On the mortality problem for matrices of low dimensions

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Abstract

In this paper, we discuss the existence of an algorithm to decide if a given set of 2×2 matrices is mortal. A set $F = \{A_1, \dots, A_m\}$ of square matrices is said to be *mortal* if there exist an integer $k \geq 1$ and some integers $i_1, i_2, \dots, i_k \in \{1, \dots, m\}$ with $A_{i_1} A_{i_2} \cdots A_{i_k} = 0$. We survey this problem and propose some new extensions. We prove the problem to be BSS-undecidable for real matrices and Turing-decidable for two rational matrices. We relate the problem for rational matrices to the entry-equivalence problem, to the zero-in-the-corner problem, and to the reachability problem for piecewise-affine functions. Finally, we state some NP-completeness results.

1 Introduction

Several undecidability results are known about problems involving matrices [6, 14]. For example, given a finite set F of matrices with integer entries, it is undecidable whether the semi-group generated by M contains a matrix having a zero in the right upper corner [17], is free [11, 8], or contains the zero matrix [20]. These problems have been proved to be undecidable when restricted to 3×3 matrices. But for both of them, the question of their decidability or undecidability when restricted to 2×2 matrices remains open [6].

In this paper, we focus on the decidability of the latter problem. A set $F = \{A_1, \dots, A_m\}$ of $d \times d$ matrices is said to be *mortal* if there exist an integer $k \geq 1$ and some integers $i_1, i_2, \dots, i_k \in \{1, \dots, m\}$ with $A_{i_1} A_{i_2} \cdots A_{i_k} = 0$. Therefore, we focus on the following decision problem denoted by $\text{MORT}_{\mathbb{Q}}(2)$.

Open Problem 1 • *Instance: a finite set F of 2×2 matrices with rational entries.*

- *Question: is F mortal?*

The decidability of problem $\textsc{Mort}_{\mathbb{Q}}(2)$ remains unknown despite a lot of interest (see [15, 16] for some references and discussions).

The question of the decidability of $\textsc{Mort}_{\mathbb{Q}}(2)$ was first mentioned as an open problem in [22] and was formulated as follows: "Find an algorithm, which given a finite set H of nonsingular linear transformations of the complex plane and lines L and M through the origin, determines whether some product from H maps L onto M."

There are at least two motivations to study the mortality problem. First, deciding whether a given set of two by two matrices is mortal is equivalent to deciding whether a switched linear system is controllable. In particular, given a system of the form $x(t+1) = A(t,u)x(t)$, where for all t the set of possible values of $A(t,u)$ is a finite set F of $d \times d$ matrices, the question of mortality of F corresponds to the controllability (to the origin) of such a system (cf. [3]).

Second, proving that $\textsc{Mort}_{\mathbb{Q}}(2)$ is decidable or undecidable, would really clarify computational-complexity issues for discrete-time and hybrid dynamical systems (cf. [12] and [9]). For example, the reachability problem for piecewise-affine dynamical systems has been proven undecidable for two-dimensional systems, but is open and related to the mortality problem (see Section 4.3) for one-dimensional systems [12].

Observe that, if $\textsc{Mort}_{\mathbb{Q}}(2)$ turned out to be undecidable, it would surely give a way, which would extend the results of [1, 12, 19, 24], to simulate a Turing machine by a dynamical system of low dimension. Indeed, most of the undecidability results known up to this date rely on simulations of Turing machines.

This paper aims at giving a global picture of the mortality problem. To do so, we will also talk about the generalization of the problem to matrices with real entries. When $K \in \{\mathbb{R}, \mathbb{Q}\}$, the problem $\textsc{Mort}_K(d)$ (resp. $\textsc{Mort}_K(d,m)$) denotes the following decision problem:

- Instance: a finite set F of $d \times d$ matrices with entries in K (resp. a set F of m $d \times d$ matrices with entries in K).

- Question: is F mortal?

The main contributions of the paper are:

- An undecidability result, already in the case of only two 2×2 matrices, for $K = \mathbb{R}$ in the Blum-Shub-Smale model of computation [5].

- A decidability result for two 2×2 matrices, in the case $K = \mathbb{Q}$ for the Turing model of computation.

  For arbitrary $|F| = m$, the question remains open.

- Reducibility relations between the mortality problem and other problems in the literature.

2 Links between dimension and number of matrices

Paterson proved in [20] that the mortality problem restricted to 3×3 matrices is not decidable.

Theorem 1 ([20]) $\text{MORT}_{\mathbb{Q}}(3)$ *is recursively unsolvable.*

More precisely, Paterson proved in [20] that, if the Post Correspondence Problem (PCP) is undecidable with p rules, then $\text{MORT}_{\mathbb{Q}}(3, 2p+2)$ is undecidable. Using the Modified Post Correspondence Problem (MPCP) instead of PCP, we improve this result.

Proposition 1 *Suppose that the Post Correspondence Problem is undecidable with p rules. Then decision problem* $\text{MORT}_{\mathbb{Q}}(3, p+2)$ *is undecidable.*

Proof. The Post Correspondence Problem (PCP) is the decision problem "given a finite set of pairs of words $\{\langle U_i, V_i\rangle \mid i = 1, \ldots, p\}$, determine if there exists a sequence of indexes $i_1, i_2, \ldots, i_k$ in $\{1, 2, \ldots, p\}$ with $U_{i_1} U_{i_2} \cdots U_{i_k} = V_{i_1} V_{i_2} \cdots V_{i_k}$".

The arguments of Paterson in [20] prove that, to any instance $\{\langle U_i, V_i\rangle \mid i = 1, \ldots, p\}$ of PCP, can be associated a finite set

$$F = \{S, T, W(U_j, V_j), W'(U_j, V_j) \mid j = 1, \ldots, p\}$$

of integer matrices, which satisfy

1. F is mortal if and only if there exists some integers $i_1, i_2, \ldots, i_k$ with $SW'(U_{i_1}, V_{i_1}) W(U_{i_2}, V_{i_2}) \cdots W(U_{i_k}, V_{i_k}) T = 0$;
2. This latter case holds if and only if $U_{i_1} U_{i_2} \cdots U_{i_k} = V_{i_1} V_{i_2} \cdots V_{i_k}$.

We replace the Post Correspondence Problem (PCP) by the Modified Post Correspondence Problem (MPCP) [10] to obtain our result. The difference between PCP and MPCP is that in the latter the first index i_1 must be equal to 1. Namely, the Modified Post Correspondence Problem is the decision problem "given a finite set of pairs of words $\{\langle U_i, V_i\rangle \mid i = 1, \ldots, p\}$, determine if there exists a sequence of indexes $i_2, \ldots, i_k$ in $\{1, 2, \ldots, p\}$ with $U_1 U_{i_2} \cdots U_{i_k} = V_1 V_{i_2} \cdots V_{i_k}$".

Since any instance of PCP can be solved by p calls to MPCP, the undecidability of PCP with p rules implies the undecidability of MPCP with p rules.

There only remains to prove that MPCP with p rules reduces to $\text{MORT}_{\mathbb{Q}}(3, p+2)$. Since in MPCP the first index i_1 is 1, the set of matrices

$$F = \{T, SW'_{U_1, V_1}, W_{U_j, V_j} \mid j = 1, \ldots, p\}$$

is mortal if and only if there exist some integers $i_2, \ldots, i_k$ with

$$SW'(U_1, V_1) W(U_{i_2}, V_{i_2}) \cdots W(U_{i_k}, V_{i_k}) T = 0.$$

By condition 2 above, this holds if and only if $\{\langle U_i, V_i\rangle \mid i = 1, \ldots, p\}$ is a positive instance of MPCP. □

The following result is proved in [2] and in [6].

Lemma 1 *[2, 6] For all $n \geq 2$, $m \geq 1$, $\text{MORT}_{\mathbb{Q}}(d, m)$ undecidable implies $\text{MORT}_{\mathbb{Q}}(dm, 2)$ undecidable.*

The minimal number p of rules for which PCP is undecidable is not known, but p is an integer between 3 and 7 (cf. [18]).

Hence, from Proposition 1, the following can be stated.

Corollary 1

- *Decision problem $\text{MORT}_{\mathbb{Q}}(3, 9)$ is undecidable.*
- *Decision problem $\text{MORT}_{\mathbb{Q}}(27, 2)$ is undecidable.*

3 On the decidability of $\text{MORT}(2, 2)$.

We now come back to the decidability of the mortality problem for two-dimensional matrices. We prove first that $\text{MORT}_{\mathbb{R}}(2, 2)$ is BSS-undecidable. Then we prove that $\text{MORT}_{\mathbb{Q}}(2, 2)$ is Turing-decidable.

We make use of the following lemma several times.

Lemma 2 *A finite set $F = \{A_1, \ldots, A_m\}$ of 2×2 matrices is mortal if and only if there exist an integer k and integers $i_1, \ldots, i_k \in \{1, \ldots, m\}$ with $A_{i_1} \cdots A_{i_k} = 0$, and*

1. $\text{rank}(A_{i_j}) = 2$ *for* $1 < j < k$,
2. $\text{rank}(A_{i_j}) < 2$ *for* $j \in \{1, k\}$.

Proof. Only the direct sense requires a proof. Assume that F is mortal. Then there exists a null product $A_{i_1} \cdots A_{i_k} = 0$ where k is minimal. Assume $k \geq 2$, because otherwise the assertion is immediate. The matrices A_{i_1} and A_{i_k} of this product are singular because otherwise a null-product with fewer matrices could be obtained by multiplying $A_{i_1} \cdots A_{i_k}$ by their inverse(s).

Let $j \geq 2$ be the smallest integer with $\text{rank}(A_{i_j}) < 2$. Since we have $A_{i_1} \cdots A_{i_k} = 0$, matrix $A_{i_1} \cdots A_{i_{j-1}}$ sends the image I of matrix $A_{i_j} \cdots A_{i_k}$ to 0. Now, I is also equal to the image of A_{i_j} and is of dimension 1. Indeed, first, I is clearly included in the image of A_{i_j}. Second, by definition of k, I cannot be of dimension 0, and third, the dimension of the image of A_{i_j} is at most 1 because $\text{rank}(A_{i_j}) < 2$. We obtain $A_{i_1} \cdots A_{i_{j-1}} A_{i_j} = 0$. This implies $j = k$, and the direct sense of the lemma. □

3.1 BSS-undecidability of $\textsc{Mort}_{\mathbb{R}}(2,2)$

Talking about the decidability or undecidability of $\textsc{Mort}_{\mathbb{R}}(2)$ requires one to talk about machines that manipulate real numbers.

One approach is to use the machine model studied in recursive analysis (e.g., see [26]). However, this model does not meet our needs because one cannot decide whether a real number is equal to zero in this model [26].

Another approach is to use the Turing machine model for real numbers proposed by Blum, Shub, and Smale in [4, 5]. Roughly speaking, a BSS-machine[1] is an extended Random Access Machine [10] that treats real numbers as basic entities; namely a BSS-machine contains an unbounded number of real registers $x_1, \dots, x_n, \dots$ each of which can hold one real number in unbounded precision. Moreover, a BSS-machine contains a finite number of built-in constants $\lambda_1, \dots, \lambda_m$. Its program is made of arithmetic operations between its real registers of type $x_i := x_j \# x_k$, for $\# \in \{+, -, *, /\}$, or of type $x_i := \lambda_j$, or of tests of type $x_i \# x_j$ with $\# \in \{>, \geq, <, \leq, =, \neq\}$. Let $\mathbb{R}^\infty = \cup_{i \in \mathbb{N}} \mathbb{R}^i$. An input $x \in \mathbb{R}^\infty$ is of type $x = (x_1, \dots, x_i)$ for some i. The input is said to be *accepted* by the machine if the program of the machine eventually halts when started with its real registers set to $(x_1, \dots, x_i, 0, \dots, 0, \dots)$. A language $L \subset \mathbb{R}^\infty$ is said to be *BSS-recursively enumerable* if it consists of the accepted inputs of some BSS-machine. The language L is said to be *BSS-recursive* if, in addition, its complement is BSS-recursively enumerable.

In other words, BSS recursive sets are those that can be decided using only arithmetical operations and tests. The reader should refer to [4, 5] for more formal descriptions. We assume that the reader is familiar with the BSS-model in the rest of this paper.

We first recall a lemma proved in [5]. A set $S \subset \mathbb{R}^n$ is said to be a *basic semi-algebraic set* if $S = \{(x_1, \dots, x_n) \mid p_1(x_1, \dots, x_n) > 0 \wedge \dots \wedge p_{n_1}(x_1, \dots, x_n) > 0 \wedge p'_1(x_1, \dots, x_n) = 0 \wedge \dots \wedge p'_{n_2}(x_1, \dots, x_n) = 0\}$ for some n-variable polynomials $p_1, p_2, \dots, p_{n_1}, p'_1, \dots, p'_{n_2}$. A *semi-algebraic set* is a finite union of basic semi-algebraic sets.

Lemma 3 *Let $L \subset \mathbb{R}^\infty$ be a BSS-recursively enumerable set. Then L is a denumerable union of semi-algebraic sets.*

Sketch of proof. Write $L = \cup_{t \in \mathbb{N}} \textsc{Acc}_t$, where $\textsc{Acc}_t$ is the subset of the inputs that are accepted by the machine at time t. Each subset $\textsc{Acc}_t$ is a semi-algebraic set. See [5] for the formal details. □

The remaining arguments of this subsection are inspired from [13]. (In fact, there seems to be a close relation between mortality and stability. Cf. [13].)

We start with the following preliminary result.

Lemma 4 *Let $a, b, \lambda \in \mathbb{R}$ be some real numbers with $a^2 + b^2 \neq 0$ and $\lambda \neq 0$. Let θ be an argument of complex number $a + ib$. The pair of matrices $F(a, b, \lambda) = \{A_1, A_2\}$ with*

[1]The BSS-model is not a realizable computation concept but is of mathematical interest for studying computations on the reals.

$$A_1 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -\lambda & 1 \\ 0 & 0 \end{pmatrix},$$

is mortal if and only if there exists an integer $n \in \mathbb{N}$ with $\lambda = \tan(n\theta)$.

Proof. From Lemma 2, we know that $F(a, b, \lambda)$ is mortal if and only if there exists an integer $n \in \mathbb{N}$ with $A_2 A_1^n A_2 = 0$. This holds if and only if there exists a nth power of A_1 which sends the image of A_2 to its kernel. Since $\mathrm{Im}\,(A_2) = \prec (1, 0) \succ$, $\mathrm{Ker}(A_2) = \prec (1, \lambda) \succ$, and since A_1 is the composition of an homothety and a rotation of angle θ, this is true if and only if there exists an integer $n \in \mathbb{N}$ with $\lambda = \tan(n\theta)$. □

The following observations are easy.

Lemma 5 *Let θ be a real number. Let $E(\theta)$ be the subset of $\mathbb{R}$ defined by*

$$E(\theta) = \{\lambda \mid \text{there exists an integer } n \in \mathbb{N} \text{ with } \lambda = \tan(n\theta)\}.$$

1. *$E(\theta)$ is a dense subset of $\mathbb{R}$ if and only if $\theta/\pi \notin \mathbb{Q}$.*
2. *There exist two rational numbers $a, b \in \mathbb{Q}$ such that any argument θ of complex number $a + ib$ satisfies $\theta/\pi \notin \mathbb{Q}$. Indeed, take for example $a = 1$ and $b = 2$ (see Lemma 6).*
3. *When $\theta/\pi \notin \mathbb{Q}$, the complement $E^c(\theta)$ of $E(\theta)$ in $\mathbb{R}$ has an uncountable number of connected components; actually, every point of $E^c(\theta)$ is its own connected component.*

We can now prove that $\mathrm{Mort}_{\mathbb{R}}(2, 2)$ is BSS-undecidable. Observe that the arguments are close to the ones in [13]. We, however, deal with a different problem and with a modified family of matrices.

Theorem 2 $\mathrm{Mort}_{\mathbb{R}}(2, 2)$ *is BSS-recursively enumerable but is not BSS-recursive.*

Proof. Building a BSS-machine that semi-recognizes $\mathrm{Mort}_{\mathbb{R}}(2, 2)$ is easy. Therefore the problem is BSS-recursively enumerable.

Representing the matrices by their coefficients, the space of the instances of problem $\mathrm{Mort}_{\mathbb{R}}(2, 2)$ is $\mathbb{R}^8$. Denote by $\mathrm{Pos} \subset \mathbb{R}^8$ (resp. by $\mathrm{Neg} \subset \mathbb{R}^8$) the subset of the positive (resp. negative) instances of the problem. Using Lemma 3, we only need to prove that Neg is not a countable union of semi-algebraic sets.

Let $a, b \in \mathbb{Q}$ with $a + ib = \rho e^{i\theta}, \theta/\pi \notin \mathbb{Q}$ as in Lemma 5. Let $\gamma : \mathbb{R} \to \mathbb{R}^8$ be the function that sends $\lambda \in \mathbb{R}$ to the pair of matrices $F(a, b, \lambda)$. By definition of γ, the image $\mathrm{Im}\,\gamma$ of γ is an algebraic subset of $\mathbb{R}^8$ and γ realizes an homeomorphism between $\mathbb{R}$ and $\mathrm{Im}\,\gamma$. By Lemma 4, we know that $\gamma^{-1}(\mathrm{Pos}) = E(\theta)$ and $\gamma^{-1}(\mathrm{Neg}) = E^c(\theta)$. Since γ is an homeomorphism, $E^c(\theta)$ and $\gamma(E^c(\theta)) = \mathrm{Neg} \cap \mathrm{Im}\,\gamma$ must have the same number of connected components. That is, by part 3 of Lemma 5, they must have an uncountable number of connected components.

Assume, by contradiction, that we can write $\text{NEG} = \cup_{i \in \mathbb{N}} S_i$ where each S_i is a semi-algebraic subset. We would have $\text{NEG} \cap \text{Im}\,\gamma = \cup_{i \in \mathbb{N}} (\text{Im}\,\gamma \cap S_i)$. Each of the $(\text{Im}\,\gamma \cap S_i)$ must be a semi-algebraic subset as the result of the intersection between an algebraic set and a semi-algebraic set. Since a semi-algebraic set has a finite number of connected components, $\text{NEG} \cap \text{Im}\,\gamma$ must have a countable number of connected components. This leads to a contradiction. □

We obtain the following immediately.

Corollary 2

- *For $n \geq 2$, $m \geq 2$, the problem $\text{MORT}_{\mathbb{R}}(n, m)$ is BSS-recursively enumerable but not BSS-recursive.*
- $\text{MORT}_{\mathbb{R}}(2)$ *is BSS-recursively enumerable but not BSS-recursive.*

However, observe that it is easy to extract the following fact from the proofs of the next section.

Theorem 3 *Problem $\text{MORT}_{\mathbb{R}}(2, 2)$ restricted to matrices with real eigenvalues is BSS-recursive.*

Let us discuss the results of Theorem 2 and Corollary 2. Deciding whether a set of matrices is mortal using *only arithmetical operations* is not possible. But it does not mean that the problem cannot be decided by an algorithm which uses non-arithmetical operations or which uses arguments about the semi-ring K of the entries for $K \neq \mathbb{R}$.

Actually, using number-theoretical arguments, we prove in the next subsection that the decision problem $\text{MORT}_{\mathbb{Q}}(2, 2)$ is Turing-decidable.

3.2 Turing-decidability of $\text{MORT}_{\mathbb{Q}}(2, 2)$

The decidability of $\text{MORT}_{\mathbb{Q}}(2, 2)$ has already been claimed [6, 21]. However, the proofs were either wrong or incomplete. More precisely, in [6], the result is claimed without proof. In [21], the result is claimed but the proof is wrong. Indeed, the proof of [21] presents an algorithm to decide $\text{MORT}_{\mathbb{Q}}(2, 2)$ which uses only arithmetical operations, and this precisely contradicts[2] Theorem 2. We present a full proof herein.

From the previous section, we know that to prove the decidability of $\text{MORT}_{\mathbb{Q}}(2, 2)$, we need the use of number-theoretical arguments. The arguments[3] we use are based on the following result extracted from [23].

Lemma 6 ([23])

- *The following decision problem is decidable.*
  - *Instance: rational numbers $p, q \in [-1, 1]$.*

[2] Concretely, the cases studied in the proof [21] do not cover all the cases. In particular, the proof forgets the case of rational matrices with complex eigenvalues.

[3] Not present in [21]. Of course, those arguments could always be patched to the (missing) cases of the proof of [21], but we prefer presenting a complete, independent, and correction-/patch-free proof.

– *Question: does there exist* $\theta \in \mathbb{R}$ *and an integer* $n \in \mathbb{N}$ *with* $\cos(\theta) = p$ *and* $\cos(n\theta) = q$*?*

- *When* $p \notin \{0, 1/2, 1\}$*, there are at most a finite number of such* n *and those* n *can be computed effectively.*

Proof. For completeness we provide the proof, which is almost cut-and-paste from [23].

Write $p = r/s, q = u/v$ where r, s, u, v are integers such that $\gcd(r, s) = \gcd(u, v) = 1$. The decidability of the problem when $p = 0$, $p = 1/2$, or $p = 1$ is trivial, since in that case the sequence $n \to \cos(n\theta)$ assumes only a computable finite number of values that can be tested against q. Suppose $p \notin \{0, 1/2, 1\}$. The function $\cos(n\theta)$ is a polynomial in $\cos(\theta)$ with integer coefficients. If this polynomial is written $\cos(n\theta) = p_n(r/s)$, then $s^n p_n(r/s)$ is some integer c_n which satisfies

$$2rc_{n+1} - s^2 c_n = c_{n+2}, \tag{1}$$

with $c_1 = r$ and $c_2 = 2r^2 - s^2$; indeed if we denote $a_n = \sin(nx)$ and $b_n = \cos(nx)$, this recurrence comes from $a_{n+1} = a_1 b_n + b_1 a_n$, $b_{n+1} = b_1 b_n - a_1 a_n$, ...

Suppose that s is not a power of 2. Write $s = 2^a b$, $v = 2^{a'} b'$ with $b > 1$, $b' \geq 1$ odd. We are searching for an integer n such that $c_n/(2^{an} b^n) = u/(2^{a'} b')$. We claim $\gcd(c_n, b^n) = 1$ for all $n \in \mathbb{N}$. Indeed, if some odd integer d divides s and c_n simultaneously, then, since $\gcd(r, s) = 1$, the assertions $d|c_{n-1}, d|c_{n-2}, \ldots, d|c_2$ imply $d|r^2$, which implies $d = 1$. As a consequence, an integer candidate n must satisfy $b' = b^n$. There are at most a finite number of such n and those n are computable.

Suppose now that s is a power of 2. Write $s = 2^k, k > 1$ (remember that we supposed $r/s \neq 1/2$). Write every c_n as $c_n = 2^{\lambda_n} v_n$ where v_n is an odd integer. Recurrence (1) becomes

$$2^{\lambda_{n+1}+1} r v_{n+1} - 2^{\lambda_n + 2k} v_n = 2^{\lambda_{n+2}} v_{n+2}. \tag{2}$$

We prove first that there exists an integer n with $\lambda_n + 1 < 2k + \lambda_{n-1}$. Indeed, if it were false, we would always have $\lambda_n + 1 \geq 2k + \lambda_{n-1}$, so that $\lambda_n + 1 \geq 2(n-1)k + \lambda_1$ would hold for all n. Since $|\cos(n\theta)| < 1$, we have $kn \geq \lambda_n$ which implies $kn \geq 2(n-1)k + \lambda_1 - 1$ for all $n \in \mathbb{N}$. Clearly this is impossible.

Let n_0 be the smallest integer such that $\lambda_{n_0+1} + 1 < 2k + \lambda_{n_0}$. Integer n_0 can be computed effectively by testing this condition for increasing n. We have $\lambda_{n_0+2} = \lambda_{n_0+1} + 1$. Indeed, from (1) we must have $rv_{n_0+1} - 2^{\lambda_{n_0}+2k-\lambda_{n_0+1}-1} v_{n_0} = 2^{\lambda_{n_0+2}-\lambda_{n_0+1}-1} v_{n_0+2}$. Considering parity of both sides, this can happen only if $\lambda_{n_0+2} = \lambda_{n_0+1} + 1$.

Since that implies $\lambda_{n_0+2} + 1 = \lambda_{n_0+1} + 2 < \lambda_{n_0+1} + 2k$, we can repeat the argument, and get by induction that for all integers $h \geq 0$, $\lambda_{n_0+2+h} = \lambda_{n_0+1+h} + 1$ holds. Hence, for each positive integer h, we must have $\lambda_{n_0+h} = \lambda_{n_0} + h$.

Now, return to the existence of an integer n with $\cos(n\theta) = u/v$. For $\cos(\theta)$ having denominator 2^k, v must be a power of 2. Suppose $v = 2^m$. It may happen that there exists a solution for $n \leq n_0$. For $n > n_0$, a solution $n = n_0 + h$ must satisfy $\cos((n_0 + h)\theta) = v_{n_0+h} 2^{\lambda_{n_0}+h}/2^{k(n_0+h)} = u/2^m$, hence $k(n_0 + h) - \lambda_{n_0} - h = m$, or $h = (m + \lambda_{n_0} - kn_0)/(k - 1)$. That is, the only integer n candidate exceeding n_0 is $n_0 + (m + \lambda_{n_0} - kn_0)/(k - 1)$. Hence, there are at most $n_0 + 2$ integer candidates n that could satisfy $\cos(n\theta) = u/v$, and those candidates are computable. □

With this we now obtain the following.

Theorem 4 $\textsc{Mort}_{\mathbb{Q}}(2, 2)$ *is decidable.*

Proof. Let $F = \{A_1, A_2\}$ be an instance of the problem. Suppose without loss of generality that the rank of A_2 is greater than the rank of A_1. If A_1 is of rank 2, then the two matrices are nonsingular and F is non-mortal by Lemma 2. If A_1 is of rank 0 then F is mortal. If the two matrices have rank 1, by Lemma 2, it suffices to test whether one of the products $A_1^2, A_1A_2, A_2A_1, A_2^2$ is null.

There remains only the case where A_2 is nonsingular and A_1 is of rank 1. By Lemma 2, F is mortal if and only if there exists an integer $n \in \mathbb{N}$ with

$$A_1 A_2^n A_1 = 0. \tag{3}$$

We want to check this relation algebraically using the Jordan forms of the matrices A_1 and A_2. Write

$$A_1 = P_1^{-1} J_1 P_1, \qquad A_2 = P_2^{-1} J_2 P_2,$$

$$J_1 = \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$J_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

with P_1 and P_2 nonsingular. Eigenvalue κ is equal to the trace of rational matrix A_1, and hence, is a rational number. Eigenvalues λ and μ are the (possibly complex) roots of the characteristic polynomial of rational matrix A_2.

Equation (3) becomes

$$P_1^{-1} J_1 P_1 P_2^{-1} J_2^n P_2 P_1^{-1} J_1 P_1 = 0$$

or, since P_1 is nonsingular,

$$J_1 P J_2^n P^{-1} J_1 = 0$$

where

$$P = P_1 P_2^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Now, after substituting P, P^{-1} and J, the problem is equivalent to testing whether there exists an integer $n \in \mathbb{N}$ with

- $ps\lambda^n - qr\mu^n = 0$, when J_2 is of the first form; or

- $(ps - qr)\lambda^n - rpn = 0$, when J_2 is of the second form.

Suppose that J_2 is of the second form. Eigenvalue λ is rational because λ is equal to the trace of rational matrix A_2 divided by two. Coefficients κ, p, q, r, and s are computable rational numbers which can easily be expressed in terms of the coefficients of the matrices A_1 and A_2 from previous considerations. Testing whether there exists an integer n with $(ps - qr)\lambda^n - rpn = 0$ is easy in that case. Indeed, the equation $(ps - qr)\lambda^x - rpx = 0$ over real variable x clearly has a unique real solution x^* in that case; any numerical method[4] can return an approximation to precision $1/2$ of x^*, namely a rational number x_{app} with $|x_{\text{app}} - x^*| < 1/2$; testing whether $(ps - qr)\lambda^n - rpn = 0$ has a solution is then equivalent to testing this equation with $n = \lfloor x_{\text{app}} \rfloor$.

Suppose that J_2 is of the first form. We want to test the existence of an integer n with $ps\lambda^n - qr\mu^n = 0$. Observe that $\lambda \neq 0, \mu \neq 0$ since A_2 is of rank 2. λ, μ and the coefficients p, q, r, s can be complex numbers but are computable elements of $\mathbb{Q}(\lambda)$. That is, they are of the form $a + \lambda b$ for some rational numbers $a, b \in \mathbb{Q}$ computable from the rational coefficients of the matrices A_1 and A_2. By computing in $\mathbb{Q}(\lambda)$, the cases $ps = 0$ or $qr = 0$ are trivial. Suppose now $ps \neq 0$ and $qr \neq 0$. The problem is equivalent to testing whether there exists an integer n with $(\lambda/\mu)^n = (pq)/(rs)$. We must have $|\lambda/\mu|^n = |pq|/|rs|$. When $|\lambda/\mu| \neq 1$, n must be equal to $|pq|/(|rs| \log |\lambda/\mu|)$; we only need to use any numerical algorithm for approximating this real quantity x^* by some rational number x_{app} with $|x_{\text{app}} - x^*| < 1/2$, and test the equation for integer $n = \lfloor x_{\text{app}} \rfloor$. When $|\lambda/\mu| = 1$ and λ and μ are real numbers, we have necessarily that $\lambda = \mu$ or $\lambda = -\mu$. In both cases, by computing in $\mathbb{Q}(\lambda)$ the problem is trivial. When $|\lambda/\mu| = 1$ and $|pq|/|rs| \neq 1$ the problem has no solution.

There remains only the case where λ and μ are two conjugated complex roots and $(pq)/(rs)$ is a complex number of modulus 1. In that case λ is a complex number with rational real part because λ is a root of the characteristic polynomial of matrix A_2 with rational coefficients. Therefore, complex numbers λ/μ and $(pq)/(rs)$ of type $a + \lambda b$ with computable $a, b \in \mathbb{Q}$ must also have rational computable real parts. If θ denotes an argument of complex number λ/μ of modulus 1, an integer n solution must satisfy $\cos(n\theta) = r'$ where r' is the real part of $(pq)/(rs)$. When the real part p' of λ/μ is equal to $1/2$, $n \mapsto (\lambda/\mu)^n$ is a periodic sequence of period 6, and it suffices to check $(\lambda/\mu)^n = (pq)/(rs)$ for $n = 0, 1, \ldots, 5$. Cases $p' = 0$ and $p' = 1$ can be dealt with similarly. Now, when $p' \notin \{0, 1/2\}$, by Lemma 6, there are at most a finite number of integers n satisfying $\cos(n\theta) = r'$ and those integers are computable. It suffices to check if equation $(\lambda/\mu)^n = (pq)/(rs)$ holds for those integers. □

We have just proved that $\text{MORT}_{\mathbb{Q}}(2, 2)$ is Turing-decidable. We do not know whether $\text{MORT}_{\mathbb{Q}}(2, 3)$ is decidable. So our knowledge of the decidability of $\text{MORT}_{\mathbb{Q}}(2)$ stops at the previous theorem. However, our proof of the BSS-undecidability of the problem shows that the problem is more a number-theoretic problem than a simple computability problem.

[4]For example, Newton's method.

In the next section, we show that $\textsc{Mort}_{\mathbb{Q}}(2)$ can be related to other open problems in the literature.

As pointed out by an anonymous referee, the previous theorem can also be proved in a much more compact (but not self-contained) way using the results of [25]. Let A_0, A_1 be two 2×2 matrices with rational entries. To the word $w = w_1 w_2 ... w_n \in \{0,1\}^*$ we associate the matrix $A_w = A_{w_1} A_{w_2} ... A_{w_n}$. The language $Z(A_0, A_1)$ is the set of words w for which $A_w = 0$. Theorem 4 says that given two 2×2 matrices A_0, A_1 with rational entries, one can effectively test if $Z(A_0, A_1)$ is empty. This could also be seen as a consequence of the following stronger claim: given two 2×2 matrices A_0, A_1 with rational entries, one can effectively compute the language $Z(A_0, A_1)$.

Indeed, if both matrices are of rank 2, then $Z(A_0, A_1)$ is empty. If $\text{rank}(A_0) = 0$ the problem is trivial, so assume $\text{rank}(A_0) = 1$. Then $A_0 = bc^T$ where b and c are nonzero column vectors with rational entries. By Lemma 2, if there exists a mortal product, then $A_0 A_1^k A_0 = 0$ for some integer k. This condition is equivalent to $c^T A_1^k b = 0$. Let $\gamma_k = c^T A_1^k b$; the sequence γ_k is a linear recursive sequence of order 2. [25] proves that there exists an algorithm for finding a semi-linear definition of the set of zeros of any linear recursive sequence of order ≤ 3 from a definition of the sequence. That implies that the set of indices k for which $\gamma_k = 0$ can be computed effectively. The language Z is equal to the set of words uvw where u and w are arbitrary words, and v is a word of the form 01111...11110 where the number of 1's is any k for which $\gamma_k = 0$.

4 Relations to other problems in the literature

In this section we prove that $\textsc{Mort}_{\mathbb{Q}}(2)$ is equivalent to the entry-equivalence problem studied in [15], to the zero-in-the-corner problem studied in [17, 6], and can be linked to the problems studied in [12].

When C is a matrix, $C_{i,j}$ denotes the entry in its ith row and jth column.

4.1 The Entry-Equivalence Problem

Here is a variation of Theorem 2 of [15] (unlike Theorem 2 of [15], we do not suppose F to be composed of only nonsingular matrices).

Lemma 7 *Let F be a finite set of 2×2 matrices. There exists an integer k and some integers $i_1, \ldots, i_k$ such that $A_{i_1} \cdots A_{i_k}$ is a matrix C satisfying $C_{2,1} = C_{2,2}$ if and only if the finite set F' composed of the matrices of F and of the matrix*

$$H = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

is mortal.

Proof. First observe that $HCH = 0$ holds if and only if $C_{2,1} = C_{2,2}$. That proves the direct sense.

Conversely, by Lemma 2, if F is mortal there exist $i_1, \ldots, i_k$ with $A_{i_1} \cdots A_{i_k} = 0$, $A_{i_j} \neq H$ for $1 < j < k$, and $\mathrm{rank}(A_{i_j}) < 2$ for $j \in \{1, k\}$. If $A_{i_1} = A_{i_k} = H$ the remark of the previous paragraph implies that $C = A_{i_2} \cdots A_{i_{k-1}}$ satisfies $C_{2,1} = C_{2,2}$. If $A_{i_1} \neq H$ and $A_{i_k} \neq H$ then $A_{i_1} \cdots A_{i_k}$ is a product of matrices from F equal to the null-matrix, and the null-matrix O satisfies $O_{2,1} = O_{2,2}$. Now, for the remaining cases, observe that equation $HC = 0$ (resp. $CH = 0$) implies $C_{2,1} = C_{2,2}$. □

We can now extend a result of [15].

Theorem 5 (Entry-Equivalence) *Let $K \in \{\mathbb{R}, \mathbb{Q}\}$.*

Problem $\textsc{Mort}_K(2)$ *is equivalent to the following decision problem:*

- *Instance: a finite set $F = \{A_1, \ldots, A_m\}$ of 2×2 matrices with entries in K.*
- *Question: does there exist an integer k and some integers $i_1, \ldots, i_k$ such that $A_{i_1} \cdots A_{i_k}$ is a matrix C satisfying $C_{2,1} = C_{2,2}$?*

and to the following decision problem:

- *Instance: a finite set $F = \{A_1, \ldots, A_m\}$ of* nonsingular *2×2 matrices with entries in K.*
- *Question: does there exist an integer k and some integers $i_1, \ldots, i_k$ such that $A_{i_1} \cdots A_{i_k}$ is a matrix C satisfying $C_{2,1} = C_{2,2}$?*

Proof. Clearly the second problem reduces to the first. The first problem reduces to the mortality problem for 2×2 matrices by Lemma 7 and a reduction from the mortality problem for 2×2 matrices to the second problem is given in [15]. □

As a corollary to our results, we obtain that the above-mentioned problems are not decidable over $\mathbb{R}$, and open and equivalent over $\mathbb{Q}$.

4.2 The Zero-in-the-Corner Problem

It is known that the problem of deciding whether the semi-group generated by a finite set of 3×3 nonsingular matrices contains an element with a zero in the right upper corner is undecidable [6, 17]. However, the decidability of the problem for 2×2 matrices is left open [6].

Nevertheless, this problem can be related to the mortality problem by the next theorem.

Theorem 6 (Zero-in-the-Corner) *Let $K \in \{\mathbb{R}, \mathbb{Q}\}$.*

Problem $\textsc{Mort}_K(2)$ *is equivalent to the following decision problem:*

- *Instance: a finite set $F = \{A_1, \ldots, A_m\}$ of 2×2 matrices with entries in K.*
- *Question: does there exist an integer k and some integers $i_1, \ldots, i_k$ such that $A_{i_1} \ldots A_{i_k}$ is a matrix C satisfying $C_{1,1} = 0$?*

and to the following decision problem:

- *Instance: a finite set $F = \{A_1, \ldots, A_m\}$ of* nonsingular 2×2 *matrices with entries in K.*

- *Question: does there exist an integer k and some integers $i_1, \ldots, i_k$ such that $A_{i_1} \cdots A_{i_k}$ is a matrix C satisfying $C_{1,1} = 0$?*

Proof. Denote

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Observing that, for all matrix C, matrix $C' = PCP^{-1}$ satisfies $C'_{1,1} = 0$ if and only if $C_{2,1} = C_{2,2}$, the above problems are equivalent to the equivalent problems of Theorem 5 by conjugations by matrix P. □

As a corollary to our results, we obtain that the above-mentioned problems are not decidable over $\mathbb{R}$, and open and equivalent over $\mathbb{Q}$.

4.3 Restriction to lower triangular matrices

It was proposed in [15] to restrict the previous problems to lower triangular matrices. Indeed, [20] also proves that the entry-equivalence problem is undecidable for lower-triangular 3×3 matrices with rational entries.

Problem $\text{MORT}_{\mathbb{Q}}(2)$ restricted to lower triangular matrices is trivially decidable [15]. Indeed, a finite set F of lower triangular matrices is mortal if and only if there exist two matrices A, B in F with $A_{1,1} = 0$ and $B_{2,2} = 0$. The zero-in-the-corner problem also becomes trivial when restricted to lower-triangular matrices.

However, the answer to the following question is not known.

Open Problem 2 (Lower triangular matrices) *Is the following decision problem decidable?*

- *Instance: a finite set $F = \{A_1, \ldots, A_m\}$ of* nonsingular lower-triangular 2×2 *matrices with rational entries.*

- *Question: does there exist an integer k and some integers $i_1, \ldots, i_k$ such that $A_{i_1} \cdots A_{i_k}$ is a matrix C satisfying $C_{2,1} = C_{2,2}$?*

We prove that this problem can be related to a non-deterministic version of the open problem mentioned in [12].

Theorem 7 *Open Problem 2 is equivalent to the decidability of the following decision problem:*

- *Instance: a finite set $F = \{f_1, \ldots, f_m\}$ of non-constant rational affine functions of dimension 1 (i.e. a set of functions of type $f_i : x \mapsto a_i x + b_i$, $a_i, b_i \in \mathbb{Q}$, $a_i \neq 0$).*

- *Question: does there exists a composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}$ of these functions that maps point 0 to point 1?*

Proof. Call this problem the *composition problem*. Suppose that a finite set $F = \{A_1, \ldots, A_m\}$ of nonsingular lower-triangular matrices is given. Without loss of generality, we can suppose $A_{2,2} = 1$ for each matrix $A \in F$. Indeed, each matrix $A \in F$ must satisfy $A_{2,2} \neq 0$ to be nonsingular, and replacing each matrix A by matrix $A/A_{2,2}$ in F does not change the mortality of set F.

Open Problem 2 reduces to the instance $F' = \{f_1, \ldots, f_m\}$ of the composition problem where $f_i : x \mapsto (A_i)_{1,1}x + (A_i)_{2,1}$. Indeed, any product $C = A_{i_1} \cdots A_{i_k}$ of lower-triangular matrices with $(A_{i_j})_{2,2} = 1$ satisfies $C_{2,2} = 1$ and $C_{2,1} = f_{i_1} \circ f_{i_2} \cdots \circ f_{i_k}(0)$.

Conversely the composition problem reduces to Open Problem 2. When a finite set $F = \{f_1, \ldots, f_m\}$ of non-constant affine rational functions is given, $f_i : x \mapsto a_i x + b_i$, it suffices to consider $F' = \{A_1, \ldots, A_m\}$ with

$$A_i = \begin{pmatrix} a_i & 0 \\ b_i & 1 \end{pmatrix}$$

and to observe that any product $C = A_{i_1} \cdots A_{i_k}$ of matrices of this form satisfies $C_{2,2} = 1$ and $C_{2,1} = f_{i_1} \circ f_{i_2} \cdots \circ f_{i_k}(0)$. □

4.4 NP-completeness results

4.4.1 K-length mortality

A set $F = \{A_1, \ldots, A_m\}$ of $d \times d$ matrices is said to be *K-length mortal* if there exist an integer $k \leq K$ and some integers $i_1, i_2, \ldots, i_k \in \{1, \ldots, m\}$ with $A_{i_1} A_{i_2} \cdots A_{i_k} = 0$.

Theorem 8 *Given a set F of m 3×3 matrices with rational entries and an integer $K \leq 1+m/2$, the decision problem "Is F K-length-mortal?" is NP-hard.*

Proof. Via the reduction of [20] (or the proof of Proposition 1) and the NP-completeness of Bounded PCP [7]. □

Observe that [2] proves that this result remains true whenever the matrices are assumed to have entries in $\{0, 1\}$.

4.4.2 Mortality without repetition

When repetitions of matrices are not allowed, the problem also becomes clearly decidable. A multi-set $F = \{A_1, \ldots, A_m\}$ of $d \times d$ matrices is said to be *mortal without repetition* if there exist integers $k \geq 1$ and some integers $i_1, i_2, \ldots, i_k \in \{1, \ldots, m\}$ such that $A_{i_1} A_{i_2} \cdots A_{i_k} = 0$ and $i_{j_1} \neq i_{j_2}$ for all $j_1 \neq j_2$.

Theorem 9 *Given a finite multi-set F of m 2×2 matrices, and an integer K, the decision problem "Is F K-length-mortal without repetition?" is NP-hard in the strong sense.*

The proof uses a reduction from subset product [7]. We restate this problem here.

Proposition 2 (Subset Product (Yao)) *Given a finite set A, a size $s(a) \in \mathbb{N}^+$ for each $a \in A$, and a positive integer B, the decision problem "Is there a subset $A' \subset A$ such that the product of the sizes of the elements in A' is exactly B?" is NP-complete in the strong sense.*

Proof.[of Theorem 9] Given an instance of subset product with $|A| = n$, define $n+3$ matrices as follows.

$$\begin{pmatrix} 1 & 0 \\ 0 & s(a) \end{pmatrix}, \text{ for } a \in A, \qquad \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Note that we have repeated the last matrix, since we are required to use it twice. Denote the last matrix by H. Check that for all 2×2 matrices A, $HAH = 0$ if and only if $A_{2,1} = A_{2,2}$. Hence, by Lemma 2, this set of matrices is mortal without repetition with length $4 \leq k \leq n+3$ steps if and only if subset product has a solution in $1 \leq k-3 \leq n$ steps. □

5 Thanks

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