Some Bounds on the Computational Power of Piecewise Constant Derivative Systems (Extended Abstract)

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Abstract. We study the computational power of Piecewise Constant Derivative (PCD) systems. PCD systems are dynamical systems defined by a piecewise constant differential equation and can be considered as computational machines working on a continuous space with a continuous time. We show that the computation time of these machines can be measured either as a discrete value, called discrete time, or as a continuous value, called continuous time. We prove that the languages recognized by PCD systems in dimension d in finite continuous time are precisely the languages of the $d-2^{th}$ level of the arithmetical hierarchy. Hence we provide a precise characterization of the computational power of purely rational PCD systems in continuous time according to their dimension and we solve a problem left open by [2].

1 Introduction

There has been recently an increasing interest in the community of control and verification theory about hybrid systems. A hybrid system is a system that combines discrete and continuous dynamics. Hybrid systems can be also be considered as computational machines: they can be seen either as machines working on a continuous space with a discrete time or as machines working on a continuous space with a continuous time.

The first point of view has been investigated in [1, 2, 4, 5]. In particular, in [1–3] the attention is focused on a very simple type of hybrid systems: Piecewise Constant Derivative Systems (PCD systems) are dynamical systems defined by a piecewise constant differential equation. It is shown that the reachability problem for PCD systems is decidable in dimension $d = 2$ and undecidable in dimension $d \geq 3$ [1, 3] . In [4], the computational power of Piecewise Constant Derivative systems is characterized as $P/poly$ in polynomial discrete time, and as unbounded in exponential discrete time.

This paper deals with the second point of view that considers hybrid systems as machines that work on a continuous space with a continuous time. The study of computational machines that work in a continuous time is only beginning: in

[6], Moore proposed a recursion theory for computations on the reals in continuous time. Recently, Asarin and Maler [2] showed, using Zeno's paradox, that every set of the arithmetical hierarchy can be recognized in finite continuous time and in finite dimension by a PCD system: every set of the arithmetical hierarchy in $\Sigma_k \cup \Pi_k$ can be recognized by a rational PCD system in dimension $5k+1$. Unfortunately, no precise characterization of the PCD recognizable sets was given in [2]. In this paper, we improve the results of Asarin and Maler and we provide a full characterization of the sets recognized by purely rational PCD systems: we show that the sets that are recognized by purely rational PCD systems in dimension d are precisely the sets of the $d-2^{th}$ level of the arithmetical hierarchy.

Section 2 is devoted to some general definitions: PCD systems, computations on PCD systems, discrete and continuous time. In section 3, we improve 5 times the result of Asarin and Maler: any arithmetical set in Σ_k can be recognized in dimension $2+k$. In section 4 we prove that this bound is optimal for purely rational PCD systems: no other set can be recognized in that dimension.

2 Definitions

A convex polyhedron of $\mathbb{R}^d$ is any finite intersection of open or closed half spaces of $\mathbb{R}$. A polyhedron of $\mathbb{R}$ is a finite union of convex polyhedral of $\mathbb{R}$. In particular, a polyhedron may be unbounded or flat. For $V \subset \mathbb{R}$, we denote by $\overline{V}$ the topological closure of V. We denote by d the Euclidean distance of $\mathbb{R}$. A rational point of $\mathbb{R}$ is a point of $\mathbb{R}$ with rational coordinates.

Definition 1 (PCD System [1, 2]). *A* Piecewise Constant Derivative (PCD) system of dimension d *is a couple* $\mathcal{H} = (\mathcal{X}, \{)$ *with* $X = \mathbb{R}$, $f : X \to X$, *where the range of* f *is a finite set* $C \subset X$*, such that for any* $c \in C$ *(c is called a* slope*)* $f^{-1}(c)$ *is a finite union of convex polyhedral sets (called* regions*). A* trajectory of $\mathcal{H}$ starting from x_0 *is a continuous solution to the differential equation* $\dot{x}_d = f(x)$*, with initial condition* x_0*, where* $\dot{x}_d$ *denotes the right derivative: that is* $\Phi : D \subset \mathbb{R}^+ \to \mathbb{X}$ *where* D *is an interval of* $\mathbb{R}^+$ *containing* 0*,* $\Phi(0) = x_0$*, and* $\forall t \in D, \dot{\Phi}_d(t) = f(\Phi(t))$*. Trajectory* Φ *is said to* continue for ever *if* $D = \mathbb{R}^+$.

In other words a PCD system consists of partionning the space into convex polyhedral regions, and assigning a constant derivative c, called *slope*, to all the points sharing the same region. The trajectories of such systems are broken lines with the breakpoints occuring on the boundaries of the regions [2]. See figure 1. The *signature* of a trajectory is the sequence of the regions that are crossed by the trajectory.

Definition 2 (Rational, purely rational PCD systems).

- *A PCD system is called* rational *if all the slopes as well as all the polyhedral regions can be described using only rational coefficients.*

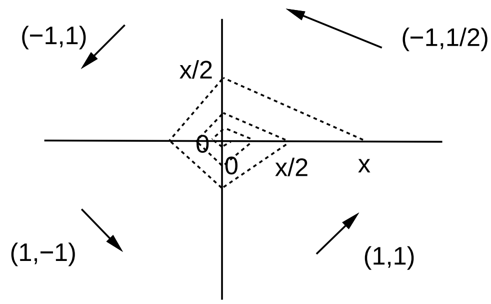

Fig. 1. A PCD system in dimension 2.

– *A PCD system is called* purely rational*, if in addition, for all trajectory Φ starting from a rational point, each time Φ enters a region in a point x, necessarily x has rational coordinates.*

Some comments are in order: one must understand that a trajectory Φ can enter a region either by a discrete transition or by converging to a point of the region: see figure 2. Thus, in other words, in a purely rational PCD system any converging process converges towards a point with rational coordinates. Note that one can construct a rational PCD system of dimension 5 that is not purely rational.

We can say some words on the existence of trajectories in a PCD system: let $x_0 \in X$. We say that x_0 is *trajectory well-defined* if there exists a $\epsilon > 0$ such that $f(x) = f(x_0)$ for all $x \in [x_0, x_0 + \epsilon * f(x_0)]$. It is clear that, for any $x_0 \in X$, there exists a trajectory starting from x_0 iff x_0 is trajectory well-defined. Given a rational PCD system $\mathcal{H}$, one can effectively compute the set $NoEvolution(\mathcal{H})$ of the points of X that are not trajectory well-defined. See that a trajectory can continue for ever iff it does not reach $NoEvolution(\mathcal{H})$.

Definition 3 (Computation [2]).

– *Let $\mathcal{H} = (\mathcal{X}, \{)$ be a PCD system of dimension d. Let $I = [0, 1]$ and let $r : \mathbb{N} \rightarrow \mathbb{I}$ be an injective coding function, let x^1, x^0 be two distinct points of $\mathbb{R}$. A computation of system $\hat{H} = (\mathbb{R}, \mho, \setminus, \mathbb{I}, \curvearrowleft^{\nVdash}, \curvearrowleft^{\nvdash})$ on entry $n \in \mathbb{N}$ is a trajectory that can continue forever (defined on all $\mathbb{R}^+$) of $\mathcal{H} = (\mathcal{X}, \{)$ starting from $(r(n), 0, \ldots, 0)$. The computation is accepting if the trajectory eventually reaches x^1, and refusing if it reaches x^0. It is assumed that the derivatives at x^1 and x^0 are zero.*

– *Language $L \subset \mathbb{N}$ is semi-recognized by $\hat{H}$ if, for every $n \in \mathbb{N}$, there is a computation on entry n and the computation is accepting iff $n \in L$. L is said to be (fully-)recognized by $\hat{H}$ when, in addition, this trajectory reaches x^0 iff $n \notin L$.*

Definition 4 (Continuous and Discrete time). *Let $\Phi_n : \mathbb{R}^+ \rightarrow \mathbb{X}$ be an accepting computation on entry $n \in \mathbb{N}$.*

- *The continuous time* $T_c(n)$ *of the computation is* $T = \min\{t \in \mathbb{R}^+ / \lneqq_{\ltimes}(\approx) = \curvearrowleft^{\nVdash}\}$.
- *Let* $T_n = \{t/\Phi_n(t)$ *crosses a boundary of a region at time* $t\}$. *It is easy to see that* T_n *is a well ordered set. The discrete time* $T_d(n)$ *of the computation is defined as the order type of well ordered set* T_n *(= the ordinal corresponding to* T_n*).*

Note that Zeno's paradox appears: to a continuous finite time can correspond a transfinite discrete time: see figure 2.

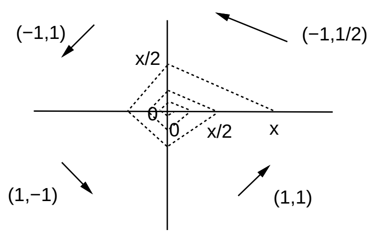

Fig. 2. Zeno's paradox: at finite continuous time $5x = 2.5(x + x/2 + x/4 + \ldots)$ the trajectory is in $(0,0)$, but it takes a transfinite discrete time ω to reach this point.

We recall the following definition:

Definition 5 (Arithmetical hierarchy [8, 7]). *The classes* $\Sigma_k, \Pi_k, \Delta_k$, *for* $k \in \mathbb{N}$, *are defined inductively by:*

- Σ_0 *is the class of the languages that are recursive.*
- *For* $k \geq 1$, Σ_k *is the class of the languages that are recursively enumerable in a set in* Σ_{k-1} *(that is semi-recognized by a Turing machine with an oracle in* Σ_{k-1}*)*
- *For* $k \in \mathbb{N}$, Π_k *is defined as the class of languages whose complement are in* Σ_k, *and* Δ_k *is defined as* $\Delta_k = \Pi_k \cap \Sigma_k$.

Several characterizations of the sets of the arithmetical hierarchy are known: see [7, 8]. In particular we will assume the reader familiar with Tarski-Kuratowski computations: assume a first order formula F, over some recursive predicates, characterizing the elements of a set $S \subset \mathbb{N}$, is given. Then S is in the arithmetical hierarchy and the Tarski-Kuratowski algorithm on formula F returns a level of the arithmetical hierarchy containing S: see [7, 8] for the full details.

3 PCD Systems can Recognize Arithmetical Sets

It was shown in [2] that every set of the arithmetical hierarchy can be recognized in finite continuous time: more precisely, it is shown that $L \in \Sigma_k \cup \Pi_k$ can be recognized by a PCD system of dimension $5k + 1$. Therefore, five dimensions are used in [2] to climb each level of the arithmetical hierarchy: one for a timer, one used for the divisions by 2, one used to do the homogenization, and two dimensions used to go from quantifier elimination to semi-recognition. We show here that only one dimension is needed (the one used to do the homogenization), and that the construction only requires purely rational PCD systems.

Theorem 1. – *Any language L of Σ_k is semi-recognized by a purely rational PCD system in dimension $2 + k$.*
– *Any language L of Δ_k is fully-recognized by a purely rational PCD system in dimension $2 + k$.*

The proof is rather technical: timers are suppressed by using machines that cross a given hyper-plane at regular time, divisions by two are done by reusing the variables defining the machines, and the two variables used in [2] to go from quantifier elimination to semi-recognition are suppressed by storing some information in the variable used to do the homogenization.

4 PCD Systems Cannot Recognize Any Other Set

4.1 Local dimension

We define:

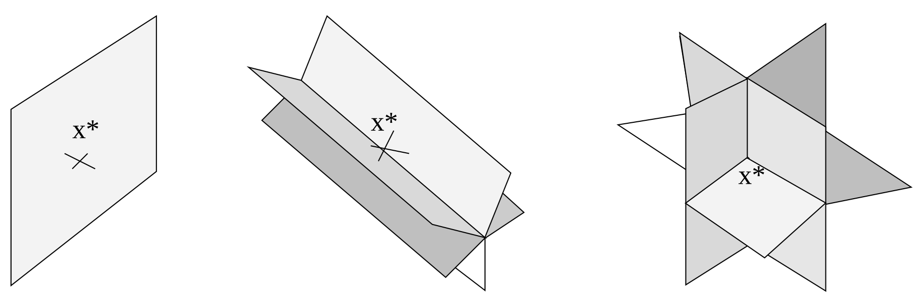

Fig. 3. From left to right: x^* is of local dimension $1^+, 2^+, 3$ in a PCD system of dimension 3.

Definition 6 (Local dimension). *Let $\mathcal{H} = (\mathcal{X}, \{)$ be a PCD system in dimension d. Let x^* be a point of X. Let Δ be a polyhedral subset $\Delta \subset X$ of maximal dimension $d - d'$ $(1 \leq d' \leq d)$ such that there exists an open convex polyhedron $V \subset X$, with $x^* \in \Delta \cap V$, and such that, for any region F of $\mathcal{H}$, $F \cap V \neq \emptyset$ implies $\Delta \subset \overline{F}$ ($\overline{F}$ is the topological closure of F).*

If $d' < d$ then x^* is said to be of local dimension d'^+. *If $d' = d$ then x^* is said to be of* local dimension d' *and we can always choose V small enough such that x^* is the only point of local dimension d' in $\overline{V}$: see figure 3.*

Note that given a rational PCD system $\mathcal{H} = (\mathcal{X}, \{)$ and $k = d'$ or $k = d'^{+}$ one can effectively compute $LocDim(\mathcal{H}, \|)$ defined as the set of the points $x \in X$ that have a local dimension equals to k.

The main idea behind definition 6 is given by the following lemma: see figure 4.

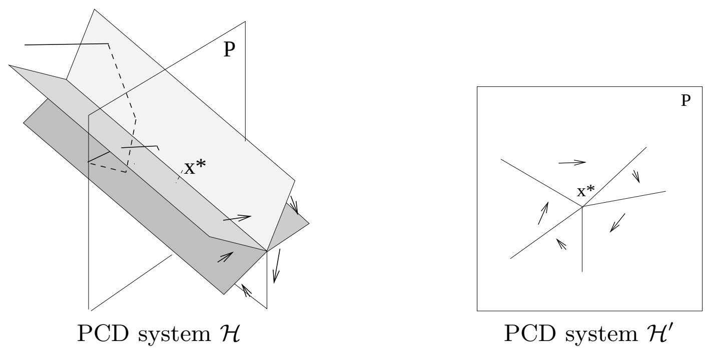

Fig. 4. Proposition 1: if x^* is of local dimension 2^+ in a PCD system $\mathcal{H}$ of dimension 3, the projections on P of the trajectories of $\mathcal{H}$ in neighborhood V of x^* are precisely the trajectories of some PCD system $\mathcal{H}'$ of dimension 2.

Proposition 1. *Let $\mathcal{H} = (\mathcal{X}, \{)$ be a PCD system in dimension d. Let x^* be a point of local dimension $(d')^+$ with $d' < d$. Call P the affine variety of dimension d' which is the orthogonal of Δ in x^*. It is possible to construct a PCD system $\mathcal{H}' = (\mathcal{X}' = \mathbb{R}^{'}, \mho')$ in dimension d' such that the trajectories of $\mathcal{H}'$ are the orthogonal projections on P of the trajectories of $\mathcal{H}$ in V.*

For any point x^*, the corresponding V is denoted by V_{x^*}. $\mathcal{H}'$, Δ are respectively denoted by $\mathcal{H}_{\S^*}$ and Δ_{x^*}. If $d' < d$ we denote by p_{x^*} and q_{x^*} the functions that map all point $x \in X$ onto its orthogonal projection on P and onto its orthogonal projection on Δ respectively. If $d' = d$, we define p_{x^*} and q_{x^*} as respectively the identity function and the null function. We assume the natural order $1 < 1^+ < 2 < 2^+ < \ldots$.

Lemma 1. *Let $\mathcal{H} = (\mathcal{X}, \{)$ be a PCD system of dimension d. Let Φ be a trajectory of $\mathcal{H}$ that reaches x^* at finite continuous time T_c. Assume that x^* is of local dimension $k = d'$ or $k = (d')^+$. For any l, denote by S_l the set of the points $x \in X$ that are reached by Φ at some time $0 \le t < T_c$ and that have local dimension l. Assume $S_l = \emptyset$, for all $l > k$.*

– *S_k is a finite set.*
– *Assume $S_k = \emptyset$. Fix the origin in x^*. Then either $S_{(d'-1)^+}$ is a finite set or there exist $y_1, y_2 \in X$ that are reached by Φ, there exists $0 < \lambda < 1$ such that $p_{x^*}(y_2) = \lambda p_{x^*}(y_1)$ and such that, for all $n \ge 1$, Φ reaches at a*

time $t_n \leq T_c$ *the point* y_n *defined by* $p_{x^*}(y_n) = \lambda^n p_{x^*}(y_1)$ *and* $q_{x^*}(y_n) = q_{x^*}(y_1) + \sum_{i=1}^{n} \lambda^i (q_{x^*}(y_2) - q_{x^*}(y_1))$.

Proof. Let $m \leq k$. We prove first that if S_m is not a finite set, then Φ reaches a point of local dimension $> m$ at some time $\leq T_c$: assume that S_m is not a finite set. $T_m = \{t | \Phi(t) \in S_m\}$ is a well ordered set. Denote its elements by $t_1^m, t_2^m, \ldots, t_\omega^m, \ldots$. Take $t_\infty^m = sup_{i \in \mathbb{N}} t_i^m$. We have $t_\infty^m \leq T_c$. Consider $x_\infty^m = \Phi(t_\infty^m)$. By continuity of Φ, there exists $t^m < t_\infty^m$ such that $t \in [t^m, t_\infty^m] \Rightarrow \Phi(t) \in V_{x_\infty^m}$. Take $t \in [t^m, t_\infty^m] \cap S_m$. From considerations of dimensions about point $\Phi(t)$ of local dimension m in $V_{x_\infty^m}$, we get that the local dimension d'' of x_∞^m is $\geq m$. From the definition of t_∞^m, we get $d'' \neq m$. Hence $d'' \geq m$ and our claim is proved: if S_m is not a finite set then Φ reaches some x_∞^m of local dimension $> m$.

The first assertion of the lemma is an easy consequence of this claim with $m = k$.

For the second assertion, take $m = (d'-1)^+$, and assume that $S_{(d'-1)^+}$ is not a finite set. From $S_k = \emptyset$, we must have $x_\infty^m = x^*$ and $t_\infty^m = T_c$. If $k < d$ denote $\mathcal{H}' = \mathcal{H}_{\S^*}$ else take $\mathcal{H}' = \mathcal{H}$. Define Φ' as $p_{x^*}(\Phi)$. From time t^m up to time T_c, Φ' is a trajectory of $\mathcal{H}' = (\mathcal{X}', \{')$ (apply proposition 1 for $k < d$), reaching $p_{x^*}(x^*)$ at time T_c. Let $\mathcal{L}$ be the set of the one-dimensional regions of $\mathcal{H}'$ that intersect $V'_{x^*} = p_{x^*}(V_{x^*})$. We claim that each time Φ' reaches a point of $S_{(d'-1)^+}$, Φ' reaches an element of $\mathcal{L}$: if Φ' reaches some point $x^{*\prime} \in X'$ of local dimension $(d-1)^+$ at some time $t \in [t^m, T_c]$, then $p_{x^*}(\Delta_{x^{*\prime}})$ is an element of $\mathcal{L}$ and contains $x^{*\prime}$. See figure 5.

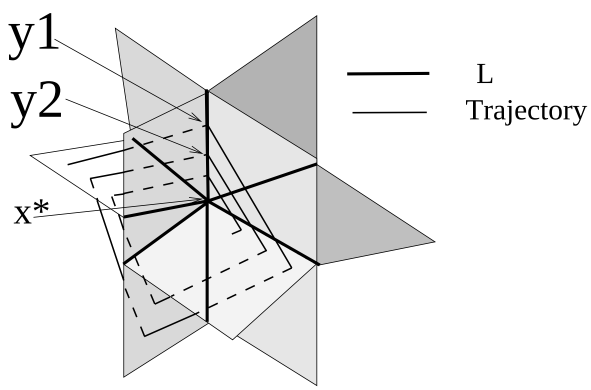

Fig. 5. Proof of lemma 1: here $d = d' = 3$. $\mathcal{L}$ defined as the set of the one dimensional regions that intersect $p_{x^*}(V_{x^*})$. $\mathcal{L}$ is made of a finite number of segments. Each time the trajectory reaches a point of local dimension 2^+, it reaches $\mathcal{L}$. If the trajectory reaches two times $\mathcal{L}$ in a same segment then the trajectory is ultimately cycling.

Since Φ' converges to $p_{x^*}(x^*)$, since $\mathcal{L}$ is a finite set, since $S_{(d'-1)^+}$ is infinite, $p_{x^*}(\Phi)$ reaches two times the same element of $\mathcal{L}$ in $p_{x^*}(y_1)$ and $p_{x^*}(y_2)$ with $p_{x^*}(y_2) = \lambda p_{x^*}(y_1)$ for some $0 < \lambda < 1$, at some times t_{y_1}, t_{y_2} with $t^m \leq t_{y_1} <$

$t_{y_2} < T_c$. Now see that by definition of V'_{x^*} all the regions of $\mathcal{H}'$ intersecting V'_{x^*} contain $p_{x^*}(x^*)$ in their topological closure. Hence we have $f'(x) = f'(\mu x)$, for all $x \in V'_{x^*}, \mu \in (0,1]$. If $\Phi'(t)$ is solution to differential equation $\dot{x}_d = f'(x)$, $\Psi'(t) = \lambda \Phi'(t/\lambda)$ is also solution. As a consequence trajectory Φ' must reach $\lambda^n p_{x^*}(y_1)$ for all n. From the definition of $\mathcal{H}'$ this implies that Φ reaches the y_n of the lemma for all n : see figure 5.

4.2 Problems *Reach* and *Conv*

Define the following problems:

Definition 7 (Problems $Reach_{d'}, Reach_{d'+}$).
Let k be either of type $k = d'$ or of type $k = d'^{+}$, where d' is an integer.

- Instance*: A purely rational PCD system $\mathcal{H} = (\mathcal{X}, \{)$ of dimension d, a polyhedral convex subset $V \subset X$, a rational polygon $x^1 \subset X$, a rational number $t_{sup} \in \mathbb{Q}$, a rational number $t_{inf} \in \mathbb{Q}$, a rational point $x_0 \in X$.*
  Question *"$Reach_k(\mathcal{H}, \mathcal{V}, \S_\prime, \S^\infty, \sqcup_{\rangle\backslash\{}, \sqcup_{\int\sqcap_\surd})$": "Do all the following conditions hold simultaneously:*
  - *trajectory Φ starting from x^0 reaches x^1 at some finite continuous time T_c*
  - *$t_{inf} < T_c \leq t_{sup}$*
  - *for any $0 \leq t \leq T_c$, $x = \Phi(t)$ is in V and is of local dimension $\leq k$."*
- Instance*: A purely rational PCD system $\mathcal{H} = (\mathcal{X}, \{)$ of dimension d, a polyhedral convex subset $V \subset X$, a rational point $x^* \in X$, a rational number $t_{sup} \in \mathbb{Q}$, a rational number $t_{inf} \in \mathbb{Q}$, a rational point $x_0 \in X$.*
  Question *"$Conv_k(\mathcal{H}, \mathcal{V}, \S_\prime, \S^*, \sqcup_{\rangle\backslash\{}, \sqcup_{\int\sqcap_\surd})$": "Do all the following conditions hold simultaneously:*
  - *the trajectory Φ starting from x_0 reaches point x^* at some finite continuous time T_c*
  - *x^* is of local dimension k and is in V*
  - *$t_{inf} < T_c \leq t_{sup}$*
  - *for any $0 \leq t < T_c$, $x = \Phi(t)$ is in V and is of local dimension $< k$."*

4.3 Case $d = 3$

Using topological considerations (the sphere of $\mathbb{R}^{\nVdash}$ verifies Jordan Theorem and the arguments of [3]) we prove:

Lemma 2. *Let $\mathcal{H} = (\mathcal{X}, \{)$ be a PCD system of dimension d. Let Φ be a trajectory of $\mathcal{H}$ of finite continuous time T_c and discrete time $T_d \geq \omega$ converging towards $x^* = \Phi(T_c)$. Assume that x^* is of local dimension $\leq 3^+$. Then necessarily the signature of Φ is ultimately cyclic.*

Lemma 3. *The following problem is decidable:*

Instance: a rational PCD system $\mathcal{H} = (\mathcal{X}, \{)$ *of dimension* d*, a finite sequence of distinct regions* $(F_0, F_1, \ldots, F_j)$ *of* $\mathcal{H}$*, a rational point* $x_0 \in X$*.*

Question: "Does the trajectory Φ *starting from* x_0 *have a periodic signature of type* $(F_0, F_1, \ldots, F_j)^\omega$ *and then reach some point* $x^* \in X$ *of local dimension* $\leq 3^+$ *at some finite continuous time* t^**"*

Moreover, given a positive instance, one can effectively compute t^* *and* x^* *as a function of the coordinates of* x_0*.*

With these lemmas, we prove:

Theorem 2. *The problems* $Reach_3$ *and* $Reach_{3^+}$ *are in* Σ_1*.*

Proof ((sketch)). We prove the assertion by providing a Turing machine algorithm that (semi-)computes the predicates: to reply to $Reach_{3^+}(\mathcal{H}, \mathcal{V}, \S_\prime, \S^\infty, t_{inf}, t_{sup})$, the general idea is the following: we simulate step by step the evolution of the trajectory Φ starting from x_0. Simultaneously, if we detect that Φ crosses for the second time a given region, we use lemma 3 to see if the signature of Φ is entering or not an infinite cycle. If it is so, still by lemma 3, we compute directly the limit of the cycle x^* and the corresponding time t^* and the simulation goes on directly from new position x^* and time t^*. We stop if we reach x^1 or the complement of V, or if the time reaches a value greater than t_{sup}. From lemma 1, we know that every point of local dimension $k = 3$ or $k = 3^+$ can only be reached using a finite number of points of local dimension k. From lemma 2 each such point x of local dimension k is reached by a cyclic signature and is dropped by the algorithm.

4.4 Case $d \geq 4$

We generalize theorem 2 to higher dimensions. We prove first:

Lemma 4. *Let* $d' \geq 4$*. Assume that* $Reach_{(d'-1)^+} \in \Sigma_p$ *and that* $Reach_{(d'-2)^+} \in \Sigma_q$ *for some integers* p, q*. Then*

- $Conv_{d'} \in \Sigma_{max(p,q+2)}$.
- $Conv_{d'^+} \in \Sigma_{max(p,q+2)}$.

Proof. Denote by $B(x^*, 1/n_1)$ the ball of radius $1/n_1$ centered in x^* for the norm of the maximum. For a subset $U \subset X$, denote its complement by U^c. Let $k = d'$ or $k = {d'}^+$. We claim:

$$
\begin{aligned}
&Conv_k(\mathcal{H}, \mathcal{V}, \S_\prime, \S^*, \sqcup_{\rangle\backslash\{}, \sqcup_{\int\sqcap_{\surd}})\\
\Leftrightarrow\quad & x^* \in LocDim(\mathcal{H}, \|) \ \wedge\ \S^* \in \mathcal{V} \ \wedge\ \sqcup_{\rangle\backslash\{} < \sqcup_{\int\sqcap_{\surd}}\\
&\wedge \exists y_1 \in \mathbb{Q}\ \exists \approx_{\not\Bbbk}, \approx_{\not\models} \in \mathbb{Q}\ \curvearrowright_{\not\Bbbk} \in \mathbb{V}_{\curvearrowleft^*} \ \wedge\ \mathbb{R}\eth\eqsim_{(\prime-\not\Bbbk)^+}(\mathcal{H}, \mathcal{V}, \S_\prime, \dagger_\infty, \sqcup_\infty, \sqcup_\in)\\
&\wedge \begin{cases}
\exists y_2 \in \mathbb{Q}\ \exists \approx_{\not\Bbbk}, \approx_{\not\trianglerighteq} \in \mathbb{Q}\ \exists \lambda \in \mathbb{R}^+\\
\begin{cases}
Reach_{(d'-1)^+}(\mathcal{H}, \mathcal{V} \cap \mathcal{V}_{\S^*}, \dagger_\infty, \dagger_\in, \sqcup_\ni, \sqcup_\triangle)\\
p_{x^*}(y_2) = \lambda p_{x^*}(y_1)\\
\lambda < 1\\
t_1 + \sum_{i=1}^{\infty} \lambda^i t_3 > t_{inf}\\
t_2 + \sum_{i=1}^{\infty} \lambda^i t_4 \leq t_{sup}\\
q_{x^*}(y_1) + \sum_{i=1}^{\infty} \lambda^i (q_{x^*}(y_2) - q_{x^*}(y_1)) = q_{x^*}(x^*)
\end{cases}\\
\vee\ \forall n_1 \in \mathbb{N}\ \mathbb{R}\eth\eqsim_{(\prime-\not\models)^+}(\mathcal{H}, \mathcal{V}, \dagger_\infty, \mathcal{B}(\S^*, \infty/\backslash_\infty), \sqcup_{\rangle\backslash\{} - \sqcup_\infty, \sqcup_{\int\sqcap_{\surd}} - \sqcup_\in)
\end{cases}
\end{aligned}
$$

Assume that we have a positive instance to formula $Conv_k$: use the notations of definition 7. Denote by S the set of the points that are reached by Φ before time T_c and that have local dimension $(d'-1)^+$. Since Φ converges to x^*, there must exist an $y_1 = \Phi(t_{y_1}) \in V_{x^*}$, $t_{y_1} < T_c$ that is reached by Φ, and such that Φ stays in V_{x^*} between time t_{y_1} and time T_c. y_1 is reached using points of local dimension $\leq (d'-1)^+$. If S is not a finite set, by lemma 1 the first clause of the disjunction is true. Assume now that S is a finite set: we can assume that t_{y_1} is chosen big enough such that Φ does not reach any point of S between time t_{y_1} and time T_c. For all $n_1 \in \mathbb{N}$ we get that the trajectory starting from y_1 reaches $B(x_*, 1/n_1)$ using only points of local dimension $\leq (d'-2)^+$. Hence the second clause of the disjunction is true.

Conversely, assume that the right hand side of the formula is true. If the first clause of the disjunction is true, the trajectory is cycling and the formula $Conv_k$ should be true. Assume now that the second clause is true. For all $n_1 \in \mathbb{N}$, we get that there exists t_{n_1} such that $\Phi(t_{n_1}) \in B(x^*, 1/n_1)$. Denote $T_c = sup_{n_1 \in \mathbb{N}} t_{n_1}$. From the continuity of Φ we get that[1] $\Phi(T_c) = x^*$. Hence Φ reaches x^* of local dimension k and formula $Conv_k$ must be true.

The result is now immediate by applying the Tarski-Kuratowski algorithm on the formula [8].

We also prove in a similar way:

Lemma 5. *Let $d' \geq 4$. Assume $Reach_{(d'-1)^+} \in \Sigma_p$ for some integer p. Then $Conv_{d'} \in \Sigma_{p+1}$.*

Proof ((sketch)). For a point $x^* \in X$ of local dimension d, define Out_{x^*} as the set of the points $x \in X$ such that the trajectory starting from x intersects the complement of V_{x^*} at a discrete time less or equal to one. We prove that, now, the following formula holds:

[1] Note that if function Φ is not defined on value T_c, since Φ is continuous with a bounded right derivative, Φ can always be extended to a continuous functions defined on value T_c.

$$
\begin{aligned}
&Conv_{d'}(\mathcal{H}, \mathcal{V}, \S_{\prime}, \S^{*}, \sqcup_{\rangle\backslash\{}, \sqcup_{\int\sqcap_{\surd}})\\
\Leftrightarrow\quad & x^{*} \in LocDim(\mathcal{H}, \|) \ \wedge\ \S^{*} \in \mathcal{V} \ \wedge\ \sqcup_{\rangle\backslash\{} < \sqcup_{\int\sqcap_{\surd}} \ \wedge\ \lceil\rangle\Updownarrow\rceil\backslash\int\rangle\wr\backslash(\mathcal{H}) = \lceil'\\
&\wedge \exists y_1 \in \mathbb{Q}\ \exists \approx_{\nVdash}, \approx_{\nvDash} \in \mathbb{Q}\ \curvearrowright_{\nVdash} \in \mathbb{V}_{\curvearrowright^{*}} \ \wedge\ \mathbb{R}\eth\eqsim_{(\prime-\nVdash)^{+}}(\mathcal{H}, \mathcal{V}, \S_{\prime}, \dagger_{\infty}, \sqcup_{\infty}, \sqcup_{\in})\\
&\left\{\begin{array}{l}
\quad Reach_{(d'-1)^{+}}(\mathcal{H}, \mathcal{X}, \dagger_{\infty}, \mathcal{X}, \sqcup_{\rangle\backslash\{} - \sqcup_{\infty}, \sqcup_{\rangle\backslash\{} - \sqcup_{\infty} + \infty)\\
\wedge\ Reach_{(d'-1)^{+}}(\mathcal{H}, \mathcal{X}, \dagger_{\infty}, \mathcal{V}^{\rfloor} \cup \mathcal{N}\wr\mathcal{E}\sqsubseteq\wr\updownarrow\sqcap\sqcup\rangle\wr\backslash(\mathcal{H}) \cup \mathcal{O}\sqcap\sqcup_{\S^{*}}, \prime, \sqcup_{\in} - \sqcup_{\int\sqcap_{\surd}})\\
\wedge\ Reach_{(d'-1)^{+}}(\mathcal{H}, \mathcal{X}, \dagger_{\infty}, \mathcal{X}, \sqcup_{\int\sqcap_{\surd}} - \sqcup_{\in}, \sqcup_{\int\sqcap_{\surd}} - \sqcup_{\in} + \infty)
\end{array}\right.
\end{aligned}
$$

We get:

Theorem 3. *Let $d' \geq 3$.*

- *$Reach_{d'}$ is in $\Sigma_{d'-2}$.*
- *$Reach_{d'+}$ is in $\Sigma_{d'-1}$ if d' is even.*
- *$Reach_{d'+}$ is in $\Sigma_{d'-2}$ if d' is odd.*

Proof. The assertion is proved by recurrence over d' using theorem 2, lemmas 5 and 4, by Tarski-Kuratowski and the fact that we have for $k = d'$ or $k = {d'}^{+}$:

$$
\begin{aligned}
&Reach_{k}(\mathcal{H}, \mathcal{V}, \S_{\prime}, \S^{\infty}, \sqcup_{\rangle\backslash\{}, \sqcup_{\int\sqcap_{\surd}})\\
\Leftrightarrow\quad & Reach_{(d'-1)^{+}}(\mathcal{H}, \mathcal{V}, \S_{\prime}, \S^{\infty}, \sqcup_{\rangle\backslash\{}, \sqcup_{\int\sqcap_{\surd}})\\
&\vee \exists n \in \mathbb{N}\ \exists < \curvearrowright^{*}_{\nvDash}, \curvearrowright^{*}_{\nVdash}, \curvearrowright^{*}_{\nvDash}, \ldots, \curvearrowright^{*}_{\ltimes} > \in \mathbb{Q}\ \exists < \approx_{\nvdash}, \ldots, \approx_{\ltimes} >\\
&\quad \exists < t'_0, \ldots, t'_n >\\
&\quad \left\{\begin{array}{l}
x^{*}_0 = x_0\\
\forall 0 \leq i < n\ Conv_{k}(\mathcal{H}, \mathcal{V}, \S^{*}_{\rangle}, \S^{*}_{\rangle+\infty}, \sqcup_{\rangle}, \sqcup'_{\rangle})\\
Reach_{(d'-1)^{+}}(\mathcal{H}, \mathcal{V}, \S^{*}_{\backslash}, \S^{\infty}, \sqcup_{\backslash}, \sqcup'_{\backslash})\\
t_0 + t_1 + \ldots + t_n > t_{inf}\\
t'_0 + t'_2 + \ldots + t'_n \leq t_{sup}
\end{array}\right.
\end{aligned}
$$

By Tarski-Kuratowski on formula $n \in L \Leftrightarrow \exists t_1 \in \mathbb{N}\ \mathbb{R}\eth\eqsim(\mathcal{H}, \mathcal{X}, \nabla(\backslash), \S^{\infty}, \prime, \sqcup_{\infty})$, we get the main result of this section:

Corollary 1.
- *If L is semi-recognized by a purely rational PCD system of dimension d, then $L \in \Sigma_{d-2}$.*
- *If L is recognized by a purely rational PCD system of dimension d, then $L \in \Delta_{d-2}$.*

And by using theorem 1:

Corollary 2.
- *The languages that are semi-recognized by purely rational PCD systems of dimension d in finite continuous time are precisely the languages of Σ_{d-2}*
- *The languages that are recognized by purely rational PCD systems of dimension d in finite continuous time are precisely the languages of Δ_{d-2}*

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