LIX, Ecole Polytechnique
Technische Universität Dresden

The Countable Boolean Vector Space and Bit Vector CSPs

PhD Defence

September 17, 2015

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Jury Members

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Constraint Satisfaction Problems

Informal definition of CSPs

- A CSP is a computational problem.
- The input consists of a finite set of variables and a finite set of constraints imposed on those variables.
- The task is to decide whether there is an assignment of values to the variables such that all the constraints are simultaneously satisfied.

Examples

- Is a propositional formula in CNF with at most three literals per clause satisfiable on $\{0, 1\}$?
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Formalisation of CSPs

Preliminaries

- Given a relational signature τ, an atomic formula is of the form $R(\overline{x})$ with R a relation in τ.
- A primitive positive (pp) formula on τ is of the form $\exists x_1 \ldots x_n(\phi_1(\overline{x}) \wedge \cdots \wedge \phi_k(\overline{x}))$ where all ϕ_i are atomic formulas.

Formal definition of *CSPs*

Given a structure Γ on a finite relational signature τ, we define the computational problem CSP(Γ):

- **Input**: a primitive positive sentence ϕ.
- **Question**: $\Gamma \models \phi$?

Natural question: what is the complexity of CSP(Γ) for a given Γ?
Proposition: it does not change when adding pp-definable relations to Γ.

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Dichotomy for finite Structures

- Schaefer'77: for any 2-element structure Γ, CSP(Γ) is either polynomially solvable or NP-complete.

Conjecture (Feder-Vardi'93)

This dichotomy holds for every finite structure Γ.

- Bulatov'03: confirmed Feder-Vardi's conjecture for domains of size 3.
- Markovic'12: confirmed for domains of size 4 (announced but not published yet).
- The conjecture is already open for domains of size ≥ 5.

What about infinite structures?

Infinite Structures

Non-Dichotomy

- Ladner'75: if $P \neq NP$, there are NP-intermediate computational decision problems, i.e., problems in NP that are neither polynomial-time tractable nor NP-complete.
- Bodirsky-Grohe'08: Every computational decision problem is polynomial-time equivalent to a CSP with an infinite template.
- Consequently: no dichotomy for CSPs on infinite structures.

Question

Can we identify large natural classes of CSPs on infinite structures whose complexity can be classified?

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The Countable Boolean Vector Space

Definition

There is up to isomorphism a unique countably infinite vector space over the field $\mathbb{F}_2$. We denote it by $(V; +)$.

Characteristics:

- fundamental structure in Model Theory
- Fraïssé limit of the class of finite $\mathbb{F}_2$-vector spaces
- **homogeneous**, i.e., any partial isomorphism between finite substructures of $(V; +)$ can be extended to an automorphism of $(V; +)$

Reducts of $(V;+)$

Definition

A reduct of a structure Δ is a relational structure with the same domain as Δ whose **relations are definable with first-order formulas over Δ**.

Examples of relations definable over $(V;+)$: let $n \geq 3$ be an integer,

- $x = 0 :\Leftrightarrow x + x = x$
- $\mathrm{Eq}_n(x_1, \ldots, x_n) :\Leftrightarrow \Sigma_{i \leq n} x_i = 0$
- $\mathrm{Ind}_n(x_1, \ldots, x_n) :\Leftrightarrow x_1, \ldots, x_n$ are linearly independent
- $\mathrm{Ieq}_n(x_1, \ldots, x_n) :\Leftrightarrow \mathrm{Eq}_n(x_1, \ldots, x_n)$ and every subfamily of size $n-1$ of $x_1, \ldots, x_n$ is linearly independent

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Bit Vector CSPs

Definition

A Bit Vector CSP is a problem $\mathrm{CSP}(\Gamma)$ where Γ is a reduct of $(V; +)$.

Examples:

1. $\mathrm{CSP}(V; \mathrm{Eq}_3, \neq)$
2. $\mathrm{CSP}(V; \mathrm{leq}_4, \mathrm{leq}_8, Z_1 \cup Z_2 \cup \mathrm{Ind}_4)$ where:
   - $Z_1(x, y, z, t) :\Leftrightarrow x = 0 \wedge \mathrm{leq}_3(y, z, t)$, and
   - $Z_2(x, y, z, t) :\Leftrightarrow x \notin \{0, y, z, t\} \wedge \mathrm{leq}_3(y, z, t)\}$
3. $\mathrm{CSP}(V; \mathrm{leq}_5, Q)$ where:
   - $Q(x, y, z, t_1, t_2, t_3) :\Leftrightarrow \mathrm{leq}_4(x, y, z, t_1) \vee \mathrm{leq}_5(x, y, z, t_2, t_3)$

Remark: 1. is in P by Gaussian elimination, but classifying the complexity of Examples 2. and 3. is not that easy.

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ω-categoricity

Conjecture

A Bit Vector CSP is either in P, or NP-complete.

How can we classify the complexity of CSP(Γ)?

- For finite structures, we can use a **universal algebraic approach**.
- To adapt it for an infinite Γ, we need a strong property on Γ: *ω-categoricity*.

Definition (Ryll-Nardzewski's form)

A countable structure of domain D is ω-categorical iff it has finitely many orbits w.r.t. the natural action of its automorphism group on D^n, for all n.

Facts:

- A reduct of an ω-categorical structure is ω-categorical.
- $(V; +)$ is ω-categorical.

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Polymorphisms

- A m-ary operation f preserves a n-ary relation R if for all n-tuples $\overline{x_1}, \ldots, \overline{x_m}$ in R, the n-tuple $(f(x_{1,i}, \ldots, x_{m,i}))_{1 \leq i \leq n}$ is again in R.
- f is called a polymorphism of a relational structure Γ if it preserves every relation of Γ.
- A unary polymorphism of Γ is called an endomorphism of Γ.

$$
\begin{array}{ccccccccc}
 & & \overset{f}{\frown} & \overset{f}{\frown} & \overset{f}{\frown} & & \overset{f}{\frown} & \\
\overline{x_1} & = & x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} & \in R \\
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 & & \smile & \smile & \smile & & \smile & \\
 & & \parallel & \parallel & \parallel & & \parallel & \\
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Universal Algebraic Approach and ω-categoricity

- Pol(Γ) (resp. End(Γ)): the set of all polymorphisms (resp. endomorphisms) of Γ.
- Inv(F): the set of all relations preserved by a set F of operations.
- $\langle\Gamma\rangle_{\text{pp}}$: the set of all relations which are **definable with a primitive positive formula** over Γ.

Theorem (Geiger'68 & Bodirsky,Nesetril'03)

For every countably infinite ω-categorical or finite structure Γ:

$$\text{Inv}(\text{Pol}(\Gamma)) = \langle\Gamma\rangle_{\text{pp}}$$

Consequently, **the complexity of** CSP(Γ) **is determined by** Pol(Γ).
We first focus on **understanding** End(Γ).

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We first focus on **understanding** $\mathrm{End}(\Gamma)$.

Classification of Endomorphism monoids of Reducts

Theorem

Let Γ be a reduct of $(V;+)$ which is not homomorphically equivalent to a reduct of $(V;0)$. Then $\mathrm{End}(\Gamma)$ belongs to a list of **27 monoids**.

Remarks:

- Homomorphic equivalence preserves the complexity of the CSP.
- CSPs of reducts of $(V;0)$ are fully classified in the thesis.

What method do we use?

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Bodirsky and Pinsker's Method using Ramsey Theory

Goal: classify End(Γ) for reducts Γ of structures with a strong combinatorial property called *Ramsey Property*.

Notation: Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be structures with same signature, and $r \in \mathbb{N}$.

- $\binom{\mathfrak{B}}{\mathfrak{A}}$ denotes the set of substructures of $\mathfrak{B}$ isomorphic to $\mathfrak{A}$.
- we write $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$ if for all colouring $\chi\colon \binom{\mathfrak{C}}{\mathfrak{A}} \to \{1, \ldots, r\}$ there exists $\mathfrak{B}' \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that χ is monochromatic on $\binom{\mathfrak{B}'}{\mathfrak{A}}$.

Ramsey Property

A structure Γ has the Ramsey property if for all finite substructures $\mathfrak{A}, \mathfrak{B}$ of Γ and all $k \in \mathbb{N}$, we have: $\Gamma \to (\mathfrak{B})^{\mathfrak{A}}_r$.

Graham-Leeb-Rothschild'71: $(V; +)$ is Ramsey.

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Generating Canonical Functions

The Ramsey property allows us to prove the existence of *canonical functions*. They play a crucial role in the classification of $\mathrm{End}(\Gamma)$.

- $f : \Delta_1 \to \Delta_2$ is canonical if the orbit of the image of a tuple $\bar{a}$ only depends on the orbit of $\bar{a}$.
- $f : \Delta \to \Delta$ generates g if g belongs to the closure of $\{f\} \cup \mathrm{Aut}(\Delta)$ under composition and pointwise convergence.
- if an endomorphism of a reduct Γ of Δ generates g, then $g \in \mathrm{End}(\Gamma)$.

Theorem (Bodirsky,Pinsker,Tsankov'11)

Let Δ be a homogeneous Ramsey structure **with finite relational signature**. For all $f : \Delta \to \Delta$ and all $C := \{c_1, \ldots, c_n\}$, f generates a canonical function g from $(\Delta, c_1, \ldots, c_n)$ to Δ such that $f{\restriction}C = g{\restriction}C$.

Fact: finitely many canonical functions from Δ_1 to Δ_2 for **finite relational signatures**.

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Theorem (Bodirsky,Pinsker,Tsankov'11)

Let Δ be a homogeneous Ramsey structure **with finite relational signature**. For all $f : \Delta \to \Delta$ and all $C := \{c_1, \ldots, c_n\}$, f generates a canonical function g from $(\Delta, c_1, \ldots, c_n)$ to Δ such that $f{\restriction}C = g{\restriction}C$.

Fact: finitely many canonical functions from Δ_1 to Δ_2 for **finite relational signatures**.

Generating Canonical Functions

The Ramsey property allows us to prove the existence of *canonical functions*. They play a crucial role in the classification of $\mathrm{End}(\Gamma)$.

- $f : \Delta_1 \to \Delta_2$ is canonical if the orbit of the image of a tuple $\bar{a}$ only depends on the orbit of $\bar{a}$.
- $f : \Delta \to \Delta$ generates g if g belongs to the closure of $\{f\} \cup \mathrm{Aut}(\Delta)$ under composition and pointwise convergence.
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Dealing with a Functional Signature

Proposition

$(V;+)$ is not first-order interdefinable with any homogeneous structure with finite relational signature.

Difficulties:

- we have to adapt Bodirsky-Pinsker's theorem to use it on $(V;+)$
- potentially infinitely many canonical functions from $(V;+)$ to $(V;+)$

Fact

Bodirsky-Pinsker's theorem can be adapted for ω-categorical homogeneous structures **with functional signatures** which have the Ramsey property.

We first study canonical functions from $(V;+)$ to $(V;+)$.

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Canonical functions of $(V; +)$ without Constants

Theorem

Let f be a unary canonical function from $(V; +)$ to $(V; +)$. There exists $h \in \mathrm{End}(V; +, \neq)$ and $a \notin h(V)$ s.t. one of the following applies:

- (id-function) $f(x) = h(x)$ for all $x \neq 0$;
- (af-function) $f(x) = h(x) + a$ for all $x \neq 0$;
- (gen-function) f sends any family of pairwise distinct elements of $V \setminus \{0\}$ to a linearly independent family;
- Degenerated case: f has an image of size at most 2.

Translation of vector $d \neq 0$: $t_d(x) := x + d$ for all x.
Fact: Translations preserve Eq_4 and are not canonical.

Useful properties

- id-functions and af-functions preserve Ieq_4 but gen-functions and translations do not.
- Let f be an injection violating Ieq_4. Then f generates a gen-function or a translation.
- $\mathrm{End}(V; \mathrm{Eq}_4, \neq)$ is generated by t_d.
- t_d together with any injection violating Eq_4 generates a gen-function.

Towards a Description of Injective Endomorphism Monoids

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Classification of Injective Endomorphism Monoids of Reducts

Theorem

Let Γ be a reduct of $(V;+)$ with only injective endomorphisms. Then one of the following holds:

- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Eq}_4, \neq)$;
- $\mathrm{End}(\Gamma)$ is contained in $\mathrm{End}(V; \mathrm{Ieq}_4, \neq)$;
- $\mathrm{End}(\Gamma)$ contains a gen-function.

Why stopping when $\mathrm{End}(\Gamma)$ contains a gen-function?

If a gen-function belongs to $\mathrm{End}(\Gamma)$, then Γ is homomorphically equivalent to a reduct of $(V; 0)$.

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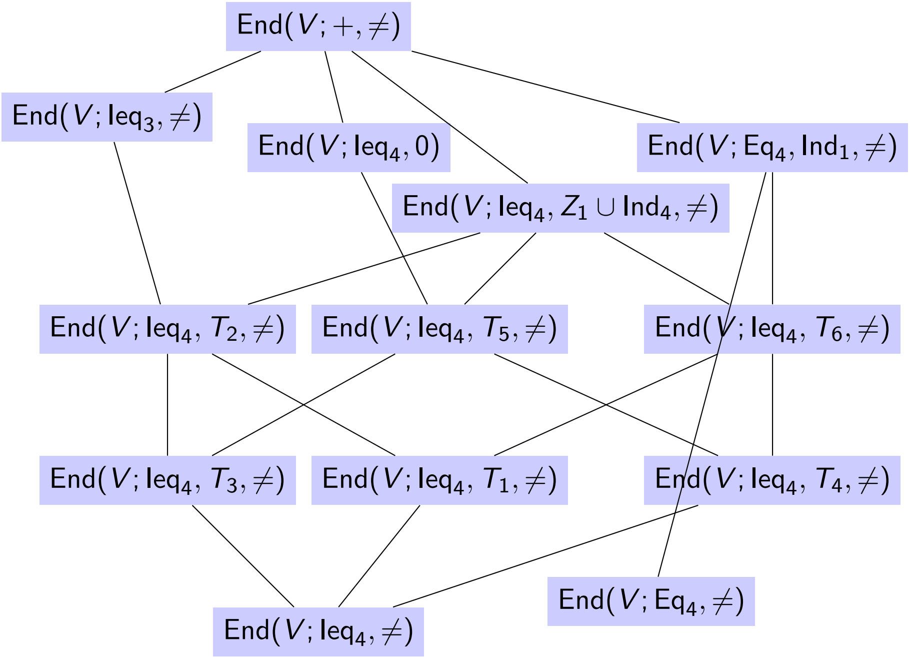
$\mathrm{End}(V; +, \neq)$
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To further simplify the study, we need a new idea!

Model-complete Cores

$\overline{\mathrm{Aut}(\Delta)}$: topological closure of $\mathrm{Aut}(\Delta)$ under pointwise convergence.

Definition

A model-complete core of a reduct Γ is a structure Δ homomorphically equivalent to Γ, and such that:

$$\mathrm{End}(\Delta) = \overline{\mathrm{Aut}(\Delta)}$$

Note that $\mathrm{CSP}(\Delta)$ and $\mathrm{CSP}(\Gamma)$ are equal by homomorphic equivalence.

Theorem (Bodirsky'06)

Every reduct Γ of an ω-categorical structure has a model-complete core Δ. All model-complete cores of Γ are isomorphic to Δ.

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Classification of Model-Complete Cores

Theorem

Let Γ be a reduct of $(V;+)$. Exactly one of the following holds:

1. $\mathrm{End}(\Gamma) = \mathrm{End}(V;+,\neq)$,
2. $\mathrm{End}(\Gamma) = \mathrm{End}(V;\mathrm{Ieq}_4,0)$, or
3. the model-complete core of Γ is isomorphic to a structure Γ' s.t.:
   a) $\mathrm{End}(\Gamma') = \mathrm{End}(V \setminus \{0\};\mathrm{Ieq}_3)$,
   b) $\mathrm{End}(\Gamma') = \mathrm{End}(V;\mathrm{Eq}_4,\neq)$,
   c) Γ' is a reduct of $(V;0)$, or
   d) Γ' is a 2-element structure.

Pathway to Bit Vector CSPs

Corollary

Let Γ be a reduct of $(V;+)$. There exists a structure Γ' with same CSP as Γ and s.t. one of the following holds:

1. $\mathrm{End}(\Gamma') = \mathrm{End}(V;+,\neq)$;
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The study of the polymorphisms is now strongly simplified.

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Case distinction of the previous corollary:

1. Γ' is a 2-element structure: P/NPc dichotomy (Schaefer'77)
2. Γ' is a reduct of $(V; 0)$: P/NPc dichotomy
   Poly. algos: **Schaefer** + ad-hoc routines
3. $\mathrm{End}(\Gamma') = \mathrm{End}(V; \mathrm{Eq}_4, \neq)$: P/NPc dichotomy
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NB: Case 1 is a joint work with Antoine Mottet and contains Bodirsky and Kara's classification of CSPs for reducts of $(\mathbb{N}; =)$ as a subcase.

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Conclusion and Perspectives

What we have achieved:

- adapt Bodirsky-Pinsker's method for **functional signatures**;
- classification of **canonical functions** from $(V; +)$ to $(V; +)$;
- classification of **endomorphism monoids of reducts** of $(V; +)$ which are not homorphically equivalent to reducts of $(V; 0)$;
- classification of **model-complete cores of reducts** of $(V; +)$ up to existential positive interdefinability. There are **finitely many**;
- **P/NPc dichotomy for Bit Vector CSPs** in 4 out of 6 listed cases.

What remains to be done:

- prove dichotomy when $\mathrm{End}(\Gamma) = \mathrm{End}(V; +, \neq)$;
- generalize the dichotomy for any vector space over **a finite field**; the **automorphism groups classification** is already established in this setting (Bodor-Kalina-Szabó'15)
- study reducts of the **atomless Boolean Algebra**;
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Thank you!