Dichotomy and logic

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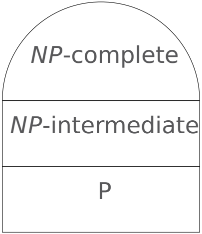
ANR δifference

CSP and dichotomy

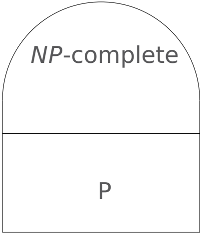
CSP = Constraint Satisfaction Problem

Real World*



Ladner's theorem ['75]

CSP world



Feder et Vardi Conjecture['93]
Bulatov and Zhuk theorem ['17]

(*)*provisio* $P \neq NP$

CSP?

CSP enjoys many definitions including

- Model checking problem.
- homomorphism problem.

Model Checking

Structure $\models$ a sentence?

primitive positive

e.g.

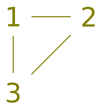
$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_4) \wedge E(x_3, x_1)$$

Model Checking

Structure $\models$ a sentence?

primitive positive

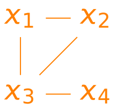
e.g.

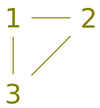


$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_4) \wedge E(x_3, x_1)$$

Homomorphism

Structure has a homomorphism to another structure?





Classification

Structure $\models$ a sentence?

Fixed Parameter
Input

Structure $\to$ Structure?

Dichotomy : for each parameter the problem is either tractable (Ptime) or hard (NP-complete).

Interesting Examples

- up to the encoding using clauses rather than boolean relations, SAT is a CSP
- graph colourability
- transitive tournament

Features of CSP

Monotonicity

If you remove a constraint from an input that is accepted, it remains accepted.

No machine characterisation

But we have algebraic characterization.

logic?

Fagin ESO = NP

Feder and Vardi :
syntactic fragment of ESO
MMSNP "=" CSP

Towards a definition of this logic MMSNP

The ESO sentence needs to describe / model check that the input structure has a homomorphism to a fixed structure.

Monadic predicates suffices to describe the element of the target assigned to an element of the input.

Monotonicity of the problem means that the sentence is also monotonic (input predicates appear always with the same polarity).

Finally $\neq$ is not needed, reflecting the fact that a homomorphism may identify two input elements.

Too much logic

One may define some syntactic restrictions of Fagin's ESO that are still expressive enough to express every problem in NP.

In particular by Skolemisation of ESO, sentences of the following form : *there are some functions* followed by some universal FO sentence.

Outline for the rest of this talk

- Define and introduce MMSNP, Feder Vardi’s logic for CSP
- State and sketch the proof of Feder Vardi theorem "equating" CSP and MMSNP

  Spoiler : it is not really equal.
- If times allows talk about other logics.

Feder and Vardi's logic for CSP

- snp consists of sentences of the form

$$\exists SO \forall FO \text{ quantifier-free formula}$$

- mmsnp is snp under 3 syntactic restrictions: Monotone, Monadic, no $\neq$
  consists of sentences of the form

$$\exists \overline{M}\, \forall \overline{x} \bigwedge_i \neg\big(\alpha_i(input, \overline{x}) \wedge \beta_i(\overline{M}, \overline{x})\big)$$

  where
  - the existential predicates $\overline{M}$ are monadic;
  - the formulae α_i and β_i are conjunction of atoms that do not contain any $\neq$; and,
  - the formulae α_i contains only positive atoms involving input symbols (monotonicity).

Remark

Each negated conjunct $\neg\big(\alpha_i(input, \overline{x}) \wedge \beta_i(\overline{M}, \overline{x})\big)$ corresponds naturally to a (partially) coloured structure / obstruction.

Examples

- triangle-free

$$\forall x \forall y \forall z \neg \big(E(x,y) \wedge E(y,z) \wedge E(z,x) \big)$$

- 3-colorability

$$\begin{aligned} \exists M_1, M_2 \forall x \forall y \neg \big(\neg M_1(x) \wedge \neg M_2(x) \big) \\ \wedge \neg \big(E(x,y) \wedge \neg M_1(x) \wedge \ \ M_2(x) \wedge \neg M_1(y) \wedge \ \ M_2(y) \big) \\ \wedge \neg \big(E(x,y) \wedge \ \ M_1(x) \wedge \neg M_2(x) \wedge \ \ M_1(y) \wedge \neg M_2(y) \big) \\ \wedge \neg \big(E(x,y) \wedge \ \ M_1(x) \wedge \ \ M_2(x) \wedge \ \ M_1(y) \wedge \ \ M_2(y) \big) \end{aligned}$$

Examples

- triangle-free (clearly not in CSP_{fin})

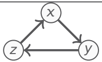
$$\forall x \forall y \forall z \neg\big(E(x,y) \wedge E(y,z) \wedge E(z,x)\big)$$

- 3-colorability

$$\begin{aligned}\exists M_1, M_2 \forall x \forall y \neg\big(&\neg M_1(x) \wedge \neg M_2(x)\big)\\ \wedge \neg\big(&E(x,y) \wedge \neg M_1(x) \wedge M_2(x) \wedge \neg M_1(y) \wedge M_2(y)\big)\\ \wedge \neg\big(&E(x,y) \wedge M_1(x) \wedge \neg M_2(x) \wedge M_1(y) \wedge \neg M_2(y)\big)\\ \wedge \neg\big(&E(x,y) \wedge M_1(x) \wedge M_2(x) \wedge M_1(y) \wedge M_2(y)\big)\end{aligned}$$

forbidden patterns problems

- triangle-free

	$\forall x \forall y \forall z$
	$\neg\big(E(x,y) \wedge E(y,z) \wedge E(z,x)\big)$

- 3-colorability

	$\exists M_1 \exists M_2 \forall x \forall y$
	$\neg\big(E(x,y) \wedge \neg M_1(x) \wedge \quad M_2(x) \wedge \neg M_1(y) \wedge \quad M_2(y)\big) \wedge$
	$\neg\big(E(x,y) \wedge \quad M_1(x) \wedge \neg M_2(x) \wedge \quad M_1(y) \wedge \neg M_2(y)\big) \wedge$
	$\neg\big(E(x,y) \wedge \quad M_1(x) \wedge \quad M_2(x) \wedge \quad M_1(y) \wedge \quad M_2(y)\big) \wedge$
	$\neg\big(\neg M_1(x) \wedge \neg M_2(x)\big)$

Expressibility of mmsnp

Every problem CSP($\mathcal{B}$) in CSP_{fin} can be expressed by a sentence of mmsnp $\Phi_{\mathcal{B}}$.

the colours code the vertices of $\mathcal{B}$	
	$\exists M_1 \exists M_2 \ldots M_{\lceil \log \lvert\mathcal{B}\rvert \rceil} \forall x \forall y$
whenever there is no arc from vertex β_1 to vertex β_2 in $\mathcal{B}$.	
	$\neg\big(E(x,y) \wedge \beta_1(\overline{M},x) \wedge \beta_2(\overline{M},y)\big) \wedge$
up to some bad colours	
	$\neg\big(\neg\beta_{bad}(x)\big)$

Question

We have seen that mmsnp can express more than CSP_{fin}. What kind of problems precisely?

Digression

forbidden patterns problems (FPP)

- FPP = restricted mmsnp where negated conjuncts are connected.
- FPP problems are closed under inverse homomorphism but also disjoint union (just like CSP problems).

Proposition

$\text{mmsnp} = \bigcup FPP$

Extension of CSP framework to infinite target structure, in particular to ω-categorical structures (CSP_ω).

Theorem 1 (Cherlin, Shelah, Shi '99)

Every problem in FPP with a single colour is in CSP_ω.

Corollary 2 (Bodirsky, Dalmau '06)

$FPP \subset CSP_\omega$.

Feder and Vardi theorem

Theorem 3 (Feder and Vardi '98)

For every sentence Φ of mmsnp*, there exists a CSP (possibly over a different signature) with some finite target $\mathcal{B}_\Phi$ such that the model checking problem of Φ reduces in polynomial time to the CSP with target $\mathcal{B}_\Phi$; and, the latter reduces to the former in (randomised) polynomial time.*

Example of no monochromatic triangle

serving as a proof sketch

- Two colours for vertices; obstruction are triangles.
- New signature : ternary, coding for a triangle.
- Reduction from binary signature to triangle signature is trivial.
- Converse reduction is non trivial : main issue deal with ternary structures that do not code triangles.
- Converse is fine if ternary structure has girth 4 or more. (no cycles of length 3 or less).

Important Lemma by Erdös

Fix two positive integers r and s.
For every structure $\mathcal{B}$, there exists a structure $\mathcal{D}$, of size polynomial in $\mathcal{B}$ such that:

- the girth of $\mathcal{D}$ is greater than r;
- there is a homomorphism from $\mathcal{D}$ to $\mathcal{B}$; and,
- for every structure $\mathcal{C}$ of size at most s, there is a homomorphism from $\mathcal{B}$ to $\mathcal{C}$ if, and only if, there is a homomorphism from $\mathcal{D}$ to $\mathcal{C}$.

This random construction can be derandomised using expanders [Kun 2007].

Feder and Vardi theorem

Theorem 4 (Feder and Vardi ’98 + Kun 2007)

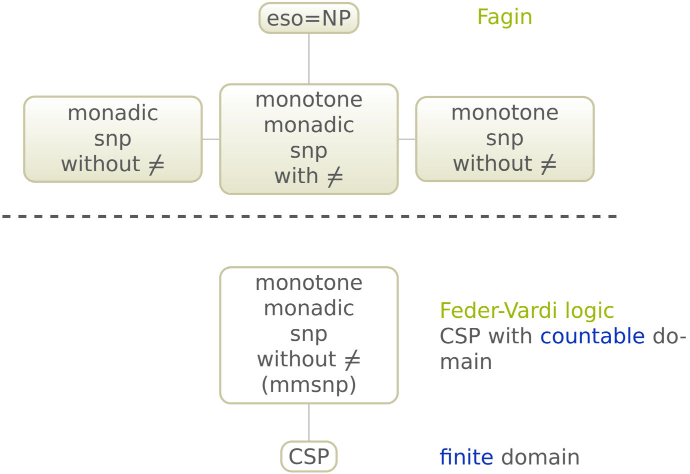
For every sentence Φ of mmsnp*, there exists a CSP (possibly over a different signature) with some finite target $\mathcal{B}_\Phi$ such that the model checking problem of Φ reduces in polynomial time to the CSP with target $\mathcal{B}_\Phi$; and, the latter reduces to the former in polynomial time.*

Corollary 5

MMSNP has a dichotomy iff CSP has a dichotomy.

Dichotomy and descriptive complexity

Non dichotomic (Ladner)



dichotomic