Equivalence Constraint Satisfaction Problems*

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Abstract

The following result for finite structures Γ has been conjectured to hold for all countably infinite ω-*categorical* structures Γ: either the model-complete core Δ of Γ has an expansion by finitely many constants such that the pseudovariety generated by its polymorphism algebra contains a two-element algebra all of whose operations are projections, or there is a homomorphism f from Δ^k to Δ, for some finite $k \geq 2$, and endomorphisms $a_1, \dots, a_n$ of Δ satisfying $\forall x, y.\, a_1(f(y, x, \dots, x)) = a_2(f(x, y, x, \dots, x)) = \cdots = a_n(f(x, \dots, x, y))$.[1] This conjecture has been confirmed for all infinite structures Γ that have a first-order definition over $(\mathbb{Q}; <)$, and for all structures that are definable over the random graph. In this paper, we verify the conjecture for all structures that are definable over an equivalence relation with a countably infinite number of countably infinite classes.

Our result implies a complexity dichotomy (into NP-complete and P) for a family of constraint satisfaction problems (CSPs) which we call *equivalence constraint satisfaction problems*. The classification for equivalence CSPs can also be seen as a first step towards a classification of the CSPs for all relational structures that are first-order definable over Allen's interval algebra, a well-known constraint calculus in temporal reasoning.

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1 Introduction

The *constraint satisfaction problem* for a fixed structure Γ with finite relational signature is the following computational problem, denoted by $\mathrm{CSP}(\Gamma)$: given a finite structure I with the same signature as Γ, decide whether there is a homomorphism from I to Γ. By selecting an appropriate structure Γ, many computational problems in various areas of theoretical computer science can be formulated as $\mathrm{CSP}(\Gamma)$, for example problems from artificial intelligence, combinatorics, finite model theory, scheduling, and database theory.

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[1] In the conference version of the paper, an incorrect conjecture has been quoted; see the erratum pdf available on the web-site of the first author.

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In 1993, Feder and Vardi conjectured that CSP(Γ) is for all finite structures Γ in P or NP-complete. There has been a considerable research activity around this *dichotomy conjecture* in constraint satisfaction, producing many results of independent interest.

One of the results that came out of the attempts to prove the dichotomy conjecture is the following *universal-algebraic dichotomy*, which essentially follows from [17, 2]; also see [4]. All concepts that appear in the statement will be defined in Section 2.

► **Theorem 1.1** (follows from [17, 21]). *Let Γ be a finite relational structure. Then either*
- *the pseudovariety generated by the polymorphism algebra of the expansion of the core of Γ by constants contains a two-element algebra all of whose operations are projections, or*
- *there is a homomorphism f from Γ^k to Γ, for some finite $k \geq 2$, that satisfies*

$$\forall x, y.\, f(y, x, \ldots, x) = f(x, y, x, \ldots, x) = \cdots = f(x, \ldots, x, y)\ .$$

It is known that when Γ satisfies the first item in Theorem 1.1, then CSP(Γ) is NP-hard. Bulatov, Jeavons, and Krokhin [13] made the conjecture that for finite structures Γ with finite relational signature that do not satisfy the first item in Theorem 1.1, the problem CSP(Γ) can be solved in polynomial time. This conjecture has been called the *tractability conjecture*, and obviously the tractability conjecture implies the dichotomy conjecture. The tractability conjecture has been verified for 2-element structures [23], 3-element structures [12], undirected graphs [11], and many other classes of finite structures.

While the tractability conjecture is open for general finite structures, it turns out that a generalized version of the tractability conjecture is true for several large classes of infinite relational structures Γ. To define those classes, we need the following concepts. In this paper we say that a relational structure Γ is *first-order definable* in Δ if Γ has the same domain as Δ, and for every relation R of Γ there is a first-order formula ϕ in the signature of Δ such that ϕ holds exactly on those tuples that are contained in R. The class of all structures with a first-order definition in $(\mathbb{Q}; <)$ has been studied in [6]; the CSPs for those structures are called *temporal constraint satisfaction problems* and they can be used to model many computational problems in temporal reasoning and scheduling. The class of all structures with a first-order definition over the countable universal homogeneous graph, aka the *random graph*, has been studied in [9]. All those structures are *ω-categorical*, that is, all countable models of their first-order theory are isomorphic.

The following has been conjectured for *all* ω-categorical structures (see Conjecture 5.3 from [4]; the formulation there is different, but equivalent by Theorem 5.5.18 in [4]).

► Conjecture 1.2. Let Γ be a countable ω-categorical relational structure. Then either
1. the model-complete core of Γ has an expansion Δ by finitely many constants such that the pseudovariety generated by the polymorphism algebra of Δ contains a two-element algebra all of whose operations are projections, or
2. the model-complete core of Γ has a polymorphism f and endomorphisms a, b satisfying

$$\forall x, y.\, a_1(f(y, x, \ldots, x)) = a_2(f(x, y, x, \ldots, x)) = \cdots = a_n(f(x, \ldots, x, y))\ .$$

This conjecture generalizes the universal-algebraic dichotomy that holds for finite structures Γ. Conjecture 1.2 has been shown for all structures Γ definable over $(\mathbb{Q}; <)$ [6], or over the random graph [9]. Moreover, the two cases of Conjecture 1.2 correspond precisely to the cases that CSP(Γ) is NP-hard, or polynomial, respectively.

In this article, we show that Conjecture 1.2 holds for all structures that are first-order definable over $(D; Eq)$, where D is a countable infinite set, and Eq is an equivalence relation on D with infinitely many infinite classes. We show that also in this case the dichotomy described in the conjecture coincides with a complexity dichotomy for the corresponding CSPs. We call them *equivalence CSPs*, since solutions to an instance I of CSP(Γ) where Γ is first-order definable over $(D; Eq)$ can be represented by exhibiting an equivalence relation on the image of a mapping from I to Γ (and thus CSP(Γ) is always in NP).

Apart from the fact that $(D; Eq)$ is, besides $(\mathbb{Q}; <)$ and the random graph, one of the fundamental ω-categorical structures, there is additional motivation to specifically study the class of structures definable over $(D; Eq)$, and we describe this motivation in the following.

1.1 Motivation and Applications

1.1.1 Composing Classification Results

Suppose Δ_1 and Δ_2 are such that we have shown Conjecture 1.2 for all structures Γ that are definable over Δ_1 or definable over Δ_2. To better understand Conjecture 1.2 in general, we would like to prove that the conjecture also holds for all structures Γ that are definable over a structure Δ that is built from Δ_1 and Δ_2 in a simple way. One of the basic ways to construct a new ω-categorical structure Δ from ω-categorical structures Δ_1 and Δ_2 is to take infinitely many copies of Δ_2, to identify each element of Δ_1 with one of those copies, and to join the copies according to the relations in Δ_1. Formally, for $i \in \{1,2\}$, write D_i for the domain and τ_i for the signature of Δ_i. Suppose that τ_1 and τ_2 are disjoint (otherwise rename the symbols). Then Δ is a $\tau_1 \cup \tau_2$ structure with domain $D_1 \times D_2$. A k-ary relation $R \in \tau_2$ denotes $\{((a,b_1),\dots,(a,b_k)) \mid (b_1,\dots,b_k) \in R^{\Delta_2}, a \in D_1\}$ in Δ; a k-ary relation $R \in \tau_1$ denotes $\{((a_1,b_1),\dots,(a_k,b_k)) \mid (a_1,\dots,a_k) \in R^{\Delta_1}, b_1,\dots,b_k \in D_2\}$ in Δ.

The simplest situation for this is when $\Delta_1 = \Delta_2 = (\mathbb{N}; =)$. Note that the structure $(D; Eq)$ is isomorphic to $\big(\mathbb{N}; \{(x,y),(u,v) \mid x = u\}\big)$; that is, the relation Eq relates exactly those elements that come from the same copy of Δ_2. So the task outlined above for $\Delta_1 = \Delta_2 = (\mathbb{N}; =)$ amounts precisely to studying the class of all structures Γ definable over $(D; Eq)$.

1.1.2 Fragments of Allen's Interval Algebra

Allen's interval algebra is a formalism introduced for temporal reasoning in Artificial Intelligence [1], and plays a central role in qualitative reasoning in general. The most fundamental computational problem for Allen's interval algebra is the so-called *network satisfaction problem*, which can be viewed as the CSP for the following structure Δ: the domain $\mathbb{I}$ of Δ are the pairs $(u,v) \in \mathbb{Q}^2$ with $u < v$, and the relations of Δ are *all* binary relations R such that the 4-ary relation $\{(x,y,u,v) \mid ((x,y),(u,v)) \in R\}$ has a first-order definition in $(\mathbb{Q}; <)$. An important achievement in temporal reasoning is the complete complexity classification of the *fragments* of Allen's interval algebra in [20, 22], that is, of the constraint satisfaction problems for structures Γ obtained from Δ by removing some of the relations.

This result has been obtained *without* the universal-algebraic approach as it is used in [13, 2, 6, 9], but by a clever case distinction and heavy use of primitive positive definitions to show hardness in cases where the known algorithms do not apply. A proof based on the universal-algebraic approach would have the advantage that it would automatically yield the much stronger classification result for all structures Γ that are first-order definable in Δ. In contrast to the classification in [20], this includes structures that have relations of arity larger than two. Such a result would be a considerable extension of the result from [20], and is currently out of reach. However, for structures Γ with a first-order definition in Δ that contain the binary relation $\big\{((x,y),(u,v)) \mid y = u\big\}$ (this relation is typically denoted by m in the literature on Allen's interval algebra), a classification of the complexity of CSP(Γ) can be derived from the classification for the structures definable in $(\mathbb{Q};<)$ (see Section 5.5.4 in [4]). Note that every structure with a definition in $(D; Eq)$ is isomorphic to a structure definable over Allen's interval algebra, by the observation that $(D; Eq)$ is isomorphic to

$$\big(\mathbb{I}; \{((x,y),(u,v)) \mid x = u\}\big)\ .$$

Hence, the classification presented here is a part of the more ambitious project to classify the CSP for all structures that are first-order definable over Allen's interval algebra.

1.2 Techniques and Outline

We give a description of our proof strategy; in this description, we freely use concepts that will be introduced in Section 2. Let Γ be a structure with a first-order definition in $(D; Eq)$. If the binary relation $E(x,y)$ defined by $Eq(x,y) \wedge x \neq y$, or the binary relation $N(x,y)$ defined by $\neg Eq(x,y)$ is not primitive positive definable in Γ, then Γ must have an endomorphism that does not preserve E or that does not preserve N. It turns out that in this case Γ is degenerate, and we use a Ramsey-theoretic analysis of the endomorphisms to reduce the classification to known results (Theorem 3.3). If E and N *are* primitive positive definable, then so is Eq, and we are in the situation that the polymorphism algebra $\mathbf{A}$ of Γ has a non-trivial congruence, namely Eq. The quotient of $\mathbf{A}$ by Eq is an algebra that contains all permutations of its domain, and for such algebras Conjecture 1.2 has already been established (Theorem 3.3). Moreover, we will consider certain algebras of $\mathbf{A}$ obtained from the congruence classes of Eq, and again they contain all permutations of their domain. The central part of the paper is a universal-algebraic argument how to combine the classification results for the quotient and the congruence classes to obtain the general classification result.

2 Tools...

2.1 ...from Model Theory

In this paper we consider two kinds of first-order structures: *relational structures* (typically ω-categorical or finite, sometimes expanded with constants) and *algebras*, that is, structures with a functional signature (see Section 2.2).

Let σ and τ be signatures with $\sigma \subseteq \tau$. When Δ is a σ-structure and Γ is a τ-structure with the same domain such that $R^\Delta = R^\Gamma$ for all $R \in \sigma$, and $f^\Delta = f^\Gamma$ for all $f \in \sigma$, then Δ is called a *reduct* of Γ, and Γ is called an *expansion* of Δ. We say that Γ is a *first-order expansion* of Δ if Γ is an expansion of Δ and all relations in Γ are first-order definable over Δ. A structure Δ is called a *finite reduct* of Γ if Δ is a reduct of Γ with a finite signature. We also write (Γ, R) for the expansion of Γ by a new relation R. Given two σ-structures Γ over the domain A and Δ over the domain B, Δ is said to be an *(induced) substructure* of Γ iff (i) $B \subseteq A$, (ii) for every n-ary function symbol f in σ the function f^Δ is a restriction of f^Γ to B^n, and (iii) for every n-ary relation symbol R in σ we have $R^\Delta = R^\Gamma \cap B^n$. For two τ-structures Γ_1 and Γ_2 the *direct product* $\Delta = \Gamma_1 \times \Gamma_2$ is the τ structure on the domain $A_1 \times A_2$, where A_1 is the domain of Γ_1 and A_2 is the domain of Γ_2 such that: (i) for every n-ary relation symbol R in τ we have $((a_1^1, a_2^1), \dots, (a_1^n, a_2^n)) \in R^\Delta$ iff $(a_1^1, \dots, a_1^n) \in R^{\Gamma_1}$ and $(a_2^1, \dots, a_2^n) \in R^{\Gamma_2}$, and (ii) for every n-ary function symbol f in τ we have that $f^\Delta\big((a_1^1, a_2^1), \dots, (a_1^n, a_2^n)\big) = \big(f^{\Gamma_1}(a_1^1, \dots, a_1^n), f^{\Gamma_2}(a_2^1, \dots, a_2^n)\big)$. The direct product $\Gamma \times \Gamma$ is also denoted by Γ^2, and the k-fold product $\Gamma \times \dots \times \Gamma$, defined analogously, by Γ^k.

We say that a map h from the domain of a τ-structures Γ to the domain of a τ-structure Δ *preserves* a first-order τ-formula ϕ with free variables $x_1, \dots, x_n$ if for all elements $a_1, \dots, a_n$ of Γ such that Γ satisfies $\phi(a_1, \dots, a_n)$, Δ satisfies $\phi(h(a_1), \dots, h(a_n))$. A map $h\colon \Gamma \to \Delta$ is a *homomorphism* if it preserves all atomic τ-formulas. An *embedding* is an injective homomorphism satisfying the stronger condition that $(t_1, \dots, t_n) \in R^\Gamma$ if and only if $(h(t_1), \dots, h(t_n)) \in R^\Delta$, for all relation symbols $R \in \tau$. An *isomorphism* is a surjective embedding, and an *automorphism* of Γ is an isomorphism between Γ and itself. The set of all automorphisms of Γ is denoted by $\mathrm{Aut}(\Gamma)$. An *orbital* of $\mathrm{Aut}(\Gamma)$ is a binary relation of the form $\{(\alpha(t_1), \alpha(t_2)) \mid \alpha \in \mathrm{Aut}(\Gamma)\}$ for elements t_1, t_2 of Γ.

A first-order theory T is *model-complete* if every embedding between models of T preserves all first-order formulas. We say that a structure is model-complete if its theory is model-complete. A homomorphism of a structure Γ into itself is called an *endomorphism*. A structure Γ is called a *core* if all endomorphisms of Γ are embeddings. A structure Δ is called a *core of* Γ if Δ is a core as well as Γ and Δ are *homomorphically equivalent*, that is, there is a homomorphism from Γ to Δ and a homomorphism from Δ to Γ.

▶ **Theorem 2.1** ([3]). *Every ω-categorical structure Γ is homomorphically equivalent to an ω-categorical model-complete core Δ. All model-complete cores of Γ are isomorphic.*

When Δ is a model-complete core with finite relational signature, c is an element of the domain of Δ, and $(\Delta, \{c\})$ is the expansion of Δ by the unary relation $\{c\}$, then there is a polynomial-time reduction from $\mathrm{CSP}((\Delta, \{c\}))$ to $\mathrm{CSP}(\Delta)$.

A structure Γ is *homogeneous* if every isomorphism between finite substructures of Γ can be extended to an automorphism of Γ. Homogeneous structures with a finite signature are ω-categorical. Good introductions to ω-categoricity can be found in [14, 18].

2.2 . . . from Universal Algebra

Let Γ be a structure. Homomorphisms from Γ^k to Γ are called *polymorphisms* of Γ. When R is a relation over the set D, we say that $f\colon D^k \to D$ *preserves* R if f is a polymorphism of $(D; R)$, and that f *violates* R otherwise. The set of all polymorphisms of a relational structure Γ, denoted by $\mathrm{Pol}(\Gamma)$, forms an algebraic object called a *clone*. A clone on some fixed domain D is a set of operations on D containing all projections and closed under composition. A clone $\mathscr{C}$ is *locally closed* iff for all natural numbers n, for all n-ary operations g on D, if for all finite $B \subseteq D^n$ there exists an n-ary $f \in \mathscr{C}$ which agrees with g on B, then $g \in \mathscr{C}$. A set of operations $\mathscr{F}$ *locally generates* an operation f if f is in the smallest locally closed clone containing $\mathscr{F}$, denoted by $\langle \mathscr{F} \rangle$.

▶ Proposition 2.2 (see e.g. Propositions 5.1.1 and 5.2.1 in [4]). Let $\mathscr{F}$ be a set of operations on some domain D. Then the following are equivalent: (i) $\mathscr{F}$ is the polymorphism clone of a relational structure; and (ii) $\mathscr{F}$ is a locally closed clone. Moreover, $\mathscr{F}$ locally generates g if and only if g preserves all relations preserved by $\mathscr{F}$.

Primitive positive formulas over a signature τ are first-order formulas built exclusively from conjunction, existential quantifiers, equality and relation symbols from τ. The first part of the following theorem is from [7], the second part is a straightforward consequence of Theorem 5.2.3 and Lemma 5.3.5 in [4].

▶ **Theorem 2.3.** *A relation R has a primitive positive definition in an ω-categorical structure Γ if and only if R is preserved by all polymorphisms of Γ. An orbital O of $\mathrm{Aut}(\Gamma)$ has a primitive positive definition in Γ if and only if O is preserved by all endomorphisms of Γ.*

An algebra $\mathbf{A}$ whose set of operations equals $\mathrm{Pol}(\Gamma)$ is called a *polymorphism algebra* of Γ; note that polymorphism algebras are not unique, since we can freely rename the operations in $\mathbf{A}$ and still obtain a polymorphism algebra; however, since such a renaming is in our context always irrelevant, we also call $\mathbf{A}$ *the* polymorphism algebra of Γ, and denote it by $\mathrm{Alg}(\Gamma)$, as if it were unique.

A *congruence* of an algebra $\mathbf{A}$ with domain A is an equivalence relation on A that is preserved by all operations in $\mathbf{A}$. Let $\mathbf{A}, \mathbf{B}$ be algebras with the same signature τ. If there is a surjective homomorphism h from $\mathbf{A}$ to $\mathbf{B}$, then $\mathbf{B}$ is called a *homomorphic image* of $\mathbf{A}$. The kernel $\{(a,b) \in A^2 \mid h(a) = h(b)\}$ of h is a congruence on $\mathbf{A}$. Any congruence of $\mathbf{A}$ gives rise to a *quotient algebra* of $\mathbf{A}$, denoted by $\mathbf{A}/\theta$, whose domain A/θ consists of the equivalence classes of θ, and which has the same signature as $\mathbf{A}$ so that $f^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = f^{\mathbf{A}}(a_1, \dots, a_n)/\theta$, for every $f \in \tau$ and all $a_1, \dots, a_n \in A$; here, a_i/θ is the equivalence class of θ containing a_i.

A class $\mathcal{V}$ of algebras with the same signature is called a *pseudovariety* if $\mathcal{V}$ contains all homomorphic images, subalgebras, and finite direct products of algebras in $\mathcal{V}$. The smallest pseudovariety containing $\mathbf{A}$ is called the pseudovariety *generated* by $\mathbf{A}$, and denoted by $\mathcal{V}(\mathbf{A})$. The relevance of pseudovarieties in constraint satisfaction comes from the following fact, which is a consequence of Theorems 5.5.6 and 5.5.15 in [4].

▶ Proposition 2.4. Let Γ and Δ be ω-categorical structures. If there exists an algebra $\mathbf{A}$ in $\mathcal{V}(\mathrm{Alg}(\Gamma))$ with the same domain as Δ and whose operations are polymorphisms of Δ, then there is for every finite reduct Δ' of Δ a finite reduct Γ' of Γ such that $\mathrm{CSP}(\Delta')$ reduces to $\mathrm{CSP}(\Gamma')$ in polynomial time.

The following lemma is a consequence that will be used several times.

▶ **Lemma 2.5.** *Let Γ be an ω-categorical structure such that $\mathcal{V}(\mathrm{Alg}(\Gamma))$ contains a two-element algebra all of whose operations are projections. Then Γ has a finite reduct Γ' such that $\mathrm{CSP}(\Gamma')$ is NP-hard.*

Proof. Suppose $\mathcal{V}(\mathrm{Alg}(\Gamma))$ contains an algebra $\mathbf{A}$ with domain $\{0,1\}$ all of whose operations are projections. The operations in $\mathbf{A}$ preserve the relation $R = \{(1,0,0),(0,1,0),(0,0,1)\}$. The problem $\mathrm{CSP}((\{0,1\};R))$ is known under the name positive 1-IN-3-3SAT, and NP-hard [16]. Now the statement follows from Proposition 2.4. ☐

We remark that the two cases in Conjecture 1.2 are always disjoint; this follows from Proposition 5.6.9 and 5.6.10 in [4]; we state it here for easy reference.

▶ Proposition 2.6. Let Γ be an ω-categorical model-complete core with a polymorphism f and an automorphism α satisfying $\forall x_1, \dots, x_n.\, f(x_1, \dots, x_n) = \alpha(f(x_2, \dots, x_n, x_1))$. Then for every expansion Δ of Γ by constants, every algebra in the pseudovariety generated by the polymorphism algebra of Δ contains an operation that is not a projection.

2.3 ... from Ramsey Theory

We use Ramsey theory to show that polymorphisms must *behave canonically* on large parts of the domain; canonical behavior will be introduced below. A wider introduction to canonical operations can be found in [8] and [4]; the definitions we present here are tailored towards applications for equivalence constraint satisfaction problems.

▶ **Definition 2.7.** Let Γ and Δ be structures over the same domain D. A *behavior* of a binary operation $f\colon D^2 \to D$ on $S \subseteq D$ is a partial function that sends a pair of orbitals (O_1, O_2) of $\mathrm{Aut}(\Gamma)$ to an orbital O_3 of $\mathrm{Aut}(\Delta)$ such that for all $(a_1, a_2) \in S^2 \cap O_1$ and $(b_1, b_2) \in S^2 \cap O_2$ we have $(f(a_1, a_2), f(b_1, b_2)) \in O_3$. A behavior is *canonical* if it is a total function. An operation $f\colon D^2 \to D$ is canonical on S as a function from Γ^2 to Δ if it has a canonical behavior on S. If a behavior of f on S sends (O_1, O_2) to O_3, then we write

$f(O_1, O_2) =_S O_3$. *Canonical unary operations* and their behavior are defined analogously. An operation f *behaves as* an operation g on S if they share the same behavior on S. In these definitions, we might omit to specify S in case that $S = D$.

Let Γ, Δ be finite τ-structures. We write $\binom{\Delta}{\Gamma}$ for the set of all substructures of Δ that are isomorphic to Γ. When Γ, Δ, Θ are τ-structures, then we write $\Theta \to (\Delta)^{\Gamma}_{r}$ if for all colorings $\chi\colon \binom{\Theta}{\Gamma} \to \{1, \ldots, r\}$ there exists $\Delta' \in \binom{\Theta}{\Delta}$ such that χ is constant on $\binom{\Delta'}{\Gamma}$.

▶ **Definition 2.8.** A class of finite relational structures $\mathcal{C}$ that is closed under isomorphisms and substructures is called *Ramsey* if for all $\Gamma, \Delta \in \mathcal{C}$ and for every finite $k \geq 1$ there exists a $\Theta \in \mathcal{C}$ such that $\Theta \to (\Delta)^{\Gamma}_{k}$.

A structure Γ is called *Ramsey* if the class of all finite structures that embed into Γ is Ramsey. A structure is called *ordered* if it carries a binary relation that denotes a linear order on its domain. When Γ is Ramsey and ordered, then the following theorem allows us to work with canonical polymorphisms of the expansion of Γ by constants.

▶ **Theorem 2.9** ([10]). *Let Γ be a homogeneous ordered Ramsey structure with finite relational signature and domain D. Let $c_1, \ldots, c_m \in D$, and let $f\colon D^2 \to D$ be any operation. Then $\{f\} \cup \mathrm{Aut}((\Gamma, c_1, \ldots, c_m))$ locally generates an operation that is canonical as a function from $(\Gamma, c_1, \ldots, c_m)^2$ to Γ, and which is identical with f on all tuples containing only values from $c_1, \ldots, c_m$.*

3 Equivalence Constraint Satisfaction Problems

We consider structures Γ with a first-order definition in $(D; Eq)$, where D is a countably infinite domain and Eq is an equivalence relation on D with infinitely many infinite equivalence classes. In the following, such structures Γ are called *equivalence constraint languages*.

We define $E(x, y) := Eq(x, y) \wedge x \neq y$ and $N(x, y) := \neg Eq(x, y)$. Note that $Eq(x, y)$ has the primitive positive definition $\exists z(E(x, z) \wedge E(z, y))$ over $(D; E)$, and it follows in particular that every operation that preserves E also preserves Eq.

▶ **Example 3.1.** An example of an equivalence constraint language is $\Gamma := (D; \{(x, y, z) \mid E(x, y) \vee N(y, z)\})$; it follows from our classification result (Corollary 7.5) that $\mathrm{CSP}(\Gamma)$ is in P. On the other hand, consider $\Delta := (D; R)$ where $R = \big\{(x, y, z) \mid (Eq(x, y) \vee Eq(y, z)) \wedge (N(x, y) \vee N(y, z))\big\}$. It follows from Corollary 7.5 that $\mathrm{CSP}(\Delta)$ is NP-complete.

When R_1, R_2 are binary relations over D and $a = (a_1, a_2) \in D^2$ and $b = (b_1, b_2) \in D^2$, we write $a\binom{R_1}{R_2}b$ to denote that $R_1(a_1, b_1)$ and $R_2(a_2, b_2)$.

▶ Observation 3.2. Let $f\colon D^2 \to D$ be a binary function that preserves E, and let $a, b, c \in D^2$ such that $a\binom{E}{=}b$ and $b\binom{=}{E}c$. Then $E(f(a), f(b))$ or $E(f(b), f(c))$.

Proof. Since f preserves Eq, we have $Eq(f(a), f(b))$ and $Eq(f(b), f(c))$. Since $a\binom{E}{E}c$ and f preserves E, we have $E(f(a), f(c))$. Thus, $f(a) \neq f(b)$ or $f(b) \neq f(c)$, which proves the statement. ☐

It is easy to see that $(D; Eq)$ is a homogeneous structure and therefore ω-categorical. Every structure with a first-order definition in an ω-categorical structure is again ω-categorical (see e.g. [18]); thus, all equivalence constraint languages are ω-categorical. All equivalence constraint languages are preserved by the automorphisms of $(D; Eq)$, and we make the following convention: a set of operations F *generates* an operation g if $F \cup \mathrm{Aut}((D; Eq))$ locally generates g (see Section 2.2). Moreover, we say that f generates g if $\{f\}$ generates g.

	id	const	e_{EE}	e_{NN}	$e_{N=}$
=	=	=	=	=	=
N	N	=	E	N	N
E	E	=	E	N	=

Figure 1 Canonical unary behaviors.

A linear order $<$ on $(D; Eq)$ is *convex* if for all $a < b < c$ in D, if $(a, c) \in Eq$, then $(a, b) \in Eq$. Expansions $(D; Eq, <)$ of $(D; Eq)$ by a convex linear order $<$ are Ramsey (see [19], Corollary 6.8).

An important subclass of equivalence constraint languages is the class of *equality constraint languages*, i.e., structures with a first-order definition over $(D; =)$. We will use the following theorem, which is due to [5], in a formulation from [4] (a combination of Theorem 6.3.3 and 5.5.18 in [4]). Note that a countably infinite relational structure is isomorphic to an equality constraint language if and only if it is preserved by all permutations of its domain [5].

▶ **Theorem 3.3** (of [5]). *Let Γ be an equality constraint language. Then exactly one of the following two cases applies.*

- *Γ is a model-complete core and $\mathcal{V}(\mathrm{Alg}(\Gamma))$ contains a two-element algebra whose operations are projections. In this case, Γ has a finite reduct Γ' such that $\mathrm{CSP}(\Gamma')$ is NP-complete.*
- *Γ has a binary polymorphism f and an automorphism α satisfying $\forall x, y.\, f(x, y) = \alpha(f(y, x))$; in fact, f can be chosen to be either constant or injective. Moreover, for all finite reducts Γ' of Γ the problem $\mathrm{CSP}(\Gamma')$ is in P.*

4 Endomorphisms

In this section we show that if an equivalence constraint language Γ has an endomorphism that violates E or N, then Γ also has one out of five canonical endomorphisms described in the following. This result will be an important first step in our complexity classification, as we will see in Section 5.

The five mentioned behaviors of canonical unary operations are denoted by *id*, *const*, e_{EE}, e_{NN}, and $e_{N=}$ and presented in Figure 1. For example, we require that $e_{EE}(N) = E$ and $e_{EE}(E) = E$. It is clear that for each of those five behaviors there exists a function from $D \to D$ with this behavior. We also use the symbols *id*, *const*, e_{EE}, e_{NN}, and $e_{N=}$ to denote a function with the respective behavior; since any two functions who have the same of these behaviors generate each other, the precise choice of those functions will not be important.

To prove the main result of this section, Theorem 4.5, we use Ramsey theory via Theorem 2.9 as follows. When e violates E or N, then there are $c_1, c_2 \in D$ such that $E(c_1, c_2)$ and $\neg E(e(c_1), e(c_2))$, or $N(c_1, c_2)$ and $\neg N(e(c_1), e(c_2))$. Let $<$ be a convex linear order on D such that $c_1 < c_2$; as mentioned before, $(D; Eq, <)$ is Ramsey. By Theorem 2.9, the operation e generates an operation f that is canonical as a function from $(D; Eq, <, c_1, c_2)$ to $(D; Eq, <)$ (and hence also canonical as a function from $(D; Eq, <, c_1, c_2)$ to $(D; Eq)$) and still violates E or N. We say that f has *behavior B between two points $x, y \in D$* if f has behavior B on $\{x, y\}$.

▶ **Lemma 4.1.** *Let $f : D \to D$ be canonical as a function from $(D; Eq, <, c_1, c_2)$ to $(D; Eq)$. If f behaves as the identity on all infinite orbits, and if it behaves as the identity between the constants c_1, c_2 and all other points, then it preserves N.*

Proof. Since f violates N we have that $Eq(f(c_1), f(c_2))$. Let c_3 be such that $E(c_1, c_3)$ and $N(c_2, c_3)$. Then $E(f(c_1), f(c_3))$ and $N(f(c_2), f(c_3))$, contradicting transitivity of Eq. ☐

▶ **Lemma 4.2.** *Let $f\colon D \to D$ be canonical as a function from $(D; Eq, <, c_1, c_2)$ to $(D; Eq)$. If f violates E and behaves as the identity on all infinite orbits, and if it behaves as the identity between the constants c_1, c_2 and all other points, then it generates $e_{N=}$.*

Proof. Since f violates E we have $E(c_1, c_2)$ and either $f(c_1) = f(c_2)$ or $N(c_1, c_2)$. We first show that the second case is impossible. There is c_3 such that $E(c_1, c_3)$ and $E(c_2, c_3)$. Hence, $E(f(c_1), f(c_3))$, $E(f(c_2), f(c_3))$, and $N(f(c_1), f(c_2))$, contradicting transitivity of Eq.

So $f(c_1) = f(c_2)$, and in particular f preserves Eq. We show by local closure that f generates $e_{N=}$. Let F be a finite subset of D. Let e be an operation generated by f such that the cardinality k of the set $\{(x, y) \in E \cap F^2 \mid e(x) = e(y)\}$ is maximal. If $k = |E \cap F^2|$, then e behaves on F as $e_{N=}$ and we are done. Otherwise, suppose there is $(x, y) \in E \cap F^2$ such that $e(x) \neq e(y)$. Since f and therefore e preserve Eq, we must have $E(e(x), e(y))$. Let α be an automorphism of $(D; Eq)$ that maps $(e(x), e(y))$ to (c_1, c_2). Then the mapping $e' := f \circ \alpha \circ e$ maps x and y to the same element, and $\{(x, y) \in E \cap F^2 \mid e'(x) = e'(y)\} > k$, contradicting the choice of e. ☐

We now analyze canonical behavior of injective functions in the case without constants.

▶ **Lemma 4.3.** *Let $f\colon D \to D$ be canonical as a function from $(D; Eq, <, c_1, c_2)$ to $(D; Eq)$. Let S be an infinite orbit of $(D; Eq, <, c_1, c_2)$ that induces a copy of $(D; Eq)$. If f does not behave as the identity on S, then it generates e_{EE}, $e_{N=}$, e_{NN}, or a constant operation.*

Proof. Since all orbitals of $(D; Eq)$ are symmetric, the unary operation f is canonical as a function from $(D; Eq, <)$ to $(D; Eq)$ if and only if it is canonical as a function from $(D; Eq)$ to $(D; Eq)$. If f does not behave as the identity on S, then f violates E or N on S. If f violates N, then either $f(N) =_S (=)$ or $f(N) =_S E$. In the first case we must have $f(E) =_S (=)$, and f is constant on S. Since S induces a copy of $(D; Eq)$, it follows by local closure that f generates a constant operation. So suppose that $f(N) =_S E$. If $f(E) =_S E$ then f behaves as e_{EE} on S, and therefore generates e_{EE}. The case that $f(E) =_S N$ is impossible, since for u, v, w with $N(u, v)$, $N(u, w)$, $E(v, w)$ this would imply $E(f(u), f(v))$, $E(f(u), f(w))$, $N(f(v), f(w))$, contradicting transitivity of Eq.

So suppose that f preserves N but violates E on S. If $f(E) =_S (=)$ then f behaves as $e_{N=}$ on S and therefore generates $e_{N=}$. Otherwise, $f(E) =_S N$; in this case f behaves as e_{NN} on S, and therefore generates e_{NN}. ☐

Next, we analyze canonical behavior of operations in the presence of two constants.

▶ **Lemma 4.4.** *Let $c_1, c_2 \in D$ be constants and let $f\colon D \to D$ be canonical as a function from $(D; Eq, <, c_1, c_2)$ to $(D; Eq)$. Let O be an infinite orbit of $(D; Eq, <, c_1, c_2)$. If f does not behave as the identity on O, or if it does not behave as the identity between one of c_1, c_2 and a point from O, then it generates e_{EE}, $e_{N=}$, e_{NN}, or a constant operation.*

Proof. Let P be an orbit of $(D; Eq, <, c_1, c_2)$ that induces a copy of $(D; Eq)$ in $(D; Eq)$. We assume that f behaves as the identity on P; otherwise, we are done by Lemma 4.3. Between any $u \in D \setminus P$ and any $v \in P$, f must behave as the identity. To see this, observe that necessarily $N(u, v)$. Suppose that $u < v$; the case that $v < u$ is analogous. Suppose for contradiction that $Eq(f(u), f(v))$. Pick a $v' \in P \setminus \{v\}$ such that $N(u, v')$ and $N(v, v')$. Then $u < v'$ because $<$ is convex. Since f is canonical we have $Eq(f(u), f(v'))$. Since f behaves as the identity on P, we have $N(f(v), f(v'))$. This contradicts transitivity of Eq. We conclude that $N(f(u), f(v))$ and hence f behaves as the identity between any $u \in D \setminus P$ and $v \in P$.

First suppose that f does not behave as the identity on O. As we have observed above, we are done if O induces in $(D; Eq)$ a structure that is isomorphic to $(D; Eq)$. Otherwise, there exists a $c \in \{c_1, c_2\}$ such that $E(u, c)$ for all $u \in O$. Since f does not behave as the identity on O we have either $f(E) =_O (=)$ or $f(E) =_O N$. In the first case, by local closure f generates $e_{N=}$. In the second case, f generates e_{NN}, again by local closure.

Now suppose that f does not behave as the identity between one of the constants $c \in \{c_1, c_2\}$ and a point p from O. We have already shown in the first paragraph that we are done when O induces in $(D; Eq)$ a structure isomorphic to $(D; Eq)$. Therefore, $E(p, c)$ for all $p \in O$. If $f(E) =_O (=)$ then f generates $e_{N=}$, and if $f(E) =_O N$ then f generates e_{NN}. ◻

▶ **Theorem 4.5.** *Any $e\colon D \to D$ violating E or N generates e_{EE}, e_{NN}, $e_{N=}$, or a constant operation.*

Proof. Since e violates E or N, there are $c_1, c_2 \in D$ such that $E(c_1, c_2)$ and not $E(e(c_1), e(c_2))$, or $N(c_1, c_2)$ and $Eq(e(c_1), e(c_2))$. By Theorem 2.9, the operation e generates an operation f that is canonical as a function from $(D; Eq, <, c_1, c_2)$ to $(D; Eq)$ and still violates E or N. Then by Lemma 4.1 and by Lemma 4.2, either

- f generates $e_{N=}$, and we are done, or
- there is an infinite orbit O such that f does not behave as the identity on O, or
- there is an infinite orbit O such that f does not behave as the identity between one of the constants $c \in \{c_1, c_2\}$ and a point from O.

In the last two cases f generates e_{EE}, e_{NN}, $e_{N=}$, or a constant operation by Lemma 4.4. ◻

5 Hardness

This section has two parts: we first use the results from the previous section to show that we can focus on equivalence constraint languages where E and N are primitive positive definable. In the second part, we use Theorem 3.3 in two different ways to isolate two groups of first-order expansions of $(D; E, N)$ that have NP-hard CSPs, and correspond to Item 1 of Conjecture 1.2. This will be complemented in the next sections by the proof that the remaining first-order expansions of $(D; E, N)$ are preserved by a binary polymorphism f satisfying Item 2 of Conjecture 1.2, and correspond to polynomial-time tractable equivalence constraint satisfaction problems.

▶ **Lemma 5.1.** *Let Γ be first-order definable in $(D; Eq)$, and let Δ be the model-complete core of Γ. Then one of the following holds:*

- *the pseudovariety generated by the polymorphism algebra of Δ contains a two-element algebra all of whose operations are projections. In this case, there exists a finite reduct Γ' of Γ such that $\mathrm{CSP}(\Gamma')$ is NP-complete;*
- *Δ has a polymorphism f and an automorphism α satisfying $\forall x, y.\, f(x, y) = \alpha(f(y, x))$. In this case, for every finite reduct Γ' of Γ we have that $\mathrm{CSP}(\Gamma')$ is in P;*
- *both E and N have a primitive positive definition in Γ.*

Proof. Consider first the case that Γ has an endomorphism f that violates E or N. By Theorem 4.5 we obtain that f generates an operation e which is from $\{e_{EE}, e_{NN}, e_{N=}\}$ or a constant operation. By Theorem 2.2, e is an endomorphism of Γ. If e is constant, then we are in Case 2 and done. Otherwise, the structure Δ induced by the image of e in Γ is infinite and preserved by all permutations, and hence an equality constraint language. Moreover, Δ is a model-complete core of Γ and the statement follows directly from Theorem 3.3.

$\min_{(E,=)}$	$=$	E	N
$=$	$=$	E	?
E	E	E	?
N	?	?	N

$\min_{(N,Eq)}$	$=$	E	N
$=$	$=$	?	N
E	?	E	N
N	N	N	N

$\min_{(N,E,=)}$	$=$	E	N
$=$	$=$	E	N
E	E	E	N
N	N	N	N

Figure 2 Important behaviors: $\min_{(E,=)}$ (left), $\min_{(N,Eq)}$ (middle), and $\min_{(N,E,=)}$ (right).

Now suppose that the orbitals E or N of $(D; Eq)$ are preserved by all endomorphisms; in particular, they are preserved by all automorphisms, and hence form orbitals of $\mathrm{Aut}(\Gamma)$. By Theorem 2.3, E and N must be primitive positive definable. $\square$

To classify first-order expansions of $(D; E)$ we use Theorem 3.3 in two different ways. The first way is via the following observation, whose proof we leave to the reader.

▶ Proposition 5.2. Let Γ be a first-order expansion of $(D; E)$. Then Eq is a congruence of $\mathbf{A} := \mathrm{Alg}(\Gamma)$ and the algebra $\mathbf{B} := \mathbf{A}/Eq$ contains all permutations of its domain.

Another way how Theorem 3.3 comes into play is as follows; again, the proof is straightforward and left to the reader.

▶ Proposition 5.3. Let Γ be a first-order expansion of $(D; E)$, and let $c \in D$ be arbitrary. Then for any $c \in D$, the set $\{d \in D \mid E(c, d)\}$ induces a subalgebra $\mathbf{B}$ of $\mathbf{A} := \mathrm{Alg}((\Gamma, c))$ that contains all permutations of its domain.

By combining those results we prove that either Γ satisfies Item 1 of Conjecture 1.2, or it has certain binary polymorphisms. Three important behaviors of binary operations, $\min_{(E,=)}$, $\min_{(N,Eq)}$, and $\min_{(N,E,=)}$ are depicted in Figure 2, which should be read analogously to Figure 1. For example, we require that $\min_{(N,Eq)}(N, E) = \min_{(N,Eq)}(E, N) = N$. The name of $\min_{(N,E,=)}$ comes from the observation that it equals the minimum operation with respect to the order $N < E < (=)$. The existence of operations with these behaviors follows from Proposition 6.1.

▶ Proposition 5.4. Let Γ be a first-order expansion of $(D; E, N)$, and let $c \in D$ be arbitrary. Then Γ is a model-complete core, and either $\mathcal{V}(\mathrm{Alg}((\Gamma, c)))$ contains a two-element algebra all of whose operations are projections, and Γ has a finite reduct Γ' such that $\mathrm{CSP}(\Gamma')$ is NP-hard, or Γ is preserved by

1. an operation f with the behavior $\min_{(E,=)}$ on $\{d \in D \mid E(c, d)\}$, and
2. an operation with the behavior $\min_{(N,Eq)}$.

Proof. Since Γ contains E and N, every endomorphism of Γ behaves as the identity. Hence, every endomorphism of Γ is locally generated by $\mathrm{Aut}(\Gamma)$. By Theorem 3.6.11 in [4], the structure Γ is a model-complete core.

The subalgebra $\mathbf{A}$ induced by $\{d \mid E(c, d)\}$ in $\mathbf{B} := \mathrm{Alg}((\Gamma, c))$ contains a unary function for each permutation of its domain (Proposition 5.3). Let Δ be the structure with the same domain as $\mathbf{A}$ that contains all the relations that are preserved by the operations in $\mathbf{A}$, and let $\mathbf{A}'$ be the polymorphism algebra of Δ. If $\mathcal{V}(\mathbf{A}')$ contains a two-element algebra all of whose operations are projections, then so does $\mathcal{V}(\mathbf{A})$ since every operation of $\mathbf{A}$ is also an operation of $\mathbf{A}'$. In this case, also $\mathcal{V}(\mathbf{B})$ contains this two-element algebra, and by Lemma 2.5 there is a finite reduct Γ' of Γ such that $\mathrm{CSP}((\Gamma', \{c\}))$ is NP-hard. Since Γ is a model-complete core, Theorem 2.1 shows that $\mathrm{CSP}(\Gamma')$ is NP-hard.

If $\mathcal{V}(\mathbf{A}')$ does not contain a two-element algebra all of whose operations are projections, then Theorem 3.3 implies that Δ has either a binary injective or constant polymorphism f. Since Γ contains the relations E and N, all its endomorphisms are injective, and therefore also the unary operations in $\mathbf{B}$, in $\mathbf{A}$, and in $\mathbf{A}'$ are injective. This implies that f is binary injective. Let τ be the signature of $\mathbf{A}$. Since the operations in $\mathbf{A}$ locally generate the operations in $\mathbf{A}'$ (Proposition 2.2), it follows that for every finite subset S of the domain of $\mathbf{A}$ there exists a $g \in \tau$ such that $g^{\mathbf{A}}$ behaves as f on S; since f is binary injective, $g^{\mathbf{B}}$ therefore has the behavior $\min_{(E,=)}$ on S as a function over Γ. By an easy compactness argument (see Lemma 3.1.8 in [4]), Γ has a polymorphism with the behavior $\min_{(E,=)}$ on all of $\{d \mid E(c,d)\}$, satisfying the condition of Item 1 in the statement.

The proof that Γ also has a polymorphism with behavior $\min_{(N,Eq)}$ is similar, based on the fact that Eq is a congruence of $\mathbf{C} := \mathrm{Alg}(\Gamma)$, and that the algebra $\mathbf{D} := \mathbf{C}/Eq$ contains all permutations of its domain. Similarly as above we argue that for every finite subset S of the domain of $\mathbf{D}$ there is an operation in $\mathbf{D}$ that behaves as $\min_{(N,Eq)}$ on the union of the classes of $\mathbf{D} = \mathbf{C}/Eq$ that correspond to elements in S (unless $\mathcal{V}(\mathbf{D})$ contains a two-element algebra all of whose operations are projections). As above, a compactness argument gives the existence of an operation in $\mathbf{D}$ that behaves as $\min_{(N,Eq)}$ on the entire domain of Γ. ◻

6 Tractability

In this section we show that equivalence CSPs that have a polymorphism with the behavior $\min_{(N,E,=)}$ can be solved in polynomial time.

▶ Proposition 6.1. There is a binary function f that is canonical as a function from $(D; Eq)^2$ to $(D; Eq)$ with the behavior $\min_{(N,E,=)}$. This function can be chosen such that there is an automorphism α of $(D; Eq)$ satisfying $\forall x, y.\, f(x, y) = \alpha(f(y, x))$.

Proof. Observe that the structure $(D; Eq)^2$ is again an equivalence relation on a countable set with infinitely many infinite classes, and by ω-categoricity there is an isomorphism i between $(D; Eq)^2$ and $(D; Eq)$. This isomorphism has the behavior $\min_{(N,E,=)}$. Let $\beta \colon D^2 \to D^2$ be defined by $(x, y) \mapsto (y, x)$. Then β is an automorphism of $(D; Eq)^2$, and $\alpha := i \circ \beta \circ i^{-1}$ is an automorphism of $(D; Eq)$ such that $i(x, y) = \alpha(i(y, x))$. ◻

The operation f whose existence is shown in Proposition 6.1 will also be denoted by $\min_{(N,E,=)}$, i.e., we use the same symbol for this operation and its behavior.

▶ Proposition 6.2. Let Δ be a structure that is first-order definable in $(D; Eq)$, with finite relational signature, and polymorphism $\min_{(N,E,=)}$. Then $\mathrm{CSP}(\Delta)$ can be solved in polynomial time.

Proof. We use Proposition 2.4, and Theorem 3.3. An operation f is called *essentially injective* if it can be obtained from an injective operation by adding dummy variables (that is, the function value of f does not depend on those additional arguments of f). Let $\mathbf{A}$ be an algebra with domain $\mathbb{N}$ whose operations are precisely the essentially injective operation over $\mathbb{N}$. It is easy to verify that the operations of $\mathbf{A}$ form a locally closed clone, and hence, by Proposition 2.2, there exists a relational structure Γ with polymorphism algebra $\mathbf{A}$. We will show there is an algebra $\mathbf{B}$ with domain D in the pseudovariety generated by $\mathbf{A}$ such that all operations of $\mathbf{B}$ preserve Δ. It then follows from Proposition 2.4 that there exists a finite signature reduct Γ' of Γ such that $\mathrm{CSP}(\Delta)$ reduces to $\mathrm{CSP}(\Gamma')$. Polynomial-time tractability of $\mathrm{CSP}(\Gamma')$ follows from Theorem 3.3.

The relation $C = \{((u_1, u_2), (v_1, v_2)) \in \mathbb{N} \mid u_1 = v_1\}$ is a congruence of $\mathbf{A}^2$, with infinitely many infinite congruence classes. Let b be any bijection between the congruence classes of C and the domain D of Δ, and let $\mathbf{B}$ be the homomorphic image of $\mathbf{A}^2$ with respect to the map b. We claim that every operation $f^{\mathbf{B}}$ of $\mathbf{B}$ preserves Δ. If there are $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that $f^{\mathbf{A}} \colon \mathbb{N}^n \to \mathbb{N}$ satisfies $f(x_1, \ldots, x_n) = g(x_{i_1}, \ldots, x_{i_k})$ for all $x_1, \ldots, x_n \in \mathbb{N}$, then it clearly suffices to verify the claim for g instead of f. Since $f^{\mathbf{A}}$ is essentially injective, we can therefore assume that $f^{\mathbf{A}}$ is injective. Then $f^{\mathbf{B}}(x^1, \ldots, x^n)$ behaves as $\min_{(N,E,=)}(x_1, \min_{(N,E,=)}(x_2, \ldots, \min_{(N,E,=)}(x_{n-1}, x_n) \ldots))$: indeed,

$$\begin{aligned} Eq(f^{\mathbf{B}}(x^1, \ldots, x^n), f^{\mathbf{B}}(y^1, \ldots, y^n)) &\Leftrightarrow f^{\mathbf{A}}(x^1_1, \ldots, x^n_1) = f^{\mathbf{A}}(y^1_1, \ldots, y^n_1) \\ &\Leftrightarrow x^i_1 = y^i_1 \text{ for all } i \le n \\ &\Leftrightarrow Eq(x^i, y^i) \text{ for all } i \le n\,. \end{aligned}$$

Since Δ is preserved by $\min_{(N,E,=)}$, it is also preserved by $f^{\mathbf{B}}$. ◻

7 Generating $\min_{(N,E,=)}$

In this section we show that when f has the behavior $\min_{(E,=)}$ on $\{d \in D \mid E(c, d)\}$ for some $c \in D$ and g has the behavior $\min_{(N,Eq)}$, then $\{f, g\}$ generates $\min_{(N,E,=)}$. Some of the proofs in this section have been omitted and can be found in the full version of the paper.

▶ **Lemma 7.1.** *Let Γ be a first-order expansion of $(D; E)$, and $c \in D$. Suppose that Γ has a polymorphism that behaves as $\min_{(E,=)}$ on $\{d \in D \mid E(c, d)\}$. Then Γ is also preserved by $\min_{(E,=)}$.*

In the proof of Lemma 7.1 we use the following lemma, which is inspired by similar statements in [9]. We remark that Item 2 in the statement below is formally unrelated to the notion of *independence* as studied in [15], but similar in spirit.

▶ **Lemma 7.2.** *Let Γ be a first-order expansion of $(D; E)$. Then the following are equivalent.*

1. *Γ has a polymorphism with the behavior $\min_{(E,=)}$.*
2. *For every primitive positive formula $\phi(x_1, \ldots, x_n)$ and $y_1, \ldots, y_4 \in \{x_1, \ldots, x_n\}$, when*
   - *$\phi(x_1, \ldots, x_n) \wedge E(y_1, y_2) \wedge y_3 = y_4$ and*
   - *$\phi(x_1, \ldots, x_n) \wedge y_1 = y_2 \wedge E(y_3, y_4)$*

   are satisfiable over Γ, then also $\phi(x_1, \ldots, x_n) \wedge E(y_1, y_2) \wedge E(y_3, y_4)$ is satisfiable over Γ.
3. *For every finite subset S of D, Γ has a polymorphism with the behavior $\min_{(E,=)}$ on S.*

▶ **Lemma 7.3.** *Let f be an operation with the behavior $\min_{(E,=)}$. Then f generates an operation with the behavior $\min_{(E,=)}$ that is canonical as a function from $(D; Eq)^2$ to $(D; Eq)$.*

▶ **Lemma 7.4.** *Let f and g be operations with the behavior $\min_{(E,=)}$ and $\min_{(N,Eq)}$, respectively. Then $\{f, g\}$ generates $\min_{(N,E,=)}$.*

Proof. By Lemma 7.3, we can assume that f is canonical as a function from $(D; Eq)^2$ to $(D; Eq)$. We will show that $h(x, y) := g(f(x, y), f(y, x))$ has the behavior $\min_{(N,E,=)}$. Consider arbitrary points $a = (a_1, a_2)$, $b = (b_1, b_2)$ in D^2. Because f and g preserve E and N, h also does, and hence $h(E, E) = E$ and $h(N, N) = N$. If $a\binom{E}{=}b$ or $a\binom{=}{E}b$, then because f has the behavior $\min_{(E,=)}$, we have both $E(f(a_1, a_2), f(b_1, b_2))$ and $E(f(a_2, a_1), f(b_2, b_1))$. Since g preserves E, we obtain that $E(h(a), h(b))$ and we are done in this case.

We now turn to the case where $a\binom{N}{Q}b$ and $Q \in \{E, =\}$, and show that $N(f(a_1, a_2), f(b_1, b_2))$ or $N(f(a_2, a_1), f(b_2, b_1))$. Assume the contrary. Let α be an automorphism of $(D; Eq)$ such that $\alpha(a_2) = a_1$. Then $(a_1, a_1)\binom{N}{Q}(b_1, \alpha(b_2))$ and $(a_1, a_1)\binom{Q}{N}(\alpha(b_2), b_1)$. By transitivity of Eq, we have that $(b_1, \alpha(b_2))\binom{N}{N}(\alpha(b_2), b_1)$. Since f is canonical as a function from $(D; Eq)^2$ to $(D; Eq)$, we have that $Eq(f(a_1, a_1), f(b_1, \alpha(b_2)))$ and $Eq(f(a_1, a_1), f(\alpha(b_2), b_1))$. Therefore, $Eq(f(b_1, \alpha(b_2)), f(\alpha(b_2), b_1))$ by transitivity of Eq. This contradicts the fact that f preserves

N. Thus we have proved that $N(f(a_1,a_2), f(b_1,b_2))$ or $N(f(a_2,a_1), f(b_2,b_1))$. Further, because g has the behavior $\min_{(N,Eq)}$, we obtain that $N(h(a), h(b))$.

The case where $a\binom{Q}{N}b$ for $Q \in \{E, =\}$ is symmetric. We have considered all the cases, and conclude that indeed h has the behavior $\min_{(N,E,=)}$. ☐

By combining Proposition 5.4, Proposition 6.2, and Lemma 7.4, we obtain the following.

▶ **Corollary 7.5.** *Let Γ be a first-order expansion of $(D; E, N)$. Then Γ is preserved by $\min_{(N,E,=)}$, and for every finite reduct Γ' of Γ the problem $\mathrm{CSP}(\Gamma')$ is in P, or Γ has a finite reduct Γ' such that $\mathrm{CSP}(\Gamma')$ is NP-hard.*

8 Conclusions and Future Work

We have shown that Conjecture 1.2 holds for all structures with a first-order definition over an equivalence relation with infinitely many infinite classes; moreover, the universal-algebraic dichotomy from Conjecture 1.2 corresponds in this case precisely to a complexity dichotomy of the corresponding constraint satisfaction problems. We obtain the following.

▶ **Theorem 8.1.** *For equivalence constraint languages Γ exactly one of the following holds:*

1. *There is an expansion Δ' of the model-complete core Δ of Γ by a constant such that $\mathrm{Alg}(\Delta')$ contains a two-element algebra whose operations are projections. In this case, for some finite reduct Γ' of Γ we have that $\mathrm{CSP}(\Gamma')$ is NP-complete.*
2. *The language Γ has a polymorphism f and an automorphism α satisfying $\forall x, y.\, f(x,y) = \alpha(f(y,x))$. In this case, $\mathrm{CSP}(\Gamma')$ is in P for every finite reduct Γ' of Γ.*

Proof. By Proposition 2.6, the two cases are mutually exclusive. By Lemma 5.1, we have that either Case 1 or Case 2 holds, or E and N are primitively positively definable over Γ. By Proposition 5.4 and Lemma 7.1 every first-order expansion Γ of $(D; E, N)$ satisfies Case 1 or Case 2 or it is preserved by an operation f with the behavior $\min_{(E,=)}$ as well as an operation g with the behavior $\min_{(N,Eq)}$. Further, Lemma 7.4 implies that Γ, and in consequence every finite reduct Γ' of Γ, is preserved by $\min_{(N,E,=)}$. The tractability of each such Γ' follows from Proposition 6.2. By Proposition 6.1 there exists an automorphism α of $(D; Eq)$ satisfying $\forall x, y.\ \min_{(N,E,=)}(x,y) = \alpha(\min_{(N,E,=)}(y,x))$; hence, we are in Case 2. ☐

Theorem 8.1 classifies also a non-trivial class of structures that are first-order definable over Allen's Interval Algebra (see Section 1.1): recall that the structure $(\mathbb{I}; R_E)$, where $R_E := \{((x,y),(u,v)) \mid x = u\}$, is isomorphic to $(D; Eq)$. In fact, we believe that the techniques of this paper can be applied to eventually classify the complexity of the CSP for all structures Γ with a first-order definition over Allen's Interval Algebra. A next step towards this goal might be to classify all such structures Γ that contain the relation R_E. In this case R_E is a congruence of $\mathrm{Alg}(\Gamma)$. Note that the quotient of $\mathrm{Alg}(\Gamma)$ by R_E, and all subalgebras corresponding to equivalence classes of R_E, contain all automorphisms of $(\mathbb{Q}; <)$. Hence, the difference to the scenario of the present paper is that one might then have to use the results from [6] about first-order expansions of $(\mathbb{Q}; <)$ instead of the dichotomy for equality constraint languages.

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A Appendix

A.1 Proof of Lemma 7.2

Proof. The implication from (1) to (2) follows directly from the definition of the behavior $\min_{(E,=)}$.

To prove that (2) implies (3), assume (2) and let $S \subset D^2$ be finite, and let Δ be the structure induced by S in Γ^2. We construct a homomorphism from Δ to Γ that has the desired behavior. Without loss of generality we can assume that S is of the form $\{b_1, \dots, b_n\}^2$, for sufficiently large n. Consider the formula ϕ with variables $x_{1,1}, \dots, x_{n,n}$, and which is the conjunction over all formulas $R(x_{i_1,j_1}, \dots, x_{i_k,j_k})$ such that $R(b_{i_1}, \dots, b_{i_k})$ and $R(b_{j_1}, \dots, b_{j_k})$ hold in Γ. So ϕ states precisely which relations hold in Γ^2 on elements from S.

Let P be the set of pairs of the form $((i_1, i_2), (j_1, j_2))$ with $1 \le i_1 < j_1 \le n$ and $1 \le i_2 < j_2 \le n$, and where $E(e_{i_1}, e_{j_1})$ and $e_{i_2} = e_{j_2}$, or $E(e_{i_2}, e_{j_2})$ and $e_{i_1} = e_{j_1}$. We show by induction on the size of $I \subseteq P$ that for the conjunction $\psi := \bigwedge_{((i_1,i_2),(j_1,j_2)) \in I} E(x_{i_1,i_2}, x_{j_1,j_2})$ the formula $\phi \wedge \psi$ is satisfiable over Γ. Note that this statement applied to the set $I = P$ gives a homomorphism h from Δ to Γ which satisfies $E(h(x_1, x_2), h(y_1, y_2))$ whenever $(x_1, x_2)\binom{=}{E}(y_1, y_2)$ or $(x_1, x_2)\binom{E}{=}(y_1, y_2)$, and hence has the behavior $\min_{(E,=)}$ on S.

For the induction beginning, let $((i_1, i_2), (j_1, j_2))$ be any element of P. Let r, s be the n^2-tuples defined as follows.

$$r := (b_1, \dots, b_1, b_2, \dots, b_2, \dots, b_n, \dots, b_n)$$

$$s := (b_1, b_2, \dots, b_n, b_1, b_2, \dots, b_n, \dots, b_1, b_2, \dots, b_n)$$

These two tuples satisfy ϕ, because the projections to the first and second coordinate, respectively, are homomorphisms from Δ to Γ. When $(e_{i_1}, e_{i_2})\binom{=}{E}(e_{j_1}, e_{j_2})$ or $(e_{i_1}, e_{i_2})\binom{E}{=}(e_{j_1}, e_{j_2})$, then either r or s satisfies $E(x_{i_1,i_2}, x_{j_1,j_2})$, proving that $\phi \wedge E(x_{i_1,i_2}, x_{j_1,j_2})$ is satisfiable in Γ.

In the induction step, let $I = \{p_1, p_2, \dots\} \subset P$ be a set of cardinality $n \geq 2$, and inductively suppose that the statement has been shown for subsets of P of cardinality $n - 1$. We write ψ for the primitive positive formula

$$\phi \wedge \bigwedge_{((i_1,i_2),(j_1,j_2)) \in I \setminus \{p_1,p_2\}} E(x_{i_1,i_2}, x_{j_1,j_2}) ,$$

since Γ contains the relation E. Write $p_1 = ((u_1, u_2), (v_1, v_2))$ and $p_2 = ((u'_1, u'_2), (v'_1, v'_2))$. Then the inductive assumption shows that $\psi \wedge E(x_{u_1,u_2}, x_{v_1,v_2})$ and $\psi \wedge E(x_{u'_1,u'_2}, x_{v'_1,v'_2})$ are satisfiable in Γ. But then, by (1), the formula $\psi \wedge E(x_{u_1,u_2}, x_{v_1,v_2}) \wedge E(x_{u'_1,u'_2}, x_{v'_1,v'_2})$ is satisfiable over Γ as well, concluding the proof.

The implication (3) $\Rightarrow$ (1) follows from a compactness argument (see Theorem 3.6.11 in [4]). ☐

A.2 Proof of Lemma 7.1

Using Lemma 7.2, we can now prove Lemma 7.1.

Proof. We use Lemma 7.2. So let $\phi(x_1, \dots, x_n)$ be a primitive positive formula and $y_1, \dots, y_4 \in \{x_1, \dots, x_n\}$ such that

- $\psi_1 := \phi(x_1, \dots, x_n) \wedge E(y_1, y_2) \wedge y_3 = y_4$ and
- $\psi_2 := \phi(x_1, \dots, x_n) \wedge y_1 = y_2 \wedge E(y_3, y_4)$

are satisfiable over Γ. We have to show that $\phi(x_1, \dots, x_n) \wedge E(y_1, y_2) \wedge E(y_3, y_4)$ are satisfiable over Γ as well. By assumption we find assignments $s_1, s_2 \colon \{x_1, \dots, x_n\} \to D$ such that t_1 satisfies ψ_1 and t_2 satisfies ψ_2. We distinguish the following cases, depending on $|\{s_1(y_1), \dots, s_1(y_4)\}| + |\{s_2(y_1), \dots, s_2(y_4)\}|$, which is either 4, 5, or 6. See Figure 3 that depicts the values of $y_1, \dots, y_4$ under s_1 on the first coordinate and its values under s_2 on the second coordinate. Solid edges indicate that Eq holds on the respective pair of points.

In each case we identify $\alpha, \beta \in \mathrm{Aut}((D; E))$ such that $x \mapsto f(\alpha(s_1(x)), \beta(s_2(x)))$ satisfies $E(y_1, y_2) \wedge E(y_3, y_4)$, which proves that $\phi(x_1, \dots, x_n) \wedge E(y_1, y_2) \wedge E(y_3, y_4)$ is satisfiable. Each case is illustrated by a figure, which is divided into two parts. On the left side we have values of $y_1, \dots, y_4$

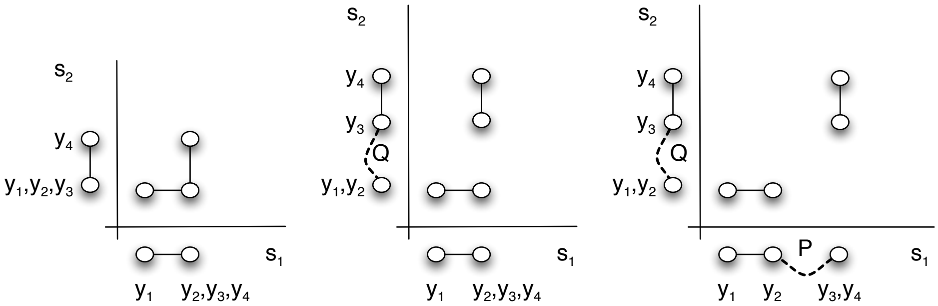

Figure 3 Illustration of the assignments s_1 and s_2 in Cases 1, 2, and 3.

under s_1 and s_2, as in Figure 3. On the right side there are values of $y_1, \ldots, y_4$ under $\alpha \circ s_1$ on the first coordinate and under $\beta \circ s_2$ on the second coordinate.

Let $d \in D$ be arbitrary such that $E(c, d)$, where c is the element from the statement.

1. First we consider the situation where $|\{s_1(y_1), \ldots, s_1(y_4)\}| + |\{s_2(y_1), \ldots, s_2(y_4)\}| = 4$. Suppose that $s_1(y_2) = s_1(y_3) = s_1(y_4)$ or $s_1(y_1) = s_1(y_3) = s_1(y_4)$, and that $s_2(y_1) = s_2(y_2) = s_2(y_3)$ or $s_2(y_1) = s_2(y_2) = s_2(y_4)$. The proof is symmetric in all cases, and we assume that $s_1(y_2) = s_1(y_3) = s_1(y_4)$ and $s_2(y_1) = s_2(y_2) = s_2(y_3)$ without loss of generality.

   Then we have $(c, d)\binom{=}{E}(c, c)$ and $(c, c)\binom{E}{=}(d, c)$, and moreover $E(f(c, d), f(c, c))$ and $E(f(c, c), f(d, c))$. So there is $\alpha \in \mathrm{Aut}((D; E))$ that maps $(s_1(y_1), s_1(y_2))$ to (d, c), and $\beta \in \mathrm{Aut}((D; E))$ that maps $(s_2(y_1), s_2(y_4))$ to (c, d). The map given by $x \mapsto f(\alpha(s_1(x)), \beta(s_2(x)))$ satisfies $E(y_1, y_2)$ since $E(f(d, c), f(c, c))$, and similarly, it satisfies $E(y_3, y_4)$ since $E(f(c, c), f(c, d))$.

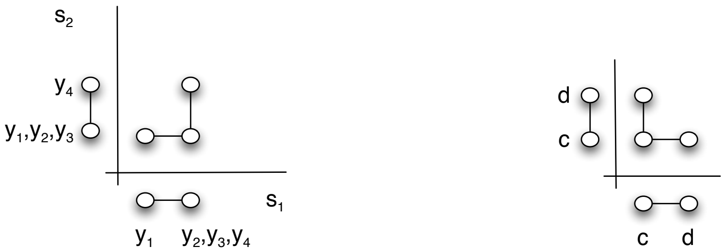

Figure 4 Illustration for Case 1.

2. Now suppose that $|\{s_1(y_1), \ldots, s_1(y_4)\}| + |\{s_2(y_1), \ldots, s_2(y_4)\}| = 5$. Again, the situation is symmetric with respect to changing the roles of s_1 and s_2, and we can assume without loss of generality that $s_1(y_2) = s_1(y_3) = s_1(y_4)$, and that $s_2(y_1) = s_2(y_2)$ (see Figure 5).

   Let us remark that by transitivity of *Eq* either $E(u, v)$ for $u = s_2(y_1) = s_2(y_2)$ and all $v \in \{s_2(y_3), s_2(y_4)\}$, or $N(u, v)$ for $u = s_2(y_1) = s_2(y_2)$ and all $v \in \{s_2(y_3), s_2(y_4)\}$. In the following we just write $Q(u, v)$, where $Q \in \{E, N\}$, since our proof does not depend on the exact value of Q. Choose a, b such that $E(a, b)$ and $Q(c, a)$ (which implies that $Q(c, b)$, $Q(d, a)$, $Q(d, b)$ by transitivity of *Eq*); clearly, such a, b exist. See Figure 5. By Observation 3.2, $E(f(c, b), f(d, b))$ or $E(f(d, b), f(d, a))$.

   In the first case, let $\alpha \in \mathrm{Aut}((D; E))$ be such that $\alpha(s_1(y_1), s_1(y_2)) = (d, c)$, and let $\beta \in \mathrm{Aut}((D; E))$ be such that $\beta(s_2(y_1), s_2(y_3), s_2(y_4)) = (b, d, c)$. In the second case, let $\alpha \in \mathrm{Aut}((D; E))$ be such that $\alpha(s_1(y_1), s_1(y_2)) = (c, d)$, and let $\beta \in \mathrm{Aut}((D; E))$ be such that $\beta(s_2(y_1), s_2(y_3), s_2(y_4)) = (c, a, b)$. In both cases, $x \mapsto f(\alpha(s_1(x)), \beta(s_2(x)))$ satisfies $E(y_1, y_2)$ and $E(y_3, y_4)$.

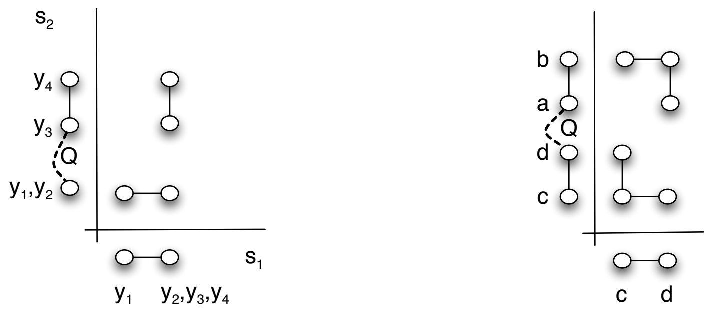

Figure 5 Illustration for Case 2.

3. Finally, suppose that $|\{s_1(y_1),\dots,s_1(y_4)\}| + |\{s_2(y_1),\dots,s_2(y_4)\}| = 6$. Again, there are $P, Q \in \{E, N\}$ such that $P(s_1(y_2), s_1(y_3))$ and $Q(s_2(y_1), s_2(y_3))$. By transitivity of Eq we then also have $P(s_1(y_1), s_1(y_3))$ and $Q(s_2(y_1), s_2(y_4))$.

   Let $a, b \in E$ be such that $E(a, b)$ and $Q(d, a)$, and let $a', b' \in E$ be such that $E(a', b')$ and $P(d, a')$. By Observation 3.2, we have that either $E(f(a', b), f(b', b))$ or $E(f(b', a), f(b', b))$. See Figure 6.

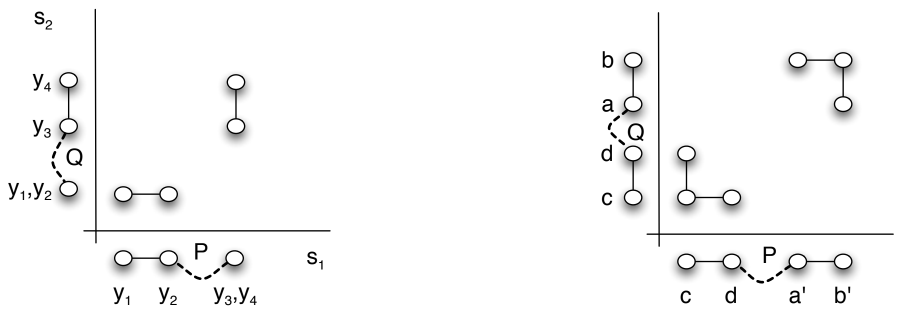

Figure 6 Illustration for Case 3.

   If $E(f(a', b), f(b', b))$, let $\alpha \in \mathrm{Aut}((D; E))$ be such that $\alpha(s_1(y_1), s_1(y_2), s_1(y_3)) = (b', a', c)$, and let $\beta \in \mathrm{Aut}((D; E))$ be such that $\beta(s_2(y_1), s_2(y_3), s_2(y_4)) = (b, d, c)$. If $E(f(b', a), f(b', b))$, then let $\alpha \in \mathrm{Aut}((D; E))$ be such that $\alpha(s_1(y_1), s_1(y_2), s_1(y_3)) = (c, d, b')$, and let $\beta \in \mathrm{Aut}((D; E))$ be such that $\beta(s_2(y_1), s_2(y_3), s_2(y_4)) = (c, a, b)$. Again, in both cases $x \mapsto f(\alpha(s_1(x)), \beta(s_2(x)))$ satisfies $E(y_1, y_2)$ and $E(y_3, y_4)$.

The statement follows from Lemma 7.2. □

A.3 Proof of Lemma 7.3

The following four relations with a first-order definition over the expansion $(D; Eq, <)$ of $(D; Eq)$ by a convex linear order appear in the proof: $E_<(x, y) := E(x, y) \wedge x < y$, $E_>(x, y) := E(x, y) \wedge x > y$, $N_<(x, y) := N(x, y) \wedge x < y$, and $N_>(x, y) := N(x, y) \wedge x > y$.

Proof. By Theorem 2.9, the operation f generates an operation f' which is canonical as a function from $(D; Eq, <)^2$ to $(D; Eq, <)$, and in particular canonical as a map from $(D; Eq, <)^2$ to $(D; Eq)$; clearly, f' must have behavior $\min_{(E,=)}$, too. We show that f' is also canonical as a function from $(D; Eq)^2$ to $(D; Eq)$. It suffices to show that $f'(=, N) = Q_1$, $f'(N, =) = Q_2$, $f'(E, N) = Q_3$, $f'(N, E) = Q_4$, for some $Q_1, Q_2, Q_3, Q_4 \in \{E, N, =\}$. We note that all $Q \in \{E, N, =\}$ are symmetric. Hence, if $Q_1(f'(a), f'(b))$ for any $a, b \in D^2$ such that $a\binom{=}{N_<}b$, then we also have $Q_1(f(a), f(b))$ for all $a, b \in D^2$ such that $a\binom{=}{N_>}b$. This implies that $f'(=, N) = Q_1$. Similarly, one can show that $f'(N, =) = Q_2$ for some $Q_2 \in \{E, N, =\}$.

The next step is to prove that $f'(E, N) = Q_1$ and $f'(N, E) = Q_2$ for some $Q_1, Q_2 \in \{E, N, =\}$. We only prove the existence of Q_1, since the argument for Q_2 is analogous. First, we show that $f'(a) \neq f'(b)$ for all $a\binom{E}{N}b$. Suppose that this is not the case. Then there are $a, b \in D^2$ satisfying $a\binom{E_<}{N_<}b$ and $f'(a) = f'(b)$. Let $c \in D^2$ be such that $b\binom{E_<}{N_<}c$ and $a\binom{E}{E}c$. Because f' is canonical as a function from $(D; Eq, <)^2$ to $(D; Eq)$, we have $f'(b) = f'(c)$, and therefore $f'(a) = f'(c)$. But this contradicts the assumption that f' preserves E.

To complete the proof, it is enough to show that $f'(E_<, N_<) = f'(E_>, N_<)$. Assume for contradiction that there are $a, b, c \in D^2$ such that $a\binom{E_<}{N_<}b$, $a\binom{E_>}{N_<}c$, $E(f'(a), f'(b))$, $N(f'(a), f'(c))$, and $b\binom{E}{E}c$, which implies $E(f'(b), f'(c))$. By transitivity of Eq we obtain that $Eq(f'(a), f'(c))$, a contradiction. $\square$