Constructing continuous systems from cellular automata

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Background: CA and chaos

Discrete dynamical systems: X, configuration space and $f : X \to X$, evolutionary rule

↪ study long term behavior of f (f^n for large n)

Cellular automata have many different behaviors

- Kůrka classification (equicontinuous, sensitive, expansive)
- Simulation relation and intrinsic universality
- Turing universality

Cellular automata

- Configurations are members of $A^{\mathbb{Z}}$, $|A| < \infty$
- Local rule $\mu : A^{2r+1} \to A$
- Global behavior applying the local rule synchronously at each site:

$$f(x)_i = \mu(x_{i-r}, \dots, x_{i+r})$$

No environment action
Same rule everywhere, anytime

Continuous systems

Disjunction of:

- Continuous time
- Continuous space
- Continuous state

If conjunction:

↪ ruled by PDE

$$\frac{\partial u}{\partial t} = F(u, u', u'', \dots)$$

Continuous systems and chaos

Usual PDE in physics

- Diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u + F(u, x, t)$$

  ↪ without environment, energy level of a single point tends to 0
- Wave equation

  ↪ simple and predictable wave propagation

Goal and research path

Looking for a PDE which is

- Somehow natural
- Has a chaotic behavior

  ↪ expansive for 1D systems, universal for 2D systems

Try to derive a PDE from a cellular automaton rule

Expansivity for continuous systems

Any change in the initial configuration is eventually impactful home

$$\exists \varepsilon, \forall c, \forall d, \exists n, c \neq d \Rightarrow d(f^n(c), f^n(d)) > \varepsilon$$

Cantor distance for CA (equivalent definition):

$$d(x, y) = \sum_{i \in \mathbb{Z}} (1 - \delta_{x_i, y_i}) 2^{-|i|} \qquad \delta\text{=Kronecker delta}$$

Natural extension to continuous systems:

$$x, y \in [0,1]^{\mathbb{R}}, d(x, y) = \int_{-\infty}^{\infty} |x(i) - y(i)| 2^{-|i|} di$$

Continuous CA, previous works

- Larger than life
- Interacting particle systems
- Probabilistic/Stochastic CA
- Interacting stochastic particle systems

Interacting stochastic particle systems

Space is discrete, time and state are continuous

- Finite alphabet A
- Configurations are from $(P_A)^{\mathbb{Z}}$

  ↪ P_A is the set of probability distributions over A
- Evolution is given by an ODE:

$$\frac{dP[c_j = s]}{dt} = i - o \qquad \begin{array}{l} i\text{=rate for entering state } s \\ o\text{=rate for leaving state } s \end{array}$$

- Rates i and o depends on the values in the neighboring cells

  ↪ defined from a local rule $\lambda : A \times A^{2r+1} \to [0, \infty)$
- The rate $\lambda(s, n)$ tells at what speed the cell changes its letter to s when the neighboring letters are n

  ↪ the higher the rate, the faster the change, proportionally

From A to P_A

As the local rule is defined over A, dealing with P_A requires an independency hypothesis

↪ the hypothesis is not true in PCA or IPS

We do this since the model is

- Sound with continuous states in P_A (stochastic particles)

  ↪ e.g. the proportions of predators and preys
- Successfully used for modeling biological and physical behavior
- Only used for time $o(1)$ and dependency radius increases with time
- Only intermediary, the goal is to find good candidate PDE

Limit system

Observe what happens when space is made continuous

↪ cells get closer and closer

We prove that for some ISPS, the limit exists and is ruled by a PDE
Idea: converse of the simulation of a PDE by the finite difference method

$$f(u, u', u'') \longleftrightarrow \mathbb{R}[u(0), u(\varepsilon), u(2\varepsilon)]$$

We consider $A = \{0, 1\}$. Then $P_A = \{(x, y) | y = 1 - x\}$. Configurations are members of $[0, 1]^{\mathbb{Z}}$.

Differential normal form

The terms i and o are polynomials, say of $\mathbb{R}[X_0, \dots, X_n]$
They can be expressed in term of $X_0, X_0', \dots, X_0^{(n)}$ where

$$X_k^{(i)} = X_{k+1}^{(i-1)} - X_{k+1}^{(i-1)}$$

- $X_1 = X_0 + X_0'$
- $X_2 = X_0 + 2X_0' + X_0''$
- $X_3 = X_0 + 3X_0' + 3X_0'' + X_0^{(3)}$
- ...

ISPS to PDE

Let $c \in [0, 1]^{\mathbb{Z}}$ be a configuration

- The terms i and o are polynomials in c_k for $k \in \{-r, \dots, r\}$
- The value $i - o$ can be put in differential normal form
- We only keep monomials which are $O(\varepsilon^k)$ for the smallest k when ε is the discretization step
  - $X_0^2 X_0^{(2)} = O(\varepsilon^2)$
  - ${X_0^{(1)}}^2 X_0^{(2)} = O(\varepsilon^4)$
- We call rank this smallest k
- We divide the polynomial by ε^k and make ε tend to 0
- The limit system is the PDE obtained keeping $O(\varepsilon^k)$ monomials in which we replace $X_0^{(k)}$ by $\frac{\partial^k u}{\partial x^k}$

Example

$$P = 2X_1^3 - 4X_1^2 X_2 + 2X_1 X_2^2 + X_0^2 - X_0 X_1 - 2X_1^2 + X_0 X_2 + X_1 X_2$$

$$\begin{aligned} P &= \underline{2X{X^{(1)}}^2} + 2{X^{(1)}}^3 + 4XX^{(1)}X^{(2)} + 4{X^{(1)}}^2 X^{(2)} \\ &+ 2X{X^{(2)}}^2 + 2X^{(1)}{X^{(2)}}^2 + \underline{2XX^{(2)}} + X^{(1)}X^{(2)} \end{aligned}$$

$$\frac{\partial u}{\partial t} = 2u \left(\frac{\partial u}{\partial x}\right)^2 + 2u \frac{\partial^2 u}{\partial x^2}$$

Positively ranked systems

- The limit PDE has spatial derivatives when the rank is positive
- This occurs when the DNF of P becomes 0 when one replaces all variables by a single one
- Given r, putting this into sagemath, one can get equations that λ must verify to be positively ranked

We make the conjecture that positively ranked λ verifies:

$$\begin{cases} \forall v \in \{0, 2r\}, \sum_{\substack{n \in \{0,1\}^{2r+1} \\ n_0 = 0, |n| = v}} \lambda(1, n) = \sum_{\substack{n \in \{0,1\}^{2r+1} \\ n_0 = 1, |n| = v}} \lambda(0, n) \\ \lambda(1, 0 \dots 0) = \lambda(0, 1 \dots 1) = 0 \end{cases}$$

CA case

- A CA rule can be transformed into a ISPS

  ↪ let the state tend to be what the CA rule requires
- No clear relation between the CA and the ISPS
- Allow to get a PDE from a CA rule

  ↪ need to do a rule by rule study to find which PDE are related to the CA

The conjecture becomes:

$$\forall v \in \{0, 2r+1\}, \sum_{\substack{n \in \{0,1\}^{2r+1} \\ |n| = v}} \mu(n) = \binom{v-1}{2r} \quad \text{with } \binom{-1}{2r} = 0$$

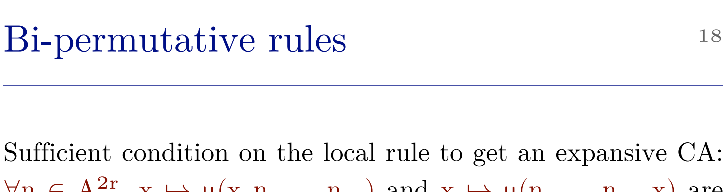
radius 1

Positively ranked elementary CA

- Shift: $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$
- Traffic: $\frac{\partial u}{\partial t} = (2u+1)\frac{\partial u}{\partial x}$
- 172: $\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x}$

No “complex” behaviors

radius 2 example



$$\frac{\partial u}{\partial t} = 18u \left(\frac{\partial u}{\partial x}\right)^2 - 5u^2 \frac{\partial^2 u}{\partial x^2} - 9 \left(\frac{\partial u}{\partial x}\right)^2 + 5u \frac{\partial^2 u}{\partial x^2}$$

Bi-permutative rules

Sufficient condition on the local rule to get an expansive CA: $\forall n \in A^{2r}$, $x \mapsto \mu(x, n_0, \dots, n_{2r})$ and $x \mapsto \mu(n_0, \dots, n_{2r}, x)$ are bijections.
A Prolog program proved that no bi-permutative rule verifies the conjecture for $r < 6$.

Futur work

- CA corresponding to a PDE
- Links between PDE, ISPS and CA
- Test more rules for higher radius
- Remove independence hypothesis (derivative of probability measure)