Analog characterization of complexity classes

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Introduction

- GPAC model by Shannon, 1941 [Sha41]
- Corresponds to solutions of polynomial differential equations [GC03]
- Implementations by means of differential analyzer [Bus31]

Analog Time Space Computable functions

- Comparison with other analog models of computation, computable analysis [BHW08]
- Problem of real time computation, introducing limit computation [BGP17b]

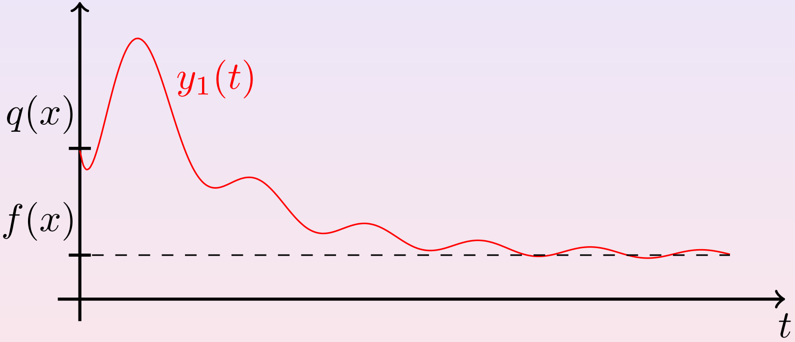
Definition 1 (ATSP)

Let $f \subseteq \mathbb{R}^n \to \mathbb{R}^m$. We say that $f \in$ATSP if and only if there exist $d \in \mathbb{N}$, $p \in \mathbb{R}^d_G[\mathbb{R}^d]$, $q \in \mathbb{R}^d_G[\mathbb{R}^n]$ and polynomials $\Pi : \mathbb{R}^2_+ \to \mathbb{R}_+$ and $\Upsilon : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that for any $x \in \operatorname{dom} f$, there exists (a unique) $y : \mathbb{R}_+ \to \mathbb{R}^d$ satisfying for all $t \in \mathbb{R}_+$:

- $y(0) = q(x)$ and $y'(t) = p(y(t))$
- $\forall \mu \in \mathbb{R}_+$ if $t \geq \Pi(\|x\|, \mu)$ then $\|y_{1..m}(t) - f(x)\| \leq e^{-\mu}$
- $\|y(t)\| \leq \Upsilon(\|x\|, t)$

Graphical example

function $f : \mathbb{R} \to \mathbb{R} \in \text{ATSP}$



$q(x)$: initial condition dependent on the input x
$y_1 : \mathbb{R} \to \mathbb{R}$ solution of the dynamical system starting from $q(x)$

Emulation and equivalences

- ATSP Class is equivalent to other formulations as showed in [BGP17a]; the AOP class, or Analog Online Polynomial Computable
- Equivalence at a computability level with computable analysis [BCGH06]
- Equivalence at a complexity level with functions computable in polynomial time, FP [BGP17b]
- Equivalence at a complexity level by means of emulation of Turing machines and real encodings of configurations

Let Γ be an alphabet and γ an injective mapping $\gamma : \Gamma \to \mathbb{N} \setminus \{0\}$, which applies letter wise over words.

Definition 2 (Discrete emulation)

Let G be a set of functions over $\mathbb{R}^2$ and let $k = 2 + \max(\gamma(\Gamma))$. The function $f : \Gamma^* \to \Gamma^*$ is called emulable under G if there exists $g \in G$ such that for any word $w \in \Gamma^*$:

$$g(\Psi_k(w)) = \Psi_k(f(w))$$

where:

$$\Psi_k(w) = \left(\sum_{i=1}^{|w|} \gamma(w_i) k^{-i}, |w| \right) \tag{1}$$

Theorem 3 (FP equivalence [BGP17b])

Let $f : \Gamma^* \to \Gamma^*$, then $f \in \mathrm{FP}$ if and only if f is emulable under ATSP.

Exponential version

Question

Is it possible to consider exponential boundaries in the ATSP definition in order to obtain a characterization of FEXPTIME?

Problems

- Missing equivalence with other classes such as AOP
- Continuous simulation of Turing Machine tailored over properties of polynomials, such as closure by composition
- unreasonable and unnatural definition of the exponential class

Example

Let Π and Υ of definition 1 be exponentials. Then given some $f \in \text{ATSE}$, to compute $f(x)$ with accuracy $e^{-\mu}$, we would need to wait a time $t^* = \Pi(\|x\|, \mu)$ exponential in $\|x\|$ and μ, due to the second condition of definition 1. Moreover, due to the third condition, at time t^*, we would have that the norm of the solution y of the ODE computing $f(x)$ is bounded by $\Upsilon(\|x\|, t^*) = \Upsilon(\|x\|, \Pi(\|x\|, \mu))$, which is a double exponential in both $\|x\|$ and μ, while what would be natural is that $\|y(t^*)\|$ is bounded by an exponential in $\|x\|$ and μ, and not a double exponential in these parameters.

Composition problem

We want our exponential class not to be closed by composition, since we want it to include exponential but not double exponentials. At the same time we need some sort of composition to be valid for the class in order to repeat the construction of [BGP17b] that leads to the equivalence result.

Our solution

Split dependence of Π and Υ in two separate factors, $\Pi_1\Pi_2$ and $\Upsilon_1\Upsilon_2$, only raising Π_1 and Υ_1 to exponentials but leaving Π_2 and Υ_2 polynomials.

Definition 4 (ATSE)

Let $f \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Let $\Pi_1, \Upsilon_1 : \mathbb{R}_+ \to \mathbb{R}_+$ be two exponentials and let $\Pi_2, \Upsilon_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two polynomials.
We say that $f \in \mathrm{ATSE}$ iff there are $d \in \mathbb{N}$, $p \in \mathbb{R}^d_G[\mathbb{R}^d]$, $q \in \mathbb{R}^d_G[\mathbb{R}^n]$ s.t. for any $x \in \operatorname{dom} f$, there is (a unique) $y : \mathbb{R}_+ \to \mathbb{R}^d$ satisfying $\forall t \in \mathbb{R}_+ :$

- $y(0) = q(x)$ and $y'(t) = p(y(t))$
- $\forall \mu \in \mathbb{R}_+$ if $t \geq \Pi_1(\|x\|)\Pi_2(\mu)$ then $\|y_{1..m}(t) - f(x)\| \leq e^{-\mu}$
- $\|y(t)\| \leq \Upsilon_1(\|x\|)\Upsilon_2(t)$

Theorem 5 (Composition of ATSE and ATSP)

Let f be a function, $f \in \mathrm{ATSE}$ and g be a function, $g \in \mathrm{ATSP}$. Let $f(\operatorname{dom} f) \subseteq \operatorname{dom} g$. Then $g \circ f \in \mathrm{ATSE}$.

Exponential result

Theorem 6 (FEXPTIME equivalence)

Let $f : \Gamma^* \to \Gamma^*$, *then* $f \in$ FEXPTIME *if and only if* f *is emulable under* ATSE.

- A similar theorem is obtained from the above to characterize the class of sets EXPTIME following the example of what was done for P in [BGP17b]
- the result is obtained by abandoning the idea of convergence of the solution $y(t)$ and instead requiring stability of $y(t) \geq 1$ for accepted inputs and of $y(t) \leq -1$ for rejected inputs once the solution of the system has reached exponential length

Generalization

Let A be a class of functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. We call $\mathrm{ATSC}(A)$ the analog class obtained from ATSE with $\Pi_1, \Upsilon_1 \in A$

Definition 7 (Sufficient conditions)

(1) if $f, g \in A$ then there exists $h \in A$ such that $f \star g(x) \leq h(x)$ for every $x \in \mathbb{R}_+$, where $\star$ denotes any operator in the list of arithmetical operations: $(+, -, \times)$

(2) if p is polynomial and $f \in A$ then there exists $g \in A$ such that $p \circ f(x) \leq g(x)$ and $f \circ p(x) \leq g(x)$ for every $x \in \mathbb{R}_+$. Moreover, the identity operator belongs to A

(3) if $f : \mathbb{N} \to \mathbb{N}$, $f \in A$, then there exists $g \in A$ such that $f(n) \leq g(n)$ for every $n \in \mathbb{N}$ and $g \in \mathrm{ATSC}(A)$

(4) if $f \in A$ then there exists $g : \mathbb{N} \to \mathbb{N}$, $g \in A$ such that $f(n) \leq g(n)$ for every $n \in \mathbb{N}$ and g is a time-constructible function.

Grzegorczyk Hierarchy, elementary functions, PR functions

FPSPACE

Question

Is it possible to adapt the same treatment for describing space complexity classes?

Problems

- The encoding in (1) is not practical for the length of the tape
- The convergence of $y(t)$ in definition 4 begins after exponential time t^*; reverse direction of the equivalence needs to compute t^*

Different encoding

Definition 8

$\hat{\Psi}(x) = w$ if $x \in \mathbb{R}$ is such that $d(x, \sum_{i=1}^{n} \gamma(w_i)k^{i-1}) < 1/2$, where $d(x, y) = \|x - y\|$, in the papers [GCB08], [BCGH06], [BCGH07];

- previous encoding in (1) can be expressed as: $\bar{\Psi}(x, y) = w$ if $(x, y) \in \mathbb{R}^2$ is s.t. $(x, y) = \left(\sum_{i=1}^{|w|} w_i k^{-i}, |w|\right)$
- By means of $\hat{\Psi}(x)$ we keep the connection with the length of the input without needing two arguments
- New encoding: many different $x \in \mathbb{R}$ encode the same word $w \in \Gamma^*$
- The encoding $\hat{\Psi}(x)$ is more robust to small perturbations.

- The norm of the encoded value x s.t. $\hat{\Psi}(x) = w$ is exponential on $|w|$, while before it was linear: $\|(x, y)\|$ s.t. $\bar{\Psi}(x, y) = w$ equals $|w|$.
- the function $f(x) = x$ is a $\bar{\Psi}$-bound, while the function $g(x) = 2^{x+1}$ is a $\hat{\Psi}$-bound
- The way the emulation of the Turing machine is done has to be different from the one used until now

Definition 9 (Encoding bounds)

Let $\Psi : D \subseteq \mathbb{R}^k \to \Gamma^*$ be an encoding and let $g : \mathbb{N} \to \mathbb{N}$. We say that a function $\phi : \mathbb{R} \to \mathbb{R}$ is a Ψ-bound if for all $v, w \in \Gamma^*$, $|v| \leq |w|$ implies that $\|x\| \leq \phi(\|y\|)$ whenever $\Psi(x) = v$ and $\Psi(y) = w$.

Space emulation I

Definition 10

Let $f : \Gamma^* \to \Gamma^*$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ be two functions. We say that f is (Ψ-)emulable in space g by an ODE

$$\begin{cases} y' = p(y, z) \\ z' = q(y, z) \end{cases} \tag{2}$$

where p, q are functions formed by polynomial components, if there are two (vector-valued) function $r, s \in \mathrm{GPVAL}$, and $\varepsilon > 0$, $\tau \geq \alpha > 0$, $j, l, k \in \mathbb{N}$ with $0 < j \leq l$ such that, for all $w \in \mathrm{dom}(f) \subseteq \Gamma^*$ and for any Ψ-bound ϕ one has that the solution of the IVP formed by (2) and the initial condition $y(0) = r(x)$, $z(0) = s(y(0)) = s \circ r(x)$, where $\Psi(x) = w$, satisfies:

Definition 10 (cont.)

Space emulation II

1. **(halting 1)** If $t_0 > 0$ is s.t. $y_1(t_0) \geq 1$, then $y_1(t) \geq 1\ \forall t \geq t_0$ and $y_1(t) \geq 3/2\ \forall t \geq t_0 + 1$;
2. (**halting 2**) There is some $t_0 \geq 0$ such that $y_1(t_0) \geq 1$;
3. (**correct output**) If $y_1(t) \geq 1$, then $\Psi(y_2(t), \ldots, y_j(t)) = f(w)$;
4. (**bound**) $\|(y(t), z(t))\| \leq \phi \circ g(x)$, $\forall t \geq 0$, $\forall x$ s.t. $\Psi(x) = w$;
5. (**robustness 1**) For any $\bar{t}_0 \geq 0$, if $z_1(\bar{t}_0) \geq 1$ and $\bar{y}_0$ is s.t. $\|\bar{y}_0 - y(\bar{t}_0)\| \leq \varepsilon$, then solution $(\bar{y}, \bar{z})$ of (2) with $\bar{y}(0) = \bar{y}_0$ and $\bar{z}(0) = s(\bar{y}_0)$ is s.t. 1–4 above hold;
6. (**robustness 2**) For any $b > a \geq 0$ s.t $|b - a| \geq \tau$, there $I = [c, d] \subseteq [a, b]$, with $|d - c| \geq \alpha$, s.t. $z_1(t) \geq 3/2\ \forall t \in I$;
7. (**robustness 3**) If $(\tilde{y}, \tilde{z})$ is a solution of (2) with $\tilde{y}(0) = \tilde{y}_0$, $\tilde{z}(0) = s(\tilde{y}_0)$ s. t. $\|\tilde{y}_0 - y(\bar{t}_0)\| \leq \varepsilon$ as in condition 5, then we can take $\tilde{y}$ in place of y in conditions 5 and 6 if $\bar{t}_0 \geq \tau$. There will always be some t_0, counted from $t = 0$ in (2), s.t. condition 1 holds.

Our result

Theorem 11

Let $f : \Gamma^* \to \Gamma^*$ be a function. Then $f \in \mathrm{FPSPACE}$ iff f is $\hat{\Psi}$-emulable in polynomial space by a polynomial ODE.

- Direct direction of the proof: emulation of Turing machine by system like Branicky's [Bra95],[GCB08] that keeps the solution bounded
- Reverse direction by computing ODE solution (FPSPACE algorithm) and restart thanks to auxiliary variables.
- It follows the same characterization for the class of sets PSPACE

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Thank you!