Enumeration: logical and algebraic approach

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Enumeration problems

Polynomially balanced predicate $A(x, y)$, decidable in polynomial time.

- $\exists ? y A(x, y)$: decision problem (class NP)
- $\sharp\{y \mid A(x, y)\}$: counting problem (class $\sharp\mathrm{P}$)
- $\{y \mid A(x, y)\}$: enumeration problem (class EnumP)

Example

Perfect matching:

- The decision problem is to decide if there is a perfect matching.
- The counting problem is to count the number of perfect matchings.
- The enumeration problem is to list every perfect matching.

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Time complexity measures for enumeration

1. the total time related to the number of solutions
   - polynomial total time: **TotalP**
2. the delay
   - incremental polynomial time: **IncP** (Circuits of a matroid)
   - polynomial delay: **DelayP** (Perfect Matching [Uno])
   - Constant or linear delay
     - A two steps algorithm: preprocessing + generation
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Enumeration problems

R: polynomially balanced binary predicate

$\textsc{Enum}{\cdot}R$
Input: $x \in \mathcal{I}$
Output: an enumeration of elements in $R(x) = \{y \mid R(x,y)\}$

Definition

The problem $\textsc{Enum}{\cdot}R$ belongs to the class $\textsc{Delay}(g,f)$ if there exists an enumeration algorithm that computes $\textsc{Enum}{\cdot}R$ such that, for all input x:

- Preprocessing in time $O(g(|x|))$,
- Solutions $y \in R(x)$ are computed successively without repetition with a delay $O(f(|x|))$

$\textsc{Constant-Delay} = \bigcup_k \textsc{Delay}(n^k, 1)$.

Enumeration complexity classes

Separation:

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Boolean combination of solutions

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Meta-algorithms for enumeration and CSP

Proposition (Creignou, Hebrard'97)

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Logic in half a slide

First order logic(FO):

- Variables: $x, y, z \ldots$
- The language σ, relations and functions: $R(x, y)$, $f(z)$
- Unary and binary connectors: $\wedge$, $\vee$, $\neg$
- Quantifiers: $\forall$, $\exists$
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Example

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Independent sets:

$$IS(T) \equiv \forall x \forall y \;\; T(x) \wedge T(y) \Rightarrow \neg E(x, y).$$

Example

Hitting sets (vertex covers) of a hypergraph represented by the incidence structure $\langle D, \{V, E, R\} \rangle$.

$$HS(T) \equiv \forall x\,(T(x) \Rightarrow V(x)) \wedge \forall y \exists x\, E(y) \Rightarrow (T(x) \wedge R(x, y))$$

First-order queries with free second order variables

This presentation

- **FO** queries with free second-order variables
- Data complexity: the query is fixed
- The complexity in term of the size of the input structure's domain
- Quantifier depth as a parameter: $\textsc{Enum}{\cdot}\Sigma_1$
- $\textsc{Enum}{\cdot}\mathrm{IS} \in \textsc{Enum}{\cdot}\Pi_1$ and $\textsc{Enum}{\cdot}\mathrm{HS} \in \textsc{Enum}{\cdot}\Pi_2$

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Previous results

1. Only first-order free variables and bounded degree structures. Durand-Grandjean'07, Lindell'08, Kazana-Segoufin'10: linear preprocessing + constant delay.
2. Only first-order free variables and acyclic conjunctive formula. Bagan-Durand-Grandjean'07: linear preprocessing + linear delay

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Enumeration of the k-cliques of a graph of bounded degree.

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Example

Typical database query. Simple paths of length k.

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Example

Enumeration of the cliques of a bounded tree-width graph.

A hierarchy result for counting functions

From a formula $\Phi(\mathbf{z}, \mathbf{T})$, one defines the counting function:

$$\#\Phi : \mathcal{S} \mapsto |\Phi(\mathcal{S})|.$$

Theorem (Saluja, Subrahmanyam, Thakur 1995)

On linearly ordered structures:
$\#\Sigma_0 \subsetneq \#\Sigma_1 \subsetneq \#\Pi_1 \subsetneq \#\Sigma_2 \subsetneq \#\Pi_2 = \sharp\mathrm{P}$.

Some $\sharp\mathrm{P}$-hard problems in $\#\Sigma_1$ (but existence of FPRAS at this level).

Corollary

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The first level: Enum$\cdot\Sigma_0$

Theorem

For $\varphi \in \Sigma_0$, $\textsc{Enum}\cdot\varphi$ can be enumerated with preprocessing $O(|D|^k)$ and delay $O(1)$ where k is the number of free first order variables of φ and D is the domain of the input structure.

Simple ingredients:

1. Transformation of a f.o. formula $\Phi(\mathbf{z}, T)$ into a propositional formula:
   - Try all values for first order variables: $\Phi(\mathbf{z}^*, T)$.
   - Replace the atomic formulas by their truth value.
   - Obtain a propositional formula with variables $T(\mathbf{w})$.
2. Gray Code Enumeration.

Bounded degree structure

Remark: The k-clique query is definable.
No hope to improve the $O(|D|^k)$ preprocessing.

Theorem

Let $d \in \mathbb{N}$, on d-degree bounded input structures, $\textsc{Enum}{\cdot}\Sigma_0 \in \textsc{Delay}(|D|, 1)$ where D is the domain of the input structure.

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Idea of proof:

- Another transformation: $\Phi(\mathbf{z}, T)$ seen as a propositional formula whose variables are the atoms of Φ.
- From each solution, create a quantifier free formula without free second order variables.
- Enumerate the solutions of this formula thanks to [DG 2007].

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Second level: Enum$\cdot\Sigma_1$

Theorem

$\textsc{Enum}\cdot\Sigma_1 \subseteq \textsc{DelayP}$. *More precisely,* $\textsc{Enum}\cdot\Sigma_1$ *can be computed with precomputation* $O(|D|^{h+k})$ *and delay* $O(|D|^k)$ *where* h *is the number of free first order variables of the formula,* k *the number of existentially quantified variables and* D *is the domain of the input structure.*

Idea of Proof: $\Phi(\mathbf{y}, T) = \exists \mathbf{x} \varphi(\mathbf{x}, \mathbf{y}, T)$

- Substitute values for $\mathbf{x}$. Collection of formulas of the form:

$$\varphi(\mathbf{x}^*, \mathbf{y}, T)$$

- Need to enumerate the (non necessarily disjoint) union.

The case Enum$\cdot\Pi_1$

Proposition

Unless $\mathrm{P} = \mathrm{NP}$*, there is no polynomial delay algorithm for* $\textsc{Enum}\cdot\Pi_1$*.*

Proof Direct encoding of SAT.

Hardness even:

- on the class of bounded degree structure
- if all clauses but one have at most two occurences of a second-order free variable

Tractable cases

Problem $\textsc{Enum}\cdot\Phi$ with $\Phi \in \Sigma_i$: transformation of Φ into a propositional formula $\tilde{\Phi}$.

Corollary

Let $\Phi(\mathbf{z}, T)$ be a formula, such that, for all σ structures, all propositional formulas $\tilde{\Phi}$ are either Horn, anti-Horn, affine or bijunctive. Then $\textsc{Enum}\cdot\Phi \subseteq \textsc{Delay}\mathrm{P}$.

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Example: independent sets and hitting sets.

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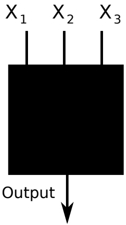
Conlusion and open problems

$\textsc{Enum}\cdot\Sigma_0 \subsetneq \textsc{Enum}\cdot\Sigma_1 \subsetneq \textsc{Enum}\cdot\Pi_1 \subsetneq \textsc{Enum}\cdot\Sigma_2 \subsetneq \textsc{Enum}\cdot\Pi_2 =$ EnumP.

- Nice but small hierarchy.
- Other tractable classes above Σ_1 (optimization operator)?
- Efficient probabilistic enumeration procedure?

Polynomial given by a black-box



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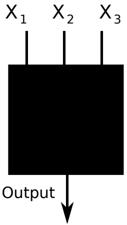

$$P(X_1, X_2, X_3) = X_1X_2 + X_1X_3 + X_2 + X_3$$

$$X_1 = 1,\ X_2 = 2,\ X_3 = 1$$

$$1 * 2 + 1 * 1 + 2 + 1$$

$$Output = 6$$

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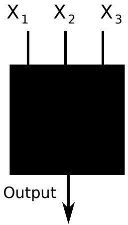

$$P(X_1, X_2, X_3) = X_1X_2 + X_1X_3 + X_2 + X_3$$

$$X_1 = -1,\ X_2 = 1,\ X_3 = 2$$

$$-1 * 1 + -1 * 2 + 1 + 2$$

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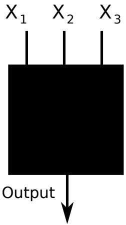

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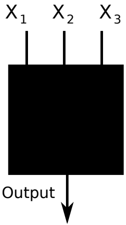

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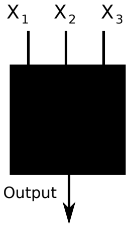

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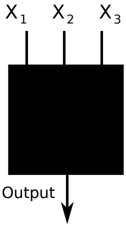

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POLYNOMIAL IDENTITY TESTING
Input: a polynomial given as a black box.
Output: decides if the polynomial is zero.

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Let P be a non zero polynomial with n variables of total degree D, if $x_1, \ldots, x_n$ are randomly chosen in a set of integers S of size $\frac{D}{\epsilon}$ then the probability that $P(x_1, \ldots, x_n) = 0$ is bounded by ϵ.

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- Garg and Schost (2009): non black-box but complexity independent from the degree of the polynomial

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Enumeration complexity: produce the monomials one at a time with a good **delay**.

Existing interpolation methods

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Multilinear polynomials

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Input: a polynomial given as a black box and two sets of variables L_1 and L_2
Output: accept if there is a monomial in the polynomial in which no variables of L_1 appear, but all of those of L_2 do.

When the polynomial is **multilinear**, this problem can be solved by finding the degree of $P_{\bar{L_1}}$ with regard to L_2: test if the degree is equal to $|L_2|$.

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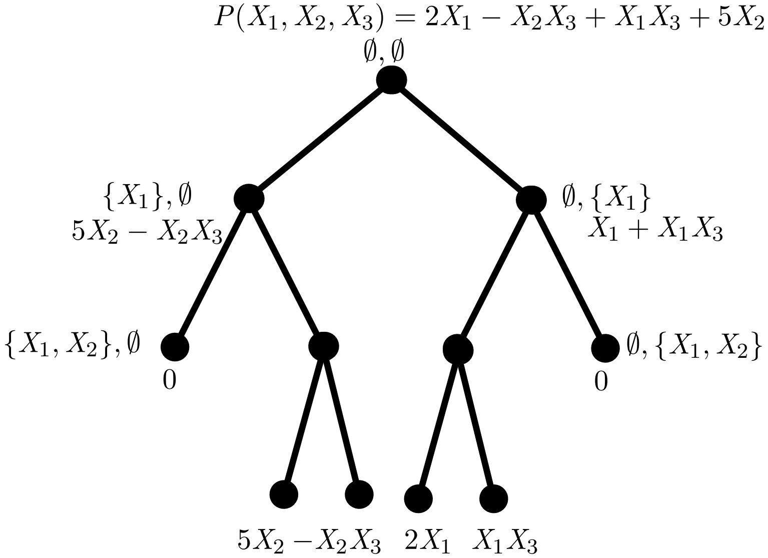

$P(X_1, X_2, X_3) = 2X_1 - X_2X_3 + X_1X_3 + 5X_2$
$\emptyset, \emptyset$
$\{X_1\}, \emptyset$
$5X_2 - X_2X_3$
$\emptyset, \{X_1\}$
$X_1 + X_1X_3$
$\{X_1, X_2\}, \emptyset$
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$\emptyset, \{X_1, X_2\}$
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$-X_2X_3$
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X_1X_3

Polynomial delay algorithm

Theorem

*Let P be a multilinear polynomial with n variables and a total degree D. There is an algorithm which computes the set of monomials of P with probability $1 - \epsilon$ and a delay **polynomial** in n, D and $\log(\epsilon)^{-1}$.*

- The algorithm can be parallelized.

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Comparison to other algorithms

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Algorithm type	Deterministic	Probabilistic	Probabilistic	Probabilistic
Number of calls	$2T$	tnD	tn^7D^4	$tnD(n + \log(\epsilon^{-1}))$
Total time	Quadratic in T	Quadratic in t	Quadratic in t	Linear in t
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Let's all do enumeration