Orthogonal Polyhedra: Representation and Computation[⋆]

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Abstract. In this paper we investigate *orthogonal polyhedra*, i.e. polyhedra which are finite unions of full-dimensional hyper-rectangles. We define representation schemes for these polyhedra based on their vertices, and show that these compact representation schemes are canonical for all (*convex and non-convex*) polyhedra in *any* dimension. We then develop efficient algorithms for membership, face-detection and Boolean operations for these representations.

1 Introduction and Motivation

Traditionally, most of the applications of computational geometry are concerned with low-dimensional spaces, motivated mainly by problems in graphics, vision and robotics. On the other hand, the analysis of dynamical systems is often done in state-spaces of higher dimension. Since geometry plays an important role in the analysis of such dynamical systems, one would expect that computational geometry will be used extensively in computer-aided design tools for control systems. Although applied mathematicians write algorithms that operate in such spaces (optimization, ODEs, PDEs) the point of view and the concerns are sometimes different from those of mainstream computational geometry. The only notable exception is the treatment of convex polyhedra in linear programming where the points of view of applied mathematics and computational geometry coincide.

Recently, attempts have been made to re-approach computer science and control theory in order to build a theory of *hybrid systems*. These are dynamical systems, defined over both discrete and continuous state variables, intended to model the interaction of computerized controllers with their physical environments (see [AKNS95,M97,HS98] for a representative sample) and to extend the scope of program verification techniques toward continuous systems. One fundamental problem in this domain is the following: *Given a dynamical system*

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defined by $\dot{x} = f(x)$, where x takes its values in the state-space $\mathbb{R}^d$, and given $P \subseteq \mathbb{R}^d$, calculate (or approximate) the set of points in the state-space reached by trajectories (solutions) starting in P. In [DM98] a method called *face lifting* was proposed, based on previous work of [KM91] [G96]. It consists of restricting P to the class of polyhedra, and iteratively "lifting" the faces of the polyhedra outward according the the maximal value of the normal component of f along the face (see [DM98] for a detailed description and Fig. 1 for an illustration).

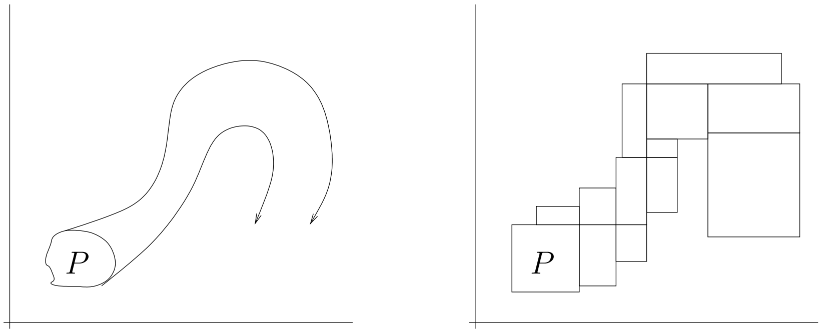


Fig. 1. A dynamical system and trajectories starting at a set P (left) and an approximation of the reachable states by polyhedra (right).

The main computational-geometric burden associated with this approach is related to the representation of intermediate polyhedra (non-convex in general), identifying their faces, decomposing them into convex subsets, and performing face lifting as well as other set-theoretic operations. Due to the complicated structure of high-dimensional non-convex polyhedra, the approach taken in [DM98] consists in restricting the class of subsets to contain only *orthogonal* (axis-parallel, isothetic) polyhedra which can be written as finite unions of full-dimensional hyper-rectangles. A special case of these polyhedra are what was called in [DM98] *griddy* polyhedra, which are generated from unit *hypercubes* with integer-valued vertices. Since arbitrary orthogonal polyhedra can be obtained from griddy ones by appropriate stretching and translation, we restrict our attention to the latter, and use the term orthogonal in order not to introduce additional terminology.

The main contribution of the paper is the definition of several canonical representation schemes for *non-convex* orthogonal polyhedra in *any* dimension. All these schemes are vertex-based and their sizes range between $O(nd)$ and $O(n2^d)$ where n is the number of vertices and d is the dimension. Based on these representations we develop relatively-efficient algorithms for membership, face detection, and Boolean operations on arbitrary orthogonal polyhedra of any dimension. The generalization of these results to more general classes of polyhedra, in particular to *timed polyhedra* used in the verification of timed automata will be reported elsewhere.

Beyond the original motivation coming from computer-aided control system design, we believe that orthogonal polyhedra and subsets of the integer grid are fundamental objects whose computational aspects deserve a thorough investigation.

The rest of the paper is organized as follows: in section 2 we define orthogonal polyhedra and their representation schemes. Section 3 is devoted to algorithms for deciding membership of a point in a polyhedron. In section 4 we discuss face detection and Boolean operations. Finally, in section 5 we mention some future research directions.

2 Orthogonal Polyhedra and Their Representation

We assume that all our polyhedra live inside a bounded subset $X = [0, m]^d \subseteq \mathbb{R}^d$ (in fact, the results will hold also for $X = \mathbb{R}_+^d$). We denote elements of X as $\mathbf{x} = (x_1 \ldots, x_d)$ and the zero and unit vector by $\mathbf{0}$ and $\mathbf{1}$. A d-dimensional grid is a product of d subsets of $\mathbb{N}$. In particular, the *elementary grid* associated with X is $\mathbf{G} = \{0, 1 \ldots, m-1\}^d \subseteq \mathbb{N}^d$. For every point $\mathbf{x} \in X$, $\lfloor \mathbf{x} \rfloor$ is the grid point corresponding to the integer part of the components of $\mathbf{x}$. The grid admits a natural partial order with $(m-1, \ldots, m-1)$ on the top and $\mathbf{0}$ as bottom. The set of subsets of the elementary grid forms a Boolean algebra $(2^{\mathbf{G}}, \cap, \cup, \sim)$ under the set-theoretic operations.

Definition 1 (Orthogonal Polyhedra). *Let* $\mathbf{x} = (x_1, \ldots, x_d)$ *be a grid point. The* elementary box *associated with* $\mathbf{x}$ *is a closed subset of* X *of the form* $B(\mathbf{x}) = [x_1, x_1+1] \times [x_2, x_2+1] \times \ldots [x_d, x_d+1]$. *The point* $\mathbf{x}$ *is called the leftmost corner of* $B(\mathbf{x})$. *The set of boxes is denoted by* $\mathbf{B}$. *An orthogonal polyhedron* P *is a union of elementary boxes, i.e. an element of* $2^{\mathbf{B}}$.

One can see that $2^{\mathbf{B}}$ is closed[1] under the following operations:

$$A \sqcup B = A \cup B$$

$$A \sqcap B = cl(int(A) \cap int(B))$$

$$\neg A = cl(\sim A)$$

(where *cl* and *int* are the topological closure and interior operations[2]) and that the bijection B between $\mathbf{G}$ and $\mathbf{B}$ which associates every box with its leftmost corner generates an isomorphism between $(2^{\mathbf{G}}, \cap, \cup, \sim)$ and $(2^{\mathbf{B}}, \sqcap, \sqcup, \neg)$. In the sequel we will switch between point-based and box-based terminology according to what serves better the intuition.

[1] It is not closed under usual complementation and intersection.

[2] See [Bro83] for definitions.

Definition 2 (Color Function). *Let P be an orthogonal polyhedron. The color function $c : X \to \{0,1\}$ is defined as follows: If $\mathbf{x}$ is a grid point than $c(\mathbf{x}) = 1$ iff $B(\mathbf{x}) \subseteq P$; otherwise, $c(\mathbf{x}) = c(\lfloor \mathbf{x} \rfloor)$.*

We say that a grid point $\mathbf{x}$ is black (resp. white) and that $B(\mathbf{x})$ is full (resp. empty) when $c(\mathbf{x}) = 1$ (resp. 0). Note that c almost coincides with the characteristic function of P as a subset of X. It differs from it only on right-boundary points (see figure 2).

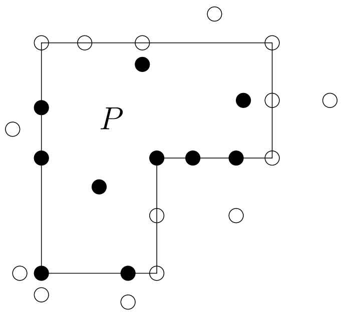


Fig. 2. An orthogonal polyhedron and a sample of the values of the color function it induces.

Definition 3 (Facets and Vertices). *In the following we consider z to be an integer in $[0, m)$, $\mathbf{x} = (x_1, \ldots, x_d)$ and a polyhedron P with a color function c.*

- *The* i-predecessor *of a point* $\mathbf{x}$ *is* $\mathbf{x}^{i-} = (x_1, \ldots, x_i - 1, \ldots, x_d)$. We use $\mathbf{x}^{ij-}$ as a shorthand for $(\mathbf{x}^{i-})^{j-}$.
- *An* i-hyperplane *is a $(d-1)$-dimensional subset $\mathcal{H}_{i,z}$ of X consisting of all points satisfying $x_i = z$.*
- *An* i-facet *of P is $F_{i,z}(P) = cl\{\mathbf{x} \in \mathcal{H}_{i,z} : c(\mathbf{x}) \mathcal{N} c(\mathbf{x}^{i-})\}$. We say that elements of $F_{i,z}(P)$ are* i-traversed.[3] Note that a facet is an orthogonal polyhedron in $\mathbb{R}^{d-1}$ rather than in $\mathbb{R}^d$.
- *A* vertex *is a (non-empty) intersection of d distinct facets. The set of vertices of P is denoted by $V(P)$.*
- *An* i-vertex-predecessor *of* $\mathbf{x}$ *is a vertex of the form $(x_1, \ldots, x_{i-1}, z, \ldots, x_d)$, $z \le x_i$. The first i-vertex-predecessor of $\mathbf{x}$, denoted by $\mathbf{x}^{i\leftarrow}$, is the one with the maximal z.*

When $\mathbf{x}$ has no i-predecessor (resp. i-vertex-predecessor) we write $\mathbf{x}^{i-} = \perp$ (resp. $\mathbf{x}^{i\leftarrow} = \perp$).

[3] A facet can be decomposed into two parts according to the orientation, that is, $F_{i,z} = F^+_{i,z} \cup F^-_{i,z}$ where $F^+_{i,z} = F_{i,z} \cap \{\mathbf{x} : c(\mathbf{x}) = 0\}$ and $F^-_{i,z} = F_{i,z} \cap \{\mathbf{x} : c(\mathbf{x}) = 1\}$. We call any such F^+ or F^- an *oriented facet*.

One can check that these definitions capture the intuitive meaning of a facet and a vertex and, in particular, that the boundary of an orthogonal polyhedron is the union of its facets. Another useful concept is that of neighborhood:

Definition 4 (Neighborhood). *The neighborhood of a grid point* $\mathbf{x}$ *is the set*

$$\mathcal{N}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \ldots \times \ldots \{x_d - 1, x_d\}$$

(the vertices of a box lying between $\mathbf{x} - \mathbf{1}$ and $\mathbf{x}$). *For every* i*,* $\mathcal{N}(\mathbf{x})$ *can be partitioned into left and right* i*-neighborhoods*

$$\mathcal{N}^{i-}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \ldots \times \{x_i - 1\} \times \ldots \times \{x_d - 1, x_d\}$$

and

$$\mathcal{N}^{i}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \ldots \times \{x_i\} \times \ldots \times \{x_d - 1, x_d\}$$

A *representation scheme* for $2^{\mathbf{B}}$ (or $2^{\mathbf{G}}$) is a set $\mathcal{E}$ of syntactic objects such that there is a surjective function ψ from $\mathcal{E}$ to $2^{\mathbf{B}}$ (i.e. every syntactic object represents at most one polyhedron and every polyhedron has at least one corresponding object). If ψ is also an injection we say that the representation scheme is *canonical* (every polyhedron has a unique representation). There are two obvious representation schemes for orthogonal polyhedra. One is the trivial *explicit* representation consisting of an enumeration of the values of c on every grid point, i.e. a d-dimensional zero-one array with m^d entries. The other is the *Boolean representation* based on all the formulae generated from inequalities of the form $x_i \geq z$ via Boolean operations. Clearly this is a representation but not a canonical one even if we restrict formulae to be in disjunctive normal form (a union of hyper-rectangles).

The *vertex representation*, around which this paper is built, consists of the set $\{(\mathbf{x}, c(\mathbf{x})) : \mathbf{x} \text{ is a vertex}\}$, namely the vertices of P along with their color. One of the main results of the paper is that this is indeed a representation scheme for $2^{\mathbf{B}}$ (canonicity is evident). Note that the set of vertices alone is *not* a representation due to ambiguity (see Fig. 3). Also notice that not every set of points and colors is a valid representation of a polyhedron.

We will also use the *neighborhood representation* in which additional information is attached to each vertex, namely the color of all the 2^d points in its neighborhood. Transforming a vertex representation into this one (whose size is $O(n2^d)$) can be performed as a pre-processing stage. Finally we extend the *extreme vertex representation*, which was proposed independently by Aguilera and Ayala in [AA97,AA98] for 3-dimensional orthogonal polyhedra, and show that it is a representation for *any* dimension.

3 Deciding Membership

In this section we show that all the abovementioned representation schemes are valid by providing decision procedures for the *membership* problem: *Given a representation of a polyhedron* P *and a grid point* $\mathbf{x}$*, determine* $c(\mathbf{x})$*, that is, whether* $B(\mathbf{x}) \subseteq P$*.*

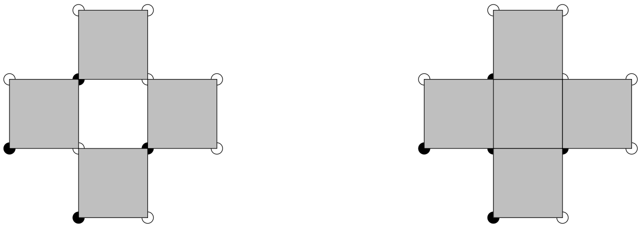


Fig. 3. Two orthogonal polyhedra and their corresponding vertex representations. Note that they have the same set of vertices and only the color of one of the vertices distinguishes one from the other.

3.1 Vertex Representation

Observation 1 (Vertex Rules).
1) A point $\mathbf{x}$ *is on an* i*-facet iff*

$$\exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}) \text{ s.t. } c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}') \tag{1}$$

2) A point $\mathbf{x}$ *is a vertex iff*

$$\forall\, i \in \{1, \ldots, d\} \exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}) \text{ s.t. } c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}') \tag{2}$$

3) A point $\mathbf{x}$ *is* not *a vertex iff*

$$\exists\, i \in \{1, \ldots, d\} \forall \mathbf{x}' \in \mathcal{N}^i(\mathbf{x})\ c(\mathbf{x}'^{i-}) = c(\mathbf{x}') \tag{3}$$

Example: Take $d = 2$ and $\mathbf{x} = (x_1, x_2)$. Then:
$\mathbf{x}$ is on a 1-facet iff $c(x_1 - 1, x_2 - 1) \neq c(x_1, x_2 - 1) \vee c(x_1 - 1, x_2) \neq c(x_1, x_2)$.
It is on a 2-facet iff $c(x_1 - 1, x_2 - 1) \neq c(x_1 - 1, x_2) \vee c(x_1, x_2 - 1) \neq c(x_1, x_2)$.
It is a vertex if both of the above hold and a *non-vertex* if
$c(x_1 - 1, x_2 - 1) = c(x_1, x_2 - 1) \wedge c(x_1 - 1, x_2) = c(x_1, x_2) \vee$
$c(x_1 - 1, x_2 - 1) = c(x_1 - 1, x_2) \wedge c(x_1, x_2 - 1) = c(x_1, x_2)$.
This is illustrated in Fig. 4.

Lemma 1 (Color of a Non-Vertex). *Let* $\mathbf{x}$ *be a non-vertex and let* j *be a direction such that for every* $\mathbf{x}' \in \mathcal{N}^j(\mathbf{x}) - \{\mathbf{x}\}$*,* $c(\mathbf{x}') = c(\mathbf{x}'^{j-})$*. Then* $c(\mathbf{x}) = c(\mathbf{x}^{j-})$*.*

Proof. Since $\mathbf{x}$ is not a vertex there exists i such that for every $\mathbf{x}' \in \mathcal{N}^i(\mathbf{x})$ $c(\mathbf{x}') = c(\mathbf{x}'^{i-})$. If $j = i$ we are done and $c(\mathbf{x}) = c(\mathbf{x}^{i-})$. Otherwise, we know that not being on an i-facet implies, in particular, $c(\mathbf{x}^{ij-}) = c(\mathbf{x}^{j-})$. In the j-direction we have $c(\mathbf{x}^{ij-}) = c(\mathbf{x}^{i-})$ and using $c(\mathbf{x}^{i-}) = c(\mathbf{x})$ and the transitivity of equality we get $c(\mathbf{x}) = c(\mathbf{x}^{j-})$ (see Fig. 5). □

Consequently we can calculate the color of a non-vertex $\mathbf{x}$ based on the color of all points in $\mathcal{N}(\mathbf{x}) - \{\mathbf{x}\}$: just find some j satisfying the conditions of Lemma 1 and let $c(\mathbf{x}) = c(\mathbf{x}^{j-})$. This gives immediately a decision procedure for the membership problem:

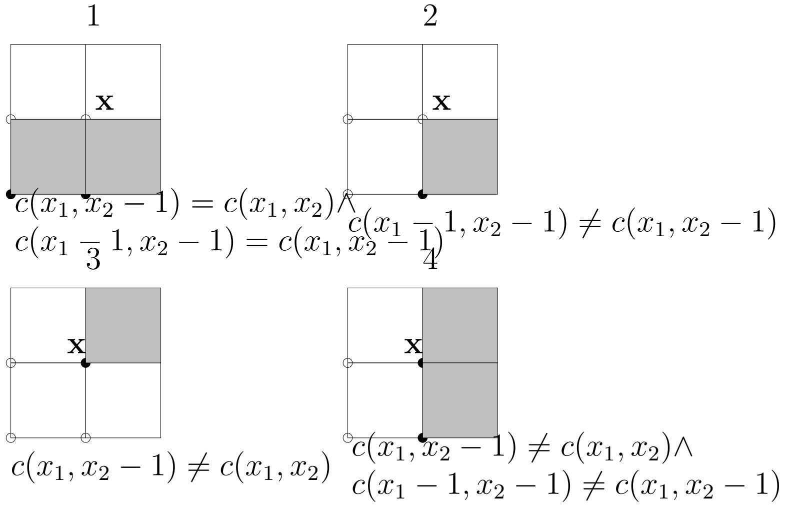

1
2
x
x
$c(x_1, x_2 - 1) = c(x_1, x_2) \wedge$
$c(x_1 - 1, x_2 - 1) = c(x_1, x_2 - 1)$
$c(x_1 - 1, x_2 - 1) \neq c(x_1, x_2 - 1)$
3
4
x
x
$c(x_1, x_2 - 1) \neq c(x_1, x_2)$
$c(x_1, x_2 - 1) \neq c(x_1, x_2) \wedge$
$c(x_1 - 1, x_2 - 1) \neq c(x_1, x_2 - 1)$

Fig. 4. Some examples of the vertex and facet conditions for a point $\mathbf{x} = (x_1, x_2)$: 1) $\mathbf{x}$ is not on a 1-facet. 2) and 3) $\mathbf{x}$ is on a 1-facet (for different reasons). In these cases it is also on a 2-facet and hence a vertex. 4) The point is on 1-facet but not on a 2-facet.

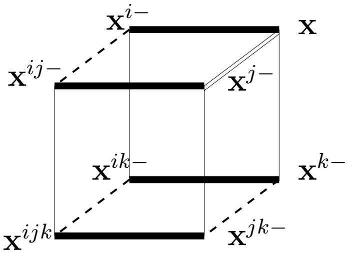


Fig. 5. An illustration of the proof of Lemma 1: horizontal lines indicate equalities in the i direction and dashed lines equalities in the j direction. The equality between $c(\mathbf{x})$ and $c(\mathbf{x}^{j-})$ is derived.

Theorem 1 (Membership for Vertex Representation). *The membership problem for vertex representation can be solved in time $O(n^d d2^d)$ using space $O(n^d)$.*

Proof. We start at $\mathbf{x}$ and call recursively the membership procedure of all the $2^d - 1$ point in $\mathcal{N}(\mathbf{x}) - \{\mathbf{x}\}$. Termination is guaranteed because we go down in the partial-order on $2^{\mathbf{G}}$ and either encounter vertices or reach the origin. We can avoid duplicate calls to the same point by memorizing the visited points and thus visit every point in the grid at most once. This gives an $O(Nd2^d)$ algorithm where N is the size of the grid.

This algorithm is not very efficient because in the worst-case one has to calculate the color of all the grid points between $\mathbf{0}$ and $\mathbf{x}$. We can improve it using the notion of an *induced grid*: let the *i-scale* of P be the set of the i-coordinates of the vertices of P and let the induced grid be the Cartesian product of its i-scales (see Fig. 6). One can see that the induced grid is the smallest (coarsest) grid containing all the vertices, that every rectangle in this grid has a uniform color and that the size of the grid is $O(n^d)$. Hence, calculating the color of a point reduces to finding its closest "dominating" point on the induced grid and applying the algorithm to that grid in $O(n^d d2^d)$ time. □

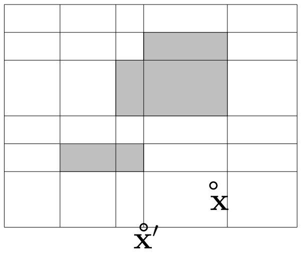


Fig. 6. A polygon, its induced grid, and a point $\mathbf{x}$ dominated by $\mathbf{x}'$.

Corollary 1 (Main Result). *The vertex representation is a canonical representation for orthogonal polyhedra.*

3.2 Neighborhood Representation

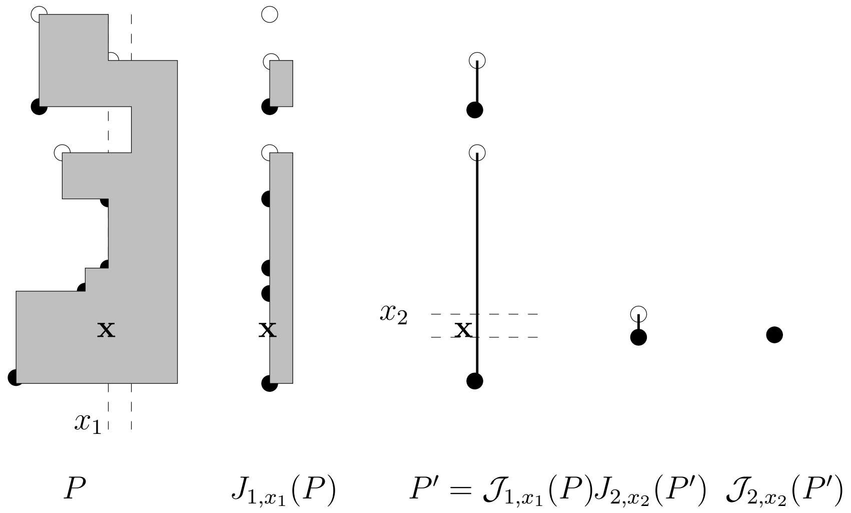
By fixing d we now suggest an $O(n \log n)$ membership algorithm for the neighborhood representation, based on successive projections of P into polyhedra of a smaller dimension.

Definition 5 (i-Slice and i-Section). *Let P be an orthogonal polyhedron and z an integer in* $[0, m)$.

- *An i-slice of P, is the d-dimensional orthogonal polyhedron $J_{i,z}(P) = P \sqcap \{\mathbf{x} : z \leq x_i \leq z+1\}$.*
- *An i-section of P, is the $d-1$-dimensional orthogonal polyhedron $\mathcal{J}_{i,z}(P) = J_{i,z}(P) \cap \mathcal{H}_{i,z}$.*

These notions are illustrated in Fig. 7.

Clearly, the membership of $\mathbf{x} = (x_1, \ldots, x_d)$ in P can be reduced into membership in $\mathcal{J}_{i,x_i}(P)$, which is a $(d-1)$-dimensional problem. By successively reducing dimensionality for every i we obtain a point whose color is that of $\mathbf{x}$. We show how the main computational activity, the calculation of i-sections, can be done using the neighborhood representation.



P $\quad J_{1,x_1}(P)$ $\quad P' = \mathcal{J}_{1,x_1}(P) J_{2,x_2}(P')$ $\quad \mathcal{J}_{2,x_2}(P')$

Fig. 7. Calculating the membership of $\mathbf{x} = (x_1, x_2)$ in P: P is transformed via its 1-slice $J_{1,x_1}(P)$, into a 1-section $P' = \mathcal{J}_{1,x_2}(P)$. Then P' is transformed, via its 2-slice $J_{2,x_2}(P')$, into its 2-section $\mathcal{J}_{2,x_2}(P')$ which is a point. The vertices of P which are $\mathbf{x}^{i\leftarrow}$ for some $\mathbf{x} \in \mathcal{H}_{1,x_1}$ are indicated.

Lemma 2 (Vertex of a Section). *Let P be a polyhedron and let P' be its i-section at $x_i = z$. A point $\mathbf{x}$ is a vertex of P' iff $\mathbf{y} = \mathbf{x}^{i\leftarrow} \neq \bot$ and for every $j \neq i$, there exists $\mathbf{x}' \in \mathcal{N}^i(\mathbf{y}) \cap \mathcal{N}^j(\mathbf{y})$ such that $c(\mathbf{x}'^{j-}) \neq c(\mathbf{x}')$. Moreover, when this condition is true, the neighborhood of $\mathbf{x}$ relative to $\mathcal{J}_{i,z}(P)$ is given by $\mathcal{N}^i(\mathbf{y})$.*

Proof. First, observe that $\mathbf{x}$ is a vertex of P' if it satisfies that condition itself, i.e. for every $j \neq i$, there exists $\mathbf{x}' \in \mathcal{N}^i(\mathbf{x}) \cap \mathcal{N}^j(\mathbf{x})$ such that $c(\mathbf{x}'^{j-}) \neq c(\mathbf{x}')$.[4]

Assume $\mathbf{x}$ satisfies the condition. There exists $\mathbf{y} = (x_1, \ldots, x_{i-1}, z, \ldots, x_d)$ such that $c(\mathcal{N}^i(\mathbf{y})) = c(\mathcal{N}^i(\mathbf{x}))$ and $c(\mathcal{N}^{i-}(\mathbf{y})) \neq c(\mathcal{N}^i(\mathbf{y}))$ with z maximal with this property. Since $c(\mathcal{N}^i(\mathbf{y})) = c(\mathcal{N}^i(\mathbf{x}))$, $\mathbf{y}$ satisfies the condition as well and since $c(\mathcal{N}^{i-}(\mathbf{y})) \neq c(\mathcal{N}^i(\mathbf{y}))$, $\mathbf{y}$ is a vertex of P. Since z is maximal with this property we have $\mathbf{y} = \mathbf{x}^{i\leftarrow}$.

Conversely assume $\mathbf{y} = \mathbf{x}^{i\leftarrow}$ exists and it satisfies the condition. Then $c(\mathcal{N}^i(\mathbf{x}) = c(\mathcal{N}^i(\mathbf{y}))$, because otherwise, by the above reasoning, there would be a vertex between $\mathbf{x}$ and $\mathbf{y}$. Hence $\mathbf{x}$ satisfies the condition as well. □

Theorem 2 (Membership for Neighborhood Representation). *The membership problem for the neighborhood representation can be solved in time* $O(nd^2(\log n + 2^d))$.

Proof. First observe that it takes $O(nd \log n)$ to determine the vertices $\mathbf{y}$ which are $\mathbf{x}^{i\leftarrow}$ for some $\mathbf{x} \in \mathcal{H}_{i,z}$. There are at most n such points. Using the previous lemma it is possible to determine, using $O(d2^d)$ time, whether each of the corresponding points on $\mathcal{H}_{i,z}$ are vertices of the section. Hence it takes $O(nd(\log n + 2^d))$ to get rid of one dimension, and this is repeated d times until P is contracted into a point. □

Remark: A similar algorithm with the same complexity can be used to calculate the color of all the points in a *neighborhood* of $\mathbf{x}$ which we describe informally. The algorithm takes double slices (which are d-dimensional thick sections of width two) of P, as illustrated in Fig. 8, and successively reduces P into the neighborhood of $\mathbf{x}$. This variation on the algorithm is used for doing Boolean operations.

3.3 Extreme Vertex Representation

The next representation scheme, inspired by the representation proposed by Aguilera and Ayala [AA97,AA98] for 3-dimensional polyhedra, can be viewed as a compaction of the neighborhood representation. Instead of maintaining all the neighborhood of each vertex, it suffices to keep only the *parity* of the number of black points in that neighborhood — in fact, it suffices to keep only vertices with odd parity. We use $\pi(\mathbf{x})$, $\pi^i(\mathbf{x})$ and $\pi^{i-}(\mathbf{x})$ to denote the parity of the number of black points in $\mathcal{N}(\mathbf{x})$, $\mathcal{N}^i(\mathbf{x})$ and $\mathcal{N}^{i-}(\mathbf{x})$, respectively. We will use the convention $\pi(\perp) = 0$.

Definition 6 (Extreme Points). *A point* $\mathbf{x}$ *is said to be* extreme *if* $\pi(\mathbf{x}) = 1$.

By enumerating all the possible configurations in dimension 1, 2 and 3, it can be checked that this definition coincides with the geometrical definition presented in [AA97,AA98] for these dimensions.

[4] Note the difference from the condition for being a vertex of P: there, the i-coordinates of the $\mathbf{x}'$s can be either z or $z - 1$ but here we insist on z. This is the reason some vertices of P disappear after making a section (see Fig. 7).

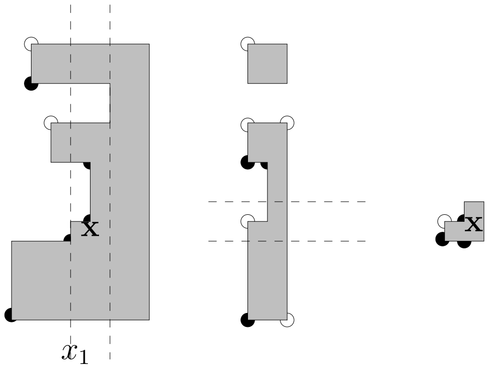


Fig. 8. Calculating the color of a neighborhood of a point.

Observation 2. *Any extreme point* $\mathbf{x}$ *is a vertex.*

Proof. By induction on the dimension d. The assertion for $d = 1$ is immediate. Now in dimension d, choose an arbitrary direction $i \in \{1, \ldots, d\}$. Exactly one of the neighborhoods $\mathcal{N}^{i-}(\mathbf{x})$ and $\mathcal{N}^{i}(\mathbf{x})$ contains an odd number of black points. Assume without loss of generality that it is $\mathcal{N}^{i}(\mathbf{x})$. By induction hypothesis such a neighborhood implies that $\mathbf{x}$ is a vertex in $\mathcal{J}_{i,x_i}(P)$. This means that for every $j \neq i$, there exists $\mathbf{x}' \in \mathcal{N}^{j}(\mathbf{x})$ such that $c(\mathbf{x}'^{j-}) \neq c(\mathbf{x}')$. Since one cannot have $c(\mathbf{x}') = c(\mathbf{x}'^{i-})$ for all $\mathbf{x}' \in \mathcal{N}^{i}(\mathbf{x})$, $\mathbf{x}$ is a vertex of P. □

The converse is not true and vertices need not be extreme as one can see in Fig. 9.

The *extreme vertex representation* consists in representing an orthogonal polyhedron by the set of its extreme vertices.[5] Note that in dimension 1 all vertices are extreme and hence the vertex and extreme vertex representations practically coincide.

In order to do successive projections on this representation we need a rule, similar to Lemma 2, for determining which points are extreme vertices of an i-section. The following is a corollary of Lemma 2:

Corollary 2. *Let* $\mathbf{x} = (x_1, \ldots, x_{i-1}, z, x_i, \ldots, x_d)$ *be a point and let* $\mathbf{y} = (\mathbf{x}^{i-})^{i\leftarrow}$ *be its (strict) i-vertex-predecessor. Then* $\pi^{i-}(\mathbf{x}) = \pi^{i}(\mathbf{y})$.

Proof. Observation 2 implies that if $\pi^{i-}(\mathbf{x}) = 1$ then $\mathbf{x}^{i-}$ must be a vertex of $\mathcal{J}_{i,z-1}(P)$. By Lemma 2, $\mathcal{N}^{i-}(\mathbf{x}) = \mathcal{N}^{i}(\mathbf{y})$.

Conversely, Observation 2 implies that if $\pi^{i}(\mathbf{y}) = 1$ then for every $j \neq i$, there exists $\mathbf{x}' \in \mathcal{N}^{i}(\mathbf{y}) \cap \mathcal{N}^{j}(\mathbf{y})$ such that $c(\mathbf{x}'^{j-}) \neq c(\mathbf{x}')$. By applying Lemma 2 to $\mathbf{x}^{i-}$ one gets that $\mathcal{N}^{i-}(\mathbf{x})$ must be equal to $\mathcal{N}^{i}(\mathbf{y})$. □

[5] To be more precise, an additional bit for the color of the origin is needed. From this information, the color of all extreme vertices can be inferred.

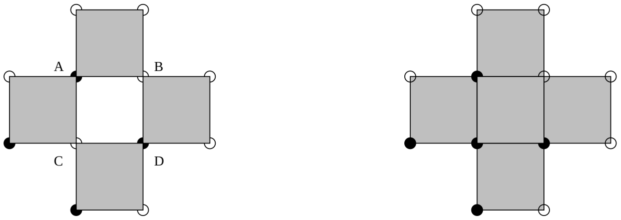


Fig. 9. All the vertices of the two polyhedra are extreme except vertices A, B, C and D.

Note that when $\pi^{i-}(\mathbf{x}) = \pi^{i}(\mathbf{x}) = 0$, $\mathcal{N}^{i-}(\mathbf{x}) = \mathcal{N}^{i}(\mathbf{y})$ need not hold.

Lemma 3 (Extreme Vertices of a Section). *Let P be a polyhedron and let $P' = \mathcal{J}_{i,z}(P)$. A point $\mathbf{x}$ is an extreme vertex of P' iff it has an odd number of extreme i-vertex-predecessors.*

Proof. First note that $\mathbf{x}$ is extreme iff $\pi^{i-}(\mathbf{x}) \neq \pi^{i}(\mathbf{x})$. We prove by induction on the number of vertex predecessors of $\mathbf{x}$. Assume $\mathbf{x}$ has no vertex predecessors. In this case $\pi^{i-}(\mathbf{x}) = 0$ and $\pi^{i}(\mathbf{x}) = 1$ iff $\mathbf{x}$ is extreme. Suppose it is true for $n-1$ vertex predecessors and let $\mathbf{x}$ have n strict vertex predecessors $\mathbf{y}^1, \ldots, \mathbf{y}^n$. By the induction hypothesis $\pi^{i}(\mathbf{y}^n)$ is equal to the number of extreme vertices among $\mathbf{y}^1, \ldots, \mathbf{y}^n$. By Corollary 2, $\pi^{i}(\mathbf{y}^n) = \pi^{i-}(\mathbf{x})$ and we have $\mathbf{x}$ not extreme if $\pi^{i}(\mathbf{x}) = \pi^{i-}(\mathbf{x}) = \pi^{i}(\mathbf{y}^n)$ and $\mathbf{x}$ extreme if $\pi^{i}(\mathbf{x}) \neq \pi^{i-}(\mathbf{x}) = \pi^{i}(\mathbf{y}^n)$. In both cases $\pi^{i}(\mathbf{x})$ coincides with the parity of the number of extreme vertices. □

One gets immediately:

Corollary 3. *Given an orthogonal polyhedron P and two integers i, z, one can compute in time $O(dn \log n)$ an extreme vertex representation of $\mathcal{J}_{i,z}(P)$.*

Applying the successive projection technique we get:

Theorem 3 (Membership for Extreme Vertex Representation). *The extreme vertex representation is canonical for orthogonal polyhedra in arbitrary dimension and the membership problem for this representation can be solved in time $O(nd^2 \log n)$.*

4 Other Operations

While our representations might be very compact, their usefulness will be measured by how much can algorithms operate on them *without* retrieving the color of every point. As it turns out, face detection is rather simple, and Boolean operations can be performed on neighborhoods of vertices and potential vertices whose number is quadratic in the number of vertices.

4.1 Face Detection

The problem of face detection is the the following: *Given a orthogonal polyhedron P, a direction i and an integer z, calculate the facet $F_{i,z}$.*

Observation 3 (Vertices of Facets). *Let $c_{i,z}$ be the color function of the facet $F_{i,z}(P)$, i.e. $c_{i,z}(\mathbf{x}) = 1$ iff $c(\mathbf{x}^{i-})\mathcal{N}c(\mathbf{x})$. Then, $\mathbf{x}$ is a vertex of $F_{i,z}$ iff it is a vertex of P with $x_i = z$ and it satisfies the vertex condition relative to $c_{i,z}$, that is, for every $j \neq i$ there exists $\mathbf{x}' \in \mathcal{N}^j(\mathbf{x}) \cap \mathcal{N}^i(\mathbf{x})$ such that $c_{i,z}(\mathbf{x}'^{j-}) \neq c_{i,z}(\mathbf{x})$.*

For the neighborhood representation one just needs to check the above condition for every vertex of P. Extreme vertices always satisfy the condition and hence one gets:

Theorem 4 (Face Detection). *The face detection problem for orthogonal polyhedra can be done in $O(nd2^d)$ using neighborhood representation and in $O(n)$ using the extreme vertex representation.*

4.2 Boolean Operations

Complementation is trivial for all our representations. Intersection and union are similar and we discuss the first (the second can be performed anyway via de-Morganization). We assume two orthogonal polyhedra P_1 and P_2 with n_1 and n_2 vertices respectively. After intersection some vertices disappear and some new vertices are created (see Fig. 10). However not every point is a candidate to be a vertex of the intersection.

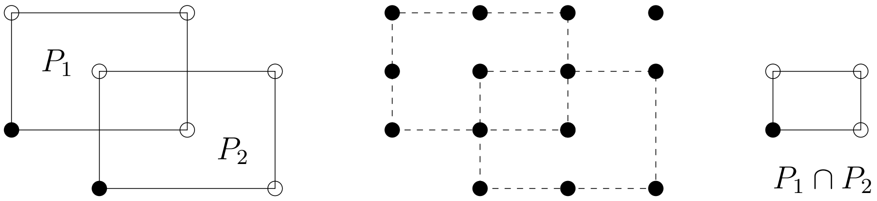


Fig. 10. Intersection of two polyhedra. In the middle one can see all the candidates for being vertices of the intersection..

Lemma 4. *A point $\mathbf{x}$ is a vertex of $P_1 \cap P_2$ only if for every i, $\mathbf{x}$ is on an i-facet of P_1 or on an i-facet of P_2.*

Proof. If there where some i such that $\mathbf{x}$ was not i-traversed in both polyhedra, it remains so after intersection. □

Lemma 5. *Let* $\mathbf{x}$ *be a vertex of* $P_1 \cap P_2$ *which is not an original vertex, and let* I_1 *(resp.* I_2*) be the set of directions* i *for which* $\mathbf{x}$ *is on an* i*-facet of* P_1 *(resp.* P_2*). Then there exists a vertex* $\mathbf{y}^1$ *of* P_1 *and a vertex* $\mathbf{y}^2$ *of* P_2*, such that* $\mathbf{x} = \max(\mathbf{y}^1, \mathbf{y}^2)$ *where* max *is applied coordinatewise.*

Proof. First we observe that if $\mathbf{x}$ is traversed at directions I_1 in P_1 then there is a vertex $\mathbf{y}$ such that it agrees with $\mathbf{x}$ on all the I_1 coordinates and is smaller than $\mathbf{x}$ in the remaining directions. The same reasoning applies to P_2. □

From this we can conclude that the candidates for being vertices of $P_1 \cap P_2$ are restricted to the following set:

$$V(P_1) \cup V(P_2) \cup \{\mathbf{x} : \exists \mathbf{y}^1 \in V(P_1)\, \exists \mathbf{y}^2 \in V(P_2) \text{ s.t. } \mathbf{x} = \max(\mathbf{y}^1, \mathbf{y}^2)\}$$

whose number is not greater then $n_1 + n_2 + n_1 n_2$. Combining this with the slicing results we have (assuming $n_1 n_2 >> n_1 + n_2$):

Theorem 5 (Boolean Operations). *The intersection of two orthogonal polyhedra with* n_1 *and* n_2 *vertices can be calculated in time* $O(n_1 n_2 d^2 2^d (n_1 + n_2))$ *using the extreme vertex representation.*

Proof. For every pair of vertices calculate their max as a potential vertex of the intersection. Then compute the color of its neighborhood (if it was not a vertex of P_1 and P_2). Finally calculate pointwise the intersection of the neighborhoods of each point and determine whether or not it is a vertex of $P_1 \cap P_2$ using the standard vertex rules. Note that when the vertices of a given polyhedron are sorted in a lexicographical order as a preprocessing step, it takes $O(nd^2)$ time to determine the color of an arbitrary point. □

5 Past and Future Directions

Orthogonal polyhedra were studied intensively by research communities such as *Computer Graphics*, *Solid Modeling*, *Computational Geometry*, etc. An elaborate survey of these disciplines, their results and methodologies is outside the scope of this paper, but it is fair to say that at least the first two, for obvious reasons, rarely look at dimensions higher than 3. The work reported in [AA97,AA98], which we extended to arbitrary dimension, is the only one we have found relevant to our approach.

We have investigated a representation scheme for orthogonal polyhedra and devised algorithms for the basic operations on them. These algorithms have been implemented and will be integrated into the system described in [DM98]. In this direction, it will be interesting to give a characterization of "typical" orthogonal polyhedra arising from continuous operations, and evaluate the average case complexity of the representation and algorithms on these. Applications of this technique to the analysis of programs with integer variables should be examined as well.

We are currently extending our results to the more general class of polyhedra manipulated by verification and synthesis algorithms for *timed automata*, generated by the (finitely many) elements of the "region graph" [AD94].

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