# Real Recursive Functions and Real Extensions of Recursive Functions

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**Abstract** Recently, functions over the reals that extend *elementarily* computable functions over the integers have been proved to correspond to the smallest class of real functions containing some basic functions and closed by composition and linear integration.
We extend this result to *all* computable functions: functions over the reals that extend total recursive functions over the integers are proved to correspond to the smallest class of real functions containing some basic functions and closed by composition, linear integration and a very natural unique minimization schema.

## 1 Introduction

The power of digital discrete time models of computations is rather well understood: all reasonable and sufficiently powerful digital discrete time models have the same power thanks to Turing's work and so-called Church thesis.

For analog models the situation is far from being so clear. Several models have been defined (e.g. the General Purpose Analog Computer (GPAC) model of Shannon [28], neural network models [29,24], hybrid systems [3,4], or theoretical physic models [11,15,23],...) but there are only few results concerning relations between their respective computational power: GPAC computable functions have been characterized mathematically as differentially algebraic functions [12,18,25,28] but this does not provide directly a way to understand the relations between the power of such machines compared to classical discrete machines. Several other analog models have been shown to exhibit super-Turing computational power: using the so-called *Zeno's paradox*, some models make it possible to compute non-Turing computable functions in a constant time: see e.g. [3,5,11,15,19]; the continuity of the space makes it sometimes possible to have models whose power is close to non-uniform complexity classes [29].

Since the progress of electronics and other domains of physics such as mechanics or optics makes the construction of some of the machines realistic, clarifying the situation becomes a crucial matter.

In [19], Moore introduced a class of functions over the reals inspired from the classical characterization of computable functions over integers: observing that the continuous analog of a primitive recursion is a differential equation, Moore proposes to consider the class of $\mathbb{R}$-*recursive functions*, defined as the smallest class of functions containing some basic functions, and closed by composition, differential equation solving (called *integration*), and minimization. The minimization schema of [19] makes it possible to use a *"compression trick"* (another incarnation of Zeno's paradox) to simulate in a bounded time an unbounded number of discrete transitions in order to recognize arithmetical (hence non-Turing-computable) reals [19].

Actually, the original definitions of [19] suffer from several technical problems that appear as soon as the minimization schema is used (see e.g. discussions in [19,9,10,20,21]), and it has been proposed to replace minimization schema by a limit schema to have well-defined classes of functions as in [20,21], or to restrict to functions defined without minimization schema as in [10,12].

Concerning second approach, in his PhD dissertation [10], Campagnolo proposes to consider a class $\mathcal{L}$ of real-functions built in analogy with the class of elementarily computable functions in classical discrete computability: class $\mathcal{L}$ is defined as the smallest class of functions containing some well-chosen basic functions and closed by composition and *linear* integration.

Class $\mathcal{L}$ is proved by Campagnolo *et al.* to be related to functions *elementarily* computable *over the integers* in classical recursion theory: any function over the integers elementary in the sense of classical recursion theory is the restriction to integers of a function that belongs to $\mathcal{L}$ [10,9]; any function in $\mathcal{L}$ that preserves integers has its restriction to integers elementarily computable [10,9].

This paper proves that this is indeed possible to define a reasonable minimization schema to get a class, that we call $\mathcal{L}+!\mu$, that corresponds in a similar way to all (i.e. not necessarily elementary) *computable* functions *over the integers*: we prove that any total recursive function over the integers is the restriction to integers of a function that belongs to $\mathcal{L}+!\mu$, and that any function in $\mathcal{L}+!\mu$ that preserves integers has its restriction to integers total recursive.

Concerning, classical discrete computability, we get a new original characterization of computable functions in terms of restrictions to integers of a natural class of functions over the reals.

Concerning analog models, our results relate the computational power of some algebraically defined classes of functions over the reals to classical discrete models, and hence contribute to understand computations over the reals, or at least to understand the computational power of $\mathbb{R}$-(sub)-recursive functions.

Furthermore the problem we solve is in some sense the definition of a minimization operator, which is strong enough to get at least Turing machine power, but not too strong to get the technical problems of [19], nor non-robust super-Turing Zeno phenomena of [3,5,11,15,19]. In that sense, we believe that our results may be a step toward understanding criteria that could guarantee "robustness" for continuous models as sought by papers like [2,14].

Moreover, we think that that our results could be a first step toward getting an algebraic characterization of functions *over the real numbers* computable *in the sense of recursive analysis*, in the spirit of [6], and alternative to [7,8].

## 2 Preliminaries

### 2.1 Mathematical preliminaries

Let $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, denote the set of natural integers, the set of rational numbers, and the set of real numbers respectively. Given $x \in \mathbb{R}^n$, we write $\overrightarrow{x}$ to emphasize that $x$ is a vector.

**Lemma 1 (Bounding Lemma for Linear Differential Equations (see e.g. [1])).** *For linear differential equation $\overrightarrow{x}' = A(t)\overrightarrow{x}$, if $A$ is defined and continuous on interval $I = [a, b]$, where $a \leq 0 \leq b$, then, for all $\overrightarrow{x}_0$, the solution of $\overrightarrow{x}' = A(t)\overrightarrow{x}$ with initial condition $\overrightarrow{x}(0) = \overrightarrow{x}_0$ is defined and unique on $I$. Furthermore, the solution satisfies $\|\overrightarrow{x}(t)\| \leq \|\overrightarrow{x}_0\| \exp(\sup_{\tau\in[0,t]} \|A(\tau)\|t)$.*

**Lemma 2 (Implicit Functions Theorem (see e.g. [26])).** *Let $f : \mathbb{R}^{k+1} \to \mathbb{R}$ be a function of class $\mathcal{C}^k$, for $k \geq 1$. Assume that for all $\overrightarrow{x}$, the equation $f(\overrightarrow{x}, y) = 0$ has exactly one solution $y$. Assume for all $\overrightarrow{x}$ that $\frac{\partial f}{\partial y}(\overrightarrow{x}, y) \neq 0$ in the corresponding root $y$. Then function $g : \mathbb{R}^k \to \mathbb{R}$ that maps $\overrightarrow{x}$ to the corresponding root $y$ is also of class $\mathcal{C}^k$.*

### 2.2 Classical Recursion Theory

Classical recursion theory deals with functions over integers. Most classes of classical recursion theory can be characterized as closures of a set of basic functions by a finite number of basic rules to build new functions [27,22]: given a set $\mathcal{F}$ of functions and a set $\mathcal{O}$ of operators on functions (an operator is an operation that maps one or more functions to a new function), $[\mathcal{F}; \mathcal{O}]$ will denote the closure of $\mathcal{F}$ by $\mathcal{O}$.

**Proposition 1 (Classical settings: see e.g. [27,22]).** *Let $f$ be a function from $\mathbb{N}^k$ to $\mathbb{N}$ for $k \in \mathbb{N}$. Function $f$ is*

- *elementary iff it belongs to $\mathcal{E} = [0, S, U, +, \ominus; \mathrm{COMP}, \mathrm{BSUM}, \mathrm{BPROD}]$;*
- *primitive recursive iff it belongs to $\mathcal{PR} = [0, U, S; \mathrm{COMP}, \mathrm{REC}]$;*
- *total recursive iff it belongs to $\mathcal{R}ec = [0, U, S; \mathrm{COMP}, \mathrm{REC}, \mathrm{MU}]$.*

*A function $f : \mathbb{N}^k \to \mathbb{N}^l$ is elementary (resp: primitive recursive, total recursive) iff its projections are elementary (resp: primitive recursive, total recursive).*

*The basic functions $0, (U_i^m)_{i,m\in\mathbb{N}}, S, +, \ominus$ and the operators BSUM, BPROD, COMP, REC, MU are given by*

1. *$0 : \mathbb{N} \to \mathbb{N}$, $0 : n \mapsto 0$; $U_i^m : \mathbb{N}^m \to \mathbb{N}$, $U_i^m : (n_1, \dots, n_m) \mapsto n_i$; $S : \mathbb{N} \to \mathbb{N}, S : n \mapsto n + 1$; $+ : \mathbb{N}^2 \to \mathbb{N}$, $+ : (n_1, n_2) \mapsto n_1 + n_2$; $\ominus : \mathbb{N}^2 \to \mathbb{N}$, $\ominus : (n_1, n_2) \mapsto max(0, n_1 - n_2)$;*
2. *BSUM : bounded sum. Given $f$, $h = \mathrm{BSUM}(f)$ is defined by $h : (\overrightarrow{x}, y) \mapsto \sum_{z<y} f(\overrightarrow{x}, z)$; BPROD : bounded product. Given $f$, $h = \mathrm{BPROD}(f)$ is defined by $h : (\overrightarrow{x}, y) \mapsto \prod_{z<y} f(\overrightarrow{x}, z)$;*

3. COMP : *composition. Given* $f_1, \ldots, f_p$ *and* $g$, $h = \mathrm{COMP}(f_1, \ldots, f_p; g)$ *is defined as the function verifying* $h(\overrightarrow{x}) = g(f_1(\overrightarrow{x}), \ldots, f_p(\overrightarrow{x}))$;
4. REC : *primitive recursion . Given* $f$ *and* $g$, $h = \mathrm{REC}(f, g)$ *is defined as the function verifying* $h(\overrightarrow{x}, 0) = f(\overrightarrow{x})$ *and* $h(\overrightarrow{x}, n+1) = g(\overrightarrow{x}, n, h(\overrightarrow{x}, n))$.
5. MU : *minimization. Given a function* $f$ *such that for all* $\overrightarrow{x}$, *there is a* $y$ *with* $f(\overrightarrow{x}, y) = 0$, *the minimization of* $f$ *is* $\mu f : \overrightarrow{x} \mapsto \inf\{y; f(\overrightarrow{x}, y) = 0\}$.

Observe that we consider here only total functions. Furthermore, observe that minimization operator can actually be reinforced into a *unique* minimization operator as follows:

**Proposition 2.** *A function* $f$ *from* $\mathbb{N}^k$ *to* $\mathbb{N}^l$, *for* $k, l \in \mathbb{N}$, *is total recursive iff its projections belong to* $[0, U, S; \mathrm{COMP}, \mathrm{REC}, \mathrm{UMU}]$ *where operator* UMU *is defined as follows:*

1. UMU*: unique minimization. Given* $f$ *such that for all* $\overrightarrow{x}$ *there is a unique* $y$ *with* $f(\overrightarrow{x}, y) = 0$, *the unique minimization of* $f$ *is defined as the function, denoted by* $!\mu(f)(\overrightarrow{x}, y)$, *that maps* $\overrightarrow{x}$ *to that unique* $y$, *for all* $\overrightarrow{x}$.

*Proof.* The inclusion $[0, U, S; \mathrm{COMP}, \mathrm{REC}, \mathrm{UMU}] \subset \mathcal{R}ec$ is immediate. Conversely, let $\phi$ be a function from $\mathcal{R}ec$. It is well known [16,27] that $\phi$ can be written as $\phi = \chi \circ \mu(\psi)$ with $\chi$ and $\psi$ in $\mathcal{E}$ and such that for all $\overrightarrow{x}$, there is at least a $y$ with $\psi(\overrightarrow{x}, y) = 0$ (recall that $\phi$ is total). Let $\sigma$ be the elementary function defined by $\sigma(m, n) = \prod_{z<n} \psi(m, z)$. Given $m$, let us note $n_0 = \mu(\psi)(m)$. We have $\forall n \leq n_0$, $\sigma(m, n) \neq 0$ and $\forall n > n_0$, $\sigma(m, n) = 0$. Let $\kappa(m, n) = 1 \ominus (1 \ominus ((1 \ominus \sigma(m, n)) + \sigma(m, n+1)))$. We have clearly $\forall n < n_0$, $\kappa(m, n) = 1$, $\kappa(m, n_0) = 0$ and $\forall n > n_0$, $\kappa(m, n) = 1$, hence $\mu(\kappa) = !\mu(\kappa) = \mu(\psi)$. $\kappa$ is an elementary function and we have $\phi = \chi \circ !\mu(\kappa)$, hence $\phi$ belongs to $[0, U, S; \mathrm{COMP}, \mathrm{REC}, \mathrm{UMU}]$.

We have $\mathcal{E} \subseteq \mathcal{PR} \subseteq \mathcal{R}ec$, and the inclusions are known to be strict [27,22]. If TIME($t$) and SPACE($t$) denote the classes of functions that are computable with time and space $t$, then, $\mathcal{PR} = \mathrm{TIME}(\mathcal{PR}) = \mathrm{SPACE}(\mathcal{PR})$ [27,22]. Class $\mathcal{PR}$ corresponds to functions computable using *For-Next programs.* Class $\mathcal{E}$ corresponds to computable functions bounded by some iterate of the exponential function [27,22].

In classical computability, more general objects than functions over the integers can be considered, in particular functionals, i.e. functions $\Phi : (\mathbb{N}^m)^{\mathbb{N}} \times \mathbb{N}^k \rightarrow \mathbb{N}^l$. A functional will be said to be *elementary* (or *primitive recursive*, *recursive*) when it belongs to the corresponding[1] class.

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[1] Formally, a function $f$ over the integers can be considered as functional $\overline{f} : (V, \overrightarrow{n}) \mapsto f(\overrightarrow{n})$. Similarly, an operator $Op$ on functions $f_1, \ldots, f_m$ over the integers can be extended to argument $\overline{Op}(F_1, \ldots, F_m) : (V, \overrightarrow{n}) \mapsto Op(f_1(V, .), \ldots, f_m(V, .))(\overrightarrow{n})$.

In that spirit, given some set $\mathcal{F}$ of basic functions $\mathbb{N}^k \rightarrow \mathbb{N}^l$ and a set $\mathcal{O}$ of operators on functions over the integers, we will still (abusively) denote by $[f_1, \ldots, f_p; O_1, \ldots, O_q]$ for the smallest class of functionals that contains basic functions $\overline{f_1}, \ldots, \overline{f_p}$, plus the functional $Map : (V, n) \rightarrow V_n$, the nth element of sequence $V$, and which is closed by the operators $\overline{O_1}, \ldots, \overline{O_q}$. For example, a functional will be said elementary iff it belongs to $\mathcal{E} = [Map, \overline{0}, \overline{S}, \overline{U}, \overline{+}, \overline{\ominus}; \overline{\mathrm{COMP}}, \overline{\mathrm{BSUM}}, \overline{\mathrm{BPROD}}]$.

## 3 Computable Analysis

The idea sustaining *computable analysis*, also called *recursive analysis*, is to define computable functions over real numbers by considering functionals over fast-converging sequences of rationals [30,17,13,31].

Let $\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q}$ be the following representation[2] of rational numbers by integers: $\nu_{\mathbb{Q}}(\langle p, r, q\rangle) \mapsto \frac{p-r}{q+1}$, where $\langle ., ., .\rangle : \mathbb{N}^3 \to \mathbb{N}$ is a computable bijection.

A sequence of integers $(x_i) \in \mathbb{N}^{\mathbb{N}}$ *represents a real number* $x$ if $(\nu_{\mathbb{Q}}(x_i))$ converges quickly toward $x$ (denoted by $(x_i) \rightsquigarrow x$) in the following sense : $\forall i, |\nu_{\mathbb{Q}}(x_i) - x| < 2^{-i}$. For $(x_i) \in (\mathbb{N}^k)^{\mathbb{N}}$, we write $(x_i) \rightsquigarrow x$ when it holds componentwise.

**Definition 1 (Recursive analysis [31]).** *A function $f : \mathbb{R}^k \to \mathbb{R}$ is said computable (or real-computable) if there exists a recursive functional $\Phi : (\mathbb{N}^k)^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that for all $\overrightarrow{x} \in \mathbb{R}^k$, for all sequence $X = (\overrightarrow{x}_n) \in (\mathbb{N}^k)^{\mathbb{N}}$, we have $(\phi(X, j))_j \rightsquigarrow f(\overrightarrow{x})$ whenever $X \rightsquigarrow \overrightarrow{x}$. A function $f : \mathbb{R}^k \to \mathbb{R}^l$, with $l > 1$, is said computable if all its projections are.*

A function $f$ will be said *elementarily computable* whenever the corresponding functional $\Phi$ is. The class of computable (respectively elementarily computable) functions over the reals will be denoted by $\mathcal{R}ec(\mathbb{R})$ (resp. $\mathcal{E}(\mathbb{R})$).

## 4 Real-sub-recursive and sub-recursive functions

Following the original ideas from [19], but avoiding the minimization schema of [19] source of many problems, Campagnolo proposed in [10] to consider the following class, built in analogy with elementarily computable functions over the integers.

**Definition 2 ([10,9]).** *Let $\mathcal{L}$ be the class of functions $f : \mathbb{R}^k \to \mathbb{R}^l$, for some $k, l \in \mathbb{N}$, defined by $\mathcal{L} = [0, 1, -1, \pi, U, \theta_3; \mathrm{COMP}, \mathrm{LI}]$ where the basic functions $0$, $1$, $-1$, $\pi$, $(U_i^m)_{i,m\in\mathbb{N}}$, $\theta_3$ and the schemata* COMP *and* LI *are the following:*

1. *$0, 1, -1, \pi$ are the corresponding constant functions; $U_i^m : \mathbb{R}^m \to \mathbb{R}$ are, as in the classical settings, projections: $U_i^m : (x_1, \ldots, x_m) \mapsto x_i$;*
2. *$\theta_3 : \mathbb{R} \to \mathbb{R}$ is defined as $\theta_3 : x \mapsto x^3$ if $x \geq 0$, $0$ otherwise;*
3. COMP*: composition is defined as in the classical settings: Given $f_1, \ldots, f_p$ and $g$, $h = \mathrm{COMP}(f_1, \ldots, f_p; g)$ is defined by $h(\overrightarrow{x}) = g(f_1(\overrightarrow{x}), \ldots, f_p(\overrightarrow{x}))$;*
4. LI*: linear integration. From $g$ and $h$, $\mathrm{LI}(g, h)$ is the maximal solution of the linear differential equation $\frac{\partial f}{\partial y}(\overrightarrow{x}, y) = h(\overrightarrow{x}, y)f(\overrightarrow{x}, y)$ with $f(\overrightarrow{x}, 0) = g(\overrightarrow{x})$.*
   *In this schema, if $g$ goes to $\mathbb{R}^n$, $f = \mathrm{LI}(g, h)$ also goes to $\mathbb{R}^n$ and $h(\overrightarrow{x}, y)$ is a $n \times n$ matrix with elements in $\mathcal{L}$.*

[2] Many other natural representations of rational numbers can be chosen and provide the same class of computable functions: see [31].

Class $\mathcal{L}$ includes common functions like +,sin,cos,−,×,exp, or $x \to r$ for all $r \in \mathbb{Q}$ (see [10,9]), but contains only total functions [9]:

**Proposition 3 ([9]).** *All functions from $\mathcal{L}$ are continuous, defined everywhere, and of class $\mathcal{C}^2$.*

Actually, observing the proofs from [10,9], schema LI can be strengthened as follows:

**Proposition 4.** *Class $\mathcal{L}$ is also the class of functions $f : \mathbb{R}^k \to \mathbb{R}^l$, for some $k, l \in \mathbb{N}$, defined by $\mathcal{L} = [0, 1, -1, \pi, U, \theta_3; \mathrm{COMP}, \mathrm{CLI}]$ where CLI is the following schema:*

1. CLI*: controlled linear integration. From $g$ and $h$, and $c$, with $h$ differentiable and entries of $h'$ bounded by $c$,* $\mathrm{CLI}(g, h, c)$ *is the maximal solution of the linear differential equation $\frac{\partial f}{\partial y}(\overrightarrow{x}, y) = h(\overrightarrow{x}, y) f(\overrightarrow{x}, y)$ with $f(\overrightarrow{x}, 0) = g(\overrightarrow{x})$. In this schema, if $g$ goes to $\mathbb{R}^n$, $f = \mathrm{CLI}(g, h, c)$ also goes to $\mathbb{R}^n$ and $h(\overrightarrow{x}, y)$ is a $n \times n$ matrix with elements in $\mathcal{L}$.*

Class $\mathcal{L}$ can be related to the class $\mathcal{E}$ of elementarily computable functions over the integers. A *real extension* $\tilde{f}$ of a function $f : \mathbb{N}^k \to \mathbb{N}^l$ over the integers is a function $\tilde{f}$ from $\mathbb{R}^k$ to $\mathbb{R}^l$ whose restriction to $\mathbb{N}^k$ is $f$. Observe that a function $\tilde{f} : \mathbb{R}^k \to \mathbb{R}^l$ over the reals is an extension of a function over the integers iff its preserves integers: $\tilde{f}(\mathbb{N}^k) \subset \mathbb{N}^l$.

**Definition 3 (Discrete Part).** *Given a class $\mathcal{C}$ of real functions, we denote by $DP(\mathcal{C})$ the class of functions over the integers that have a real extension in $\mathcal{C}$.*

**Proposition 5 ([10,9]).** *$\mathcal{E} = DP(\mathcal{L})$. I.e.:*

- *If a function from $\mathcal{L}$ extends some functions over the integers, this latter function is elementarily computable.*
- *Any elementarily computable function over the integers, has a real extension that belongs to $\mathcal{L}$.*

Actually, class $\mathcal{L}$ can also be partially related to the class $\mathcal{E}(\mathbb{R})$ of functions over the real numbers elementarily computable in the sense of recursive analysis: any function from $\mathcal{L}$ is in $\mathcal{E}(\mathbb{R})$ [10,9]. We proved in [6] that the inclusion is actually strict, but that adding a limit schema to class $\mathcal{L}$, allows us to capture whole class $\mathcal{E}(\mathbb{R})$ for functions defined over a compact domain.

## 5 Real-recursive and recursive functions

We are now going to extend the class $\mathcal{L}$ with a minimization schema in order to get a class whose discrete part correspond to total recursive functions over the integers.

To do so, we need to introduce a zero-finding operator that permits to simulate the classical discrete minimization schema over the integers. However, this operator needs to be stricter than a simple "return the smallest root" since this idea, investigated in [19], has shown to be the source of numerous problems, including ill-defined problems and super-Turing Zeno phenomena [10,9,21,20,19].

Our idea is to use the alternative UMU schema which is equivalent to schema MU for classical computability, but has real counterparts which turn out to preserve real computability.

Indeed, motivated by Proposition 2, by Lemma 2, and by results from recursive analysis about the computability of zeros (see e.g. [31]), we define our unique-zero-finding operator UMU as follows (observe that we also take schema CLI instead of schema LI, which is equivalent when schema UMU is not present):

**Definition 4.** *Given a differentiable function $f$ from $\mathbb{R}^{k+1}$ to $\mathbb{R}$ , if for all $\overrightarrow{x}$, $y \mapsto f(\overrightarrow{x}, y)$ is a non-decreasing function with a unique root $y_0$, on which $\frac{\partial f}{\partial y}(\overrightarrow{x}, y_0) > 0$, then* UMU$(f)$ *is defined as follows:*

$$\mathrm{UMU}(f) : \begin{cases} \mathbb{R}^k \longrightarrow \mathbb{R} \\ \overrightarrow{x} \mapsto y_0 \text{ such that } f(\overrightarrow{x}, y_0) = 0 \end{cases}$$

*Let $\mathcal{L}{+}!\mu$ be the set of functions defined by*

$$\mathcal{L}{+}!\mu = [0, 1, U, \theta_3; \mathrm{COMP}, \mathrm{CLI}, \mathrm{UMU}].$$

**Lemma 3.** *$\mathcal{L} \subset \mathcal{L}{+}!\mu$.*

*Proof.* (sketch) We only need to prove that constant functions $-1$ and $\pi$ are in $\mathcal{L}{+}!\mu$. Indeed, $-1$ is the unique root of $x \mapsto x + 1$, and $\pi = 4\arctan(1)$, where $\arctan(x)$ is the solution of linear differential equation $\arctan(0) = 0$ and $\arctan'(x) = \frac{1}{1+x^2}$, and $x \mapsto \frac{1}{1+x^2}$ can be obtained by applying UMU on $x, y \mapsto (1 + x^2)y - 1$.

**Lemma 4.** *All functions from $\mathcal{L}{+}!\mu$ are of class $\mathcal{C}^2$ and total.*

*Proof.* By structural induction. Basic functions 0, 1, $U$, $\theta_3$ are total and of class $\mathcal{C}^2$. Now, class $\mathcal{C}^2$ and totality are preserved by composition, by linear integration (see e.g. [1]), and by schema UMU by Lemma 2.

Now, observe that operator UMU preserves real computability:

**Lemma 5.** *Given $f : \mathbb{R}^{k+1} \longrightarrow \mathbb{R}$ real computable, if* UMU$(f)$ *is defined, then* UMU$(f)$ *is also real computable.*

*Proof.* Given $\overrightarrow{x} \in \mathbb{R}^k$, let $y_0$ be the unique $y_0$ with $f(\overrightarrow{x}, y_0) = 0$. Since $f(\overrightarrow{x}, .)$ is continuous, non-decreasing, and with a unique root, we have $f(\overrightarrow{x}, y) < 0$ for $y < y_0$, and $f(\overrightarrow{x}, y) > 0$ for $y > y_0$.

There exists $m \in \mathbb{N}$, such that $f(\overrightarrow{x}, -m) < 0$ and $f(\overrightarrow{x}, m) > 0$: one just need to take any integer $m$ with $-m < y_0 < m$. Actually, such an $m$ can be computed as follows:

$m = 1$
**Repeat**
  Compute $f_1 = f(\overrightarrow{x}, m)$ **and** $f_2 = f(\overrightarrow{x}, -m)$ at precision $\pm 2^{-m}$
  $m = m + 1$
**Until** $( f_1 > 2^{-m}$ **and** $f_2 < -2^{-m} )$
Return $m$

Indeed, given any integer $m_0 \in \mathbb{N}$ with $-m_0 < y_0 < m_0$, (take for example $\lfloor |y_0| \rfloor + 1$), we have for all $m \geq m_0$, $f(\overrightarrow{x}, m) \geq f(\overrightarrow{x}, m_0) > 0$ and $f(\overrightarrow{x}, -m) \leq f(\overrightarrow{x}, -m_0) < 0$. Now, for $m$ big enough (i.e. $m \geq m_0$, $2^{-m} \leq |f(\overrightarrow{x}, -m_0)|$, and $2^{-m} \leq |f(\overrightarrow{x}, m_0)|$) we have $f_1 > 2^{-m}$ and $f_2 < -2^{-m}$ and the algorithm stops with an $m$ such that $f(\overrightarrow{x}, -m) < 0$ and $f(\overrightarrow{x}, m) > 0$.

Computing $y_0$ then reduces to compute the unique root of function $f(\overrightarrow{x}, .)$ over a compact $[-m, m]$. The fact that this is indeed computable can be seen as a consequence of the results in [31].

Here is a direct proof: given $n$, we have to find an approximation of $y_0$ at precision $2^{-n}$. Let us slice $[-m, m]$ in $2^i$ closed intervals: $[-m, m] = \cup_{0 \leq j < 2^i} [y_j, y_{j+1}]$ where $y_j = -m + j\frac{2m}{2^i}$. Let $z_j$ be an approximation of $f(\overrightarrow{x}, y_j)$ computed at precision $2^{-i}$. We know that for a root to exist in $[y_j, y_{j+1}]$, the only possibilities are that $|z_j| < 2^{-i}$ or $|z_{j+1}| < 2^{-i}$ or $z_j z_{j+1} < 0$[3]. Then, let $m_i$ be the $y_j$ (resp. $M_i$ be the $y_{j+1}$) where index $j$ is the smallest (resp. greatest) integer $0 \leq j < 2^i$ with $|z_j| < 2^{-i}$ or $|z_{j+1}| < 2^{-i}$ or $z_j z_{j+1} < 0$.

The sequences $(m_i)$ and $(M_i)$ have range in compact sets, so there exist subsequences $(m_{\phi(i)})$ and $(M_{\phi(i)})$ that converge, thanks to Bolzano-Weierstrass theorem. Let $m^*$ and $M^*$ be the limits of those sequences. For all $i$, either $|f(\overrightarrow{x}, m_i)| \leq |f(\overrightarrow{x}, m_i) - z_j| + |z_j| < 2^{-i} + 2^{-i}$, or $|f(\overrightarrow{x}, m_i + 2^{-i})| \leq |f(\overrightarrow{x}, m_i) - z_{j+1}| + |z_{j+1}| < 2^{-i} + 2^{-i}$, or $f(\overrightarrow{x}, m_i) f(\overrightarrow{x}, m_i + 2^{-i}) < 0$. Since $f$ is continuous, we can deduce that $f(\overrightarrow{x}, m^*) = 0$. For the same reason, $f(\overrightarrow{x}, M^*) = 0$ and since $y \mapsto f(\overrightarrow{x}, y)$ has only one root, $m^* = M^*$. So, there exists $i$ such that $M_i - m_i < 2^{-n}$. When this holds, $m_i$ is an approximation at precision $2^{-n}$ of the root. This means that the following algorithm terminates and returns an approximation of $y_0$ at precision $2^{-n}$.

$i = 0$
**Repeat**
  Compute $m_i$ **and** $M_i$
  $i = i + 1$
**Until** $M_i - m_i < 2^{-n}$
Return $m_i$

**Lemma 6.** *Given $h$, $g$ and $c$ real computable, then $f = \mathrm{CLI}(g, h, c)$ is also real computable.*

*Proof.* Observing carefully [10,9], if given $\overrightarrow{x} \in \mathbb{R}^k$ and some $\overline{y} \in \mathbb{Q}$ one can bound effectively the norms of $h(\overrightarrow{x}, y)$, $f(\overrightarrow{x}, y)$, $\frac{\partial^2 f}{\partial y^2}(\overrightarrow{x}, y)$ for $|y| \leq \overline{y}$, then

[3] In fact, since the function we are investigating is non-decreasing, we could have more accurate constraints, however these ones are sufficient.

$f$ will be real computable: use the constructions and bounds based on Euler's method to prove preservation of elementarily computability by linear integration in [10,9], but replacing elementary bounds by computable bounds.

Now, from [31], it is known that one can bound effectively the norm of any real computable function on a compact domain, and so we only need to care about $f(\overrightarrow{x}, y)$ and $\frac{\partial^2 f}{\partial y^2}(\overrightarrow{x}, y)$. But the norm of $f(\overrightarrow{x}, y)$ can be bounded effectively by Lemma 1 from bounds on the norms of $g(\overrightarrow{x})$ and $h(\overrightarrow{x}, y)$ on the corresponding domain, which are computable by previous argument. Now, $\|\frac{\partial^2 f}{\partial y^2}(\overrightarrow{x}, y)\| = \|(h^2(\overrightarrow{x}, y)+\frac{\partial h}{\partial y}(\overrightarrow{x}, y))f(\overrightarrow{x}, y)\|$, hence is bounded by $(\|h^2(\overrightarrow{x}, y)\|+\|c(\overrightarrow{x}, y))\|) \times \|f(\overrightarrow{x}, y)\|$. First factor can still be bounded effectively since $h^2(\overrightarrow{x}, y)$ and $c(\overrightarrow{x}, y)$ are particular real computable functions, and we just see that second factor can be bounded effectively.

From previous two Lemmas, the fact that basic functions are real computable and observing that composition is known to preserve real computability for total functions (see [31]), we obtain:

**Theorem 1.** *Every function belonging to $\mathcal{L}+!\mu$ is real computable.*

We now prove the converse direction. Following lemma is a weaker form of a Lemma that we proved in [6]:

**Lemma 7.** *Given $f : \mathbb{R}^2 \to \mathbb{R}$ in $\mathcal{L}$, there exists $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ in $\mathcal{L}$ such that $\forall (m, n) \in \mathbb{N}^2$, $\forall (x, y) \in \mathbb{R}^2$,*

- $\tilde{f}(m, n) = f(m, n)$
- $\tilde{f}(m, y) \in [f(m, \lfloor y \rfloor), f(m, \lfloor y + 1 \rfloor)]$ *(or* $[f(m, \lfloor y + 1 \rfloor), f(m, \lfloor y \rfloor)]$*).*
- $\tilde{f}(x, n) \in [f(\lfloor x \rfloor, n), f(\lfloor x + 1 \rfloor, n)]$ *(or* $[f(\lfloor x + 1 \rfloor, n), f(\lfloor x \rfloor, n)]$*).*

*Proof.* Let $\zeta = \frac{3\pi}{2}$. Let $\omega : x \mapsto \zeta\theta_3(\sin(2\pi x))$. $\forall i$, $\int_i^{i+1} \omega = 1$ and $\omega$ is equal to 0 on $[i+\frac{1}{2}, i+1]$ for $i \in \mathbb{N}$. Let $\Omega$ its primitive equal to 0 in 0, and $int : x \mapsto \Omega(x-\frac{1}{2})$. Function $int$ is a function similar to the integer part: $\forall i \in \mathbb{N}$, $\forall x \in [i, i + \frac{1}{2}]$, $int(x) = i = \lfloor x \rfloor$. Figure 1 shows graphical representations of $\omega$ and $int$.

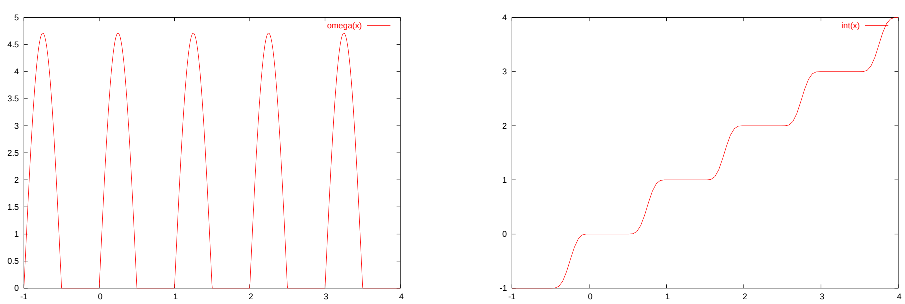


**Figure 1.** Graphical representation of $\omega$ and $int$

Let $\Delta(i,y) = f(i,y+1) - f(i,y)$. Then for all $i \in \mathbb{N}$, $y \in \mathbb{R}$, we have $\omega(y)\Delta(i,int(y)) = \begin{cases} 0 & \textit{whenever } y - \lfloor y \rceil \geq 1/2 \\ \omega(y)\Delta(i,\lfloor y \rceil) & \textit{otherwise.} \end{cases}$

Let $G$ be the solution of the linear differential equation $G(x,0) = f(x,0)$, $\frac{\partial G}{\partial y}(x,y) = \omega(y)\Delta(x,int(y))$. An easy induction on $j$ then shows that $G(i,j) = f(i,j)$ for all integer $j$. Furthermore, by construction, $\forall i \in \mathbb{N}$, $G(i,y)$ belongs to the interval delimited by $G(i,\lfloor y \rceil) = f(i,\lfloor y \rceil)$ and $G(i,\lfloor y+1 \rceil) = f(i,\lfloor y+1 \rceil)$.

Now, let $\tilde{f}$ be the solution of the linear differential equation $\tilde{f}(0,j) = G(0,j)$, $\frac{\partial \tilde{f}}{\partial x}(x,y) = \omega(x)(G(int(x+1),y) - G(int(x),y))$. We have $\forall (i,j) \in \mathbb{N}^2$, $\tilde{f}(i,j) = f(i,j)$. And $\forall i \in \mathbb{N}$, $\tilde{f}(i,y)$ belongs to the interval delimited by $\tilde{f}(i,\lfloor y \rceil) = f(i,\lfloor y \rceil)$ and $\tilde{f}(i,\lfloor y+1 \rceil) = f(i,\lfloor y+1 \rceil)$. And also, $\forall j \in \mathbb{N}$, $\tilde{f}(x,j)$ belongs to the interval delimited by $\tilde{f}(\lfloor x \rceil,j) = f(\lfloor x \rceil,j)$ and $\tilde{f}(\lfloor x+1 \rceil,j) = f(\lfloor x+1 \rceil,j)$.

**Theorem 2.** *Every recursive function over the integers has a real extension in* $\mathcal{L}+!\mu$.

*Proof.* Let $\phi$ be a function from $\mathcal{R}ec$. We have $\phi = \chi \circ !\mu(\kappa)$ as in the proof of Proposition 2. Let $\iota(m,n) = 2 \times (1 \ominus \sigma(m,n)) + (1 \ominus \kappa(m,n))$ where $\sigma$ is the same as in the proof of Proposition 2. $\forall m \in \mathbb{N}$, for $n = n_0 = !\mu(\kappa)(m,n)$, we have $\iota(m,n_0) = 1$, and before this $n_0$, $\iota(m,n)$ is equal to 0 and after this $n_0$, $\iota(m,n)$ is equal to 2. Let $i$ be a real extension of $\iota$ in $\mathcal{L}$ given by Proposition 5. Let $\tilde{i}$ be the function from $\mathcal{L}$ obtained by Lemma 7 on $f(m,x) : m,x \mapsto i(m,x) - 1$.

$\forall m \in \mathbb{N}$, there exists exactly one $y \in \mathbb{R}$ (given by $y_0 = !\mu(\kappa)(m,n)$) such that $\tilde{i}(m,y) = 0$. But, we can not directly apply schema UMU, since we have no assurance[4] that it also holds for non integer values $m$. However, from the constructions in the proof of Lemma 7, given $m \in \mathbb{N}$, we have $\tilde{i}(m,y)$ equal to $-1$ for $y \leq y_0 - 1$, and equal to $\Omega(y)$ for $y \in [y_0 - 1, y_0 + 1]$, where $\Omega$ is defined in that proof.

Consider $\mathcal{M}(x) = \theta_3(x+1)$. We have $\mathcal{M}(x) = 0$ if $x \leq -1$ and $\mathcal{M}(x) \geq 1$ if $x \geq 0$. Let us define $\tilde{g}$ as the solution of the differential equation $\tilde{g}(\overrightarrow{x},0) = -1$, $\frac{\partial \tilde{g}}{\partial y}(\overrightarrow{x},y) = \alpha \mathcal{M}(\tilde{i}(\overrightarrow{x},y))$. Let us choose $\alpha$ (maple says $\alpha = \frac{1024}{2609}$) such that $\alpha \int_{-1}^{0} \mathcal{M}(\Omega(x))\mathrm{d}x = 1$. We have $\forall m \in \mathbb{N}$, $\tilde{g}(m,y) = 0 \Leftrightarrow y = !\mu(\kappa)(m,n)$.

Then define $g$ as the solution of the linear differential equation $g(\overrightarrow{x},0) = -1$, $\frac{\partial g}{\partial y}(\overrightarrow{x},y) = \beta \mathcal{M}(\tilde{g}(\overrightarrow{x},y))$. If we choose $\beta$ adequately[5] (maple says $\beta = \frac{a\pi^4}{b\pi^4 - c\pi^2 + d}$ for some integers $a,b,c,d$) , we will still have $\forall m \in \mathbb{N}$, $g(m,y) = 0 \Leftrightarrow y = !\mu(\kappa)(m,n)$.

The point is that, since $\mathcal{M}$ is always non-negative, we know that $\forall x \in \mathbb{R}$, $y \mapsto \tilde{g}(x,y)$ is non-decreasing, and, because of Lemma 7, and from the definition of function $\mathcal{M}(x)$, it must go to infinity when $y$ goes to infinity. Actually, it must be equal to $-1$ up to a certain value $y_-$, then be strictly increasing, and since it goes to infinity, it must have a root $y_0$ strictly greater than $y_-$. Now the derivative in this root $y_0$ cannot be 0 since $\mathcal{M}(x)$ is zero only when $x \leq -1$.

[4] Actually, another problem is that the derivative relative to the second variable in the root point is 0.

[5] This $\beta$ is in $\mathcal{L}$ since it can be obtained as $a * \pi^4 * \mathrm{UMU}(x \mapsto (b\pi^4 - c\pi^2 + d)x - 1)$.

This $g$ is such that $\forall \overrightarrow{x}$, $\exists ! y_0$ such that $g(\overrightarrow{x}, y_0) = 0$ and $\frac{\partial g}{\partial y}(\overrightarrow{x}, y_0) \neq 0$ and for all $\overrightarrow{x}$, $y \mapsto g(\overrightarrow{x}, y)$ is non-decreasing. We can thus apply UMU to this $g$. Now if we extend $\chi$ in a real function $h$ belonging to $\mathcal{L}$ using Proposition 5, we have $h \circ \text{UMU}(g)$ extending $\phi = \chi \circ \mu(\psi)$ and belonging to $\mathcal{L}+!\mu$.

From previous two theorems, we obtain the main result of this paper:

**Theorem 3.** *$\mathcal{R}ec = DP(\mathcal{L}+!\mu)$. I.e:*

- *If a function from $\mathcal{L}+!\mu$ extends some function over the integers, this latter function is total recursive.*
- *Any total recursive function over the integers, has a real extension that belongs to $\mathcal{L}+!\mu$.*

*Proof.* The second item is Theorem 2. The first item is immediate from Theorem 1: if a function $f$ belonging to $\mathcal{L}+!\mu$ preserves integers, then a recursive function that equals $f$ on $\mathbb{N}^k$ can easily be obtained from the functional computing $f$.

**Corollary 1.** *$\mathcal{L}$ is strictly included in $\mathcal{L}+!\mu$.*

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