Chapter 1

On the Richness of Continuous Dynamical Systems

1.1 Introduction

This chapter and the following one present several dynamical systems coming from models from physics, biology, bioinformatics, computer virology, game theory and distributed algorithmic. This chapter focuses on systems that do not involve explicitly some concurrency or competition between agents, whereas the following will present some of these models.

As we announced in the introduction, this discussion is motivated by several purposes.

First, we try to show the richness of dynamical systems, of their behaviors, and the difficulties behind their study. Almost all the examples from this chapter are from the excellent monograph [Hirsch et al., 2003].

Second, we try to demonstrate the interest of the class of polynomial Cauchy problems, both for computability over the reals, and for the fact that it really covers most of the systems seen in nature.

Finally, we try to show by all these examples that several devices are intrinsically continuous and can be used as such to do some computations.

We also discuss, thanks to the very pedagogical paper [Krivine et al., 2006], several problems that arise when one tries to discretize continuous systems. Hence we argue that the continuous abstraction is often the good way to do.

1.2 Mathematical preamble

1.2.1 Dynamical Systems

In this chapter, a continuous time dynamical system corresponds to a solution of an ordinary differential equation, with some initial condition. That is to say, to a solution of a Cauchy problem of type

$$y' = f(y), \qquad y(t_0) = x \tag{1.1}$$

where $f : E \to \mathbb{R}^n$, with $E \subset \mathbb{R}^n$ open.

A trajectory $y(t)$ is a differentiable function $y : I \subset \mathbb{R} \to E$ that satisfies the equation.

A discrete time dynamical system corresponds to a sequence solution of a recurrence equation: the analogue of Cauchy problem (1.1) for discrete time systems is a recurrence equation of type

$$y_{t+1} = f(y_t), \qquad y_0 = x. \tag{1.2}$$

A more complete discussion about what is called in general a dynamical system can be found in Chapter 3.

A semi-algebraic set is a subset of $\mathbb{R}^n$ that can be represented by a finite number of polynomial equalities and inequalities. More precisely, let $g \in \mathbb{R}[X_1, \ldots, X_n]$ be a polynomial. Denote $U(g) = \{\mathbf{x} \in \mathbb{R}^n | g(x) > 0\}$. A set E is semi-algebraic if it belongs to the smallest set that contains the $U(g)$ and that is closed by complementations, unions and intersections.

By Tarksi-Seidenberg theorem, this corresponds to all the sets that can be defined in the first order theory over $(\mathbb{R}, +, -, \times)$: any set that can be defined from real constants, operations $+$, $-$, and $\times$, negations, conjunctions, disjunctions, and real existential and universal quantifications, is semi-algebraic.

As a preamble to our discussion, let us say that, if the modelling of systems by dynamical systems is rather ancient, one must understand that the richness of a part of the obtained behaviors is discussed only since recently.

In particular, as Hirsch, Smale and Devaney write in the second edition of their monograph [Hirsch et al., 2003], in 1970, when the first edition was edited, the world *chaos* had never been used in a mathematical context, about dynamical systems. They write

> "The discovery of such complicated dynamical systems as the horseshoe map, homoclinic tangles, and the Lorenz system, and their mathematical analyzes, convinced scientists that simple stable motions such as equilibria or periodic solutions were not always the most important behavior of solutions of differential equations"

1.2.2 Linear systems

Linear systems consist of continuous time dynamical systems $\mathbf{x}' = f(\mathbf{x}, t)$, where f is a linear function in $\mathbf{x} \in \mathbb{R}^n$. In other words, there exists a matrix A such that

$$\mathbf{x}' = A(t)\mathbf{x}. \tag{1.3}$$

For example, a particle of mass m attached to a spring has its position $x(t)$ at time t, linked to its second derivative by the differential equation of second order

$$x'' + p^2 x = 0,$$

for a constant p that represents the stiffness of the spring.

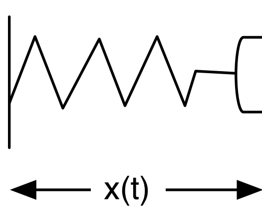

Figure 1.1: Harmonic Oscillator

This model is called the model of the harmonic oscillator, and its solutions are of type $x(t) = A\cos(pt+t_0)$.

This is indeed a linear system, since according to the classical technique, x can be obtained by projecting the solutions of system

$$\begin{cases} x' &= y \\ y' &= -p^2 x \end{cases}$$

over first coordinate.

What makes the strength of linear systems is their interest in practice. Indeed, it is well known that the study in a point of the solutions of non-linear system can be realized by working on its linearization in this point [Hirsch et al., 2003].

However, their modelling power is rather limited. A simple dynamic like $x' = x(1-x)$ is not linear for example.

Linear systems are rather well understood. For example, over $\mathbb{R}^2$, according to the determinant $\det(A)$ and the trace $\mathrm{tr}(A)$ of matrix A of size 2×2, we have the classification of possible behaviors of Figure 1.2: see [Hirsch et al., 2003].

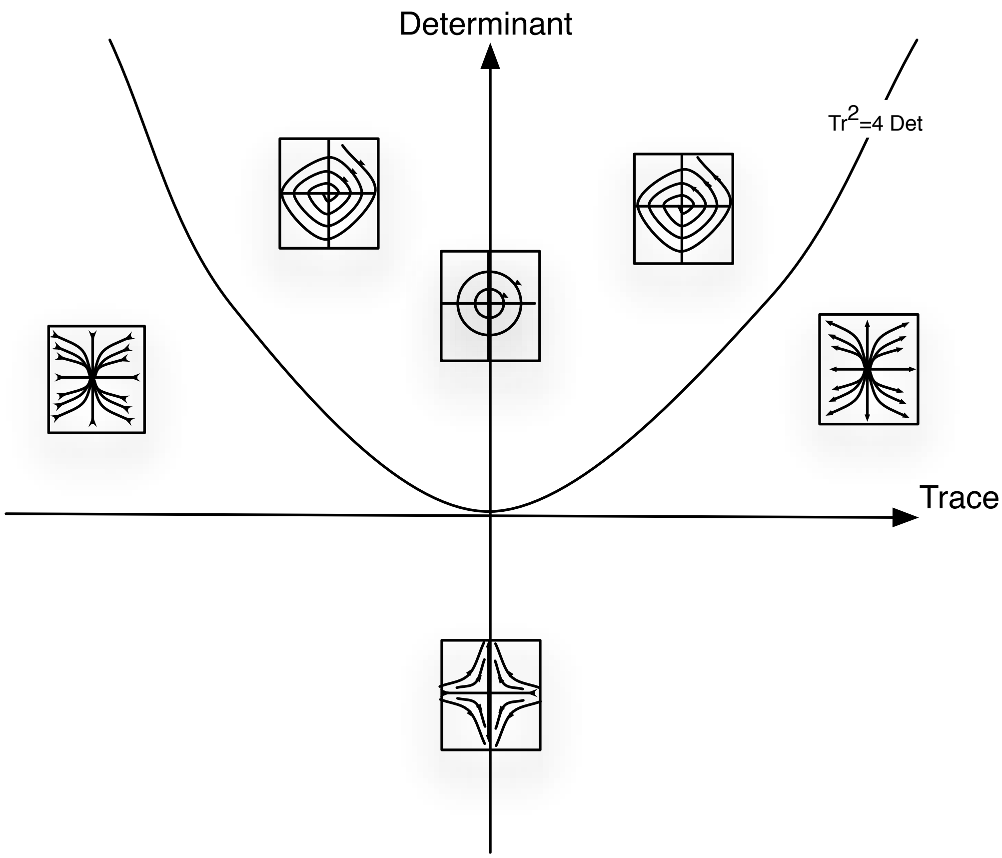

Figure 1.2: Symbolic representation of the dynamics of $\mathbf{x}' = A\mathbf{x}$, in dimension 2, according to the trace and the determinant of A: see [Hirsch et al., 2003].

This classification points out an interesting phenomenon. First, even if the system is linear, the type of obtained dynamics is actually function of the determinant and of the trace of the matrix A, that is to say a semi-algebraic condition on the coefficients of A, and not a linear condition.

In higher dimensions, the analysis is subtler than the simple consideration of trace and determinant, but the observation is still true: the good tools to analyze linear dynamics are not linear functions and conditions, but semi-algebraic conditions.

That semi-algebraic conditions are sufficient can be seen as a consequence of Tarksi-Seidenberg theorem.

But on can often be more explicit on the involved conditions. For example, Routh-Hurwitz theorem claims the following result.

Theorem 1 (Routh-Hurwitz) *The origin is an attracting stable point of linear system* $\mathbf{x}' = A\mathbf{x}$ *over* $\mathbb{R}^n$ *iff* $\Delta_1 > 0$, $\Delta_2 > 0, \cdots, \Delta_n > 0$ *with*

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 & \cdots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4} & b_{2k-5} & b_{2k-6} & \cdots & b_k \end{vmatrix}$$

where the characteristic polynomial of A *is written* $\lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n$, *in other words* $b_i = \frac{1}{i!}\frac{d^i|A-\lambda I|}{d\lambda^i}(0)$.

Thus, the good tool to discuss linear systems is actually the manipulation of sign conditions on polynomials.

1.2.3 Polynomial Cauchy Problems

The systems

$$\mathbf{x}' = p(\mathbf{x}, t) \tag{1.4}$$

where p is a vector of polynomials (in $\mathbf{x}$ and t) are not linear systems, but the previous remark still hold for these systems: the study of their stability in a point can be realized by linearization in this point, and hence with polynomials. In some way, polynomials allow to talk about polynomials, whereas linear functions do not allow talking about linear functions in the general case. Observe, that in a certain way, this argument is at the heart[1] of the Blum Shub Smale model [Blum et al., 1989, Blum et al., 1998], upon which we will come back (in Chapter 5).

According to [Graça, 2006], we propose to distinguish a particular class of ordinary differential equations that we will call in this document *polynomial Cauchy problems*.

Definition 1 (Polynomial Cauchy problem) *A polynomial Cauchy problem is a Cauchy problem of type*

$$\begin{cases} \mathbf{x}' & = & p(\mathbf{x}, t) \\ \mathbf{x}(0) & = & \mathbf{x}_0 \end{cases}$$

where $p(\mathbf{x}, t)$ *is a vector of polynomials, and* $\mathbf{x}_0$ *is some initial condition.*

What makes the true interest of this class of systems is its generality and its power. Immediately, all linear systems fall in this class, as well as all dynamics where explicitly the equation is polynomial. But this is also true for all systems that involve functions such as sin, cos and actually functions that can be defined in turn as (projections of) solutions of a polynomial Cauchy problem. This permits to say that all the examples in monographs like [Hirsch et al., 2003, Murray, 2002], and actually all examples from this document are in this class.

For example, consider the dynamic of a pendulum. The laws of physics give immediately a dynamic of type

$$x'' + p^2 \sin(x) = 0.$$

If the angle $x(t)$ is small, one can approach this dynamic by the equation of the harmonic oscillator, which is a linear dynamic. If one does not make this hypothesis, à priori, the sinus function seems to imply that the dynamic is not a solution of a polynomial differential equation.

[1] At least of the existence of universal machine of the model of the initial paper [Blum et al., 1989].

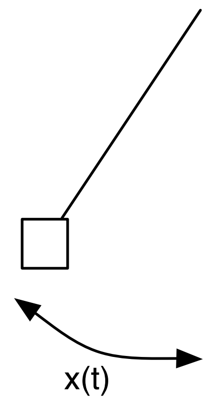

Figure 1.3: Pendulum

However, define $y = x'$, $z = \sin(x)$, $u = \cos(x)$. A simple computation of derivatives show that we must have

$$\begin{cases} x' &= y \\ y' &= -p^2 z \\ z' &= yu \\ u' &= -yz \end{cases} .$$

That shows that solution $x(t)$ can be obtained as the projection of a solution of a polynomial Cauchy problem.

Actually, our reader can get convinced that all the systems considered in books like [Hirsch et al., 2003], or [Murray, 2002] can be put in the form of ordinary differential equations defined by polynomial Cauchy problems. That proves their interest in practice, at least for modelling.

One can find in [Graça, 2006] the proof of the closure of functions (projections of) solutions of polynomial Cauchy problems by addition, subtraction, multiplication, division, composition, differentiation, composition, and inverse composition. It is even proved there that an equation of type $\mathbf{x}' = f(\mathbf{x}, t)$, where f is a vector of functions (projections of) solutions of polynomial Cauchy problems correspond to a polynomial Cauchy problem.

1.2.4 GPAC and polynomial Cauchy problems

These remarks are not so alleviating that they may appear, since they are incarnations of deep remarks about computability over the reals.

Indeed, this class of dynamical systems becomes even more interesting if one realizes that it captures all what can be computed by some models of continuous time machines, such as the General Purpose Analog Computer of Shannon [Shannon, 1941].

This model is a theoretical abstraction of the continuous machines that existed at the time of Shannon, such as the Differential Analyser built for the first time in 1931 under the supervision of Vannevar Bush at MIT [Bush, 1931]. If digital computers finally won the competition against analog computers, don't forget their ancestors. We will come back in a future chapter (Chapter 3) on the history of analog machines.

Indeed, it has recently been proved (correcting and simplifying the articles [Shannon, 1941], [Pour-El, 1974], [Lipshitz and Rubel, 1987]) that:

Theorem 2 ([Graça and Costa, 2003]) *A function is GPAC-generated (i.e. computable by the General Purpose Analog Computer from Shannon) if and only if it is the projection of a solution of a polynomial Cauchy problem.*

Actually, the GPAC, or polynomial Cauchy problems, allow capturing almost all the functions in practice. In some sense, in other terms, Shannon was right in calling its model "General Purpose" analog computer, since its has a universality property similar to the property that has Turing machines with respect to discrete machines, and computable discrete functions.

It is interesting to realize that we have almost the same phenomenon as in classical computability with some classes of functions such as elementary functions introduced by [Kalmár, 1943], or primitive recursive functions.

Indeed, in classical computability, almost all usual functions are elementary (respectively primitive recursive). The only few counter-examples, such as the Ackermann function, correspond in some way to a diagonalization over this property [Kalmár, 1943], [Rose, 1984]. Furthermore, in classical computability these classes are very robust, and stable by many operations.

Here, almost all the functions are generable by a GPAC. The previous properties are a proof of the robustness of the class of GPAC generable functions. The only few counter-examples of non-GPAC computable functions that are known are obtained by considering functions that are not differentially algebraic: an unary function y is differentially algebraic on interval I, if there exists a non-null polynomial p with real coefficients such that

$$p\left(t, y, y', ..., y^{(n)}\right) = 0$$

on I. A function that is not differentially algebraic is said *transcendentally transcendental*.

Question 1 *Is it possible to formalize that? Can we relate elementary functions, primitive recursive functions, and GPAC computable functions more formally than by only this analogy?*

Among transcendentally transcendental functions, there is:

Theorem 3 *The functions*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

and the Riemmann zeta function

$$\zeta(x) = \sum_{k=0}^{\infty} \frac{1}{k^x}$$

are not differentially algebraic.

These functions constitute somehow the analogue to GPAC computable functions of Ackermann function with respect to elementary functions. The proof of the transcendentally transcendence of function Γ is due to Hölder in 1887 [Hölder, 1887]. The one for function Γ to Hilbert, written by Stadigh [Stadigh, 1902].

1.3 Discrete Versus Continuous

1.3.1 Don't be too discrete

After these mathematical preambles, let's come to the heard of the subject.

Some of our colleagues, mostly computer scientists, often ask us why we focus on continuous systems, with arguments of type

- in computer science, nothing is continuous, all the processes are discrete
- the continuum is only an unrealistic abstraction of discrete world
- continuous systems must be discretized to be simulated
- ...
- what is the link with computer science?
- ...

Our answer to these objections will be constituted by a series of examples, to motivate the discussion.

1.3.2 Some Discrete Dynamical Systems

Let's first start by presenting several examples of discrete time dynamical systems that raised to a lot of interest in literature. The first class was mostly discussed in computer science community. The second class mostly by mathematicians.

The Turing Machine

First, the Turing machine. A Turing machine on alphabet Σ, with set of internal states Q, corresponds to a discrete time dynamical system. The state of the machine at a given time corresponds to the data of the tape, of the position of the head, as well as the internal state of the machine. All that can be coded for example by an element of the set $\Sigma^\omega \times \mathbb{Z} \times Q$. The program of the machine corresponds to some evolution rules that can be translated immediately to a discrete transition function that gives the state of the machine at time $t+1$ from its state at time t.

Seeing a Turing machine as a discrete time dynamical system does not really change things, but we think this is important to realize that when one discusses discrete time dynamical systems in whole generality, there is at least the richness of this class: possibility of self-simulation, undecidability results, ... The possibility that a Turing machine can compute any computable function, and in particular can simulate other Turing machines, implies immediately that this is the class of dynamical systems with the worst behaviors among all discrete time systems (at least if we restrict to systems with a computable transition function). In particular, the class of Turing machines is a class of chaotic systems that can exhibit strange attractors, ...

These considerations appear also for all classes of continuous time dynamical systems, as soon as they allow the simulation of discrete time dynamical systems such as Turing machines.

The logistic map

After having considered the worse, let's try to consider the simplest. The simplest would be a linear dynamic of type $x(t+1) = ax(t)$, i.e. $x(t) = x(0)a^t$, that is the dynamic of a geometric sequence, not really interesting.

Let's consider a polynomial dynamic over $\mathbb{R}$ of type

$$x(t+1) = \lambda x(t)(1 - x(t)), \tag{1.5}$$

that is to say *the logistic map*. We will suppose $\lambda > 0$.

This dynamic is actually motivated by a population dynamic. Indeed, if we suppose that the birth rate of each individual is constant at each generation, the number of individual follows at time t the Malthus law

$$x(t+1) = ax(t),$$

and hence an exponential growth, without any limits.

This is more reasonable, as suggested by Verhulst in 1838, to consider that the dynamic is actually of type

$$x(t+1) = \lambda x(t)(1 - x(t)/M),$$

where M is the maximal population that can be supported by the environment. By doing the change of variable $x(t) = X(t)/M$, one falls on the previous dynamic.

Since article [May, 1976], it is well known that a very rich variety of behaviors can be generated according to the value of constant λ. Let's recall main properties.

For $1 < \lambda < \lambda_1 = 3$, the fix point $\lambda^* = 1 - 1/\lambda$ is stable, globally attracting, and attracts all trajectories that start from $x(0) \in]0,1[$. For $\lambda = \lambda_1$, a cycle of length 2 appears through a fork bifurcation. This cycle of length 2 stays stable and globally attracting for $x(0) \in]0,1[$, and $3 < \lambda < \lambda_2 = 1 + \sqrt{6}$. This continues with a sequence $\lambda_3, \lambda_4, \cdots, \lambda_k$ such that for $\lambda_k < \lambda < \lambda_{k+1}$, there is an attracting cycle of length 2^k. The sequence $\lambda_1, \cdots, \lambda_k, \cdots$ converges toward $\lambda^\infty \approx 3.5699$. For $\lambda^\infty < \lambda$ the dynamic becomes chaotic.

The dynamic of this system is now relatively well known, but before 1976, no one thinks that a discrete time dynamical system of dimension 1, as simple as this one, can exhibit such a richness of possible behaviors.

The picture of Figure 1.4 is now very often present in recent books about dynamical systems.

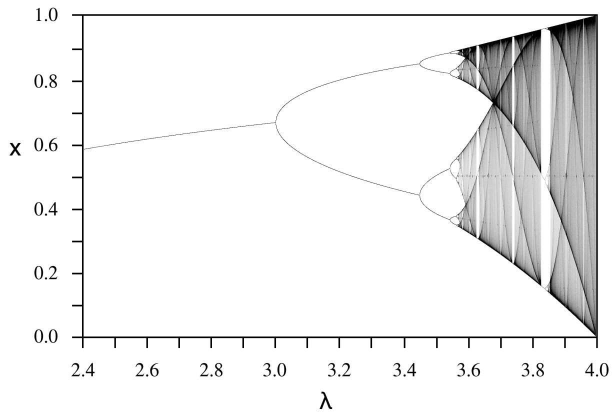

Figure 1.4: Orbit diagram for the logistic map for $2.4 < \lambda < 4$ (image from Wikipedia common image database)

1.3.3 On the Misdeeds of Discretization

Let's repeat the very pedagogical and very instructive statements of [Krivine et al., 2006]. This is not possible to give an analytic solution to dynamic (1.5) (except for $\lambda = 4$). Hence, as this is the limit when t is big that is of interest, it is interesting to focus on corresponding continuous problem. The corresponding continuous system is

$$y' = y(\lambda(1-y)-1). \tag{1.6}$$

The solutions of (1.5) correspond to the Euler method discretization with a step of 1 of the continuous equation (1.6). However, it turns out that (1.6) has an explicit analytic solution

$$y(t) = \frac{(\lambda-1)y_0}{\lambda y_0 + (\lambda(1-y_0)-1)e^{-(\lambda-1)t}}$$

that converges toward $\lambda^* = 1 - 1/\lambda$. In other words, the continuous dynamic is not chaotic, and behaves better than its discretizations.

The discretizations by Euler's method of the continuous version reflect the behavior of the continuous version when the step of discretization is smaller that some characteristic value, and become chaotic for bigger discretization steps [Krivine et al., 2006].

This constitutes a first example of systems, where a discretization can introduce complications, and can make their analysis harder than in the continuous world.

This example can be considered as artificial, and completely disconnected from any reality. This is true that the logistic map can be considered as a toy for mathematicians. However, if systems with so simple dynamics are problematic, one cannot expect too much of optimism when equations become more complicated.

For the most sceptical about the contributions of continuous systems for computer science, let's come back to well known things. Let's first be convinced that one knows how to compute otherwise than with a digital computer. To that purpose, let's first discuss how to realize easily continuous operations.

1.4 Realizing Continuous Operations

Each generation focus on the power of the machines that it has. So let's see what was computability in XIX century. We are not trying to promote two century old's computer science, but we are just trying to outline that that this computer science gave birth in the past to very nice mathematical results.

1.4.1 Planar mechanisms

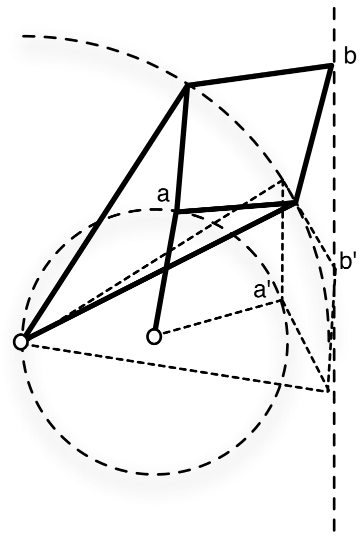

Figure 1.5: Peaucellier's mechanism. The circular motion of a is transformed into a circular motion of b.

The power of planar mechanisms made of rigid bars linked by their end by rivets attracted much attention in England and in France in the late 19th century, with a new birth of interest in Russia at the end of the forties. See for example [Artobolevskii, 1964], [Svoboda, 1948].

Everybody knows the pantograph, which allows realizing dilatations. The Peaucellier's mechanism allows transforming a linear motion into a circular motion.

More generally, this is natural to ask what is the power of such devices.

This is given by the following very nice result (see for e.g. [Artobolevskii, 1964], [Svoboda, 1948]) attributed to Kempe [Kempe, 1876], formulated here following [Smith, 1998]: the power of such devices corresponds to semi-algebraic sets.

Theorem 4 (Completeness of planar mechanism)

- *For any non-empty semi-algebraic set S, there exists a mechanism with n points that move on linear segments, but that are free to move on these segments, and that forces the relation $(x_1, \dots, x_n) \in S$, where x_i are the distances on the linear segments.*
- *Conversely, the domain of evolution of any finite planar mechanism is semi-algebraic.*

Again polynomials and semi-algebraic sets appear.

Since these mechanisms can be considered as very antique, let's go to modern electronic.

1.4.2 Curves of Lissajous

Our next example is only pedagogical, and is here only because we like it a lot. However, it raises a basic question, deeper than it appears at first sights.

Question 2 *What is reasonable and what is not when talking about computations over the reals?*

This is easy to generate a sinusoid with an R, L, C circuit, and hence a solution to equation $x'' = -p^2 x$. The constant p, directly related with the period of the signal, can be expressed easily in terms of variables R, L, C.

Every student that played with an oscilloscope, have tried to plug on X entry a sinusoid, and on Y entry another sinusoid of distinct period, corresponding to another constant q.

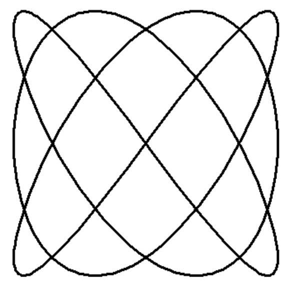

Figure 1.6: A curve obtained for $p/q = 4/3$.

By varying the value of q, any good student knows that what is obtained are called Lissajous's curves: see Figure 1.6.

Theorem 5 (Lissajous's curves)

- *For p/q rational, the curve is stable graphically on the oscilloscope. The value of fraction p/q can be read on the screen from the number of oscillations of x and y on a cycle.*
- *For p/q irrational, the curve is dense on the screen.*

In some way, we have a physical device of computation that is able to determine if the ratio of two values is rational or irrational. Theoretically, this is impossible, for models of computations such a recursive analysis. Is this not astonishing?

Observe that even if one doesn't believe in such a possibility of testing rationality, given p, q, the device returns the reduced expression of fraction p/q, i.e. reduce fractions.

Right, most sceptic people will not like our example of model of computation based on oscilloscopes, and will argue about how one can determine when a curve is dense or not in a screen, or how to read the number of oscillations on x and y when this number is high, ... This said, can they tell me what is reasonable and what is not, when talking formally about this model? And can we characterize what is its power if we restrict to "reasonable" operations, without referring to some external Church-Turing thesis, or a kind of Church-Turing thesis for oscilloscopes.

Well, Ok, let's focus on more realistic things.

1.4.3 Realizing integrations

This is a classical exercise (at least in France) for students to express the output voltage V in terms of the input voltage U in the electronic assembly of Figure 1.7 with an operational amplifier.

We have

$$V(t) = -1/RC \int_0^t U(t)dt,$$

i.e. the system computes an integral. The output tension is the integral of the input tension.

This proves, if needed, that there is no need to discretize to compute an integral. This is indeed possible to realize continuous operations, without using classical digital computers.

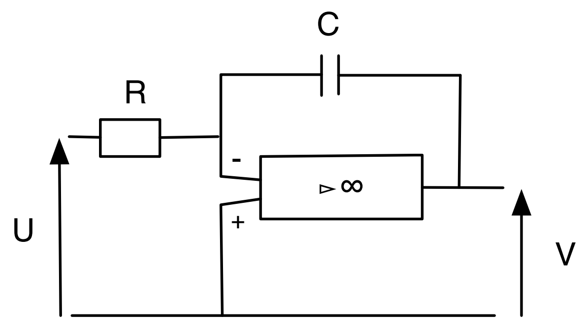

Figure 1.7: An integration realized by an operational amplifier.

1.4.4 A continuous computer

If one knows how to realize integrations with an operational amplifier, one can probably know how to realize additions, and generate constants.

In other words, one can be convinced that this is possible to realize each of the operations of Figure 1.8. For each of the units of this figure, the exercise is to realize an electronic device that constraints output (at right) to be written function of inputs (at left).

One then know how to realize circuits as in Figure 1.9, by connecting different elements, and by allowing feedback connections.

Shannon proved that, what can be read at the output of a unit of such a system corresponds precisely to computable functions by his machine, the General Purpose Analog Computer [Shannon, 1941]. Of course, at that time there were no operational amplifier, nor electronic, but one knew how to realize mechanically all the operations of Figure 1.8, and this was effectively used to build mechanical computers such as the Differential Analyser, built for the first time in 1931 at MIT.

In other words, we have

Theorem 6 ([Shannon, 1941]) *The following three conditions are equivalent*

- *Function f is computable by such an electronic assemby (i.e. can be read as a function of time at the output of a unit of such an assembly)*
- *Function f is GPAC-generable (i.e. computable by the Differential Analyser)*
- *Function f is the projection of a solution of polynomial Cauchy problem.*

The proof of Shannon (corrected by [Pour-El, 1974, Lipshitz and Rubel, 1987, Graça and Costa, 2003]) is completely constructive, and gives the assembly of basic units in terms of the description of the polynomial Cauchy problem.

In other words, the theorem can be read as "we can compute electronically any function that is a (projection of a) solution of a polynomial Cauchy problem, in real time, without using any digital computer. Conversely, no more can be computed using such units."

Related with our remarks about the fact that usual functions are projections of solutions of polynomial Cauchy problems, except some few examples that look like diagonalization, aren't we are saying that GPAC captures the good model of computation over the reals?

Question 3 *Is there a Church-Turing thesis for continuous time computations?*

People from recursive analysis could say no, since function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is computable in the sense of recursive analysis, but is not GPAC computable. But this is two distinct notions of computation: in the recursive analysis model, one talks about computations at the limit, whereas, in the GPAC one talks about real time generation of functions. This leads to the following question, less ambitious, about which we will come back.

Question 4 *What is the exact power of GPAC, if one does not restrict to real time generation? Is GPAC really less powerful than Turing machines?*

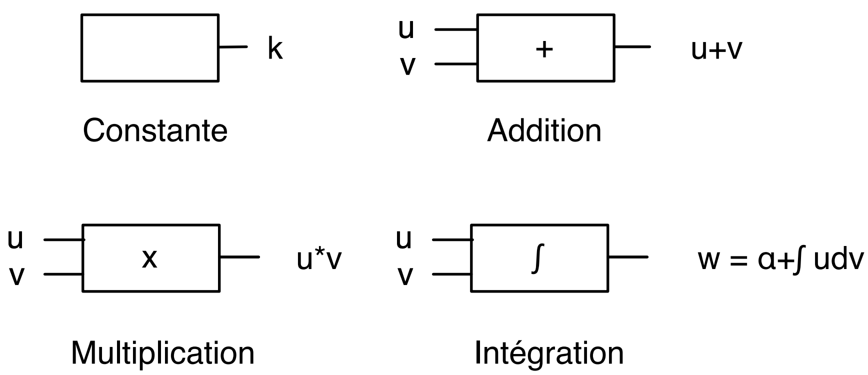

Figure 1.8: The basic units of a GPAC (the output w of an integration operator satisfies $w'(t) = u(t)v'(t)$, $w(t_0) = \alpha$ for some initial condition α).

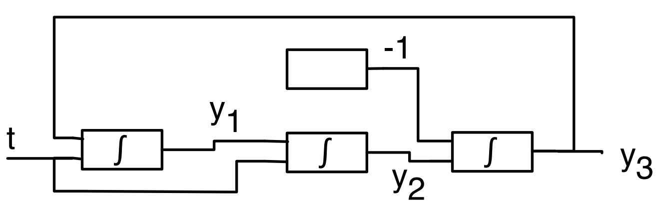

Figure 1.9: Generating cos and sin by a GPAC. In form of a system of equations, we have $y_1' = y_3$, $y_1(0) = 1$, $y_2' = y_1$, $y_2(0) = 0$, $y_3' = -y_1$, $y_3(0) = 0$. It follows that $y_1 = \cos$, $y_2 = \sin$, $y_3 = -\sin$.

1.5 Some Remarkable Dynamical Systems

Let's stop for some pages to see in all dynamical systems some computational models, and let's discuss several remarkable dynamical systems from literature.

Our discussion aims only at showing the richness of the possible behaviors of dynamical systems, by discussing sometimes-related discretization problems. This discussion will continue in the next chapter, with models with some notions of concurrency, or competitions between agents.

1.5.1 In Meteorology

The most famous chaotic dynamical system is without contest the system formulated by Lorenz in 1963 as a (over simplified) model of atmospheric convection.

The Lorenz system of [Lorenz, 1963] can be written

$$\begin{cases} x' &= \sigma(y-x) \\ y' &= \rho x - y - xz \\ z' &= xz - bz \end{cases}$$

where σ, ρ et b are three real parameters.

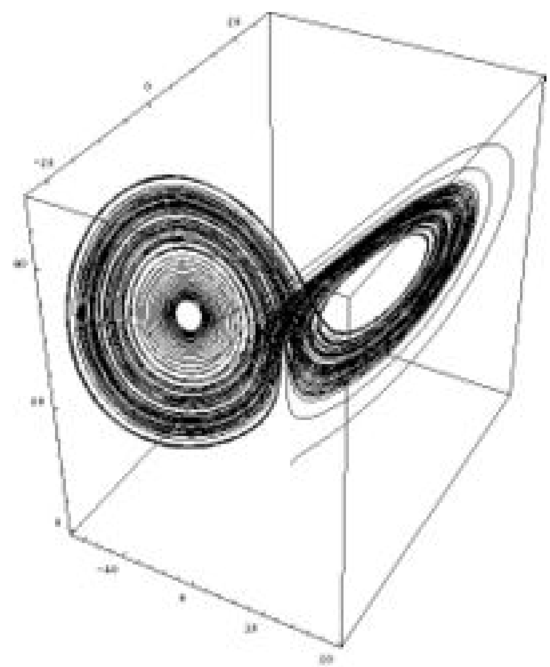

Figure 1.10: A dynamic of Lorenz system (picture from the Wikipedia common image database).

For $\sigma = 10$, $\rho = 28$, $b = 8/3$, the system exhibits what is called a strange attractor. Before this model becomes of research interest for scientists, the only known attractors for dynamical systems where fixed points and closed orbits [Hirsch et al., 2003]. Trajectories converge toward the attractor, however two arbitrary close initial conditions ultimately differ strongly in they way they converge toward it. Similar phenomena have been exhibited for several other dynamics, like the Rosler attractor.

Refer to [Hirsch et al., 2003], for an introduction. We will observe that to study this continuous dynamical system, the authors of [Hirsch et al., 2003], replace it by a discrete dynamic, actually by a hybrid dynamic. This shows that discrete systems, and in particular hybrid systems, are pertinent to analyze continuous systems.

Observe that the system is of dimension 3. Chaotic phenomena of same type in dimension 2 are not possible because of Poincaré-Bendixon theorem.

1.5.2 In Chemistry

Lotka-Volterra's Model

Lotka observed in [Lotka, 1920] that the following set of coupled reactions have remarkable dynamical properties.

$$\begin{cases} \overline{X} + Y_1 & \rightarrow & 2Y_1 \\ Y_1 + Y_2 & \rightarrow & 2Y_2 \\ Y_3 & \rightarrow & Z \end{cases}$$

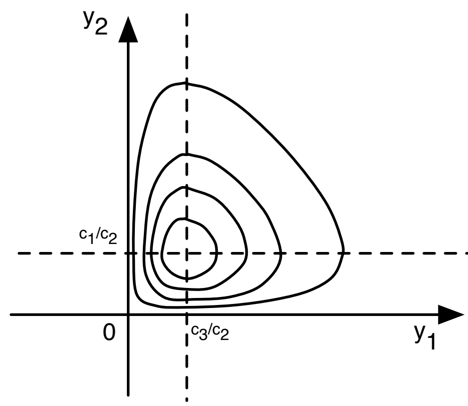

Figure 1.11: Dynamics of Lotka-Volterra's models

Put in equations, the dynamic is simply

$$\begin{cases} y_1' & = & c_1 x y - c_2 y_1 y_2 \\ y_2' & = & c_2 y_1 y_2 - c_3 y_2 \end{cases}$$

where c_1, c_2, c_3 are kinetic constants of the reactions.

This dynamic coincides with a dynamic proposed by Volterra in 1925 [Volterra, 1931] as a simple model of prey predator models, about which we will come back.

Theorem 7 *The system has a stable point $y_1 = c_3/c_2$, $y_2 = c_1/c_2$. All trajectories starting from another point that this stable point form close curves, letting invariant the quantity $H(u,v) = \alpha(u - \log u) + (v - \log v)$, where $u = c_2 y_1/c_3$, $v = c_2 y_2/c_1$, $\alpha = c_3/c_2$. In other words, the trajectories are closed level curves of function H: see Figure 1.11*

This system is instructive for several reasons. First, it shows in a theoretical way that some chemical mechanisms can exhibit oscillations. However, the model is purely formal, without link with any true system, and to be right, before the reaction discovered by Belousov, discussed later on, most of the chemists thought this was impossible to observe physically oscillations, because of the laws of thermodynamic.

Mostly, this is an example that is well discussed in [Krivine et al., 2006], about problems linked with discretizations.

Indeed, [Krivine et al., 2006] shows that any Euler's explicit schema of simulation of the dynamic does not preserve function H, and hence that oscillations diverge in practice in any numerical simulation of by an Euler's method. This is possible to build an implicit Euler's method that works [Krivine et al., 2006], but this shows once more than discretizing a continuous system is often problematic, and can lead to systems that do not simulate in a correct way the underlying system.

Belousov-Zhabothinksy's Reaction

The most famous oscillating chemical reaction is without contest the reaction discovered by Belousov in 1950.

Belousov was focusing on Krebs's cycle, and in particular in the role of citric acid in this cycle. The Cycle of Krebs is a complex biochemical mechanism that takes place in the metabolism of sugar such as glucose. Belousov wanted to understand the role of citric acid in this cycle. He tried to proportion thanks to bromate of potassium, but the reaction was too slow, and he added a catalyser. But he discovered that the color of the obtained mixture was changing periodically. He then started to study it more deeply by varying pH of the solution and by adding a colored indicator. However, all its attempts of publications of his discovery were rejected by referees, arguing with thermodynamic that such oscillations in chemistry are impossible. He only succeeded to publish a note in the rather unknown journal *Sbornik Referatov po Radiaconi Medecine*, which is reprinted in [Field and Burger, 1985].

In 1961, Anatol Zhabotinsky, who was a student in biophysics at Moscow University, devoted its PhD to a thorough study of Belousov's reaction [Zaikin and Zhabotinsky, 1970]. Following suggestion of professor S. E. Schnoll, he replaced citric acid by malonic acid, and got a system in which the amplitude of oscillation is even higher than in the original system.

During several years, Belousov-Zhabotinsky's reaction was a laboratory curiosity, rather unknown, in particular because of the political context of that time, before attracting attention [Dupuis and Berland, 2004].

Several oscillating reactions are now known. Refer to [Murray, 2002, Field and Burger, 1985] for more discussions.

1.5.3 In Physic

Van der Pol's equation

Consider a R, L, C circuit as in the figure below, where R is not a perfect resistance: a perfect resistance would introduce a linear relation between the voltage at its extremities and the intensity that crosses it. Suppose that the relation is actually some function $V = f(i)$.

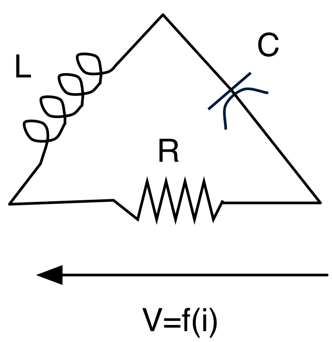

Figure 1.12: R,L,C Circuit

Choosing $L = C = 1$, x the intensity that crosses the solenoid, y the voltage between extremities of condenser, the system must satisfy

$$\begin{cases} x' &= y - f(x) \\ y' &= -x. \end{cases}$$

This is called *Lienard's equation*.

In the case $f(x) = x^3 - x$, one gets *van der Pol's equation*.

$$x' = y - x^3 + x$$

The following result is a nice and classical exercise of mathematics. The result is nice, since the proof does not rely in exhibiting the periodic solution, but in using topological arguments (fixed points) on a Poincaré section of the system.

Theorem 8 *There is a non-trivial periodic solution to the van der Pol's equation, and any other solution (except the instable equilibrium point at origin) converges towards this solution. In other terms, the system oscillates.*

If we consider

$$\begin{cases} x' &= y - f_\mu(x) \\ y' &= -x \end{cases}$$

with $f_\mu(x) = x^3 - x$, $\mu \in [-1, 1]$, we find again the van der Pol's system for $\mu = 1$. A Hopf's bifurcation can be observed: for $\mu < 0$, all solutions converge towards origin: the system is inert. When μ becomes positive, the periodic solution appears, in some sense life appears, and the system starts to oscillate.

The n-body problem

The n-body problem consists in solving the equation of motion of n bodies in gravitational interaction, knowing their mass, positions, and initial velocities.

All non-amnesic student knows that n-body problem is completely analytically solvable, using Kepler's laws, for $n = 2$.

In opposition to a common belief, 3-body problem has an exact analytic solution, discovered by Sundman in 1909 [Henkel, 2001]. Unfortunately, this solution is in the form of an infinite series whose convergence is very slow, and hence is useless in practice.

For the case $n > 2$, with the exception of very specific cases where an exact solution is known, approximate numerical resolution methods are used.

Following [Krivine et al., 2006], observe that any discretization of the schema of motion of the 2-body problem by an explicit Euler's method does not simulate correctly the motion of the two bodies. Indeed, these schemas doesn't conserve the energy of motion.

As observed in [Coullet et al., 2004], [Krivine et al., 2006], for the anecdote, observe that Newton in its *Principia* in 1687, who didn't know differential calculus invented later on, did a discrete reasoning to establish the theory of universal attraction, using an implicit discretization schema, from Robert Hooke. It is remarkable that its schema has the property of conserving energy [Coullet et al., 2004], [Krivine et al., 2006], whereas an explicit discretization Euler's explicit schema would not have allowed a so simple and elegant reasoning.

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