Chapter 1

On matrix mortality in low dimensions

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1.1 Description of the problem

A set $F = \{A_1, \ldots, A_m\}$ of $n \times n$ matrices is said to be *mortal* if there exist integers $k \geq 1$ and $i_1, i_2, \ldots, i_k \in \{1, \ldots, m\}$ such that $A_{i_1} A_{i_2} \cdots A_{i_k} = 0$. In that case F is also said to be *k-length mortal*.

We use MORTALITY(n) to denote the class of decision problems "Is a given set F consisting of $n \times n$ matrices mortal?"and MORTALITY(n, m) to denote "Is a given set F of m $n \times n$ matrices mortal?". We also use PAIR-MORTALITY(n) as a synonym for MORTALITY($n, 2$). Unless otherwise noted, all matrices are assumed to have integer-valued entries. But MORTALITY($n, m; \mathbb{R}$), for example, denotes the third problem class for matrices with real-valued entries.

Evidently, MORTALITY(1) and MORTALITY($n, 1$) are efficiently decidable. However, the general complexity of MORTALITY(2) and

Pair-Mortality(n), $n < 27$, remains unknown—despite a lot of interest (see [5, 6], which contain some related results, and the references therein).

1.2 Motivation

Such problems arise as follows:

1. *Controllability of switched linear systems.* Given a system of the form $x(t+1) = A(t,u)x(t)$, where for all t the set of possible values of $A(t,u)$ is a finite set F, the questions above correspond to the controllability (to the origin) of such a system. Cf. [2].
2. Mortality(2) is also equivalent to the following problem [8]: Find an algorithm which, given a finite set H of non-singular linear transformations of the complex plane, and lines L and M through the origin, determines whether some product from H maps L onto M.

1.3 Available results

1. Mortality(3) is recursively unsolvable [7]: the proof relies on a reduction of this problem to the Post Correspondence Problem (PCP). It is constructive, using $2p + 2$ matrices if PCP is undecidable with p "rules." By considering Modified PCP it is possible to prove undecidability using only $p + 2$ matrices [3]. Current bounds on p lie in $\{3, \ldots, 7\}$ (see [1, p. 12] for references and a discussion).
2. Mortality and pair-mortality can be related: if Mortality(n, m) is undecidable, then Pair-Mortality(nm) is undecidable [1, 4].
3. Pair-Mortality(2) is decidable [3, 4]. However, the proof uses elementary number theoretic arguments for matrices with complex eigenvalues that do not generalize to matrices with real entries: Pair-Mortality($2; \mathbb{R}$) has been proved BSS-undecidable [3], yielding Mortality(n, m) BSS-undecidable for all $n \geq 2$, $m \geq 2$. Nevertheless, Pair-Mortality($2; \mathbb{R}$) is BSS-decidable for matrices with real eigenvalues [3].
4. Pair-Mortality(n) is decidable and NP-complete when restricted to matrices with non-negative entries [1]. The same argument can be used to show that Mortality(n, m) restricted to non-negative matrices is decidable. The problem of deciding whether a given pair of $n \times n$ matrices is k-length mortal, with integer k encoded in unary, is NP-complete; it remains so when the matrices are restricted to have entries in $\{0, 1\}$ [1]. The conclusion of NP-completeness in [1] can be more easily obtained using Paterson's construction and reduction to Bounded PCP [3]. The boolean entry case does then not follow, but NP-completeness of "Given a set F of 3×3 matrices and positive integer $K \leq |F|$, is F k-mortal for some $k \leq K$?" does.

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