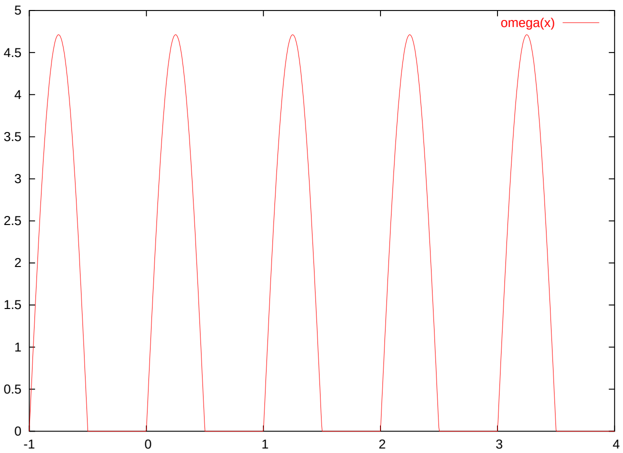

Figure 1. Graphical representation of ω

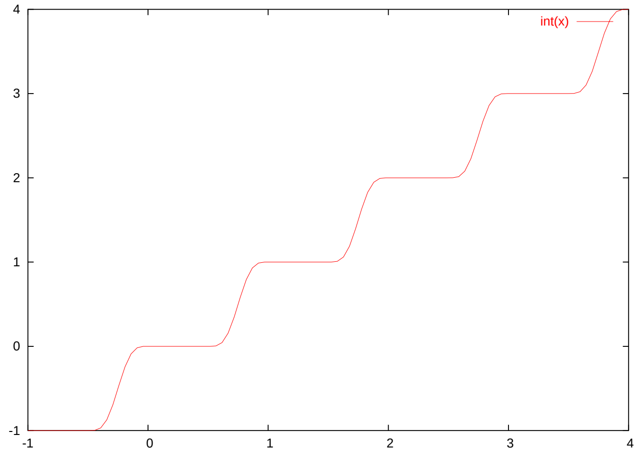

Figure 2. Graphical representation of int

Let G be the solution of the linear differential equation

$$\begin{cases} G(x,0) & = & f(x,0) \\ \frac{\partial G}{\partial y}(x,y) & = & \omega(y)\Delta(x, int(y)). \end{cases}$$

An easy induction on j then shows that $G(i,j) = f(i,j)$ for all integer j. Furthermore, by construction, $\forall i \in \mathbb{Z}$, $G(i,y)$ belongs to the interval delimited by $G(i,\lfloor y \rfloor) = f(i,\lfloor y \rfloor)$ and $G(i,\lfloor y+1 \rfloor) = f(i,\lfloor y+1 \rfloor)$.

Now, let $\tilde{f}$ be the solution of the linear differential equation

$$\begin{cases} \tilde{f}(0,j) & = & G(0,j) \\ \frac{\partial \tilde{f}}{\partial x}(x,y) & = & \omega(x)(G(int(x+1),y) - G(int(x),y)). \end{cases}$$

We have $\forall (i,j) \in \mathbb{Z}^2, \tilde{f}(i,j) = f(i,j)$. And $\forall i \in \mathbb{Z}$, $\tilde{f}(i,y)$ belongs to the interval delimited by $\tilde{f}(i,\lfloor y \rfloor) = f(i,\lfloor y \rfloor)$ and $\tilde{f}(i,\lfloor y+1 \rfloor) = f(i,\lfloor y+1 \rfloor)$. And also, $\forall j \in \mathbb{Z}$, $\tilde{f}(x,j)$ belongs to the interval delimited by $\tilde{f}(\lfloor x \rfloor, j) = f(\lfloor x \rfloor, j)$ and $\tilde{f}(\lfloor x+1 \rfloor, j) = f(\lfloor x+1 \rfloor, j)$.

□

Theorem 5.2. Every total recursive function over the integers has a real extension in $\mathcal{L}+!\mu$.

This also holds more generally for recursive functions defined on $[a,b] \cap \mathbb{N}$ for some a, b possibly infinite.

Proof:

Let ϕ be a function from $\mathcal{R}ec$. Let $\mathcal{D} = [a,b]$ be its domain, taking a and b infinite if ϕ is total.

We have $\phi = \chi \circ !\mu(\kappa)$ as in the proof of Proposition 2.2. Let

$$\iota(m,n) = 2 \times (1 \ominus \sigma(m,n)) + (1 \ominus \kappa(m,n))$$

where σ is the same as in the proof of Proposition 2.2. $\forall m \in \mathbb{N} \cap \mathcal{D}$, for $n = n_0 = !\mu(\kappa)(m,n)$, we have $\iota(m,n_0) = 1$, and before this n_0, $\iota(m,n)$ is equal to 0 and after this n_0, $\iota(m,n)$ is equal to 2. Let i be a real extension of ι in $\mathcal{L}$ given by Proposition 4.3. Let $\tilde{i}$ be the function from $\mathcal{L}$ obtained by Lemma 5.5 on $f(m,x) : m, x \mapsto i(m,x) - 1$.

$\forall m \in \mathcal{D} \cap \mathbb{N}$, there exists exactly one $y \in \mathbb{R}$ (given by $y_0 = !\mu(\kappa)(m,n)$) such that $\tilde{i}(m,y) = 0$. But, we can not directly apply schema UMU, since we have no assurance[7] that it also holds for non integer values m. However, from the constructions in the proof of Lemma 5.5, given $m \in \mathbb{N}$, we have $\tilde{i}(m,y)$ equal to -1 for $y \leq y_0 - 1$, and equal to $\Omega(y)$ for $y \in [y_0 - 1, y_0 + 1]$, where Ω is defined in that proof.

Consider $\mathcal{M}(x) = \theta_3(x+1)$. We have $\mathcal{M}(x) = 0$ if $x \leq -1$ and $\mathcal{M}(x) \geq 1$ if $x \geq 0$. Let us define $\tilde{g}$ as the solution on $[a,b] \times \mathbb{R}$ of the differential equation

$$\begin{cases} \tilde{g}(x,0) & = & -1 \\ \frac{\partial \tilde{g}}{\partial y}(x,y) & = & \alpha \mathcal{M}(\tilde{i}(x,y)). \end{cases}$$

[7]Actually, another problem is that the derivative relative to the second variable in the root point is 0.

Let us choose α (maple says $\alpha = \frac{1024}{2609}$) such that $\alpha \int_{-1}^{0} \mathcal{M}(\Omega(x))\mathrm{d}x = 1$. We have $\forall m \in \mathbb{N}$, $\tilde{g}(m,y) = 0 \Leftrightarrow y = !\mu(\kappa)(m,n)$.

Then define g as the solution on $[a,b] \times \mathbb{R}$ of the linear differential equation $g(x,0) = -1$, $\frac{\partial g}{\partial y}(x,y) = \beta\mathcal{M}(\tilde{g}(x,y))$. If we choose β adequately[8] (maple says $\beta = \frac{a\pi^4}{b\pi^4 - c\pi^2 + d}$ for some integers a, b, c, d) , we will still have $\forall m \in \mathbb{N}$, $g(m,y) = 0 \Leftrightarrow y = !\mu(\kappa)(m,n)$.

The point is that, since $\mathcal{M}$ is always non-negative, we know that $\forall x \in \mathbb{R}$, $y \mapsto \tilde{g}(x,y)$ is non-decreasing, and, because of Lemma 5.5, and from the definition of function $\mathcal{M}(x)$, it must go to infinity when y goes to infinity. Actually, it must be equal to -1 up to a certain value y_-, then be strictly increasing, and since it goes to infinity, it must have a root y_0 strictly greater than y_-. Now the derivative in this root y_0 cannot be 0 since $\mathcal{M}(x)$ is zero only when $x \leq -1$.

This g is such that $\forall x$, $\exists! y_0$ such that $g(x,y_0) = 0$ and $\frac{\partial g}{\partial y}(x,y_0) \neq 0$ and for all x, $y \mapsto g(x,y)$ is non-decreasing. We can thus apply UMU to this g. Now if we extend χ in a real function h belonging to $\mathcal{L}$ using Proposition 4.3, we have $h \circ \mathrm{UMU}(g)$ extending $\phi = \chi \circ \mu(\psi)$ and belonging to $\mathcal{L}+!\mu$. □

From previous two theorems, we obtain:

Theorem 5.3. For total functions $\mathcal{R}ec = DP(\mathcal{L}+!\mu)$. I.e:

- If a function from $\mathcal{L}+!\mu$ extends some total function over the integers, this latter function is total recursive.
- Any total recursive function over *the integers*, has a real extension that belongs to $\mathcal{L}+!\mu$.

Proof:
The second item is Theorem 5.2. The first item is immediate from Theorem 5.1: if a function f belonging to $\mathcal{L}+!\mu$ preserves integers, then a recursive function that equals f on $\mathbb{N}^k$ can easily be obtained from the functional computing f. □

Corollary 5.1. $\mathcal{L}$ is strictly included in $\mathcal{L}+!\mu$.

6. Alternative schemata

6.1. An alternative: searching a unique root on some domain

Slight modifications of the proofs in this paper can easily prove that schema UMU could be actually be replaced by the following schema.

Proposition 6.1. We have also

$$\mathcal{R}ec = DP(\mathcal{L}+!\mu_{[m,M]}).$$

where $\mathcal{L}+!\mu_{[m,M]} = [0, 1, U, \theta_3; \mathrm{COMP}, \mathrm{CLI}, !\mu_{[m,M]})]$ where $!\mu_{[m,M]}$ is the following schema:

[8]This β is in $\mathcal{L}$ since it can be obtained as $a * \pi^4 * \mathrm{UMU}(x \mapsto (b\pi^4 - c\pi^2 + d)x - 1)$.

1. Given a function f from $\mathbb{R}^{k+1}$ to $\mathbb{R}$, and two functions $m, M : \mathbb{R}^k \to \mathbb{R}$, if for all $\overrightarrow{x}$, $y \mapsto f(\overrightarrow{x}, y)$ has a unique root y on $[m(\overrightarrow{x}), M(\overrightarrow{x})]$, on which $\frac{\partial f}{\partial y}(\overrightarrow{x}, y) \neq 0$, then $!\mu_{[m,M]} f$ is defined as the function that maps $\overrightarrow{x}$ to that root for all $\overrightarrow{x}$.

This schema is obviously more difficult to apply than UMU since it requires to give bounds on the researched zero. However, it is more straightforward considering known theorems of zero-searching in recursive analysis.

6.2. An other alternative: searching the minimum of a convex function

Observing that the (always unique) minimum of non-monotone convex function is real computable, and that the zero of a non-decreasing function is the minimum of its primitive, and that this latter primitive is a non-monotone convex function, schema UMU can actually be also replaced by the following schema.

Proposition 6.2. We have also

$$\mathcal{R}ec = DP(\mathcal{L} + min\_convex).$$

where $\mathcal{L} + min\_convex = [0, 1, U, \theta_3; \text{COMP}, \text{CLI}, min\_convex)]$ and min_convex is the following schema:

1. Given a function f from $\mathbb{R}^{k+1}$ to $\mathbb{R}$, such that for all $\overrightarrow{x}$, $y \mapsto f(\overrightarrow{x}, y)$ is a convex non-monotone function, whose second derivative exists and is non-null on its minimum, then $min\_convex(f)$ is defined as the function that maps $\overrightarrow{x}$ to the minimum of $y \mapsto f(\overrightarrow{x}, y)$.

Note that the idea of considering minimum of convex functions comes partly from discussions with several people, including Manuel Campagnolo, Felix Costa and Cris Moore. The question whether this precise schema would be equivalent to previous one was raised by Manuel Campagnolo (private discussion).

7. Link with computable analysis

In [4, 6], we proved that by adding a well chosen limit operator to the class $\mathcal{L}$, it was possible to capture not only discrete elementar functions but also elementary functions in the sense of computable analysis.

If we carefully add this limit operator to the here-defined $\mathcal{L}+!\mu$ class, we now prove that one similarly obtains a class that captures not only discrete recursive functions but real recursive functions (in the sense of recursive analysis).

To do so, we consider the following schema, already considered in [6]: a polynomial β over $x \in \mathbb{R}$ is a function of the form $\beta : \mathbb{R} \to \mathbb{R}$, $\beta : x \mapsto \sum_{i=0}^{n} a_i x^i$ for some $a_0, \ldots, a_n \in \mathbb{R}$. A polynomial β over $\overrightarrow{x} = (x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1}$ is a function of the form $\beta : \mathbb{R}^{k+1} \to \mathbb{R}$, $\beta : \overrightarrow{x} \mapsto \sum_{i=0}^{n} a_i x_{k+1}^i$ for some $a_0, \ldots, a_n$ polynomial over $(x_1, \ldots, x_k) \in \mathbb{R}^k$.

Definition 7.1. (LIM_w **schema)**
Let $f : \mathbb{R} \times \mathcal{D} \subset \mathbb{R}^{k+1} \to \mathbb{R}^l$, $K : \mathcal{D} \to \mathbb{R}$ and $\beta : \mathcal{D} \to \mathbb{R}$ a *polynomial* with the following hypothesis: for all $\overrightarrow{x}$, for all $t \geq \|\overrightarrow{x}\|$,

$$\|\frac{\partial f}{\partial t}(t, \overrightarrow{x})\| \leq K(\overrightarrow{x}) \exp(-t\beta(\overrightarrow{x})).$$

Then, on every product of closed intervals $I \subset \mathbb{R}^k$ on which $\beta(\overrightarrow{x}) > 0$, $\lim_{t\to+\infty} f(t, \overrightarrow{x})$ exists by Lemma 2.3. We define F by $F(\overrightarrow{x}) = \lim_{t\to\infty} f(t, \overrightarrow{x})$. If F is of class $\mathcal{C}^2$, then we define $\mathrm{LIM}_w(f, K, \beta)$ as this function $F : I \to \mathbb{R}$.

Let us now define a new class $\mathcal{L}^*_{!\mu}$.

Definition 7.2.

$$\mathcal{L}^*_{!\mu} = [0, 1, U, \theta_3; \mathrm{COMP}, \mathrm{CLI}, \mathrm{UMU}, \mathrm{LIM}_w]$$

We have the following theorem:

Theorem 7.1. For functions of class $\mathcal{C}^2$ defined on a compact domain,

$$\mathcal{L}^*_{!\mu} = \mathcal{R}ec(\mathbb{R}).$$

The proof of this theorem will be done for one direction by structural induction, the other by applying a more general property linked to the LIM_w schema in the rest of this section.

7.1. Proof of Theorem 7.1: Upper bounds

We will now prove the first direction of the theorem, namely $\mathcal{L}^*_{!\mu} \subset \mathcal{R}ec(\mathbb{R})$.

Proposition 7.1. Every function belonging to $\mathcal{L}^*_{!\mu}$ is real computable in the sense of computable analysis.

Proof:
We have already proved that 0, 1, U, θ_3 belong to $\mathcal{R}ec(\mathbb{R})$ and that COMP, CLI and UMU preserve $\mathcal{R}ec(\mathbb{R})$. Hence, we only need to show that LIM_w also preserves $\mathcal{R}ec(\mathbb{R})$.

Let $g = \mathrm{LIM}_w(f, K, \beta)$, with f computed by recursive functional ϕ. We give the proof for f defined on $\mathbb{R} \times \mathcal{C}$ to $\mathbb{R}$ where $\mathcal{C}$ is a closed interval of $\mathbb{R}$. The general case is easy to obtain.

Let $x \in \mathbb{R}$, with $\beta(x) > 0$. Since $\beta(x)$ is a polynomial, $1/\beta(x)$ can be bounded by some integer N computable from x. Similarly, $K(x)$ can be bounded by some integer K computable from x. In a same way, the norm of x can be bounded by some integer X computable from x.

Let $(x_n) \rightsquigarrow x$. For all $i, j \in \mathbb{N}$, if we write abusively i for the constant sequence $k \mapsto i$, we have $|\nu_{\mathbb{Q}}(\phi(((i, x_n), j)) - f(i, x)| < \exp(-j)$.

By Lemma 2.3, if i is big enough ($i > \|x\|$), we have

$$\begin{aligned} |f(i, x) - g(x)| &\leq \tfrac{K \exp(-\beta(x) i)}{\beta(x)} \\ &\leq KN \exp(-\beta(x) i). \end{aligned}$$

Hence,

$$|\nu_{\mathbb{Q}}(\phi((i,x_n),j)) - g(x)| < \exp(-j) + KN\exp(-\beta(x)i).$$

If we take $j' \geq j+1$, $i' \geq N(j+1+\lceil\ln(KN)\rceil)$, we have $\exp(-j') \leq \frac{1}{2}\exp(-j)$, and $KN\exp(-\beta(x)i') \leq \frac{1}{2}\exp(-j)$. Hence g is computed by the functional

$$\psi : ((x_n), j) \mapsto \phi((\max(X, N(j+1+\lceil\ln(KN)\rceil), x_n)), j+1).$$

since for all j,

$$\|\nu_{\mathbb{Q}}(\psi((x_n),j)) - g(x)\| \leq \frac{exp(-j)}{2} + \frac{exp(-j)}{2} \leq exp(-j).$$

□

7.2. Proof of Theorem 7.1: Lower bounds

We will now prove the converse direction of the theorem: We are going to prove that for functions of class $\mathcal{C}^2$ defined on a compact domain, $\mathcal{R}ec(\mathbb{R}) \subset \mathcal{L}^*_{!\mu}$.

In fact, we are going to prove a more general proposition that more or less states that given a discrete class $C \supset \mathcal{E}$ that has some basic properties of closure and a class of real functions $\mathcal{C}$ such that $C \subseteq DP(\mathcal{C})$, then the class of recursive analysis defined with functions from C, denoted by $C(\mathbb{R})$ is included in $\mathcal{C}+\mathrm{LIM}_w$ (defined as being the class $\mathcal{C}$ with LIM_w as additional operator.

The researched inclusion $\mathcal{R}ec(\mathbb{R}) \subseteq \mathcal{L}^*_{!\mu}$ for functions of class $\mathcal{C}^2$ over compact domains will follow considering $C = \mathcal{R}ec$ and $\mathcal{C} = \mathcal{L}+!\mu$.

Formally,

Definition 7.3. Let C be a class of functions over the integers, with $\mathcal{E} \subset C$. A function $f : \mathbb{R}^k \to \mathbb{R}$ is said to belong to $C(\mathbb{R})$ as in Definition 3.1: There exists a functional[9] $\Phi \in C$ such that for all $\overrightarrow{x}$, for all sequence X, $X \rightsquigarrow \overrightarrow{x} \implies (\Phi(X,j))_j \rightsquigarrow f(\overrightarrow{x})$.

A function $f : \mathbb{R}^k \to \mathbb{R}^l$, with $l > 1$, is said to belong to $C(\mathbb{R})$ if all its projections are.

Definition 7.4. Given a class of real function $\mathcal{C} = [\mathcal{F}; \mathcal{O}]$, we will denote $\mathcal{C}+\mathrm{LIM}_w$ for the class $[\mathcal{F}; \mathcal{O}, \mathrm{LIM}_w]$.

Definition 7.5. A *modulus of continuity* of a function $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ defined over a compact domain is a function $M : \mathbb{N} \to \mathbb{N}$ such that for all $i \in \mathbb{N}$, for all x, y,

$$\|x - y\| < \exp(-M(i)) \Rightarrow \|f(x) - f(y)\| < \exp(-i).$$

More generally (the modulus of continuity of a function defined over a compact domain gives clearly a uniform modulus of continuity).

[9] In the sense of Footnote 3.

Definition 7.6. A *uniform modulus of continuity* of a function $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ defined over a closed domain is a function $M : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all integer K, $i \in \mathbb{N}$, for all $x, y \in [-K, K]^k$,

$$\|x - y\| < \exp(-M(K, i)) \Rightarrow \|f(x) - f(y)\| < \exp(-i).$$

Proposition 7.2. Let C be a class of functions over the integers, closed by composition, that contains $\mathcal{E}$, and $\mathcal{C}$ be a class of real functions that contains $\mathcal{L}$, that is closed under composition, and by taking primitives, such that $C \subseteq DP(\mathcal{C})$.

Then for functions of class $\mathcal{C}^2$ defined on a compact domain, whose derivatives have a modulus of continuity in C, $C(\mathbb{R}) \subseteq \mathcal{C} + \mathrm{LIM}_w$.

The researched lower bound $\mathcal{R}ec(\mathbb{R}) \subseteq \mathcal{L}^*_{!\mu}$ for functions of class $\mathcal{C}^2$ over compact domains indeed follows: indeed, consider $C = \mathcal{R}ec$, $\mathcal{C} = \mathcal{L}+!\mu$, and the following two well-known results, that can be seen as slight generalizations of Corollary 6.4.8 and Theorem 6.2.7 of [36], or of Theorem 2 of Chapter 1 of [29].

Lemma 7.1. Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be a function of class $\mathcal{C}^2$ defined over compact domain $\mathcal{D}$.

If f is in $C(\mathbb{R})$, then its partial derivatives also are.

Proof:

We give the proof for a function f defined on interval $[0, 1]$ to $\mathbb{R}$. The general case is easy to obtain.

Since f'' is continuous on a compact set, f'' is bounded by some constant M. By mean value theorem, we have $|f'(x) - f'(y)| \leq M|x - y|$ for all x, y.

Given $x \in [0, 1]$, and $i \in \mathbb{N}$, an approximation z of $f'(x)$ at precision $\exp(-i)$ can be computed as follows: compute n with $M \exp(-n) \leq \exp(-i)/2$. Compute y_1 a rational at most $\exp(-i-n-2)$ far from $f(x)$, and y_2 a rational at most $\exp(-i-n-2)$ far from $f(x+\exp(-n))$.

Take $z = (y_1 - y_2)/\exp(-n)$.

This is indeed a value at most $\exp(-i)$ far from $f'(x)$ since by mean value theorem there exists $\chi \in [x, x + \exp(-n)]$ such that $f'(\chi_j) = \frac{f(x+\exp(-n))-f(x)}{\exp(-n)}$. Now

$$\begin{aligned}
|z - f'(x)| &\leq \tfrac{|y_1 - f(x)|}{\exp(-n)} + \tfrac{|y_2 - f(x+\exp(-n))|}{\exp(-n)} + |\tfrac{f(x+\exp(-n))-f(x)}{\exp(-n)} - f'(x)| \\
&\leq \exp(-i-n-2)\exp(n) + \exp(-i-n-2)\exp(n) \\
&\quad +|f'(\chi_j) - f'(x)| \\
&\leq 2\exp(-i-2) + M\exp(-n) \\
&\leq \exp(-i)/2 + \exp(-i)/2 \\
&\leq \exp(-i).
\end{aligned}$$

□

The following lemma is easy.

Lemma 7.2. If $f : \mathbb{R}^k \to \mathbb{R}^l$ is defined over a product of compact intervals, f is of class $\mathcal{C}^1$, and $f \in C(\mathbb{R})$, then f has a modulus of continuity in C.

Proof:
The norm of any derivative of f, as a continuous function over a compact is bounded by some integer m. By mean value theorem, function $M(i) = m + i$ is easily shown to be a modulus of continuity of f. As it is a linear function, it belongs to $\mathcal{E} \subset C$. □

Remark 7.1. The proof is non-constructive: m can not be obtained from the code of f in the general case, and hence the modulus of continuity can not be obtained from the code of f.

But, for all function satisfying our hypotheses, there is a modulus of continuity in C.

7.3. Proof of Proposition 7.2

To prove Proposition 7.2, we use arguments similar to [4] and some properties from [36].

Indeed, the following lemma is used to get functions for which we know the behavior everywhere given only their values over a discrete set of points. It is a refined version of lemma 5.5.

Lemma 7.3. Let $\epsilon : \mathbb{R} \to \mathbb{R}$ be some decreasing function of $\mathcal{C}$, with $\epsilon(x) > 0$ for all x and going to 0 when x goes to $+\infty$, and $1/\epsilon(x) \in \mathcal{C}$. Write ϵ_i for $\epsilon(\lfloor i \rfloor)$, $\mathbb{Z}\epsilon_i$ for $\{j\epsilon_i; j \in \mathbb{Z}\}$, and $\lfloor x \rfloor_{\epsilon_i}$ for $\max\{y \in \mathbb{Z}_{\epsilon_i}; y < x\}$.

Given $f : \mathbb{R}^2 \to \mathbb{R}^l$ in $\mathcal{C}$, there exists $F : \mathbb{R}^2 \to \mathbb{R}^l$ in $\mathcal{C}$ with the following properties:

- For all $i \in \mathbb{N}$, $x \in \mathbb{Z}\epsilon_i$, $F(i,x) = f(i,x)$
- For all $i \in \mathbb{N}$, $x \in \mathbb{R}$, $\|F(i,x) - f(i, \lfloor x \rfloor_{\epsilon_i})\| \leq \|f(i, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(i, \lfloor x \rfloor_{\epsilon_i})\|$
- For all $i \in \mathbb{R}^{\geq 0}$, $x \in \mathbb{R}$,

$$\begin{aligned}\|\tfrac{\partial F}{\partial i}(i,x)\| \leq\ & 5\|f(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_i}) - f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \\ & +25\|f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \\ & +25\|f(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}} + \epsilon_{i+1}) - f(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}})\|.\end{aligned}$$

Proof:
We are going to reuse the functions ω and int defined in the proof of lemma 5.5 whose graphical representations are shown in Figures 1 and 2.

Let $\Delta(i,x) = f(i, x + \epsilon(i)) - f(i,x)$. For all i,x, we have

$$\begin{aligned}\frac{\omega(x/\epsilon(i))}{\epsilon(i)}\Delta(i, \epsilon(i)\ int(x/\epsilon(i))) &= 0 \textit{ whenever } x - \lfloor x \rfloor_{\epsilon(i)} \geq \epsilon(i)/2 \\ &= \frac{\omega(x/\epsilon(i))}{\epsilon(i)}\Delta(i, \lfloor x \rfloor_{\epsilon(i)}) \textit{ otherwise.}\end{aligned}$$

Let G be the solution of the linear differential equation

$$\begin{cases} G(i,0) & = & f(0) \\ \frac{\partial G}{\partial x}(i,x) & = & \frac{\omega(x/\epsilon(i))}{\epsilon(i)}\Delta(i,\epsilon(i)int(x/\epsilon(i))) \end{cases}$$

An easy induction on j then shows that $G(i, j\epsilon(i)) = f(i, j\epsilon(i))$ for all $j \in \mathbb{Z}$.

On $[j\epsilon(i),(j+1)\epsilon(i))$,

$$G(i,x) - f(i, \lfloor x \rfloor_{\epsilon(i)}) = \int_{j\epsilon(i)}^{x} \frac{\omega(t/\epsilon(i))}{\epsilon(i)}\Delta(i, \lfloor t \rfloor_{\epsilon(i)})dt,$$

hence, for all $i \in \mathbb{N}$,

$$\|G(i,x) - f(i, \lfloor x \rfloor_{\epsilon_i})\| \leq \|\Delta(i, \lfloor x \rfloor_{\epsilon_i})\| = \|f(i, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(i, \lfloor x \rfloor_{\epsilon_i})\|.$$

Now, let $\Delta'(i,x) = G(i+1,x) - G(i,x)$. For all i,x we have

$$\begin{array}{rcl} \omega(i)\Delta'(int(i),x) & = & 0 \textit{ whenever } i - \lfloor i \rfloor \geq 1/2 \\ & = & \omega(i)\Delta'(\lfloor i \rfloor, x) \textit{ otherwise} \end{array}$$

Let F be the solution of linear differential equation

$$\begin{cases} F(0,x) & = & G(0,x) \\ \frac{\partial F}{\partial i} & = & \omega(i)\Delta'(int(i),x) \end{cases}$$

An easy induction on i shows that $F(i,x) = G(i,x)$ for all integer i, and all $x \in \mathbb{R}$. Hence $F(i,x) = f(i,x)$ for all $i \in \mathbb{N}$, $x \in \mathbb{Z}\epsilon_i$ and

$$\|F(i,x) - f(i, \lfloor x \rfloor_{\epsilon_i})\| \leq \|f(i, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(i, \lfloor x \rfloor_{\epsilon_i})\|$$

for all $i \in \mathbb{N}$, $x \in \mathbb{R}$.

Now, $\frac{\partial F}{\partial i}$ is either 0 or $\omega(i)\Delta'(\lfloor i \rfloor, x) = \omega(i)(G(\lfloor i+1 \rfloor, x) - G(\lfloor i \rfloor, x))$. In any case, it is derivable in x, and hence $\frac{\partial^2 F}{\partial x \partial i}$ is either 0 or $\omega(i)(\frac{\partial G}{\partial x}(\lfloor i+1 \rfloor, x) - \frac{\partial G}{\partial x}(\lfloor i \rfloor, x))$.

When $x \in \mathbb{Z}\epsilon_i$, bounding ω by 5 ($\zeta \leq 5$),

$$\|\frac{\partial F}{\partial i}\| \leq 5\|f(\lfloor i+1 \rfloor, x) - f(\lfloor i \rfloor, x)\|.$$

When $x \in \mathbb{R}$,

$$\|\frac{\partial^2 F}{\partial x \partial i}\| \leq \|\frac{\partial G}{\partial x}(\lfloor i+1 \rfloor, x\| + \|\frac{\partial G}{\partial x}(\lfloor i \rfloor, x)\|.$$

The term $\|\frac{\partial G}{\partial x}(\lfloor i \rfloor, x)\|$ can be either 0 or

$$\begin{array}{rcl} 5\|\frac{\omega(x/\epsilon_i)}{\epsilon_i}\Delta(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| & \leq & \frac{25}{\epsilon_i}\|\Delta(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \\ & \leq & \frac{25}{\epsilon_i}\|f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\|. \end{array}$$

A similar bound holds for the other term, replacing i by $i+1$.

Using mean value theorem,

$$\begin{array}{rcl}\|\frac{\partial F}{\partial i}(i,x)\| & \leq & \|\frac{\partial F}{\partial i}(i,\lfloor x\rfloor_{\epsilon_i})\| + \|\frac{\partial^2 F}{\partial x \partial i}(i,x)\|(x-\lfloor x\rfloor_{\epsilon_i}) \\ & \leq & \|\frac{\partial F}{\partial i}(i,\lfloor x\rfloor_{\epsilon_i})\| + \epsilon(i)\|\frac{\partial^2 F}{\partial x \partial i}(i,x)\|\end{array},$$

which yields the expected bound. □

Lemma 7.4. If $f : \mathcal{D} \subset \mathbb{R} \to \mathbb{R}$ is defined over a closed interval containing 0, with bounds either rational or infinite, belongs to $C(\mathbb{R})$, of class $\mathcal{C}^1$, with a uniform modulus of continuity in C, then the primitive $\int(f)$ that is equal to 0 at 0 is in $\mathcal{C} + \mathrm{LIM}_w$.

Proof:
Let $M_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$ be the uniform modulus of continuity in C of function f: given some integer, K, $M_{\mathbb{N}}(K, \_)$ is a modulus of continuity of function f on $[-K, K]$.

For all $i, j \in \mathbb{N}$, let $x_j = j\exp(-M_{\mathbb{N}}(i+1,i))$, so that for all $x, y \in [x_j, x_{j+1}] \cap [-i-1, i+1]$, we have

$$|f(x) - f(y)| \leq exp(-i).$$

For all j, let p_j and q_j two integers such that $p_j \times \exp(-q_j)$ is at most $\exp(-i)$ far from $f(x_j)$. The functions $p_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$, and $q_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$ that map (i,j) to corresponding p_j and q_j are in C.

Since $C \subset DP(\mathcal{C})$ these functions as well as function $M_{\mathbb{N}}$ can be extended to function $p : \mathbb{R}^2 \to \mathbb{R}$, $q : \mathbb{R}^2 \to \mathbb{R}$, $M : \mathbb{R}^2 \to \mathbb{R} \in \mathcal{L}$. Consider function $g : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ defined on all $(i,x) \in \mathbb{R} \times \mathcal{C}$ by $g(i,x) = p(i, \exp(M(i+1,i))x)e^{-q(i,\exp(M(i+1,i))x)}$. By construction, for i, j integer, we have

$$g(i, x_j) = p_j \exp(-q_j).$$

Consider the function F given by Lemma 7.3 for function g and $\epsilon : i \mapsto \exp(-M(i+1,i))$. We have

$$F(i, x_j) = g(i, x_j)$$

and

$$\|g(i, x_j) - f(x_j)\| \leq \exp(-i)$$

for all i, j.

For all $x \in \mathcal{C}$, and all integer $i \geq \|x\|$ we have (observe that if x_j denotes $\lfloor x\rfloor_\epsilon$, we have $x_j, x_{j+1} \in [-i-1, i+1]$)

$$\begin{array}{rcl}\|F(i,x) - f(x)\| & \leq & \|F(i,x) - F(i,\lfloor x\rfloor_\epsilon)\| + \|F(i,\lfloor x\rfloor_\epsilon) - g(i,\lfloor x\rfloor_\epsilon)\| \\ & & +\|g(i,\lfloor x\rfloor_\epsilon) - f(\lfloor x\rfloor_\epsilon)\| + \|f(\lfloor x\rfloor_\epsilon) - f(x)\| \\ & \leq & \|F(i,\lfloor x\rfloor_\epsilon + \epsilon) - F(i,\lfloor x\rfloor_\epsilon)\| + 0 + \exp(-i) + \exp(-i) \\ & \leq & \|g(i,x_{j+1}) - g(i,x_j)\| + 2\exp(-i) \\ & \leq & \|g(i,x_{j+1}) - f(x_{j+1})\| + \|g(i,x_j) - f(x_j)\| \\ & & +\|f(x_{j+1}) - f(x_j)\| + 2\exp(-i) \\ & \leq & 5 \times \exp(-i).\end{array}$$

Consider the function $G : \mathbb{R}^2 \to \mathbb{R}$ defined for all $i, x \in \mathbb{R}$ by the linear differential equation

$$\begin{cases} G(i,0) & = & 0 \\ \frac{\partial G}{\partial x}(i,x) & = & F(i,x) \end{cases}$$

Hence

$$G(i,x) = \int_0^x F(i,u)du.$$

We get

$$\|\frac{\partial G}{\partial x}(i,x) - f(x)\| = \|F(i,x) - f(x)\| \leq 5 \times \exp(-i)$$

and by mean value theorem on function $G(i,x) - f(x)$, we get

$$\|G(i,x) - \int_0^x (f)(x)\| \leq (5 \times \exp(-i))|x|,$$

when $i \geq \|x\|$.

Hence, $\int(f)(x)$ is the limit of $G(i,x)$ when i goes to $+\infty$ with integer values. We just need to check that schema LIM_w can be applied to function G of $\mathcal{L}^*$ to conclude: indeed, the limit of $G(i,x)$ when i goes to $+\infty$ will exist and coincide with this value, i.e. $\int(f)(x)$.

Since $\frac{\partial G}{\partial i} = \int_0^x \frac{\partial F}{\partial i}(i,u)du$ implies

$$\|\frac{\partial G}{\partial i}\| \leq \int_0^x \|\frac{\partial F}{\partial i}\|du \leq |x| \times \|\frac{\partial F}{\partial i}\| \leq (x^2+1) \times \|\frac{\partial F}{\partial i}\|,$$

we only need to prove that we can bound $\|\frac{\partial F}{\partial i}\|$ by $K(x) \times \exp(-i)$ for some function $K \in \mathcal{L}^*$, and $i \geq \|x\|$.

But from Lemma 5.5, we know that for all i, x,

$$\begin{aligned} \|\tfrac{\partial F}{\partial i}(i,x)\| \leq\ & 5\|g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_i}) - g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \\ & +25\|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \\ & +25\|g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}} + \epsilon_{i+1}) - g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}})\|. \end{aligned}$$

First term can be bounded by $5 \times \exp(-i) + 5 \times \exp(-i) = 10 \times \exp(-i)$.

Second term can be bounded by

$$25(\|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor x \rfloor_{\epsilon_i} + \epsilon_i)\| + \|f(\lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor x \rfloor_{\epsilon_i})\| + \|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i}) - f(\lfloor x \rfloor_{\epsilon_i})\|).$$

Hence by $25 \times \exp(-i) + 25 \times \exp(-i) + 25 \times \exp(-i) = 75 \times \exp(-i)$, for $i \geq \|x\|$.

Similarly for third term, replacing i by $i+1$.

Hence, when $i \geq \|x\|$,

$$\|\frac{\partial F}{\partial i}(i,x)\| \leq 160 \times \exp(-i),$$

and

$$\|\frac{\partial G}{\partial i}(i,x)\| \leq 160 \times (x^2+1) \times \exp(-i),$$

and so schema LIM_w can be applied on function G of $\mathcal{L}^*$ to get function $\int(f)$. This ends the proof.

□

Actually, the previous lemma can easily be extended a little bit to get any primitive (clearly this implies Proposition 7.2 for functions from a closed subset of $\mathbb{R}$ to $\mathbb{R}$, considering a function as the primitive of one of its derivative. The case $\mathbb{R}^k$ to $\mathbb{R}^l$ can be obtained by adapting our arguments to functions of several variables.).

Lemma 7.5. Let h be a function of $C(\mathbb{R})$ defined at 0.

If $f : \mathcal{D} \subset \mathbb{R} \to \mathbb{R}$ is defined over a closed interval containing 0, with bounds either rational or infinite, belongs to $C(\mathbb{R})$, of class $\mathcal{C}^1$, with a uniform modulus of continuity in C, then the primitive of f equal to $h(0)$ at 0 is in $\mathcal{C} + \mathrm{LIM}_w$.

Proof:
Replace in previous proof the initial condition $G(i, 0) = 0$ of the differential equation defining function G, by $G(i, 0) = g(i)$ where $g : \mathbb{R} \to \mathbb{R}$ is a function converging to $h(0)$, obtained by extending a suitably chosen function $g : \mathbb{N} \to \mathbb{N}$. □

This ends the proof that $\mathcal{R}ec(\mathbb{R}) = \mathcal{L}^*_{!\mu}$ for functions defined over product of compact intervals, which is an analog characterization of recursive functions in the sense of computable analysis. In fact, we obtained a more general result concerning the power of our LIM_w operator that can indeed be seen as a description of the missing link between discrete computability theory and computable analysis.

7.4. Other results

From those results and those proofs, we can derive some other results of interest.

We can apply Proposition 7.2 to other classes than $\mathcal{L}+!\mu$. For example, is presented in [21] a class

$$\bar{\mathcal{D}} = [0, 1, -1, U; \mathrm{COMP}, \bar{I}]$$

that contains extensions of all recursive primitive functions. In other words,

$$\mathcal{PR} \subset DP(\bar{\mathcal{D}}).$$

Where $\bar{I}$ is defined as an integration operator: given functions $f_1, ..., f_m$ of arity n and $g_1, ..., g_{m+1}$ of arity $n + 1 + m$, if there is a unique set of functions $h_1, ..., h_m$ such that

$$\begin{array}{rcl} h_i(x, 0) & = & f_i(x) \\ \frac{\partial h_i}{\partial y}(x, y) & = & g_i(x, y, \overrightarrow{h}(x, y)) \text{ for all } y \in I - S \end{array}$$

on an interval I containing 0 where $S \subset I$ is a countable set of isolated points and h and $\frac{\partial h}{\partial y}$ are both continuous, then $h = h_1$ is defined.

Since $\bar{\mathcal{D}}+\theta_3$ contains $\mathcal{L}$, is stable under composition and integration, we have by Proposition 7.2 the following result:

Proposition 7.3. For functions of class $\mathcal{C}^2$ defined on a compact domain,

$$\mathcal{PR}(\mathbb{R}) \subset \bar{\mathcal{D}} + \theta_3 + \mathrm{LIM}_w$$

Remark 7.2. Observe that θ_3 was not considered as a basic function in [21], since this integration operator allows to get non-analytic functions that can play the role of function θ_3. For the same reasons, we believe that actually $\mathcal{PR}(\mathbb{R}) \subset \bar{\mathcal{D}} + \mathrm{LIM}_w$.

We have also some results for closed intervals non necessarily compact:

Proposition 7.4. Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ product of closed intervals.

If f and the derivatives of f are in $\mathcal{R}ec(\mathbb{R})$ then $f \in \mathcal{L}^*_{!\mu}$.

Proof:
For $k = l = 1$, this follows from Lemma 7.5, considering function f as the primitive of one of its derivative, and observing that it is known that any computable function over the reals has a computable uniform modulus of continuity [36]. The case $\mathbb{R}^k$ to $\mathbb{R}^l$ can be obtained by adapting arguments to functions of several variables. □

Remark 7.3. The previous arguments holds for any classes C, $\mathcal{C}$ for which it is known that any function over the reals in $C(\mathbb{R})$ has a uniform modulus of continuity in C. In particular, for elementar functions, all levels of the Grzegorczyk hierarchy [6] or for primitive recursive functions.

In particular:

Proposition 7.5. Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ product of closed intervals.

If f and the derivatives of f are in $\mathcal{PR}$ then $f \in \bar{\mathcal{D}} + \theta_3 + \mathrm{LIM}_w$.

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