

Algebraic Characterization of Complexity Classes of Computable Real Functions

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Abstract. Several algebraic machine-independent characterizations of computable functions over the reals have been obtained recently. In particular nice connections between the class of computable functions (and some of its sub and sup-classes) over the reals and algebraically defined (sub- and sup-) classes of \mathbb{R} -recursive functions à la Moore 96 have been obtained.

We provide in this paper a framework that allows to relate these classes to classical computability and complexity classes over the integers.

While our setting provides a new reading of some of the existing characterizations, this also provides new results: in particular, we provide an algebraic characterization of polynomial time computable real functions.

1 Introduction

Building a well founded theory of computations over the reals is a crucial task. However, computation theory over the reals is not as well understood as classical discrete computation theory. In particular, unlike what happens for discrete computations where the Church-Turing thesis yields a clear equivalence between models, when talking about real or analog computations, several approaches have been developed, with various motivations, but with not so-clear relations. This includes the Blum-Shub-Smale model [3, 4], Shannon's General Purpose Analog Computer *GPAC* [21], algebraically defined classes of functions over the reals à la Moore 96 [19] (\mathbb{R} -recursive functions), as well as the classical recursive analysis approach present from the beginning of computation theory [22].

While there is no hope to unify the Blum-Shub-Smale approach with recursive analysis point of view, some recent works have shown strong connections between recursive analysis, Shannon's General Purpose Analog Computer model, and \mathbb{R} -recursive functions. These results basically say that all these paradigms of computations are more or less equivalent: see [6, 7] or survey [5].

However, up till now discussions have mainly been restricted to the computability level, and not to complexity level. While there is a well developed and

rather well understood theory of complexity for real computations (over compact domains) [17], talking about complexity for analog models of computations is often very problematic. One reason is that there is no robust and well defined notions of time and space for these models [19, 1, 20, 5].

In this paper, our aim is to provide a framework to relate the recursive analysis approach to algebraically defined classes of functions over the reals at both the computability and complexity level, yielding a substantial first step towards this direction.

To do so, we actually relate (polynomial-time) computable functions over the reals to (polynomial-time) computable functions over the integers, and provide some statements that allow to state that a class of functions over the reals that approximates in some defined sense computable functions over the integers yields naturally an algebraic characterization of the corresponding class over the reals.

First, our results provide a new understanding of some of the recent constructions relating recursive analysis to Shannon's General Purpose Analog Computer, and to \mathbb{R} -recursive functions [6, 7].

Second, our results provide, up to our knowledge, the first algebraic (machine-independent) characterization of polynomial time computable functions over the reals.

More concretely, we first express how recursive analysis relates the notion of (polynomial-time) computable functions over the integers to the corresponding notion over the reals. While the obtained results can be seen as rather basic, they yield a direct understanding of how algebraic characterizations of computability and complexity classes over the integers can be lifted to algebraic characterizations of corresponding classes over the reals.

Indeed when talking about computability and complexity over the integers, several machine-independent characterizations of computable functions are known [11]: in particular, Kleene's functions are well known to be exactly the functions computable by Turing machines. Cobham [12], and later Bellantoni and Cook [2] were among the first to propose algebraically defined characterizations of polynomial time computable functions. See survey [11] for some other machine independent characterizations of classical computability and complexity classes over the integers.

Our setting is actually proved to be robust to approximations: one does not need to be able to compute exactly the corresponding class over the integers, but only some defined approximation of it to be able to compute the corresponding class over the reals. This provides a way to reformulate/reprove/reread very nicely some constructions already used in [6, 7].

Notice that compared to the framework proposed by Campagnolo and Ojakian in [10], our framework is definitively different, and has the main advantage to allow to talk not only about the computability level but also about the complexity level. Algebraic characterizations of functions over more general domains, including the reals, have been obtained in [8]. However, the obtained characterization in this latter paper is rather different in spirit to the ones discussed here: on one hand, a more abstract setting that is not restricted to real functions is

considered there, but on the other hand the discussion is also restricted to the computability level, and less close in spirit to above mentioned models of analog computations.

Potential applications include the possibility of proving whether a given function can be computed in polynomial time without resorting to effectively program it, as well as the possibility of building methods to automatically derive computational properties of programs/systems, in the lines of [15, 16, 18] for discrete programs.

We also believe in the pedagogical value of our characterizations. They yield ways to define computability and complexity over the reals without any resort to any kind of machinery in the spirit of (Type 1 or Type 2) Turing machines, or any kind of discrete machinery. This might be considered as very natural to avoid discrete machinery when talking about computation of real functions.

2 Basic Definitions from Recursive Analysis

We first recall some definitions from recursive analysis [23, 17].

Let $\mathbb{D} = \{r \in \mathbb{Q} : r = \frac{a}{2^b} \text{ for integers } a, b\}$ (these are the rationals with finite binary representation). We adopt the following representation of real numbers.

Definition 1. *Assume $x \in \mathbb{R}$. Then x can be represented by a Cauchy sequence $\varphi_x : \mathbb{N} \rightarrow \mathbb{D}$ that converges at a binary rate:*

$$\forall n \in \mathbb{N} : |x - \varphi_x(n)| \leq 2^{-n}. \quad (1)$$

Given $x \in \mathbb{R}$, let CF_x denote the class of Cauchy functions that represent x .

Definition 2. *(Modulus of continuity)*

1. *Assume a function $f : [0, 1] \rightarrow \mathbb{R}$. Then f has a modulus of continuity if there exists a function $m : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $x, y \in [0, 1]$ and for all $n \in \mathbb{N}$: if $|x - y| \leq 2^{-m(n)}$, then $|f(x) - f(y)| \leq 2^{-n}$.*
2. *Assume a function $f : [0, \infty) \rightarrow \mathbb{R}$. Then f has a modulus of continuity if there exists a function $m : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that for all $k, n \in \mathbb{N}$ and for all $x, y \in [0, 2^k]$ the following holds: if $|x - y| \leq 2^{-m(k, n)}$, then $|f(x) - f(y)| \leq 2^{-n}$.*

The notions of computability and complexity of real functions have been characterized as will be shown below.

Definition 3. *Assume a function $f : I \rightarrow \mathbb{R}$ where I is either $[0, 1]$ or $[0, \infty)$. We say that f is computable if there exists a function-oracle Turing machine $M^{(\cdot)}$ such that for every $x \in I$, for every $\varphi_x \in CF_x$, and for every $n \in \mathbb{N}$ the following holds:*

$$|M^{\varphi_x}(n) - f(x)| \leq 2^{-n}. \quad (2)$$

Notice that we need to talk about functions defined over unbounded (non-compact) domains, unlike what is done for example in [17]: Computability over compact domains is characterized by the existence of a computable modulus and a computable approximation function (see corollary 2.14 in [17]). Computability over unbounded domains can be characterized in a similar way as indicated by the following proposition.

Proposition 1. *Assume a function $f: [0, \infty) \rightarrow \mathbb{R}$. Then f is computable iff there exist two computable functions $m: \mathbb{N}^2 \rightarrow \mathbb{N}$ and $\psi: \mathbb{D} \cap [0, \infty) \times \mathbb{N} \rightarrow \mathbb{D}$ such that*

1. m is a modulus of continuity for f ,
2. ψ is an approximation function for f , that is, for every $d \in \mathbb{D} \cap [0, \infty)$ and every $n \in \mathbb{N}$ the following holds: $|\psi(d, n) - f(d)| \leq 2^{-n}$.

Proof. The proof is a simple extension of the proof of Corollary 2.14 in [17].

Definition 4. *(Polynomial time complexity of real functions)*

1. Assume a function $f: [0, 1] \rightarrow \mathbb{R}$. Then f is polynomial time computable if it is computable in the sense of Definition 3, and there exists an oracle machine $N^0(n)$ that computes f whose computation time is bounded by $p(n)$ for some polynomial function p .
2. Assume a function $f: [0, \infty) \rightarrow \mathbb{R}$. Then f is polynomial time computable if it is computable in the sense of Definition 3, and there exists an oracle machine $N^0(n)$ that computes $f(x)$ whose computation time is bounded by $q(k, n)$ for some polynomial q where $k = \min\{j: x \in [0, 2^j]\}$.

For a domain that is a combination of compact and unbounded components computability/complexity can be defined and characterized in the obvious way.

The following proposition characterizes polynomial time computability of real functions over unbounded domains. Using an extension of Theorem 2.19 in [17] the proof is similar to that of Proposition 1.

Proposition 2. *Assume a function $f: [0, \infty) \rightarrow \mathbb{R}$. Then f is polynomial time computable iff there exist two functions $m: \mathbb{N}^2 \rightarrow \mathbb{N}$ and $\psi: \mathbb{D} \cap [0, \infty) \times \mathbb{N} \rightarrow \mathbb{D}$ such that*

1. m is a modulus function for f and it is polynomial with respect to both the extension parameter k and the precision parameter n , that is, $m(k, n) = O((k + n)^a)$ for some $a \in \mathbb{N}$.
2. ψ is an approximation function for f , that is, for every $d \in \mathbb{D} \cap [0, \infty)$ and for every $n \in \mathbb{N}$ the following holds:

$$|\psi(d, n) - f(d)| \leq 2^{-n} \tag{3}$$

3. $\psi(d, n)$ is computable in time $p(|d| + n)$ for some polynomial p .

3 Transferring Computability and Complexity over The Integers to the Reals

Proofs of some of the results in this section can be found in the appendix.

3.1 General Case

We now state some results that yield ways to transfer notions of computability/complexity over the integers to corresponding notions over the reals.

We first give a way to relate computability over some compact domain to computability over this domain times reals.

Proposition 3 (Computability over I vs Computability over $I \times \mathbb{R}$). *The following are equivalent.*

1. a function $f : [0, 1] \rightarrow \mathbb{R}$ is computable
2. there exists a computable function $g : [0, 1] \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}$ such that

$$\forall x \in [0, 1], \forall y \in \mathbb{R}^{\geq 0}: |g(x, y) - yf(x)| \leq 1 \quad (4)$$

3. there exists a computable function $g : [0, 1] \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}$ such that,

$$\forall x \in [0, 1], \forall y \in \mathbb{N}: |g(x, y) - yf(x)| \leq 1 \quad (5)$$

These statements works fine also at the complexity level¹:

Proposition 4 (Complexity over I vs Complexity over $I \times \mathbb{R}$). *The following are equivalent:*

1. a function $f : [0, 1] \rightarrow \mathbb{R}$ is polynomial time computable,
2. there exists a polynomial time computable function $g : [0, 1] \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}$ such that (4) holds.
3. there exists a polynomial time computable function $g : [0, 1] \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}$ such that (5) holds.

3.2 Lipschitz Functions

For Lipschitz functions, it is possible to relate computability over compact domains to computability over \mathbb{R}^2 .

Proposition 5 (Computability over I vs Computability over \mathbb{R}^2). *Fix an arbitrary constant ϵ . Assume a function $f : [0, 1] \rightarrow \mathbb{R}$ that is Lipschitz. Then the following are equivalent:*

1. f is computable

¹ Or at any level above polynomial time. That is to say, polynomial time computable can be replaced by computable in elementary time in following proposition for example.

2. there exists some computable function $g: \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}$ such that

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}^{\geq 1}, x \leq y: |g(x, y) - yf(\frac{x}{y})| \leq \epsilon \quad (6)$$

This can still be transferred at the complexity level¹:

Proposition 6 (Complexity over I vs Complexity over \mathbb{R}^2). Fix any arbitrary constant ϵ . Assume a function $f: [0, 1] \rightarrow \mathbb{R}$ that is Lipschitz. Then the following are equivalent:

1. f is polynomial time computable,
2. there exists a polynomial time computable function $g: \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}$ such that (6) holds.

We propose then to consider the following notion of approximation.

Definition 5 (Approximation). Let \mathcal{C} be a class of functions from \mathbb{R}^2 to \mathbb{R} . Let \mathcal{D} be a class of functions from \mathbb{N}^2 to \mathbb{N} . We say that \mathcal{C} approximates \mathcal{D} if for any function $g \in \mathcal{D}$, there exists some function $\tilde{g} \in \mathcal{C}$ such that for all $x, y \in \mathbb{N}$ we have

$$|\tilde{g}(x, y) - g(x, y)| \leq 1/4.$$

This yields the following result.

Theorem 1 (Computability over I vs Approximate Computability over \mathbb{N}^2). Consider a class \mathcal{C} of computable real functions that approximates total (discrete) recursive functions. Assume that $f: [0, 1] \rightarrow \mathbb{R}$ is Lipschitz. Then the following are equivalent:

1. f is computable,
2. there exists some function $g \in \mathcal{C}$ such that

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}^{\geq 1}, x \leq y: |g(x, y) - yf(\frac{x}{y})| \leq 3 \quad (7)$$

This can also be stated at the complexity level:

Theorem 2 (Complexity over I vs Approximate Complexity over \mathbb{N}^2). Consider a class \mathcal{C} of polynomial time computable functions that approximates polynomial time computable discrete functions. Assume that $f: [0, 1] \rightarrow \mathbb{R}$ is Lipschitz. Then the following are equivalent:

1. f is polynomial time computable,
2. there exists a function $g \in \mathcal{C}$ such that (7) holds.

3.3 Avoiding the Lipschitz Hypothesis

This is indeed possible to avoid the Lipschitz hypothesis, and to yet talk about complexity, at the price of a little bit of complications.

Definition 6 ($\#_k$). Fix some integer k . Denote by $\#_k$ the function $\#_k : \mathbb{R}^{\geq 1} \rightarrow \mathbb{R}^{\geq 0}$ defined by $\#_k[x] = 2^{(\log_2 x)^k}$.

Definition 7 (Polynomial Time Computable Integer Approximation). A function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to have a polynomial time computable integer approximation if there exists some polynomial time computable function $h : \mathbb{N}^d \rightarrow \mathbb{N}$ with $|h(\bar{x}) - \lfloor g(\bar{x}) \rfloor| \leq 1$ for all $\bar{x} \in \mathbb{N}^d$.

We then have.

Proposition 7 (Complexity over I vs Complexity over \mathbb{R}^2). Fix an arbitrary constant ϵ . The following are equivalent.

1. a function $f : [0, 1] \rightarrow \mathbb{R}$ is computable
2. there exists some function $g : \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}$ such that
 - (a) g has a polynomial time computable integer approximation
 - (b) for some integer k ,

$$\forall x \in [0, 1], \forall y \in \mathbb{R}^{\geq 1} : |g(x \cdot \#_k[y], y) - yf(x)| \leq \epsilon, \quad (8)$$

(c) for some integer M ,

$$\forall x_1, x_2 \in \mathbb{R}^{\geq 0}, y \in \mathbb{R}^{\geq 1} : |x_1 - x_2| \leq 1 \Rightarrow |g(x_1, y) - g(x_2, y)| \leq M \quad (9)$$

Notice that (8) is equivalent to (6) for $k = 1$, by some obvious change of variable.

Proof. (2) \Rightarrow (1) : For simplicity, assume $\epsilon = 1$. Assume there exists a function g that satisfies the above conditions. Assume some $x \in [0, 1]$ and $n \in \mathbb{N}$. Let $y = 2^n$. From condition (2b) we have

$$\begin{aligned} |g(2^{n^k} x, 2^n) - 2^n f(x)| &\leq 1 \\ |2^{-n} g(2^{n^k} x, 2^n) - f(x)| &\leq 2^{-n} \end{aligned} \quad (10)$$

Let h be some polynomial time computable function with $|h(x, y) - \lfloor g(x, y) \rfloor| \leq 1$ for all $x, y \in \mathbb{N}$ that exists by (2a).

Then

$$|g(\lfloor 2^{n^k} x \rfloor, 2^n) - h(\lfloor 2^{n^k} x \rfloor, 2^n)| \leq 2 \quad (11)$$

From (2c) we have

$$|g(\lfloor 2^{n^k} x \rfloor, 2^n) - g(2^{n^k} x, 2^n)| \leq M \quad (12)$$

From the previous two equations

$$\begin{aligned} |g(2^{n^k}x, 2^n) - h(\lfloor 2^{n^k}x \rfloor, 2^n)| &\leq M + 2 \\ |2^{-n}g(2^{n^k}x, 2^n) - 2^{-n}h(\lfloor 2^{n^k}x \rfloor, 2^n)| &\leq 2^{-n}(M + 2) \end{aligned} \quad (13)$$

From Equations (10) and (13)

$$|f(x) - 2^{-n}h(\lfloor 2^{n^k}x \rfloor, 2^n)| \leq 2^{-n}(M + 3) \quad (14)$$

This gives an approximation to $f(x)$ to within the desired precision 2^{-n} . The computation time of $h(\lfloor 2^{n^k}x \rfloor, 2^n)$ is bounded by $p(n)$ for some polynomial p , hence, $f(x)$ is polynomial time computable.

(1) \Rightarrow (2) : Assume that $f: [0, 1] \rightarrow \mathbb{R}$ is polynomial time computable. Hence f has a polynomial modulus $m(n) = n^k$ for some constant $k \in \mathbb{N}$. Define g as follows:

$$g(x, y) = \begin{cases} yf(\frac{x}{\#_k[y]}) & x \leq \#_k[y], y \geq \epsilon \\ yf(1) & \text{otherwise} \end{cases} \quad (15)$$

Then for every $x \in [0, 1]$ and $y \in \mathbb{R}^{\geq 1}$ we have

$$|g(x.\#_k[y], y) - yf(x)| = 0 \leq 1,$$

hence condition (2b) is satisfied.

Now assume $x_1, x_2 \in \mathbb{R}^{\geq 0}, y \in \mathbb{R}^{\geq 1}$ such that $|x_1 - x_2| \leq 1$. There are three cases.

case 1: $x_1 \leq \#_k[y]$ and $x_2 \leq \#_k[y]$, then

$$\begin{aligned} |g(x_1, y) - g(x_2, y)| &= |yf(\frac{x_1}{\#_k[y]}) - yf(\frac{x_2}{\#_k[y]})| \\ &= y|f(\frac{x_1}{\#_k[y]}) - f(\frac{x_2}{\#_k[y]})| \end{aligned}$$

We have $|\frac{x_1}{\#_k[y]} - \frac{x_2}{\#_k[y]}| = \frac{1}{\#_k[y]}|x_1 - x_2| \leq \frac{1}{\#_k[y]} = 2^{-(\log_2 y)^k}$. Hence, using the modulus of continuity of f , $|f(\frac{x_1}{\#_k[y]}) - f(\frac{x_2}{\#_k[y]})| \leq 2^{-\log_2 y} = \frac{1}{y}$ implying $|g(x_1, y) - g(x_2, y)| \leq 1$ and condition (2c) is satisfied for $M = 1$.

case 2: $x_1 > \#_k[y]$ and $x_2 > \#_k[y]$, then

$$|g(x_1, y) - g(x_2, y)| = |yf(1) - yf(1)| = 0$$

and condition (2c) is satisfied for $M = 1$.

case 3: $x_1 \leq \#_k[y]$ and $x_2 > \#_k[y]$, then

$$\begin{aligned} |g(x_1, y) - g(x_2, y)| &\leq |g(x_1, y) - g(\#_k[y], y)| + |g(\#_k[y], y) - g(x_2, y)| \\ &\leq 1 + 0 = 1 \end{aligned}$$

by above two cases, and hence condition (2c) is satisfied for $M = 1$.

Note that all division, multiplication, and f are all computable. Hence, g is computable.

Assume $i, j \in \mathbb{N}$ such that $i \leq \#_k[j]$. Then $g(i, j) = jf(\frac{i}{\#_k[j]})$. The computation of the floor of the last function involves the following:

1. Shift left the binary representation of i by $|j|^k$ positions. The result would be a dyadic rational d .
2. Simulate the computation of $f(d)$ assuming large enough though fixed precision. When simulating the oracle d is presented exactly.
3. Multiply the output of the previous step by j . Finally, truncate the result to extract the integer part.

All of these steps can be performed in polynomial time. The fixed precision of step 2 can be calculated from the modulus of f . Since the output of step 2 is an approximation there is a possibility that the floor is computed with an error that can be bounded by 1. The case when $i > \#_k[j]$ is similar. Hence, Condition (2a) is satisfied.

For ease of notation, let's consider that interval $[a, b]$ is $[b, a]$ if $b < a$.

Definition 8 (Peaceful Functions).

- A function $g : \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ is said to be peaceful if

$$\forall x \in \mathbb{R}^{\geq 0}, \forall y \in \mathbb{N}^{\geq 1}, : g(x, y) \in [g(\lfloor x \rfloor, y), g(\lceil x \rceil, y)]. \quad (16)$$

- We will say that a class \mathcal{C} of functions peacefully approximates some class \mathcal{D} of functions, if the class of peaceful functions from \mathcal{C} approximates \mathcal{D} .

Theorem 3 (Complexity over I vs Approximate Complexity over \mathbb{N}^2).

Consider a class \mathcal{C} of functions that peacefully approximates polynomial time computable discrete functions, and whose functions have polynomial time computable integer approximation².

Then the following are equivalent.

1. a function $f : [0, 1] \rightarrow \mathbb{R}$ is polynomial time computable,
2. there exists some peaceful function $g \in \mathcal{C}$ such that
 - (a) for some integer k ,

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}^{\geq 1}, x \leq \#_k[y] : |g(x, y) - yf(\frac{x}{\#_k[y]})| \leq 3, \quad (17)$$

- (b) for some integer M ,

$$\forall x, x' \in \mathbb{R}^{\geq 0}, \forall y \in \mathbb{R}^{\geq 1}, x, x' \leq \#_k[y], |x - x'| \leq 1 : \\ y|f(\frac{x}{\#_k[y]}) - f(\frac{x'}{\#_k[y]})| \leq M \quad (18)$$

² A sufficient condition for that is that restrictions to integers of functions from \mathcal{C} are polynomial time computable.

Remark 1. Condition (2b) is a property of function f , and hence is a necessary condition to get computability of f .

Proof. (1) \Rightarrow (2) : by Proposition 7 for $\epsilon = 3/4$, there exists some function g with a polynomial time computable integer approximation h such that (8) holds.

Now, by hypothesis, there exists some peaceful $\tilde{h} \in \mathcal{C}$ such that

$$\forall x, y \in \mathbb{N} : |\tilde{h}(x, y) - h(x, y)| \leq 1/4$$

Hence

$$\forall x, y \in \mathbb{N} : |\tilde{h}(x, y) - \lfloor g(x, y) \rfloor| \leq 1 + \frac{1}{4} = \frac{5}{4} \quad (19)$$

We have $|g(x, y) - \lfloor g(x, y) \rfloor| \leq 1$, then $|\tilde{h}(x, y) - g(x, y)| \leq \frac{9}{4}$, for all $x, y \in \mathbb{N}$. Finally, we have (through change of variables in Eq. (8))

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}^{\geq 1}, x \leq \#_k[y] : |\tilde{h}(x, y) - yf(\frac{x}{\#_k[y]})| \leq \frac{9}{4} + \frac{3}{4} = 3 \quad (20)$$

Now, by (2c) of Proposition 7, we know that for all $x, y \in \mathbb{N}$, $\delta \in [0, 1]$, $|g(x + \delta, y) - g(x, y)| \leq M$ for some integer M . From Eq. (8), clearly condition (2b) must hold.

(2) \Rightarrow (1) : This follows from Proposition 7 with $\epsilon = 3$, observing that the peacefulness of g and condition (2b) implies condition (2c) of Proposition 7.

The previous theorems can be generalized to any complexity class as indicated by the following corollary.

Corollary 1. *Let \mathcal{D} be some class of functions from \mathbb{N} to \mathbb{N} above the class of polynomial functions and closed under composition. For a function $T \in \mathcal{D}$, consider $\#_T$ as the function $\#_T : \mathbb{R}^{\geq 1} \rightarrow \mathbb{R}^{\geq 0}$ defined by $\#_T[x] = 2^{(T(\log_2 x))}$.*

Consider a class \mathcal{C} of functions that peacefully approximates discrete functions computable in time \mathcal{D} ; and whose functions have integer approximations computable in time \mathcal{D}^3 .

Then the following are equivalent.

1. *a function $f : [0, 1] \rightarrow \mathbb{R}$ is computable in time \mathcal{D} ,*
2. *there exists some peaceful function $g \in \mathcal{C}$ such that*
 - (a) *for some $T \in \mathcal{D}$*

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}^{\geq 1}, x \leq \#_T[y] : |g(x, y) - yf(\frac{x}{\#_T[y]})| \leq 3, \quad (21)$$

(b) *for some integer M ,*

$$\forall x, x' \in \mathbb{R}^{\geq 0}, \forall y \in \mathbb{R}^{\geq 1}, x, x' \leq \#_T[y], |x - x'| \leq 1 : \\ y|f(\frac{x}{\#_T[y]}) - f(\frac{x'}{\#_T[y]})| \leq M \quad (22)$$

³ A sufficient condition for that is that restrictions to integers of functions from \mathcal{C} are computable in time \mathcal{D} .

Proof. The proof is similar to that of the previous theorem. It should be noted that if f is computable in time bounded by \mathcal{D} then it has a modulus in \mathcal{D} . This is a direct consequence of Theorem 2.19 in [17].

4 Applications

We now present applications of our previous theorems and corollaries to characterize algebraically some computability and complexity classes of functions over the reals. The main new results of this sections are theorems 4 and 5 which give characterizations of polynomial-time computable functions on the reals as the functions described by a specific class \mathcal{W} .

To do so, we propose the following terminology.

Definition 9 (Definability). *Let \mathcal{C} be a class of functions from $\mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$. A function $f : [0, 1] \rightarrow \mathbb{R}$ is said to be \mathcal{C} -definable if there exists some total function $g \in \mathcal{C}$ such that:*

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}^{\geq 1}, x \leq y: |g(x, y) - yf(\frac{x}{y})| \leq 3 \quad (23)$$

Let T be a function from \mathbb{N} to \mathbb{N} .

Definition 10 (T, M -Bounded). *A function $f : [0, 1] \rightarrow \mathbb{R}$ is said to be T, M -bounded if*

$$\begin{aligned} \forall x, x' \in \mathbb{R}^{\geq 0}, \forall y \in \mathbb{R}^{\geq 1}, x, x' \leq \#_T[y], |x - x'| \leq 1: \\ y|f(\frac{x}{\#_T[y]} - f(\frac{x'}{\#_T[y]})| \leq M \end{aligned} \quad (24)$$

Definition 11 (T -Definability). *Let \mathcal{C} be a class of functions from $\mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$. A function $f : [0, 1] \rightarrow \mathbb{R}$ is said \mathcal{C} - T -definable if there exists some total function $g \in \mathcal{C}$ such that*

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}^{\geq 1}, x \leq \#_T[y]: |g(x, y) - yf(\frac{x}{\#_T[y]})| \leq 3 \quad (25)$$

4.1 GPAC, Polynomial IVP, and computable functions

The General Purpose Analog Computer, introduced by Claude Shannon in [21] to model a mechanical device, consists of circuits interconnecting basic blocks that can be constants, adders, multipliers, and integrators. GPAC computable functions have been characterized in different ways since the creation of this model. In the following, we will use Graça and Costa's characterization by PIVP (Polynomial Initial Value Problems) [14].

Definition 12. A function is said to be PIVP if it is a component of the solution of a differential equation of the following form:

$$\begin{cases} y(t_0) = y_0 \\ y'(t) = p(t, y) \end{cases}$$

with $y : \mathbb{R}^n \rightarrow \mathbb{R}$ and p is a vector of polynomials.

Lemma 1 ([14]). A function is generated by a GPAC iff it is PIVP.

Lemma 2 ([13]). PIVP functions is a class of computable functions that peacefully approximates total (discrete) recursive functions.

Proof. It is proved in [13] than Turing machines can be robustly simulated by PIVP functions. Robust means that even with noise (that does not exceed $1/4$) the computation is conducted satisfactorily. This lemma implies that discrete functions that are computable can be generated by GPACs with precision $\frac{1}{4}$. Hence the class of PIVP functions approximates the discrete recursive functions.

We get the following corollaries (that can be seen as a new reading of [6]) as a direct application of Theorem 1 and Corollary 1.

Proposition 8. A Lipschitz function $f : [0, 1] \rightarrow \mathbb{R}$ is computable iff it is PIVP-definable.

Proposition 9. Let $f : [0, 1] \rightarrow \mathbb{R}$ be some T, M -bounded function, for some total recursive function T . Then f is computable iff it is PIVP- T -definable.

4.2 Real recursive functions revisited

Some characterizations of computable functions in the context of real recursive functions [19] are already known, see for example [7, 10]. They make use of limits that have restrictive constraints. We can dispense with those limits by using theorem 1 with a simpler class of functions.

Definition 13. Let \mathcal{L}_μ be the smallest set of functions containing 0, 1, projections, $\theta_3 : \max(0, x^3)$ and closed under the operations of composition, controlled linear integration (CLI) and unique minimization (UMU).

CLI is defined for functions g, h and c , with the norm of the first partial derivative of h bounded by c , as a solution to the following differential equation $\partial_y f(x, y) = h(x, y)f(x, y)$ with initial condition $f(x, 0) = g(x)$.

Given a differentiable function $f : \mathcal{D} \times \mathcal{I} \subseteq \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ where $\mathcal{D} \times \mathcal{I}$ is a product of closed intervals, if for all $\mathbf{x} \in \mathcal{D}$, $y \mapsto f(\mathbf{x}, y)$ is a non-decreasing function with a unique root y_0 that lies in the interior of \mathcal{I} and $\partial_y f|_{(\mathbf{x}, y_0)} > 0$,

then UMU(f) is defined as $\begin{cases} \mathcal{D} \longrightarrow \mathbb{R} \\ \mathbf{x} \mapsto y_0 \end{cases}$

Lemma 3 ([7]). Functions in \mathcal{L}_μ is a class of computable functions that peacefully approximate total recursive functions.

We get from Theorem 1 and Corollary 1.

Proposition 10. *A Lipschitz function $f : [0, 1] \rightarrow \mathbb{R}$ is computable iff it is \mathcal{L}_μ -definable.*

Proposition 11. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be some T, M -bounded function, for some total recursive function T . Then f is computable iff it is \mathcal{L}_μ - T -definable.*

Similarly, we know that this class without the UMU operator closely matches discrete elementary functions.

Definition 14. *Let \mathcal{L} be the smallest set of functions containing $0, 1, -1, \pi$, projections and θ_3 and closed under composition and controlled linear integration.*

Lemma 4 ([9]). *\mathcal{L} is a class of total functions computable in elementary time that peacefully approximates total discrete elementary computable functions.*

We get from a generalization of Theorem 2 and Corollary 1.

Proposition 12. *A Lipschitz function $f : [0, 1] \rightarrow \mathbb{R}$ is computable in elementary time iff it is \mathcal{L} -definable.*

Proposition 13. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be some T, M -bounded function, for some elementary function T . Then f is computable in elementary time iff it is \mathcal{L} - T -definable.*

4.3 Polynomial Time Computable Functions

We define a class of polynomial time computable real functions. These functions are essentially extensions to \mathbb{R} of the Bellantoni and Cook class [2].

Definition 15. *Define functions algebra*

$$\mathcal{W} = [0, U, s_0, s_1, pr_0, pr_1, \theta_1, e, o; SComp, SI, Lin]$$

1. a zero-ary function for the constant 0, $0(;) = 0$,
2. a set of projection functions $U_i^{m+n}(x_1, \dots, x_m; x_{m+1}, \dots, x_{m+n}) = x_i$,
3. successor functions, $s_i(; x) = 2x + i$ for $i \in \{0, 1\}$,
4. two predecessor functions

$$pr_0(; x) = \begin{cases} n & 2n \leq x \leq 2n + 1, n \in \mathbb{N} \\ n + (\epsilon - 1) & 2n + 1 \leq x \leq 2n + \epsilon, 1 \leq \epsilon \leq 2 \end{cases}$$

$$pr_1(; x) = \begin{cases} n + \epsilon & 2n \leq x \leq 2n + \epsilon, 0 \leq \epsilon \leq 1 \\ n + 1 & 2n + 1 \leq x \leq 2n + 2 \end{cases}$$

5. a continuous function to sense inequalities, $\theta_1(; x) = \max\{0, x\}$,

6. parity distinguishing functions

$$e(; x) = \frac{\pi}{2}\theta_1(\sin\pi x)$$

$$o(; x) = \frac{\pi}{2}\theta_1(-\sin\pi x)$$

7. safe composition operator *SComp*: assume a vector of functions $\bar{g}_1(\bar{x};)$, a vector of functions $\bar{g}_2(\bar{x}; \bar{y})$, and a function h of arity $|\bar{g}_1| + |\bar{g}_2|$. Define new function f

$$f(\bar{x}; \bar{y}) = h(\bar{g}_1(\bar{x};); \bar{g}_2(\bar{x}; \bar{y}))$$

8. safe integration operator *SI*: assume functions g, h_0, h_1 , define a new function f satisfying

$$f(0, \bar{y}; \bar{z}) = g(\bar{y}; \bar{z})$$

$$\partial_x f(x, \bar{y}; \bar{z}) = e(x;)[h_1(pr_0(x;), \bar{y}; \bar{z}, f(pr_0(x;), \bar{y}; \bar{z}))$$

$$\quad - h_0(pr_0(x;), \bar{y}; \bar{z}, f(pr_0(x;), \bar{y}; \bar{z}))]$$

$$+ o(x;)[h_0(pr_1(x;), \bar{y}; \bar{z}, f(pr_1(x;), \bar{y}; \bar{z}))$$

$$\quad - h_1(pr_1(x;)-1, \bar{y}; \bar{z}, f(pr_1(x;)-1, \bar{y}; \bar{z}))]$$

Notice that for simplicity we misused the basic functions so that their arguments are now in normal positions (the alternative is to redefine a new set of basic functions with arguments in normal positions).

9. A linearization operator *Lin*: given functions g, h , define a new function f by

$$f(x, \bar{y}; \bar{z}) = \begin{cases} \delta h(2pr_0(x;)+1, \bar{y}; \bar{z}) + (1-\delta)g(2pr_0(x;), \bar{y}; \bar{z}) & e(; x) \geq o(; x) \\ \delta' g(2pr_1(x;), \bar{y}; \bar{z}) + (1-\delta')h(2pr_1(x;)-1, \bar{y}; \bar{z}) & o(; x) \geq e(; x) \end{cases}$$

where $\delta = x - 2pr_0(x;)$, $\delta' = x + 1 - 2pr_1(x;)$.

By induction the following can easily be shown.

Proposition 14. *Every function in \mathcal{W} is polynomial time computable.*

From the fact that the built functions contain piecewise linear extensions to \mathbb{R} of the Bellantoni and Cook class [2], we have.

Proposition 15. *Every polynomial time computable discrete function has a peaceful extension in \mathcal{W} .*

We get from Theorem 2 and and Corollary 1.

Theorem 4. *A Lipschitz function $f : [0, 1] \rightarrow \mathbb{R}$ is polynomial-time computable iff it is \mathcal{W} -definable.*

Theorem 5. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be some n^k, M -bounded function for some k . Then f is polynomial-time computable iff it is \mathcal{W} - n^k -definable.*

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A Proofs

A.1 Proof of Proposition 3

Proof. (1) \Rightarrow (2) : is obtained directly by letting $g(x, y) = yf(x)$.

(2) \Rightarrow (1) : Since g is computable, there exists an oracle machine N^0 which computes g . Assume an input $x \in [0, 1]$ and a Cauchy sequence $\varphi_x \in CF_x$. Assume $n \in \mathbb{N}$ and consider an oracle machine $M^{\varphi_x}(n)$ that does the following:

1. Simulate the operation of $N^{\varphi_x, \varphi_y}(0)$ (for any oracle φ_y),
2. Whenever N^0 queries $\varphi_x(i)$, M^0 queries its own oracle and returns $d = \varphi_x(i)$. Whenever N^0 queries $\varphi_y(j)$, M^0 returns 2^{n+1} .
3. Repeat the last step as long as N^0 keeps querying,
4. Let e be the output of N^0 . Output $2^{-(n+1)}e$.

From this procedure we have

$$\begin{aligned} |e - g(x, 2^{n+1})| &\leq 1 \\ |2^{-(n+1)}e - 2^{-(n+1)}g(x, 2^{n+1})| &\leq 2^{-(n+1)} \end{aligned} \quad (26)$$

From the proposition hypothesis:

$$\begin{aligned} |g(x, 2^{n+1}) - 2^{n+1}f(x)| &\leq 1 \\ |2^{-(n+1)}g(x, 2^{n+1}) - f(x)| &\leq 2^{-(n+1)} \end{aligned} \quad (27)$$

Then

$$\begin{aligned} |M^{\varphi_x}(n) - f(x)| &\leq |M^{\varphi_x}(n) - 2^{-(n+1)}g(x, 2^{n+1})| + |2^{-(n+1)}g(x, 2^{n+1}) - f(x)| \\ &= |2^{-(n+1)}e - 2^{-(n+1)}g(x, 2^{n+1})| + |2^{-(n+1)}g(x, 2^{n+1}) - f(x)| \\ &\leq 2^{-(n+1)} + |2^{-(n+1)}g(x, 2^{n+1}) - f(x)| \quad \text{from Inequality 26} \\ &\leq 2^{-(n+1)} + 2^{-(n+1)} \quad \text{from Inequality 27} \\ &\leq 2^{-n} \end{aligned}$$

(2) \Rightarrow (3) : Obvious.

(3) \Rightarrow (1) : Same as the proof (2) \Rightarrow (1) since the value of y used in this proof, namely 2^{n+1} , is a natural number.

A.2 Proof of Proposition 4

Proof. The proof is similar to that of Proposition 3 only replacing computability by polynomial time computability and observing: (1) multiplication can be done in polynomial time and (2) the procedure given in Proposition 3 is actually polynomial time computable in terms of the precision parameter n .

A.3 Proof of Proposition 5

Proof. (1) \Rightarrow (2) : Define g on integer points as follows,

$$g(x, y) = \begin{cases} 0 & y = 0 \\ yf(\frac{x}{y}) & x \leq y, y \geq 1 \\ yf(1) & \text{otherwise} \end{cases} \quad (28)$$

and piecewise linear for non-integer values. Clearly, g is computable and satisfies (6).

(2) \Rightarrow (1) : For simplicity assume $\epsilon = 1$. Since f is Lipschitz there exists a non-negative constant M such that for all $x, y \in [0, 1]$ the following holds: $|f(x) - f(y)| \leq M|x - y|$. Let $a \in \mathbb{N}$ such that $M \leq 2^a$. Hence, for all $x, y \in [0, 1]$ the following holds: $|f(x) - f(y)| \leq 2^a|x - y|$. Since g is computable, there exists an oracle machine N^0 which computes g . Assume an input $x \in [0, 1]$ and a Cauchy function $\varphi_x \in CF_x$. Assume $n \in \mathbb{N}$ and consider an oracle machine $M^{\varphi_x}(n)$ that does the following:

1. Let $n' = n + 2 + a$ and let $d = \varphi_x(n')$,
2. Then $|d - x| \leq 2^{-n'}$, hence, it can be assumed without loss of generality that (for example by truncating extra bits), $d = \frac{k_1}{2^{n'}}$ for some $k_1 \in \mathbb{N}$,
3. Simulate the operation of $N^0(0)$,
4. Whenever N^0 queries $\varphi_x(i)$, M^0 returns k_1 . Whenever N^0 queries $\varphi_y(j)$, M^0 returns $2^{n'}$.
5. Repeat the last step as long as N^0 keeps querying,
6. Let e be the output of N^0 . Output $2^{-n'}e$.

Then

$$\begin{aligned} |e - g(k_1, 2^{n'})| &\leq 1 \\ |2^{-n'}e - 2^{-n'}g(k_1, 2^{n'})| &\leq 2^{-n'} \end{aligned} \quad (29)$$

From (6) with $\epsilon = 1$

$$\begin{aligned} |g(k_1, 2^{n'}) - 2^{n'}f(\frac{k_1}{2^{n'}})| &\leq 1 \\ |2^{-n'}g(k_1, 2^{n'}) - f(\frac{k_1}{2^{n'}})| &\leq 2^{-n'} \end{aligned} \quad (30)$$

From Inequalities (29) and (30) we have

$$|2^{-n'}e - f(\frac{k_1}{2^{n'}})| \leq 2^{-(n'-1)} \quad (31)$$

From the fact that f is Lipschitz we have

$$|f(\frac{k_1}{2^{n'}}) - f(x)| \leq 2^a 2^{-n'} = 2^{-(n+2)} \quad (32)$$

From Inequalities (31) and (32) we have

$$|2^{-n'}e - f(x)| \leq 2^{-(n'-1)} + 2^{-(n+2)} \leq 2^{-n} \quad (33)$$

This completes the proof that f is computable.

A.4 Proof of Proposition 6

Proof. Similar to the proof of Proposition 5 replacing computability by polynomial time computability.

A.5 Proof of Theorem 1

Proof. (1) \Rightarrow (2) : by Proposition 5, there exists some computable g such that (6) holds with $\epsilon = \frac{3}{4}$. Computing g with precision $1/2$, one can easily build some total recursive function $h : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that $|h(x, y) - \lfloor g(x, y) \rfloor| \leq 1$. Then h is total recursive, and hence by hypothesis there exists some $\tilde{g} \in \mathcal{C}$ such that

$$\forall x, y \in \mathbb{N} : |\tilde{g}(x, y) - h(x, y)| \leq 1/4$$

Hence

$$\forall x, y \in \mathbb{N} : |\tilde{g}(x, y) - \lfloor g(x, y) \rfloor| \leq 1 + \frac{1}{4} = \frac{5}{4} \quad (34)$$

We have $|g(x, y) - \lfloor g(x, y) \rfloor| \leq 1$, then $|\tilde{g}(x, y) - g(x, y)| \leq \frac{9}{4}$. Finally, we have the desired result

$$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}^{\geq 1}, x \leq y : |\tilde{g}(x, y) - yf(\frac{x}{y})| \leq \frac{9}{4} + \frac{3}{4} = 3 \quad (35)$$

(2) \Rightarrow (1) : This follows from Proposition 5 with $\epsilon = 3$, observing that functions from \mathcal{C} are assumed computable.

A.6 Proof of Theorem 2

Proof. Similar to the proof of Theorem 1 replacing computability by polynomial time computability.