Proposal for M2 Internship for ENS Lyon for 2020

Continuous Time Models of Computations and Computations Over the Ordinals
November 6, 2019
Olivier Bournez

Keywords: Analog Models of Computation. Ordinal computation. Continuous Time Models of Computation. Computability. Infinite Time Machines.

Supervisor: Olivier Bournez.
Administrative Position: Professor (of Computer Science) at Ecole Polytechnique.
Location: Laboratoire d’Informatique de l’X, LIX, Ecole Polytechnique , 91128 Palaiseau Cedex
Phone: +33 (0)1 77 57 80 78
Email: olivier.bournez@lix.polytechnique.fr.

General Introduction
Classical computability theory deals with computation over a finite time. From Church-Turing thesis, all sufficiently powerful models of computations are basically equivalent, and hence equivalent to Turing machines. Computability theory is motivated by understanding what is computable, but also by understanding what is provable or not in mathematics.

Recent years and months have seen a clear renewal of interest for models of computation over the ordinals. That is to say, for models where the time is not necessarily finite, and can be transfinite. This is a very old idea, already discussed in [10, 9], but the model of Infinite Time Turing Machines, popularized by [6] in 2000, brought clearly new lights on this field.

In particular, some very surprising results have been proved for this model. This includes the fact that there exists some (ordinal) time that do not correspond to a halting time, and there are even interval of such times: there are called gaps in this context.

Actually, the model has been developed meanwhile in many directions, and have demonstrated to be very useful to revisit fundamentals questions of foundations of mathematics with a new computability light. In particular, constructions from set theory [1]. The approach is also alternative to classical higher order recursion theory [11] in several important aspects, in particular by the fact that it is closer to aspects related to machines.

The purpose of this work is in a long term to contribute to understand the related computability theory, in relation with classical approaches developed in reference books such as [1, 11].

Description of the work
Classical computability theory has strong links with polynomial Ordinary Differential Equations. It turns out that generalized computability, in particular transfinite (ordinal time) computability has links with more general classes of ODEs than polynomial ODES.

We recently considered Continuous ODEs $y' = f(y)$, where $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a continuous function. From Cauchy-Peano’s theorem, a corresponding IVP always has at least a solution, but unicity does not hold unlike in Cauchy-Lipschitz’s theorem. We recently established that such ODEs can be seen as a model of computation over the ordinals: a transfinite time computation of a Turing machine can be embedded into a dynamic over a compact domain in a finite time using a suitable simultaneous space and time-contraction [2]. However, to do so, unicity of solutions is lost, and one has to restrict to a class of continuous functions that always admit unique greedy solutions, i.e. going in a greedy way in some fixed direction.
The purpose of the current work is to explore these connections between ODEs and ordinal time computations. In particular, to characterize up to which ordinal transfinte time computations with Turing machines can be simulated by ODEs, and the relations to models such as the infinite Time Turing machine model [5].

In addition to this connection, one clear motivation of this work is to go to possibly new proofs for theorems in constructible and descriptive set theory, or in ODE theory. Indeed, recent developments of ordinal computability have demonstrated various convincing alternatives to the classical proofs of set theory [3, chapter 9]. This includes [8, 7, 12, 4]. In a dual way, we believe this may provide a new perspective on the analysis of the various solutions of a continuous ODEs.

Comments

The true topic of the work is related to computability theory. This requires only common and basic knowledge in ordinary differential equations. Most of the intuitions of our today's constructions come from classical computability.

There is no specific prerequisite for this internship. This subject can be extended to a PhD. Possibilities of funding according to the administrative situation of candidates.

The supervisor is also open on variations on these questions according to the preferences of the candidate.

References