Algorithmic Information Theory and Foundations of probability

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Acknowledgements and apology

Natural Science: a simplistic view

Theoretician

Experimenter

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predictions (theory)	observations (data)

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match or not

Probability theory

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OK, but this is the case for *any* outcome!

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- Shuffling factory: quality control that blocks decks that are not well shuffled.
- But what does it mean? All orderings are equiprobable

Two practical questions

- How do we use probabilistic hypothesis in practice?
- How do we select a plausible probabilistic hypothesis?

Cournot principle

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E. Borel: …Fewer than a million people live in Paris. Newspapers daily inform us about strange events or accidents that happen to some of them. Our life would be impossible if we were afraid of all adventures we read about. So one can say that from a practical viewpoint one can ignore with probability less than one over million… Often trying to avoid something bad we are confronted with even worse…

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- Mathematician: if a very unprobable simple event has happened

Frequency interpretation

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Bernoulli distribution B_p on n-bit sequences

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Bernoulli distribution B_p on n-bit sequences
Event: $|\text{frequency} - p| > \varepsilon$ has small probability (and is simple)

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- for most strings of length n the complexity is close to n: it is less than $n - d$ for 2^{-d}-fraction only

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- Kolmogorov complexity does not take into account resources used by the program that generates x

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for non-uniform distribution:
$d(x) = \log_2 P(x) - K(x)$

An incompressibility paradox

we do not think that fair coin never produces (or less frequently produces) compressible sequences, but they discredit the fairness hypothesis (unlike others)

⟨...⟩ the very Calculus of Probabilities to which I have referred, forbids all idea of the extension of the parallel ⟨...⟩ This is one of those anomalous propositions which, seemingly appealing to thought altogether apart from the mathematical, is yet one which only the mathematician can fully entertain. Nothing, for example, is more difficult than to convince the merely general reader that the fact of sixes having been thrown twice in succession by a player at dice, is sufficient cause for betting the largest odds that sixes will not be thrown in the third attempt. A suggestion to this effect is usually rejected by the intellect at once. It does not appear that the two throws which have been completed, and which lie now absolutely in the Past, can have influence upon the throw which exists only in the Future. The chance for throwing sixes seems to be precisely as it was at any ordinary time—that is to say, subject only to the influence of the various other throws which may be made by the dice. And this is a reflection which appears so exceedingly obvious that attempts to controvert it are received more frequently with a derisive smile than with any thing like respectful attention. The error here involved — a gross error redolent of mischief — I cannot pretend to expose within the limits assigned me at present. (Edgar Poe)

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Less philosophical version:

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Less philosophical version:

Imagine we have a dice (nonsymmetric), know the position of its center of gravity, and have unlimited computation power. Can we compute probabilities of different outcomes using mechanical laws?

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Theoretically they can be computed

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what happens: initial condition is revealed bit by bit

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dynamical laws + one more: the world was created in an incompressible state

this law together with mixing property implies that outcomes for a fair coin form an incompressible sequence

Pseudorandom number generators (Yao–Micali)

$$G : \mathbb{B}^{1000} \longrightarrow \mathbb{B}^{1000000}$$

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"things seem random because we do not know they are not"

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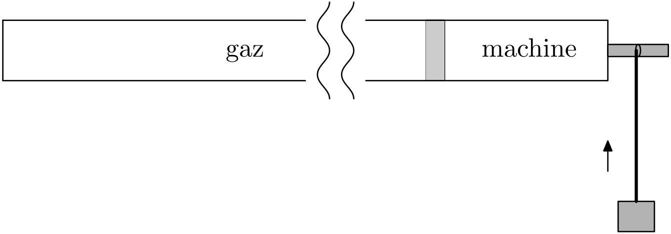
- the first one cannot be a corollary of dynamic laws since it is not time-symmetric

Perpetuum mobile of the second kind



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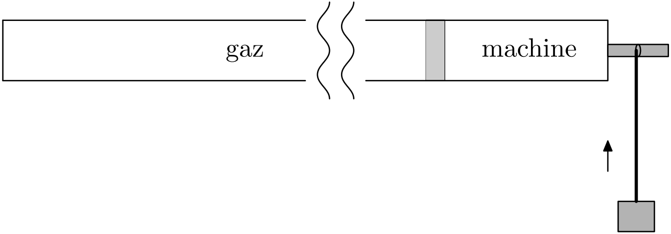

moves the weight arbitrary high if the reservoir is large enough (for most states of the gaz in the reservoir)

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Large set cannot be mapped into a small one

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q-Cournot principle: the events with negligible amplitude do not happen