A NEW OPERATION ON PARTIALLY ORDERED SETS

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Abstract. Recently it has been shown that all non-trivial closed permutation groups containing the automorphism group of the random poset are generated by two types of permutations: the first type are permutations turning the order upside down, and the second type are permutations induced by so-called rotations. In this paper we introduce rotations for finite posets, which can be seen as the poset counterpart of Seidel-switch for finite graphs. We analyze some of their combinatorial properties, and investigate in particular the question of when two finite posets are rotation-equivalent. We moreover give an explicit combinatorial construction of a rotation of the random poset whose image is again isomorphic to the random poset. As an corollary of our results on rotations of finite posets, we obtain that the group of rotating permutations of the random poset is the automorphism group of a homogeneous structure in a finite language.

1. Operations from generic objects

Switching of a graph was introduced by van Lint and Seidel in connection with a problem of finding equilateral n-tuples of points in elliptic geometry in 1966 [14]. The operation of *switch* or *Seidel-switch* on a graph with respect to a set X of vertices works as follows: interchange edges and non-edges between X and its complement, leaving edges within and outside X unaltered. Two graphs are called *switch-equivalent* iff one can be obtained from the other one up to isomorphism by application of a Seidel-switch [17].

Since then the switch operation on graphs has gained applications in several areas of mathematics. In [18] the Seidel-switch is applied to construct cospectral non-isomorphic graphs; for more details on the role of the Seidel-switch in the theory of spectra of graphs see the monograph [1]. Switches are also used in the classification of root systems and Weyl-groups [2], or in coding theory [12]. In [10] new strongly regular and amorphic association schemes are constructed using switch operations. In geometry isomorphic invariants for translation planes are defined via the Seidel-switch [15]. Several authors have considered the complexity of deciding whether or not a given graph can be switched to a graph having some particular property. Polynomial-time algorithms are known for switching to a triangle-free graph [6, 5], to a claw-free graph [8], to an Eulerian graph [5], to a bipartite graph [5], and to a planar graph [3, 11]. On the hard side, Kratochvil [11] has shown NP-completeness of deciding whether or not a given graph can be switched to a regular one. In [9] the complexity of parameterized problems related to switching are investigated. It is also proved in [13] that deciding whether or not two given finite graphs are switch-equivalent is isomorphism-complete.

A priori the switch operation might look quite arbitrary among all operations that one could imagine on graphs, and its popularity surprising, but its special role is reflected in the following result from [19] concerning the *random graph*. The random graph $\mathbb{G}$ [4] is the unique countably infinite graph which is *universal* in the sense that any finite graph is isomorphic with

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an induced subgraph of $\mathbb{G}$ and which is *homogeneous*, that is, any isomorphism between finite induced subgraphs of $\mathbb{G}$ extends to an automorphism of $\mathbb{G}$. The random graph is obtained as the *Fraïssé limit* of the class of all finite graphs, and can thus be seen as the generic object representing the class of finite graphs (see e.g. [7]). Moreover, if the edges of a countably infinite graph are chosen independently with probability $\frac{1}{2}$, then this graph is isomorphic to $\mathbb{G}$ with probability 1. It is not difficult to see that if one applies the Seidel-switch to $\mathbb{G}$ with respect to a finite subset of $\mathbb{G}$, then the resulting graph is isomorphic to $\mathbb{G}$. Hence, there is a permutation on $\mathbb{G}$ witnessing this isomorphism; call such permutations *switching*. The mentioned result from [19], due to Thomas, states the following.

Theorem 1.1 (from [19]). *The closed subgroups of the full symmetric group* $\mathrm{Sym}(\mathbb{G})$ *on* $\mathbb{G}$ *which contain the automorphism group* $\mathrm{Aut}(\mathbb{G})$ *of* $\mathbb{G}$ *are precisely the following.*

(1) $\mathrm{Aut}(\mathbb{G})$*;*
(2) *the group of automorphisms and anti-automorphisms of* $\mathbb{G}$*;*
(3) *the group of switching permutations of* $\mathbb{G}$*;*
(4) *the join of (2) and (3);*
(5) $\mathrm{Sym}(\mathbb{G})$.

This theorem can be interpreted as follows: just like $\mathrm{Aut}(\mathbb{G})$ is the group of all symmetries of $\mathbb{G}$, the closed permutation groups containing $\mathrm{Aut}(\mathbb{G})$ stand for all symmetries of $\mathbb{G}$ if we are willing to give up some of the structure of $\mathbb{G}$. For example, it is obvious that flipping edges and non-edges of $\mathbb{G}$ yields a graph isomorphic with $\mathbb{G}$; this symmetry is reflected by one of the closed groups containing $\mathrm{Aut}(\mathbb{G})$, namely the group of all automorphisms and anti-automorphisms of $\mathbb{G}$. Now the theorem implies that the generic graph $\mathbb{G}$ has only one other symmetry of this kind – the one reflected by the Seidel-switch.

In this paper we initiate the investigation of the analogue of the Seidel-switch for partial orders. Similarly to the situation with graphs, there exists a generic partial order $\mathbb{P}$, called the *random partial order*, which represents the class of finite partial orders. More precisely, $\mathbb{P}$ is the Fraïssé limit of the class of all finite partial orders, or the unique countable partial order which is homogeneous and universal for the class of finite partial orders. In the companion paper [16] it is shown that this generic order has only two symmetries in the above sense: one symmetry is given by reversing the order, and the second one by so-called *rotations* (see Definition 3.1).

Theorem 1.2 (from [16]). *The closed subgroups of* $\mathrm{Sym}(\mathbb{P})$ *containing* $\mathrm{Aut}(\mathbb{P})$ *are precisely the following.*

(1) $\mathrm{Aut}(\mathbb{P})$*;*
(2) *the group of automorphisms and anti-automorphisms of* $\mathbb{P}$*;*
(3) *the group of rotating permutations of* $\mathbb{P}$*;*
(4) *the join of (2) and (3);*
(5) $\mathrm{Sym}(\mathbb{P})$.

Any function on the generic partial order yields functions on all finite partial orders by restriction – in particular, by restricting rotating permutations of the generic partial order, we obtain a notion of rotation for finite partial orders. The study of such rotations of finite posets is worthwhile; the very same combinatorial questions which have been considered for the Seidel-switch over the years do make perfect sense for these new operations.

We introduce the notion of a rotation of a poset and obtain several combinatorial results concerning rotations of finite posets. We first show that rotations can be decomposed into so-called *cuts*, and that rotations of finite posets can be further decomposed into cuts with respect to a single element. We then give a description of *rotation-equivalent* finite posets via their three-element subsets and maximal elements, and further investigate the equivalence classes of rotation equivalence, in particular the classes of chains and antichains. We show that deciding for two given finite posets $\mathcal{P}, \mathcal{Q}$ whether $\mathcal{P}$ is rotation-equivalent to a poset isomorphic with $\mathcal{Q}$ is graph-isomorphism complete. In Section 4 we turn to rotations of $\mathbb{P}$. Using a one-point extension of $\mathbb{P}$ we explicitly construct a non-trivial rotation of the random poset $\mathbb{P}$ whose image is isomorphic to $\mathbb{P}$, thus showing that group (3) of Theorem 1.2 properly contains $\mathrm{Aut}(\mathbb{P})$. We also show that rotations of finite posets are precisely the restrictions of rotating permutations of $\mathbb{P}$. Finally, as an application of our results on rotations of finite posets, we obtain that the group of rotating permutations of $\mathbb{P}$ is the automorphism group of a homogeneous relational structure with one ternary relation.

2. Notation

Before starting our investigations we introduce some notation concerning partially ordered sets. Let $(P, \leq)$ be a poset. For $x, y \in P$ we write $x < y$ iff $x \leq y$ and $x \neq y$; $x \geq y$ iff $y \leq x$; and $x \perp y$ iff $x \not\leq y$ and $y \not\leq x$. For $X, Y \subseteq P$ we write $X < Y$ iff for all $x \in X$ and all $y \in Y$ we have $x < y$. Instead of $\{x\} < Y$ we write $x < Y$. The set of maximal elements of $(P, \leq)$ is denoted by $\max(P, \leq)$. For two posets with disjoint domains P and Q let $P \oplus Q$ denote the *linear sum* of P and Q, i.e., the poset with domain $P \cup Q$ whose order is defined in the following way: the respective orders on P and on Q are kept, and $P < Q$. A subset $X \subseteq P$ is a *downset* iff for any $x \in X$ and $y \in P$ if $y < x$, then $y \in X$. An *up-set* is defined dually. When $X \subseteq P$, then we write $(X, \leq)$ for what really is $(X, X^2 \cap \leq)$.

3. Rotations

3.1. **Defining rotations and cuts.** Seidel's switch assigns a graph to a graph. Rotations and cuts assign a poset to a poset.

Definition 3.1. Let $\mathcal{P} = (P, \leq)$ be a poset, and let A and C be disjoint subsets of P such that A is a downset, C is an up-set and $A < C$. Let $B := P \setminus (A \cup C)$. We assign to $\mathcal{P}$ a new structure $\mathfrak{R}_{A,C}(\mathcal{P}) = (P, \preceq)$ by setting $x \preceq y$ if and only if one of the following hold:

- $x, y \in A$ or $x, y \in B$ or $x, y \in C$, and $x \leq y$;
- $x \in B, y \in A$ and $x \perp y$ in $(P, \leq)$;
- $x \in C$ and $y \in A$;
- $x \in C$, $y \in B$ and $x \perp y$ in $(P, \leq)$.

The assignment $\mathfrak{R}_{A,C}$ is called a *rotation*. If C is empty, then $\mathfrak{R}_{A,\emptyset} =: \mathfrak{C}_A$ is called a *cut*. If $A = \{a\}$ for some $a \in P$ then we write $\mathfrak{C}_a$ for $\mathfrak{C}_{\{a\}}$.

In what follows, when we write $\mathfrak{R}_{A,C}$ or $\mathfrak{C}_A$, we shall assume that A and C satisfy the conditions of Definition 3.1. Let us first observe that under these conditions, rotations of posets yield posets.

Proposition 3.2. *Let $\mathcal{P} = (P, \leq)$ be a poset, and let $A, C \subseteq P$ be as in Definition 3.1. Then $\mathfrak{R}_{A,C}(\mathcal{P})$ is a poset. In particular, $\mathfrak{C}_A(\mathcal{P})$ is a poset.*

Proof. Reflexivity and antisymmetry of $\preceq$ follow easily from Definition 3.1. To check transitivity suppose that $x \preceq y \preceq z$. If x, y and z belong to the same set in the partitioning $P = A \cup B \cup C$, then $\preceq$ and $\leq$ are equal, hence $x \preceq z$. If $x \in A$, then $y, z \in A$, as A is an up-set in $\mathfrak{R}_{A,C}(\mathcal{P})$. Hence $x \preceq z$. Now suppose that $x \in B$. If $z \in B$ as well, then $y \in B$ and $x \preceq z$ is proved. If $y \in B$ and $z \in A$, then $y \perp z$ in $\mathcal{P}$. Now, $x \geq z$ would imply $y \geq z$ by the transitivity of $\leq$, hence $x \perp z$ in $\mathcal{P}$, so $x \preceq z$. If $y, z \in A$, then $x \perp y$ in $\mathcal{P}$, hence $x \perp z$ as before and $x \preceq z$ holds, again. Finally, let $x \in C$. If $z \in C$ or $z \in A$, then $x \preceq z$ is trivial. The case $z \in B$ is handled as the case $x \in B$.

□

3.2. **Rotations via one-point extensions.** The following proposition shows that any rotation of a poset $\mathcal{P}$ corresponds to a certain operation on a one-point extension of $\mathcal{P}$.

Proposition 3.3. *Let $\mathcal{P} = (P, \leq)$ be a poset.*

- *Consider any one-point extension of $\mathcal{P}$ with domain $P \cup \{a\}$. Set $A := \{x \in P \mid x < a\}$, $C := \{x \in P \mid x > a\}$ and $B := \{x \in P \mid x \perp a\}$. Then the subsets A, B, C of P satisfy the conditions of Definition 3.1 for $\mathcal{P}$, and so the rotation $\mathfrak{R}_{A,C}$ of $\mathcal{P}$ is defined.*
- *Conversely, whenever subsets A, B, C of P satisfy the conditions of Definition 3.1, and hence the rotation $\mathfrak{R}_{A,C}$ of $\mathcal{P}$ is defined, then these sets can be defined in this way via a one-point extension to domain $P \cup \{a\}$ by setting $a < C$, $a > A$ and $a \perp B$.*

Proof. The first part of the statement is implied by the fact that A is a downset, C is an up-set, and that $A < x < C$ implies $A < C$. Hence the conditions of a rotation are satisfied. For the second part we need to show that we obtain a poset this way. To check transitivity we only need to examine those triples that contain a. If $x < y < a$, then $x \in A$, hence $x < a$. If $x < a < z$, then $x \in A$ and $z \in C$, hence by the assumptions $x < z$. If $a < y < z$, then $z \in C$, hence $x < z$. □

By the preceding proposition, the "rotated" sets A, B and C of a rotation of a partial order $(P, \leq)$ can be imagined as being definable in a one-point extension with domain $P \cup \{a\}$ by the parameter a. This justifies the following definition.

Definition 3.4. Let $(P, \leq)$ be a poset and A, B, C pairwise disjoint subsets of P satisfying the conditions Definition 3.1, i.e.,

- A is a downset,
- C is an up-set,
- $A < C$,
- $A \cup B \cup C = P$.

Then we call the triple A, B, C an *extendible triple.*

3.3. **Decomposing rotations.** Switching a graph with respect to a finite subset X of its vertices can be done by switching the graph consecutively with respect to $\{x\}$ for all $x \in X$ in any order. It turns out that we have a similar phenomenon for cuts and rotations, although we have to be a bit more careful.

Proposition 3.5. *Let $A, B, C \subseteq P$ be an extendible triple of a poset $\mathcal{P} = (P, \leq)$, and let $E \subseteq F \subseteq P$ be downsets. Then the following hold:*

(1) $\mathfrak{R}_{A,C} = \mathfrak{C}_{A \cup B} \mathfrak{C}_A$,
(2) $\mathfrak{C}_F = \mathfrak{C}_{F \setminus E} \mathfrak{C}_E$.

Moreover, if A is finite, and $a_1, \ldots, a_k \in A$ enumerate A in such a way that $a_j \not< a_i$ for all $1 \leq i < j \leq k$, then

(3) $\mathfrak{C}_A = \mathfrak{C}_{a_k} \mathfrak{C}_{a_{k-1}} \cdots \mathfrak{C}_{a_2} \mathfrak{C}_{a_1}$.

Proof. The first part of the statement follows from the definition observing that the relationship between the elements of A and C changes twice by cuts, hence their ordering is reversed. The elements of B alter their relationship to A and C once, satisfying Definition 3.1. Item (2) is shown similarly. Item (3) follows from the iterated use of (2): observe that $\{a_i\}$ is a one-element downset in $\mathfrak{C}_{a_{i-1}} \cdots \mathfrak{C}_{a_2}\mathfrak{C}_{a_1}(\mathcal{P})$. □

Corollary 3.6. *Any rotation is the composition of two cuts. In a finite poset any cut or rotation can be obtained as a composition of cuts with respect to one-element sets.*

3.4. **Rotation-equivalence.** Similarly to the Seidel-switch for graphs, rotations define a partition of the finite posets on the same domain.

Definition 3.7. We say that two posets $(P, \leq)$ and $(P, \preceq)$ are *rotation-equivalent* iff $(P, \leq)$ can be mapped to $(P, \preceq)$ by using a series of rotations. *Cut-equivalence* is defined similarly.

Proposition 3.8. *Rotation-equivalence and cut-equivalence are identical relations.*

Proof. This follows from Corollary 3.6. □

Proposition 3.9. *Rotation-equivalence is an equivalence relation.*

Proof. Reflexivity is implied by $\mathfrak{R}_{\emptyset,\emptyset}(\mathcal{P}) = \mathcal{P}$ for every poset $\mathcal{P}$. Transitivity is a consequence of the definition. For symmetry, let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (P, \preceq)$ be two posets, and suppose first that for some rotation $\mathfrak{R}_{A,C}$ of $\mathcal{P}$, $\mathfrak{R}_{A,C}(\mathcal{P})$ is isomorphic to $\mathcal{Q}$. Assume without loss of generality that $\mathfrak{R}_{A,C}(\mathcal{P}) = \mathcal{Q}$. Then A is an up-set and C is a downset in $\mathcal{Q} = \mathfrak{R}_{A,C}(\mathcal{P})$, and $C \prec A$. Hence $\mathfrak{R}_{C,A}(\mathcal{Q})$ is defined and equal to $\mathcal{P}$. Now if $\mathcal{Q}$ can be obtained from $\mathcal{P}$ applying more than one rotation, then every step can be reversed as in the above argument. □

Rotations divide the three-element posets with domain $\{a, b, c\}$ into the following three equivalence classes.

$\mathcal{O}_1$: the class of the 3-element antichain $a \perp b, b \perp c, c \perp a$;
  $a < b, a < c, b \perp c$; $b < a, b < c, a \perp c$; $c < a, c < b, b \perp c$;
  $a > b, a > c, b \perp c$; $b > a, b > c, a \perp c$; $c > a, c > b, b \perp c$;

$\mathcal{O}_2$: the class of a 3-element chain:
  $a < b < c$; $b < c < a$; $c < a < b$;
  $a < b, c \perp a, c \perp b$; $b < c, a \perp b, a \perp c$; $c < a, b \perp a, b \perp c$;

$\mathcal{O}_3$: the dual of $\mathcal{O}_2$:
  $a > b > c$; $b > c > a$; $c > a > b$;
  $a > b, c \perp a, c \perp b$; $b > c, a \perp b, a \perp c$; $c > a, b \perp a, b \perp c$.

Interestingly, the rotation classes of the three-element subsets of a finite partial order, together with its maximal elements, determine the whole partial order.

Proposition 3.10. *Let $(P, \leq)$ and $(P, \preceq)$ be two finite posets. Then the following are equivalent:*

(1) $(P, \leq) = (P, \preceq)$.
(2) $\max(P, \leq) = \max(P, \preceq)$*, and* $(P, \leq)$ *and* $(P, \preceq)$ *are rotation-equivalent.*
(3) $\max(P, \leq) = \max(P, \preceq)$*, and for all* $a, b, c \in P$ *the posets* $(\{a, b, c\}, \leq)$ *and* $(\{a, b, c\}, \preceq)$ *are rotation-equivalent.*

Proof. The implications (1) $\implies$ (2) and (2) $\implies$ (3) are obvious. To see that (3) implies (1), let $M = \max(P, \leq) = \max(P, \preceq)$. If $M = P$ then both posets are antichains and the statement holds. At first we examine the relationship of the maximal elements to the other elements of the poset. If $|M| = 1$, then we have a unique maximal element m, and $a \leq m$ for every $a \in P$. Now, let $|M| \geq 2$, $a \in P \setminus M$ and m_1 and m_2 be two distinct elements of M. Let us assume that $(a, m_1, m_2) \in \mathcal{O}_2$. As $m_1 \perp m_2$, the triple (a, m_1, m_2) is not a chain. The three element antichain is not in $\mathcal{O}_2$, hence there is a comparability among the elements of $\{a, m_1, m_2\}$. By the maximality of m_1 and m_2 either $a < m_1$ or $a < m_2$. Only the first case can happen in $\mathcal{O}_2$, thus we must have $a < m_1$ and $a \perp m_2$. Similarly, if $(a, m_1, m_2) \in \mathcal{O}_3$, then $a < m_2$ and $a \perp m_1$. Finally assume that $(a, m_1, m_2) \in \mathcal{O}_1$. Then either $a < m_1, m_2$ or $a \perp m_1, m_2$. The element a is below at least one maximal element, say $a < m$. If $(a, m, m') \in \mathcal{O}_1$ for every $m' \in M$, then $a < m'$ for every $m' \in M$. Now we can determine the relationship between the elements of M and P: if $(a, m_1, m_2) \in \mathcal{O}_1$ for every $m_1, m_2 \in M$, then $a < m$ for every $m \in M$. Otherwise, $a < m$ for some $m \in M$ if and only if there is an element $m' \in M$ such that $(a, m, m') \in \mathcal{O}_2$.

Now let $a, b \in P \setminus M$. Let us choose an $m \in M$ satisfying $b < m$. As $a \not> m$, we have that $a < b$ if and only if $(a, b, m) \in \mathcal{O}_2$.

We have shown that the maximal elements and the rotation classes of the 3-element subsets uniquely determine $\leq$. They determine $\preceq$ in the same way. Hence, $(P, \leq) = (P, \preceq)$. $\square$

We remark that the equivalence of (1) and (2) of this proposition does not hold for infinite posets. Consider the order of the rationals $(\mathbb{Q}, \leq)$ and apply the rotation $\mathfrak{R}_{A,C}$ to this poset, where A is the downset of negative numbers and C its complement: while Item (2) of Proposition 3.10 holds, (1) fails. We shall see in Section 4 that the equivalence of (2) and (3) also holds for infinite posets.

Corollary 3.11. *Let $(P, \leq)$ be a finite poset. Then there are at most $2^{|P|}$ posets with domain P which are rotation-equivalent to $(P, \leq)$.*

Proof. Any poset with domain P in the rotation-equivalence class of $(P, \leq)$ is determined by its set of maximal elements by Proposition 3.10. There are at most $2^{|P|}$ many choices to the set of the maximal elements. $\square$

Note that Corollary 3.11 is a little misleading. In most cases posets are considered up to isomorphism; the number of non-isomorphic rotation-equivalent posets can be much less. For example, we shall see below that in the class of an n-element antichain there are 2^n posets, but only a set of $n+1$ non-isomorphic posets.

Problem. *Find the possible sizes of rotation-equivalence classes of the posets of size n. In particular, what is the maximum and minimum size of such a class?*

We will now have a look at the rotation-equivalence classes of finite chains and antichains. In the following, let $\mathcal{C}_n$ denote the n-element chain $1 < 2 < \cdots < n$ and $\mathcal{AC}_n$ denote the n-element antichain on $\{1, \ldots, n\}$.

Proposition 3.12. *A finite poset $\mathcal{P}$ with domain $\{1, \ldots, n\}$ is rotation-equivalent to*

1. *$\mathcal{C}_n$ iff $\mathcal{P}$ is isomorphic to the disjoint union of two (possibly empty) chains $\mathcal{C}_s$ and $\mathcal{C}_t$, where $s + t = n$.*
2. *$\mathcal{AC}_n$ iff $\mathcal{P}$ is isomorphic to a linear sum of two (possibly empty) antichains $\mathcal{AC}_s \oplus \mathcal{AC}_t$, where $s + t = n$.*

Proof. We prove (1). Clearly, the disjoint union $\mathcal{D}$ of $\mathcal{C}_s$ and $\mathcal{C}_t$, where $s+t=n$, is rotation-equivalent to $\mathcal{C}_n$: just apply $\mathfrak{C}_A$ to $\mathcal{C}_n$, for an s-element downset A of $\mathcal{C}_n$. For the converse, by Proposition 3.5 it suffices to verify that applying a cut with respect to a minimal element of $\mathcal{D}$, we again obtain a poset which is a disjoint union of two chains. Let a be a minimal element of $\mathcal{D}$, say $a \in \mathcal{C}_s$. Then $\mathfrak{C}_a(\mathcal{D})$ is the disjoint union of $\mathcal{C}_{s-1}$ and $\mathcal{C}_{t+1}$, because a is removed from $\mathcal{C}_s$ and added on top of $\mathcal{C}_t$.

The argument for (2) is equally simple and left to the reader. □

Lemma 3.13. *Let $\mathcal{P} = (P, \leq)$ be a poset and $S, T \subseteq P$ be antichains with $S < T$ and such that $S \cup T \neq \emptyset$. Then there is a rotation $\mathfrak{R}_{A,C}$ of $\mathcal{P}$ such that $\max(\mathfrak{R}_{A,C}(\mathcal{P})) = S \cup T$.*

Proof. Let S and T be given. By replacing S by T and T by $\emptyset$ in case S is empty, we may assume that S is non-empty. Set

$$A = \{x \in P \mid x \leq s \text{ for some } s \in S\}$$

and

$$C = \{x \in P \mid x > S \text{ and } x \not\leq t \text{ for any } t \in T\}.$$

Since $S \neq \emptyset$ we have $A \neq \emptyset$. Setting $B = P \setminus A \cup C$, and hence it follows from the definition of a rotation that

$$\max(\mathfrak{R}_{A,C}(\mathcal{P})) = \max(A, \leq) \cup \{b \in \max(B, \leq) \,|\, b > A\}.$$

It remains to show that this set equals $S \cup T$. Clearly, $\max(A, \leq) = S$; we show $T = \{b \in \max(B, \leq) \,|\, b > A\}$. Let $t \in T$. Then $t \notin A$ since $S < T$; moreover, $t \notin C$ by the definition of C, so $t \in B$. No other element of B can be above t by the definition of C, hence $t \in \max(B, \leq)$. Since $t > S$ we derive $t \in \{b \in \max(B, \leq) \,|\, b > A\}$. For the converse, let b be an element of the latter set; then since $b > S$ but $b \notin C$, we have $b \leq t$ for some $t \in T$. We already know $t \in B$, and so $b = t \in T$. □

Theorem 3.14. *Let $(P, \leq)$ and $(P, \preceq)$ be finite posets. Then the following are equivalent:*

1. *$(P, \leq)$ and $(P, \preceq)$ are rotation-equivalent.*
2. *There is a rotation mapping $(P, \leq)$ to $(P, \preceq)$.*
3. *For all $a, b, c \in P$ the posets $(\{a, b, c\}, \leq)$ and $(\{a, b, c\}, \preceq)$ are rotation-equivalent.*

Proof. The implication from (2) to (1) is trivial, and (1) $\implies$ (3) is obvious. To see (3) $\implies$ (2), let $M = \max(P, \preceq)$. We claim that M is a linear sum of at most two antichains in $(P, \leq)$. To see this, we distinguish two cases. If M is an antichain in $(P, \leq)$ then there is nothing to show. Otherwise, pick $a, b \in M$ such that $a < b$. Now let $c \in M$ be arbitrary. Then, since $\{a, b, c\}$ forms an antichain in $(P, \preceq)$ and since $(\{a, b, c\}, \leq)$ and $(\{a, b, c\}, \preceq)$ are rotation-equivalent, c must be comparable to precisely one element in $\{a, b\}$, say without loss of generality a. Then $a < c$ and $b \perp c$ in $(P, \leq)$. It follows that M is the linear sum of two antichains in $(P, \leq)$, namely those elements of M which are incomparable with a and those which are incomparable with b. By Lemma 3.13 there is a rotation that maps $(P, \leq)$ to a poset $(P, \leq')$ such that $M = \max(P, \leq')$. We have that the three-element subsets of $(P, \leq')$ and $(P, \preceq)$ are rotation-equivalent, and so Proposition 3.10 implies $(P, \leq') = (P, \preceq)$, proving (2). □

Corollary 3.15. *If $(P, \leq)$ and $(P, \preceq)$ are rotation-equivalent finite posets, then there exists a single rotation which maps $(P, \leq)$ to $(P, \preceq)$. Consequently, the composition of two rotations is a rotation.*

Proof. The statement follows from Theorem 3.14. □

We now show that the decision problem whether or not two given finite posets are isomorphic and the decision problem whether or not these posets are rotation-equivalent have the same computational complexity.

Theorem 3.16. *It is poset-isomorphism complete to decide for two given finite posets $\mathcal{P}, \mathcal{Q}$ whether or not $\mathcal{P}$ is rotation-equivalent to a poset isomorphic to $\mathcal{Q}$.*

Proof. At first we reduce the poset isomorphism problem to the rotation-equivalence problem. Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preceq)$ be two finite posets. Since isomorphic posets have the same size, we may assume that $|P| = |Q| = n$. Let $\mathcal{S}$ be an n-element antichain on some set S which is disjoint from P and Q. Let $\mathcal{P}'$ and $\mathcal{Q}'$ be the disjoint unions of $\mathcal{P}$ and $\mathcal{Q}$ with $\mathcal{S}$. To prove the theorem it is enough to show that $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic if and only if $\mathcal{P}'$ and $\mathcal{Q}'$ are rotation-equivalent. One direction being trivial, we assume that $\mathcal{P}'$ and $\mathcal{Q}'$ are rotation-equivalent, i.e., there exists a rotation $\mathfrak{R}_{A,C}$ of $\mathcal{P}'$ such that $\mathfrak{R}_{A,C}(\mathcal{P}')$ is isomorphic to $\mathcal{Q}'$. Suppose that $A \cup C \neq \emptyset$. If $A \cap S \neq \emptyset$, then the symmetric closure of the poset relation of $\mathfrak{R}_{A,C}(\mathcal{P}')$ is connected, a contradiction since the latter is not the case for $\mathcal{Q}'$. The same contradiction arises when $C \cap S \neq \emptyset$, so we must have $A \cap S = C \cap S = \emptyset$. In that case, all elements of S belong to the same connected component of symmetric closure of the poset relation of $\mathfrak{R}_{A,C}(\mathcal{P}')$, hence there are at most n such components, again a contradiction. Hence, $A \cup C = \emptyset$, and so $\mathcal{P}'$ and $\mathcal{Q}'$ are in fact isomorphic. But then $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic as well.

We now reduce the rotation-equivalence problem to the isomorphism problem. Let $\mathcal{P} = (P, \leq)$ and $\mathcal{Q} = (Q, \preceq)$ be two finite posets. Pick any $p \in P$; by Lemma 3.13, there exists a unique poset $\mathcal{P}_p$ on P which is rotation-equivalent to $\mathcal{P}$ and whose only maximal element is p. Similarly, for every $q \in Q$ there exists a unique poset $\mathcal{Q}_q$ on Q which is rotation-equivalent to $\mathcal{Q}$ and whose only maximal element is q. By Proposition 3.10, $\mathcal{P}$ and $\mathcal{Q}$ are rotation-equivalent if and only if $\mathcal{P}_p$ is isomorphic to $\mathcal{Q}_q$ for some $q \in Q$. □

In the following proposition we only observe that poset-isomorphism completeness is equivalent to graph-isomorphism completeness.

Proposition 3.17. *It is graph-isomorphism complete to decide for two given finite posets $\mathcal{P}, \mathcal{Q}$ whether or not $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic.*

Proof. Let (V, E) and (W, F) be two finite graphs. Let V' be the set of one- and two-element subsets of V. Define a poset $\mathcal{P} = (V', \leq)$ by setting $\{u\} \leq \{u\}$, $\{u\} \leq \{u, v\}$ iff $(u, v) \in E$ and $\{u\} \geq \{u, v\}$ iff $(u, v) \notin E$. Define $\mathcal{Q}$ from (W, F) is the same manner. Then $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic if and only if (V, E) and (W, F) are isomorphic. This reduces the graph-isomorphism problem to the poset-isomorphism problem.

For the converse, let $(P, \leq)$ and $(Q, \preceq)$ be two finite posets. We may assume that $|P| = |Q| = n$. Let s be the number of levels of P. For every level $1 \leq i \leq s$, pick elements $p_1^i, \ldots, p_{n+i}^i$ outside P, and define a graph on $P' := P \cup \{p_j^i \mid 1 \leq i \leq s \ \wedge \ 1 \leq j \leq n + i\}$ as follows. All p_j^i, p_k^i are connected by an edge; moreover, all p_j^i are connected with all elements of the i-th level of $(P, \leq)$. We connect furthermore $x, y \in P$ by an edge iff they are comparable with respect to $\leq$, and they are in adjacent levels. Define a graph from $(Q, \preceq)$ in the same manner. Then the two graphs obtained are isomorphic if and only if $(P, \leq)$ and $(Q, \preceq)$ are isomorphic. □

We finish this section with an extension of our results on finite posets to infinite posets via compactness. We would like to remark that we do not dispose of a "direct" proof of the following proposition which does not make use of our results on finite posets.

Proposition 3.18. *Theorem 3.14 and hence also Corollary 3.15 hold for all infinite posets $(P,\leq)$ and $(P,\preceq)$ as well.*

Proof. We only need to show that (3) implies (2) in Theorem 3.14. Let $\{p_i \,|\, i \in \omega\}$ be an enumeration of P, and set $P_n := \{p_i \,|\, i \leq n\}$, for all $i \in \omega$. By Theorem 3.14, there exist rotations $\mathfrak{R}_n$ such that $\mathfrak{R}_n(P_n,\leq) = (P_n,\preceq)$. Let S consist of all restrictions of some $\mathfrak{R}_n$ to some set P_m, where $m \leq n$. For $\mathfrak{R}, \mathfrak{R}' \in S$, set $\mathfrak{R} \sqsubseteq \mathfrak{R}'$ iff the domain of the poset rotated by $\mathfrak{R}$ is contained in the corresponding domain for $\mathfrak{R}'$, and $\mathfrak{R}'$ agrees with $\mathfrak{R}$ on the set where they are both defined. Then $\sqsubseteq$ defines a finitely branching tree on S. By König's tree lemma, this tree has an infinite branch. This branch defines a rotation which sends $(P,\leq)$ to $(P,\preceq)$. $\square$

4. The random poset

In this section we turn our attention to rotations of the random poset $\mathbb{P} = (P,\leq)$. Up to isomorphism, $\mathbb{P}$ is the unique countably infinite partial order satisfying the following *extension property*:

(EXT) For every finite subset $Q \subseteq P$ and every one-point extension $(Q \cup \{s\},\leq)$ of $(Q,\leq)$ there exists $a \in P$ such that $(Q \cup \{s\},\leq)$ and $(Q \cup \{a\},\leq)$ are isomorphic via the mapping between these orders which fixes all $q \in Q$ and sends s to a.

The image of a rotation of $\mathbb{P}$ is not necessarily isomorphic to $\mathbb{P}$. For example, if $A = \{x \in \mathbb{P} \,|\, x \leq a\}$ for some $a \in \mathbb{P}$ and $C > A$ is arbitrary, then $\mathfrak{R}_{A,C}(\mathbb{P})$ has a maximal element (namely the element a), hence it cannot be isomorphic to the random poset. In the companion paper [16], we showed the existence of non-trivial rotations of $\mathbb{P}$ which send $\mathbb{P}$ to a partial order isomorphic to $\mathbb{P}$ using model-theoretic methods; here, we give a combinatorial proof of this fact using one-point extensions. The existence of such a rotation of $\mathbb{P}$ implies that the group of rotating permutations in Theorem 1.2 really is a proper supergroup of $\mathrm{Aut}(\mathbb{P})$: for, if $\mathfrak{R}_{A,C}(\mathbb{P}) \simeq \mathbb{P}$ for $A, C \subseteq \mathbb{P}$ for which either A or C is non-empty, then this isomorphism is witnessed by a permutation in $\mathrm{Sym}(\mathbb{P}) \setminus \mathrm{Aut}(\mathbb{P})$, and this permutation separates the group of rotating permutations from $\mathrm{Aut}(\mathbb{P})$.

Theorem 4.1. *There is a rotation of the random poset $\mathbb{P}$ whose image is isomorphic to $\mathbb{P}$.*

Proof. Let $a \in \mathbb{P}$ and let $A = \{x \in \mathbb{P} \,|\, x < a\}$, $C = \{x \in \mathbb{P} \,|\, x > a\}$ and $B = \{x \in \mathbb{P} \,|\, x \perp a\}$. Then the rotation $\mathfrak{R}_{A,C}$ is well-defined. Now omit a from $\mathbb{P}$. Then $\mathbb{P} \setminus \{a\}$ is isomorphic to $\mathbb{P}$ – this is well-known and easy to see by verifying the extension property. Thus we can denote $\mathbb{P} \setminus \{a\}$ by $\mathbb{P}$. Then $A \cup B \cup C = \mathbb{P}$ and $\mathfrak{R}_{A,C}$ is a rotation of $\mathbb{P}$. Let $\mathcal{Q} = (P,\preceq) := \mathfrak{R}_{A,C}(\mathbb{P})$. We show that $\mathcal{Q}$ is isomorphic to $\mathbb{P}$.

To do this, we check that (EXT) holds for $\mathcal{Q}$. Let $S \subseteq P$ be finite and $A', B', C' \subseteq S$ be such that $A' \prec C'$, A' is a $\preceq$-downset, C' is an $\preceq$-up-set in $S = A' \cup B' \cup C'$. Then the following relations hold:

$$(C \cap C') > (A \cap C') \cup (B \cap B') \cup (C \cap A') \tag{1}$$

$$(B \cap C') \cup (C \cap B') > (A \cap B') \cup (B \cap A') \tag{2}$$

$$(A \cap C') \cup (B \cap B') \cup (C \cap A') > (A \cap A') \tag{3}$$

For Item (1), $C \cap C' > A \cap C'$ holds because $A < C$. In $\mathcal{Q}$ we have $A' \prec C'$, and $\mathfrak{R}_{A,C}$ does not change the relations inside C, hence $C \cap C' > C \cap A'$. For $c \in C$ and $b \in B$ either $b < c$ or $b \perp_< C$. Thus either $b \perp_\prec c$ or $b \succ c$. In $\mathcal{Q}$ for any $c \in C'$ and $b \in B'$ either $b \prec c$ or $b \perp_\prec c$. Thus for any $b \in B \cap B'$ and $c \in C \cap C'$ we must have $b < c$. Hence (1) holds. Item (3) is proved dually.

For Item (2), $C \cap B' > A \cap B'$ because $A < C$, and $(B \cap C') > (B \cap A')$ holds because $A' \prec C'$. For $y \in A$ and $x \in B$ either $y < x$ or $y \perp_< x$. Thus either $y \perp_\prec x$ or $x \prec y$. In $\mathcal{Q}$ for any $x \in C'$ and $y \in B'$ either $y \prec x$ or $y \perp_\prec x$. Thus for any $x \in B \cap C'$ and $y \in A \cap B'$ we must have $x > y$, so $B \cap C' > A \cap B'$. The case $C \cap B' > B \cap A'$ is proved similarly.

At first, let $C \cap C' \neq \emptyset$. Now, C' is an up-set in S with respect to $\preceq$. As $C > A$ in $\mathbb{P}$ we have $C \prec A$ in $\mathcal{Q}$. Thus $c \prec A$ for any element $c \in C \cap C'$, so $C' \supseteq A$ in $\mathcal{Q}$. The subsets A', B', C' are disjoint, hence $A \cap B' = \emptyset$ and $A \cap A' = \emptyset$. As C is a $\prec$-downset and $c \succ A'$ for any element $c \in C \cap C'$, we have $A' \subset C$. Thus, as the subsets A, B, C are disjoint, $A \cap A' = B \cap A' = \emptyset$, as well. Now let

$$
\begin{aligned}
X_0 =&(A \cap A') \cup (B \cap C') \cup (C \cap B') = (B \cap C') \cup (C \cap B'),\\
X_1 =&(A \cap B') \cup (B \cap A') \cup (C \cap C') = (C \cap C'),\\
X_2 =&(A \cap C') \cup (B \cap B') \cup (C \cap A').
\end{aligned}
$$

We claim that X_2, X_0, X_1 is an extendible triple with respect to the order $\leq$, and there is a $q \in C$ extending it. Moreover, q extends S in $\mathcal{Q}$. At first, as C and C' are up-sets in $\mathbb{P}$ and $(S, \preceq)$, respectively, X_1 is an up-set in $(S, \leq)$. Secondly, we show that X_2 is a downset in $(S, \leq)$. For this we need to show that no element of X_2 is above an element of X_0.

As A, B, C is an extendible triple in $\mathbb{P}$, considering the relationships of the elements in $\mathbb{P}$ we have that $x \not> y$, if

- $x \in (A \cap C')$ and $y \in (C \cap B')$,
- $x \in (B \cap B')$ and $y \in (C \cap B')$,
- $x \in (A \cap C')$ and $y \in (B \cap C')$.

If x and y are both in A, B or C, then their relationship is the same in $\mathbb{P}$ and $\mathcal{Q}$. Hence $x \not> y$, if

- $x \in (B \cap B')$ and $y \in (B \cap C')$,
- $x \in (C \cap A')$ and $y \in (C \cap B')$.

Finally, let

- $x \in (C \cap A')$ and $y \in (B \cap C')$.

Then $x \prec y$ and $x > y$ or $x \perp y$. Hence, by the definition of the rotation $x \perp y$ holds. Thus X_2 is a downset in $(S, \leq)$ and X_2, X_0, X_1 is an extendible triple with respect to $\leq$.

Now we show that X_2, X_0, X_1 can be extended in C. Let us include again the element a that we have omitted from $\mathbb{P}$. Now, $\mathbb{P} \cup \{a\}$ is isomorphic to $\mathbb{P}$ and $X_2 \cup \{a\}$ is a downset in $(S, \leq)$, X_1 is an up-set in $(S, \leq)$ and $X_2 \cup \{a\} < X_1$. As $\mathbb{P} \cup \{a\}$ is the random poset, there exists a point $q \in \mathbb{P}$ satifying the conditions $X_2 \cup \{a\} < q, X_0 \perp q, X_1 > q$. Hence, q is in C and q extends X_2, X_0, X_1 in $\mathbb{P}$.

We need to show that q extends S in $\mathcal{Q}$. The relationship of q to the elements of C is not altered, hence they remain the same in $\mathcal{Q}$. If $x \in (B \cap B')$, then $x > q$ turns to $x \perp_\prec q$. If $x \in (A \cap C')$, then $x < q$ turns to $x \succ q$. Finally, if $x \in (B \cap C')$, then $x \perp q$ turns to $x \succ q$, hence q extends S in $\mathcal{Q}$.

The dual case when $A \cap A' \neq \emptyset$ can be handled similary. In this case

$$\begin{aligned}
X_0 =&(A \cap A') \cup (B \cap C') \cup (C \cap B') = (A \cap A'),\\
X_1 =&(A \cap B') \cup (B \cap A') \cup (C \cap C') = (A \cap B') \cup (B \cap A')\\
X_2 =&(A \cap C') \cup (B \cap B') \cup (C \cap A')
\end{aligned}$$

X_0, X_1, X_2 will be the extendible triple with respect to $\leq$, and we can find a point $q \in A$ such that q extends S in $\mathcal{Q}$.

Finally, assume that $A \cap A' = C \cap C' = \emptyset$. In this case

$$\begin{aligned}
X_0 =&(A \cap A') \cup (B \cap C') \cup (C \cap B') = (B \cap C') \cup (C \cap B'),\\
X_1 =&(A \cap B') \cup (B \cap A') \cup (C \cap C') = (A \cap B') \cup (B \cap A')\\
X_2 =&(A \cap C') \cup (B \cap B') \cup (C \cap A').
\end{aligned}$$

We are going to prove that X_1, X_2, X_0 is an extendible triple with respect to $\leq$, and we can find a point $q \in B$ such that q extends S in $\mathcal{Q}$. The arguments from the first two cases imply that X_0 is an up-set and X_1 is a downset in $(S, \leq)$. Consider $\mathbb{P} \cup \{a\}$, as before. The triple $X_1, X_2 \cup \{a\}, X_0$ is an extendible triple with respect to $\leq$. Hence, there exists $q \in B$ extending it in $\mathbb{P}$. Now, q extends S in $\mathcal{Q}$. It can be checked similarly to the previous cases: The relationship of q to the elements of B is not altered at q, hence they remain the same in $\mathcal{Q}$. If $x \in (A \cap B')$, then $x < q$ turns to $x \perp_{\prec} q$. If $x \in (C \cap B')$, then $x > q$ turns to $x \perp_{\prec} q$. Finally, if $x \in (A \cap C')$, then $x \perp q$ turns to $x \succ q$, and if $x \in (C \cap A')$, then $x \perp q$ turns to $x \succ q$. Hence q extends S in $\mathcal{Q}$.

We obtained that (EXT) holds for $\mathcal{Q}$. Therefore $\mathcal{Q}$ is isomorphic to the random poset. $\square$

We now show that any rotation of a finite poset can be interpreted as the restriction of a rotating permutation of $\mathbb{P}$, showing that our notion of rotation of a finite poset really is the analogue of the Seidel-switch for posets.

Proposition 4.2. *Let $(X, \leq)$ be a finite poset, and let $\mathfrak{R}_{U,V}$ be a rotation of $(X, \leq)$. Let $e_1 : (X, \leq) \to \mathbb{P}$ and $e_2 : \mathfrak{R}_{U,V}(X, \leq) \to \mathbb{P}$ be embeddings. Then there exists a rotating permutation α of $\mathbb{P}$ such that $e_2 = \alpha \circ e_1$.*

Proof. Set $A' := e_1[U]$, $C' := e_1[V]$, and $B' := e_1[X] \setminus A' \cup C'$. By the extension property of $\mathbb{P}$, there exists $a \in P$ such that $A' < a$, $a < C'$, and $a \perp B'$. Set $A = \{x \in \mathbb{P} \,|\, x < a\}$, $C = \{x \in \mathbb{P} \,|\, x > a\}$ and $B = \{x \in \mathbb{P} \,|\, x \perp a\}$. By the proof of Theorem 4.1, $(P \setminus \{a\}, \leq)$ and $\mathfrak{R}_{A,C}(P \setminus \{a\}, \leq)$ are isomorphic; let β be a rotating permutation of $(P \setminus \{a\}, \leq)$ witnessing this. Pick any isomorphism $i : \mathbb{P} \to (P \setminus \{a\}, \leq)$. Then $\gamma := i^{-1} \circ \beta \circ i$ is a rotating permutation of $\mathbb{P}$ for the rotation $\mathfrak{R}_{i^{-1}[A], i^{-1}[C]}$. Since $A' \subseteq A$, $C' \subseteq C$, and $B' \subseteq B$, we have that $i^{-1} \circ \beta \circ e_1$ changes the relations between the elements of $(X, \leq)$ just like the rotation $\mathfrak{R}_{U,V}$. Thus, by the homogeneity of $\mathbb{P}$, there exists $\delta \in \mathrm{Aut}(\mathbb{P})$ such that $\delta \circ i^{-1} \circ \beta \circ e_1 = e_2$. Again by the homogeneity of $\mathbb{P}$, there exists $\varepsilon \in \mathrm{Aut}(\mathbb{P})$ such that $i^{-1} \circ e_1 = \varepsilon \circ e_1$. Set $\alpha := \delta \circ \gamma \circ \varepsilon$. Composed of rotating permutations and automorphisms of $\mathbb{P}$, α is itself a rotating permutation of $\mathbb{P}$. Moreover, $\alpha \circ e_1 = \delta \circ \gamma \circ \varepsilon \circ e_1 = \delta \circ i^{-1} \circ \beta \circ i \circ i^{-1} \circ e_1 = e_2$, proving the proposition. $\square$

The remainder of this paper is devoted to showing that the rotating permutations of $\mathbb{P}$ can be described as the automorphisms of a homogeneous structure with one ternary relation. In the following, for $i \in \{1, 2, 3\}$ we identify $\mathcal{O}_i$ with the ternary relation on $\mathbb{P}$ which consists of all triples in $\mathbb{P}$ which induce a poset isomorphic with a poset in $\mathcal{O}_i$. The following has already been observed in [16], but here we provide a much shorter proof which draws on our results about finite rotations.

Proposition 4.3. *The rotating permutations of* $\mathbb{P}$ *are precisely the automorphisms of the structure* $(P, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$.

Proof. Clearly, if α is a rotating permutation of $\mathbb{P}$, then $(\{a, b, c\}, \leq)$ and $(\{\alpha(a), \alpha(b), \alpha(c)\}, \leq)$ are rotation-equivalent for all $a, b, c \in P$, and hence α is an automorphism of $(P, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$. Conversely, if a permutation α has the latter property, then setting $\alpha(x) \preceq \alpha(y)$ if and only if $x \leq y$ we get that $(P, \preceq)$ and $\mathbb{P}$ satisfy condition (3) of Theorem 3.14; hence, by Proposition 3.18 there is a rotation $\mathfrak{R}$ such that $\mathfrak{R}(\mathbb{P}) = (P, \preceq)$. By the definition of $\preceq$, the permutation α is an isomorphism from $\mathbb{P}$ to $(P, \preceq)$, and hence it is a rotating permutation with respect to the rotation $\mathfrak{R}$. □

As another application of our results, we shall see that $(P, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ is homogeneous. This is a very strong property for a structure to have and implies many nice other properties. Among these is *quantifier elminination*, i.e., every first-order formula over a homogeneous structure is equivalent to a formula without quantifiers. We remark moreover that although $\mathbb{P}$ is homogeneous and $(P, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ has a first-order definition in $\mathbb{P}$, it does not automatically follow that $(P, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ is homogenous itself or first-order interdefinable with a homogeneous structure in a finite language: there exist counterexamples in similar situations (see the introduction of [19]).

Theorem 4.4. $(P, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ *is homogeneous.*

Proof. Let $S, T \subseteq P$ be finite and let $i : S \to T$ be a partial isomorphism between the structures induced by these sets in $(P, \mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_2)$. Pick any $s \in S$ and set $t := i(s)$. Let $\mathfrak{R}$ and $\mathfrak{R}'$ be rotations of $(S, \leq)$ and $(T, \leq)$, respectively, with the property that s and t are the unique maximal elements of $\mathfrak{R}(S, \leq)$ and $\mathfrak{R}'(T, \leq)$. Let $u : \mathfrak{R}(S, \leq) \to \mathbb{P}$ and $v : \mathfrak{R}'(T, \leq) \to \mathbb{P}$ be embeddings. Then by Proposition 4.2 there exist rotating permutations α, β of $\mathbb{P}$ whose respective restrictions to S and T are equal to u and v. Now set $i' := v \circ i \circ u^{-1}$. Then i' preserves $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$, and sends the unique maximal element of $(u[S], \leq)$ to the unique maximal element of $(v[S], \leq)$. Thus, by Proposition 3.10, it is an isomorphism between these posets, which by the homogeneity of $\mathbb{P}$ extends to an automorphism γ of $\mathbb{P}$. Hence, $\beta \circ \gamma \circ \alpha^{-1}$ is an extension of i to an automorphism of $(P, \mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_2)$. □

Observe that $\mathcal{O}_3$ can by defined from $\mathcal{O}_2$ by $(a, b, c) \in \mathcal{O}_3 \leftrightarrow (c, b, a) \in \mathcal{O}_2$. Moreover, $\mathcal{O}_1$ can be defined from $\mathcal{O}_2$ since it is the complement of $\mathcal{O}_2 \cup \mathcal{O}_3$ in P^3. Hence, the automorphism groups of the structures $(P, \mathcal{O}_2)$, $(P, \mathcal{O}_3)$, and $(P, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ are all identical: they consist of the rotating permutations.

Corollary 4.5. $(P, \mathcal{O}_2)$ *and* $(P, \mathcal{O}_3)$ *are homogeneous, and the automorphisms of any of these structures are precisely the rotating permutations of* $\mathbb{P}$.

Proof. We show homogeneity for $(P, \mathcal{O}_2)$; the argument for $(P, \mathcal{O}_3)$ is identical. Observe that the definition of $\mathcal{O}_3$ from $\mathcal{O}_2$ given above do not use quantifiers; in other words, for any finite $S \subseteq P$ we have that the triples of elements in S which are elements of $\mathcal{O}_2$ determine those triples which are elements of $\mathcal{O}_3$. Hence, any partial isomorphism between finite induced substructures of $(P, \mathcal{O}_2)$ is a partial isomorphism between the structures induced in $(P, \mathcal{O}_2, \mathcal{O}_3)$. For the same reason we have that any partial isomorphism between finite induced substructures of $(P, \mathcal{O}_2)$ is a partial isomorphism between the structures induced in $(P, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$. By Theorem 4.4, any such partial isomorphism extends to an automorphism of $(P, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$, which is also an automorphism of $(P, \mathcal{O}_2)$. □

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