Differential equations in proof-theory: an introduction through linear logic

Marie Kerjean

CNRS & LIPN, Université Paris Sorbonne Nord
kerjean@lipn.fr

ANR ∂ifférence Kick-Off meeting, December 2020

- I know nothing about complexity theory, very little about computer algebra or discret ODEs.
- But : I have used differential operators as a logical connective.
- I will sketch the use of differential operators in proof theory, through linear logic, and speak of possible applications for programming languages.

Teaser

We know how to speak about differentiation in proof theory:

$$\text{usual-proof} \rightsquigarrow \text{linear proof}$$

We have a specific connective for linear implication: $A \multimap B$.

$A \multimap B$ represents exactly the type of programs characterized by differentiation.

- We just need to find a sensible way to work generalize $\multimap$ to general differential operators.

Proof theory and Type theory

Programmes

Terme $P = \lambda x^A . y^B$

Type A

Évaluation
$Pt \rightsquigarrow_\beta t'$

Logique

$$\dfrac{\begin{matrix}[\pi] \\ \vdots\end{matrix}}{A \vdash B}\ \text{Preuve}$$

Formula: A,
$A \Rightarrow B$, $\neg A$

Cut-Elimination

A functional program has type $A \Rightarrow B$.
Proofs are constructed via inference rules:

$$\dfrac{A \vdash B \qquad B \vdash C}{A \vdash C}\ \text{cut} \qquad\qquad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

Linear logic

Usual implication — Linear Implication

Linear Logic

$$A \Rightarrow B = !\, A \multimap B$$

Exponential

A proof is linear when it uses only once its hypothesis A.

A linear proof is in particular non-linear.

$$\frac{A \vdash B \text{ is linear}}{!A \vdash B \text{ is non-linear}}\ \text{dereliction}$$

Applications in programming languages, complexity theory, quantum computing, category theory ...

Working tool: semantics

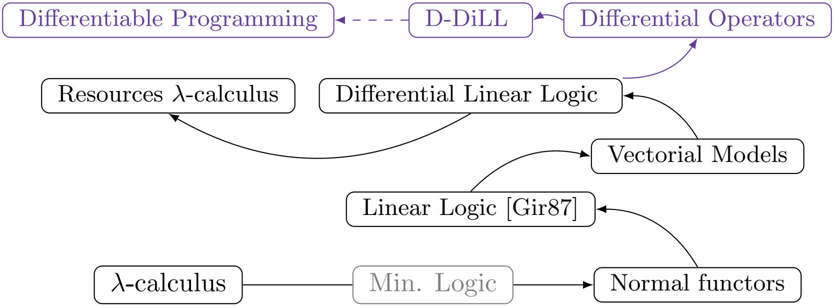
The syntax mirrors the semantics.

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Types	Formulas	Objects
Execution	Cut-elimination	Equality

Working tool: semantics

The syntax mirrors the semantics.

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Types	Formulas	Objects
Execution	Cut-elimination	Equality



Differential Linear Logic

[Ehrhard Regnier ∼ 2005]

Differential Linear Logic

new rules for constructing proofs

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}$$

linear $\hookrightarrow$ *non-linear.*

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}$$

non-linear $\hookrightarrow$ *linear*

Differential Linear Logic

[Ehrhard Regnier ∼ 2005]

Differential Linear Logic

new rules for constructing proofs

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}$$

linear $\hookrightarrow$ *non-linear.*

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}$$

non-linear $\hookrightarrow$ *linear*

Cut-elimination:

$$\dfrac{\dfrac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\ \bar{d} \qquad \dfrac{\vdash \Delta, \ell : A^{\perp}}{\vdash \Delta, \ell : ?A^{\perp}}\ d}{\Gamma, \Delta}\ \text{cut}$$

$$\rightsquigarrow \qquad \dfrac{\vdash \Gamma, x : A \qquad \vdash \Delta, \ell : A^{\perp}}{\Gamma, \Delta, D_0(\ell)(x) = \ell(x) : \mathbb{R} = \perp}\ \text{cut}$$

The dynamic of proofs/programs computes the differential

Solving differential equations in proofs

Differential Linear Logic

new rules for constructing proofs

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}$$

linear $\hookrightarrow$ non-linear.

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}$$

non-linear $\hookrightarrow$ linear

From differentiation to differential operators

$$\frac{g : !_D A \vdash B}{g * E_D : !A \vdash B}$$

parameter $\hookrightarrow$ solution.

$$\frac{f : !A \vdash B}{D(f) : !_D A \vdash B}$$

differentiation $\hookrightarrow$ Diff. operator.

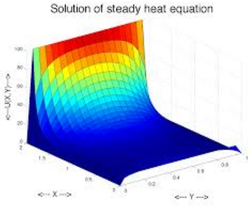
The dynamic of proofs/programs computes the solution to a differential equations

Linear Partial Differential Equations with constant coefficient

Consider D a LPDO with constant coefficients:

$$D = \sum_{\alpha, |\alpha| \leq n} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}.$$

The heat equation in $\mathbb{R}^2$
$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$
$u(x, y, 0) = f(x, y)$



Theorem (Malgrange 1956)

For any D LPDOcc, there is $E_D \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R})'$ such that $DE_D = \delta_0$, and thus for any $\phi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$:

$$D(E_D * \phi) = \phi$$

Linear Partial Differential Equations with constant coefficient

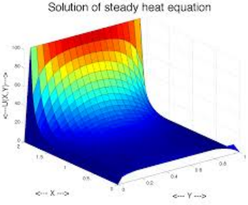
Consider D a LPDO with constant coefficients:

$$D = \sum_{\alpha, |\alpha| \leq n} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}.$$

The heat equation in $\mathbb{R}^2$

$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

$u(x, y, 0) = f(x, y)$



Theorem (Malgrange 1956)

For any D LPDOcc, there is $E_D \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R})'$ such that $DE_D = \delta_0$, and thus for any $\phi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$:

$$\text{output } D(E_D * \phi) = \phi \text{ input}$$

All these development on proofs and their *syntax* is justified by their *semantics*: their mathematical interpretation

Category theory often serves as an *intermediate layer* between proofs and their interpretation as functions between mathematical objects.

Interpreting proofs in a certain category

- A formula A: an object (e.g a vector space) $[A]$.
- A proof of $A \vdash B$: a function from A to B.
- Cut-elimination = composition of functions.

One main law needs to be interpreted: currying.

$$A \wedge B \Rightarrow C \equiv A \Rightarrow B \Rightarrow C$$

Monoïdal closedeness (linear version):

$$\mathcal{L}(A \otimes B, C) \simeq \mathcal{L}(A, \mathcal{L}(B, C))$$

Cartesian closedeness (non-linear version):

$$\mathcal{C}^\infty(A \times B, C) \simeq \mathcal{C}^\infty(A, \mathcal{C}^\infty(B, C))$$

Also: we're linear classical. $\neg\neg A \equiv A$ $A'' \simeq A$

Interpreting in a real mathematical structure

Combining notions as : *topological tensor products*, *higher-order smooth functions*, *Mackey/Weak/quasi complete spaces*, *reflexive spaces* ...

Historically: discrete models and *quantitative semantics* : $!A := \sum_n A^{\otimes^n}$

Exponentials as distributions [LICS2018]

A *smooth* and classical model of Differential Linear Logic where:

$$!A = \mathcal{C}^\infty(A, \mathbb{R})'.$$

Result: functional programming $\Leftrightarrow$ functional analysis.

Currying for smooth functions	*Kernel theorems*
$!A \otimes !B \simeq !(A \times B)$	$\mathcal{C}^\infty(E,\mathbb{R})' \hat{\otimes} \mathcal{C}^\infty(F,\mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$
Currying for linear functions	Grothendieck's topological problem

Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int fg.$$

Théorie des distributions, Schwartz, 1947.

- In a classical and Smooth model of Differential Linear Logic, the exponential is a space of distributions with compact support.

$$!A \multimap \bot = A \Rightarrow \bot$$
$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$
$$(!E)'' \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$
$$!E \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

- Seely's isomorphism corresponds to the *Kernel theorem*:

$$\mathcal{C}^\infty(E, \mathbb{R})' \tilde{\otimes} \mathcal{C}^\infty(F, \mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$$

Exponentials as spaces of solutions

Hypothesis: a *classical* logic interpreted by reflexive spaces: $A \simeq A''$.

$$
\begin{aligned}
f \in \mathcal{C}^\infty(A,\mathbb{R}) \text{ is linear} \quad & \textit{iff } \forall x, f(x) = D_0(f)(x) \\
& \textit{iff } \exists g \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}), f = \bar{d}g \\
\phi \in A'' \simeq A \quad & \textit{iff } \exists \psi \in\, !A, D_0(\phi) = \psi \\
\phi \in\, !_D A \quad & \textit{iff } \exists \psi \in\, !A, D(\phi) = \psi
\end{aligned}
$$

Generalizing differentiation

$$!_D A = \{ \text{ distribution } \phi \text{ solution to a differential equation } D\phi = \psi \ \}$$

e.g.: $!_{Id} A \simeq\, !A$ $\qquad$ $!_{D_0} A \simeq A$.

Works only when D *is a LPDO with constant coefficient.*

Linearity/Non-linearity ⇝ Solutions/Parameter of a differential equation.
Reduction of proofs/programs ⇝ resolution of an equation.

The same cut-elimination

$$\bar{d}_D : \begin{cases} !_D E \to \mathcal{C}^\infty(E,\mathbb{R})' \\ \phi \mapsto D\phi \end{cases} \qquad d_D : \begin{cases} \mathcal{C}^\infty(E,\mathbb{R})' \to !_D E \\ \psi \mapsto \psi \end{cases}$$

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !_D A}{\vdash \Gamma, D\phi : !A}\,\bar{d}_D \qquad \dfrac{\vdash \Delta, g : ?_D A^\perp}{\vdash \Delta, g : ?A^\perp}\,d_D}{\vdash \Gamma, \Delta, D(\phi)(E_D * g) : \mathbb{R} = \perp}\,\text{cut} \rightsquigarrow$$

$$\dfrac{\vdash \Gamma, \phi : !_D A \qquad \vdash \Delta, g : ?_D A}{\vdash \Gamma, \Delta, D(\phi(g)) \neq \phi(g) : \mathbb{R} = \perp}\,\text{cut}$$

The same cut-elimination

$$\bar{d}_D : \begin{cases} !_D E \to \mathcal{C}^\infty(E,\mathbb{R})' \\ \phi \mapsto D\phi \end{cases} \qquad d_D : \begin{cases} \mathcal{C}^\infty(E,\mathbb{R})' \to !_D E \\ \psi \mapsto \psi * E_D \end{cases}$$

$$\dfrac{\dfrac{\vdash \Gamma, \phi : !_D A}{\vdash \Gamma, D\phi : !A}\,\bar{d}_D \qquad \dfrac{\vdash \Delta, g : ?_D A^\perp}{\vdash \Delta, g * E_D : ?A^\perp}\,d_D}{\vdash \Gamma, \Delta, D(\phi)(E_D * g) = \phi(g) : \mathbb{R} = \perp}\,\text{cut} \rightsquigarrow$$

$$\dfrac{\vdash \Gamma, \phi : !_D A \qquad \vdash \Delta, g : ?_D A}{\vdash \Gamma, \Delta, D(\phi * E_D)(g)) = \phi(g) : \mathbb{R} = \perp}\,\text{cut}$$

D-DiLL

DiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, !A}\, \bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x)!A}\, \bar{d}$$

$D - DiLL$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \int D : ?_D A}\, w_D \qquad \frac{\vdash \Gamma, f : ?A, g : ?_D A}{\vdash \Gamma, f.g : ?_D A}\, c \qquad \frac{\vdash \Gamma, f : ?_D A}{\vdash \Gamma, f * E_D : ?A}\, d_D$$

$$\frac{\vdash}{\vdash E_D : !_D A}\, \bar{w}_D \qquad \frac{\text{param. } \vdash \Gamma, \phi : !A \qquad \text{sol. } \vdash \Delta, \psi : !_D A}{\vdash \Gamma, \Delta, \phi * \psi : !_D A}\, \bar{c}_D \qquad \frac{\vdash \Gamma, \psi : !_D A}{\vdash \Gamma, D\psi : !A}\, \bar{d}_D$$

A deterministic cut-elimination.

D-DiLL

DiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ c \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, !A}\ \bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \bar{c} \qquad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x)!A}\ \bar{d}$$

$D - DiLL$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \int D : ?_D A}\ w_D \qquad \frac{\vdash \Gamma, f : ?A, g : ?_D A}{\vdash \Gamma, f.g : ?_D A}\ c \qquad \frac{\vdash \Gamma, f : ?_D A}{\vdash \Gamma, f * E_D : ?A}\ d_D$$

$$\frac{\vdash}{\vdash E_D : !_D A}\ \bar{w}_D \qquad \frac{\text{input} \vdash \Gamma, \phi : !A \qquad \text{output} \vdash \Delta, \psi : !_D A}{\vdash \Gamma, \Delta, \phi * \psi : !_D A}\ \bar{c}_D \qquad \frac{\vdash \Gamma, \psi : !_D A}{\vdash \Gamma, D\psi : !A}\ \bar{d}_D$$

A deterministic cut-elimination.

Conclusion:

Stay home message:

- Logical formulas can represent solutions to differential equations.
- Cut-elimination can represent the computation of a solution to a differential equation.

Perspectives:

- Can we incorporate the real-life computation/approximation of a PDE/ODE in this system ?
- The concept of linearity in proof theory was a game-changing discovery. What's the computational concepts defined by more general equations ?