CSPs and Datalog

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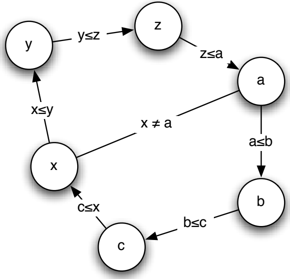
Example

An instance of CSP$(\mathbb{Q}; \leq, \neq)$:



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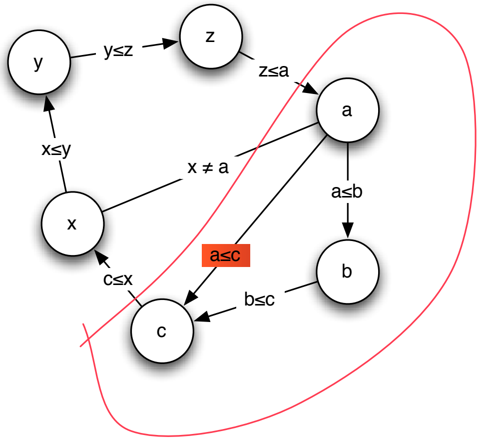

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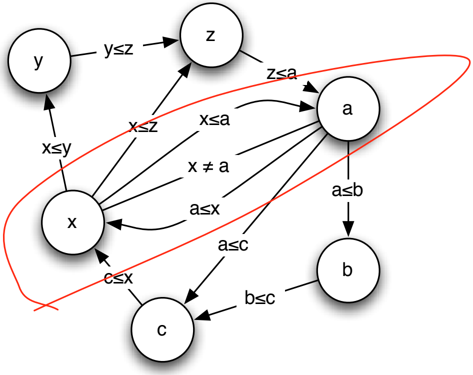

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Main algorithmic technique studied in more applied AI literature

(l, k)-Datalog

Example of a Datalog program

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\begin{aligned}
tc(x, y) &:- \quad x < y \\
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Advantage of this perspective: can also be applied to infinite Γ
However, need: Γ has only finitely many binary primitive positive definable unary relations.

Completeness

Every Datalog program can be evaluated on an input structure A in polynomial time in the size of A.

Definition

Say that Datalog solves $\text{CSP}(\Gamma)$ if there exists a Datalog program Π such that for all finite A:
Π derives *false* list if and only if there is no homomorphism from A to Γ.

Facts:

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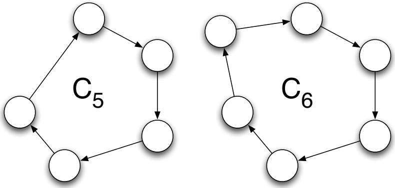
Question: Which CSPs can be solved by Datalog?

Exercise

Write a Datalog program that solves graph 2-colorability, $\mathrm{CSP}(K_2)$.

The existential pebble game



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Two players: Spoiler and Duplicator

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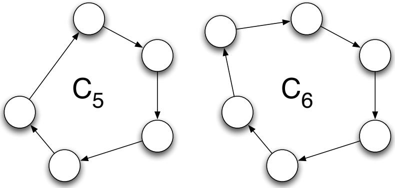

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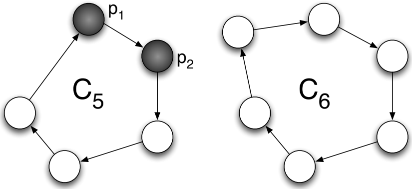

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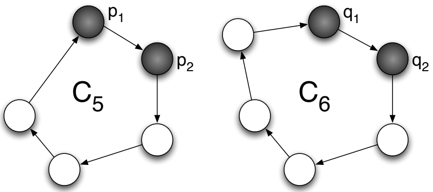

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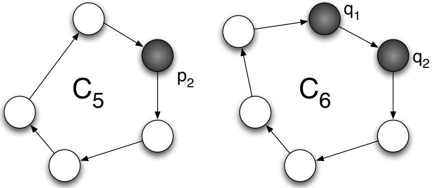

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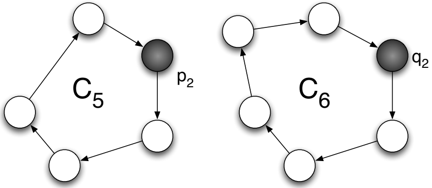

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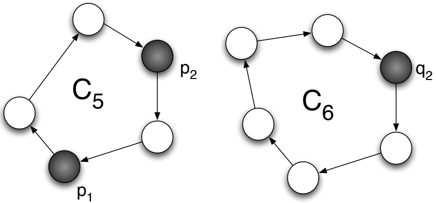

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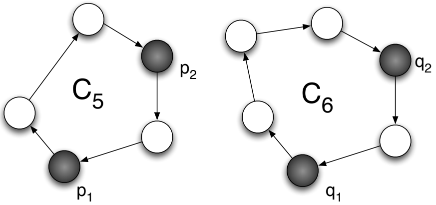

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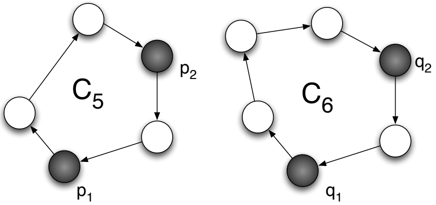

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Spoiler wins if eventually $p_i \mapsto q_i$ is not a partial homomorphism from A to B, otherwise Duplicator wins.

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Let Γ be an arbitrary relational structure.

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- Suppose that Duplicator has a winning strategy: that is, no matter how Spoiler plays, Duplicator can always play such that she wins.
  Then Duplicator also has a memoryless winning strategy: she can win in such a way that her moves only depend on the current positions of the pebbles, and not on the history of the game.

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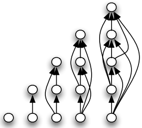

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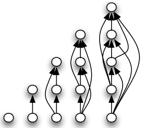

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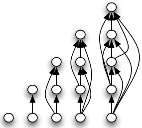

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But: Duplicator looses the $(1, 2)$-pebble game on (G, T_∞) for all graphs G with a directed cycle.

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Claim (proof comes later): $(\mathbb{Q}; <)$ and $(\mathbb{Q}; \{(x, y, z) \mid x > y \vee x > z\})$ have only finitely many inequivalent primitive positive definable relations, for all n.

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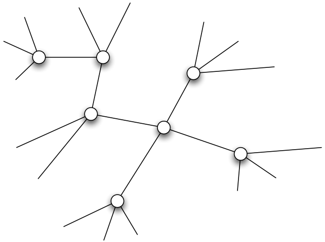

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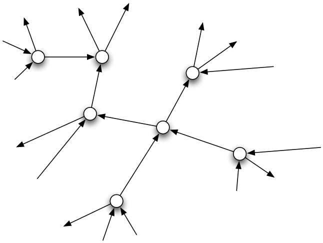

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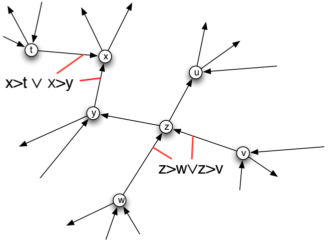

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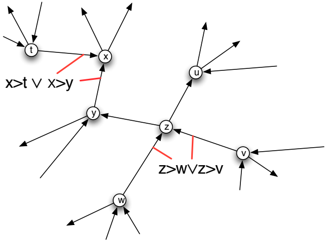

Transform into an instance of the CSP
Facts:

- the resulting instance A of CSP$(\mathbb{Q}; x > y \vee x > z)$ is unsatisfiable
- Duplicator wins the existential pebble game on A, $(\mathbb{Q}; x > y \vee x > z)$

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Clearly, this is satisfied at the beginning of the game.

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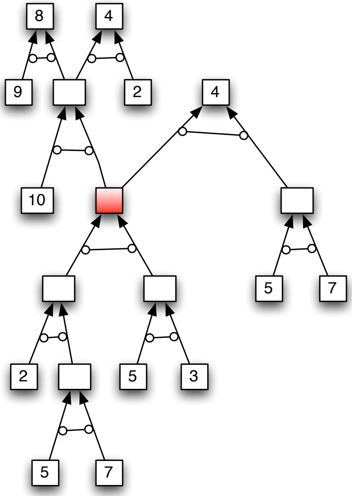

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