Arc Consistency

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Local Consistency Techniques

Central idea: ‘constraint propagation’

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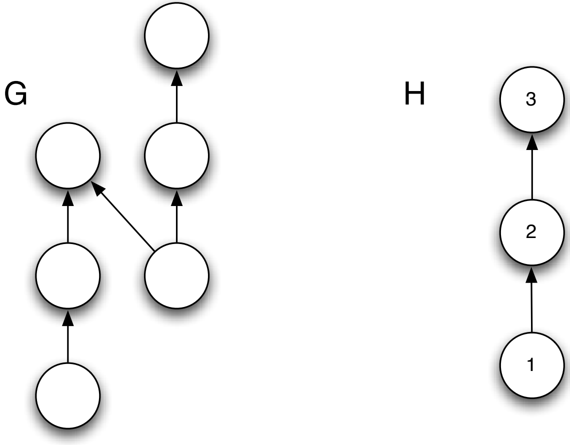
- One of the earliest algorithmic approaches to the CSP
  Montanari’74, Mackworth’77, Freuder’82
- Very important in the AI literature
- Comes close to what humans naively try when confronted with a CSP

Example

Does G homomorphically map to $H := \vec{P}_2$?



The Arc-Consistency Procedure (AC)

Fix finite digraph H.

$\text{AC}_H(G)$
Input: a finite digraph G.
Data structure: a list $L(x) \subseteq V(H)$ for each vertex $x \in V(G)$.

Set $L(x) := V(H)$ for all $x \in V(G)$.
Do
    For each $(x, y) \in E(G)$:
        Remove u from $L(x)$ if there is no $v \in L(y)$ with $(u, v) \in E(H)$.
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Claim: if algorithm derives empty list, there is no homomorphism from A to B!

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It might be that AC_H does not derive empty list on G, even though there is no homomorphism from G to H.

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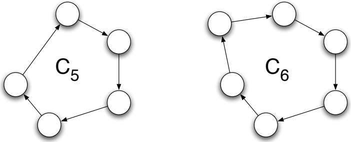

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Question: for which H does AC solve CSP(H)?

The Power Graph

Definition (Power-Set Graph)

The power-set graph $P(H)$ of a digraph H has non-empty subsets of $V(H)$ as vertices. Put an edge between S_1 and S_2 iff

- for every $u \in S_1$ there exists a $v \in S_2$ such that $(u, v) \in E(H)$, and
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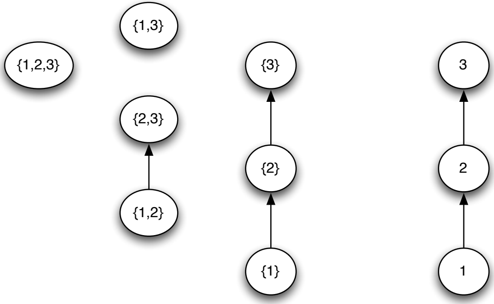

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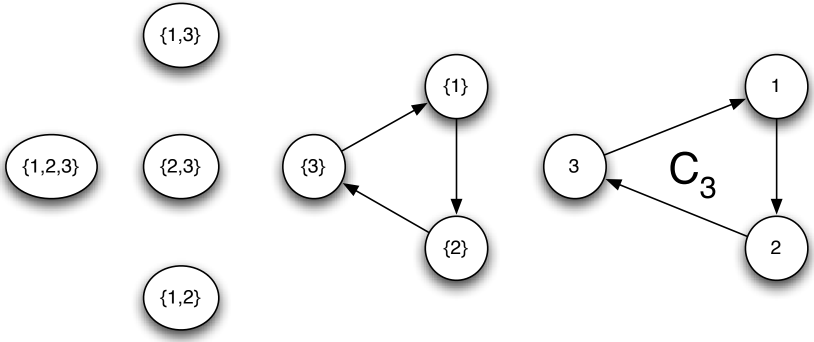

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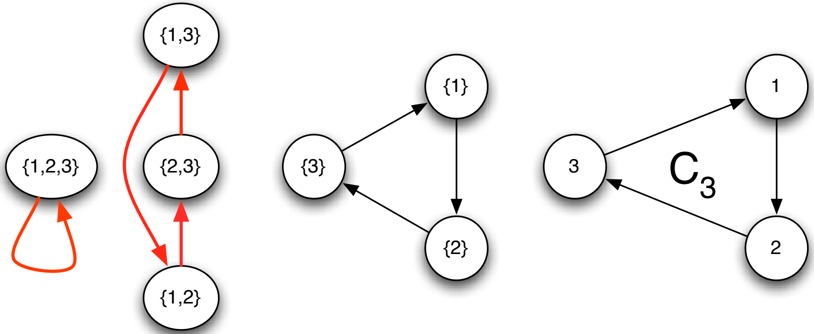

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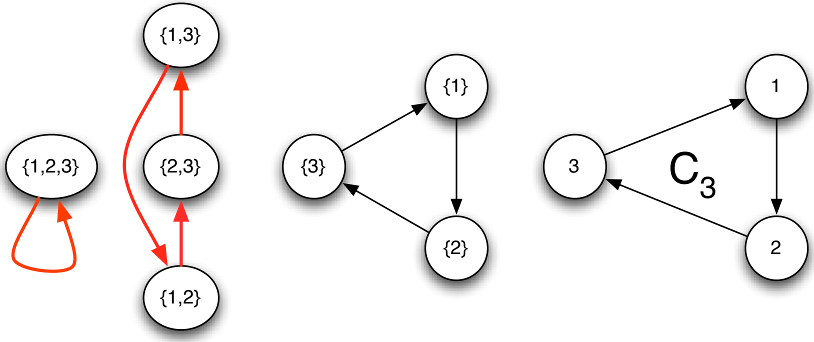

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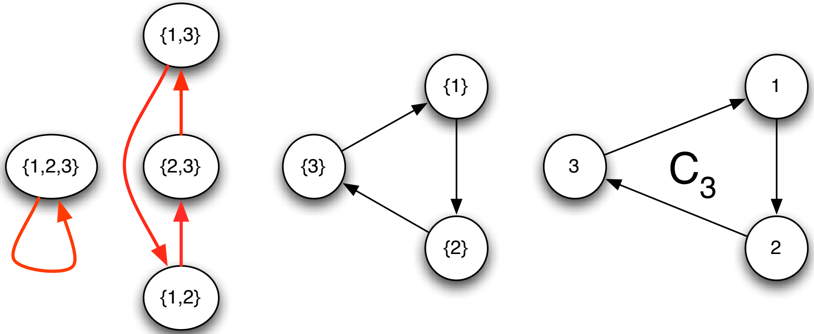

- For all H we have $H \to P(H)$
- Not for all H we have $P(H) \to H$.

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Lemma

AC_H does not derive the empty list on G if and only if $G \to P(H)$.

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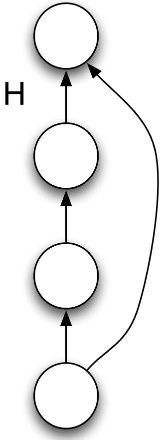
- If AC does not derive the empty list on G, by Lemma $G \rightarrow P(H)$.
  Composing homomorphisms, we obtain $G \rightarrow H$.

QED.

Exercise

Does Arc Consistency solve CSP(H) for the following graph H:



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Facts.

For all digraphs G,

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Definition

A digraph H has *tree duality* if there is a (not necessarily finite) set $\mathcal{N}$ of orientations of trees such that

$$\forall G: \quad G \to H \quad \Leftrightarrow \quad \mathcal{N} \not\to G$$

Tree Duality, Power Graph, Arc-Consistency

Theorem 2.

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3 $\Rightarrow$ 2: suffices to show that for all trees T,

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1 $\Rightarrow$ 3: If AC derives the empty list on G, then the derivation has a tree-like structure.

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- Power-set graph can be generalized to general relational structures.
- Today's proofs still work.
- In this more general setting, Boolean structures preserved by min or by max can be solved by Arc-Consistency.