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# Topological dynamics of unordered Ramsey structures

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**Abstract**

In this paper we investigate the connections between Ramsey properties of Fraïssé classes $\mathcal{K}$ and the universal minimal flow $M(G_{\mathcal{K}})$ of the automorphism group $G_{\mathcal{K}}$ of their Fraïssé limits. As an extension of a result of Kechris, Pestov and Todorcevic [14] we show that if the class $\mathcal{K}$ has finite Ramsey degree for embeddings, then this degree equals the size of $M(G_{\mathcal{K}})$. We give a partial answer to a question of Angel, Kechris and Lyons [1] showing that if $\mathcal{K}$ is a relational Ramsey class and $G_{\mathcal{K}}$ is amenable, then $M(G_{\mathcal{K}})$ admits a unique invariant Borel probability measure that is concentrated on a unique generic orbit.

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# 1 Introduction

With a Fraïssé class of finite structures $\mathcal{K}$ one can associate in a natural way a topological group $G_{\mathcal{K}}$, namely, the automorphism group of the Fraïssé limit of $\mathcal{K}$. For example, the Fraïssé limit of finite dimensional vector spaces over a fixed finite field $F$ is the $\aleph_0$-dimensional vector space $V_{\infty,F}$ over $F$ with automorphism group $\mathrm{GL}(V_{\infty,F})$. The groups of the form $G_{\mathcal{K}}$ are precisely the Polish groups that are non-archimedian in the sense that they have a basis at the identity consisting of open subgroups ([2]).

In [14] Kechris, Pestov and Todorcevic developed a "duality theory" [13, §4(A)] linking finite combinatorics of $\mathcal{K}$ with topological dynamics of $G_{\mathcal{K}}$, more precisely, it links combinatorial properties of $\mathcal{K}$ with properties of the universal minimal $G_{\mathcal{K}}$-flow $M(G_{\mathcal{K}})$. For groups of the form $G_{\mathcal{K}}$ the flow $M(G_{\mathcal{K}})$ is an inverse limit of metrizable $G_{\mathcal{K}}$-flows (cf. [14, T1.5]), and in many interesting cases is metrizable itself. If so, $M(G_{\mathcal{K}})$ either has the size of the continuum or else is finite [14, §1(E)]. An extreme case is that $M(G_{\mathcal{K}})$ is a single point, that is, $G_{\mathcal{K}}$ is extremely amenable. It is shown in [14] that for ordered $\mathcal{K}$ this happens if and only if $\mathcal{K}$ is Ramsey. For example, $V_{\infty,F}$ together with the so-called "canonical order" has an extremely amenable automorphism group.

We give a characterisation of $M(G_{\mathcal{K}})$ having an arbitrary finite cardinality in terms of Ramsey properties of $\mathcal{K}$. Namely, we use Fouché's Ramsey degrees [8, 9, 10] and show that $M(G_{\mathcal{K}})$ has finite size $d$ if and only if $\mathcal{K}$ has Ramsey degree $d$ (Theorem 3.1). We do not assume $\mathcal{K}$ to be ordered, but use Ramsey degrees *for embeddings* instead (see e.g. [17, 3]). These coincide with the usual Ramsey degrees on rigid structures, so our characterisation generalises the mentioned result of [14] and so does its proof. As a corollary we get (Corollary 3.15) that Ramsey degrees for embeddings are asymptotic in the sense that all structures in $\mathcal{K}$ have degree at most $d$ if all *large enough* structures have degree at most $d$ (i.e. every structure embeds into one of degree at most $d$).

Given an appropriate (unordered) class $\mathcal{K}$ one can first produce a so-called reasonable order expansion $\mathcal{K}^*$ whose Fraïssé limit expands the limit of $\mathcal{K}$ by a (linear) order $<^*$. The group $G_{\mathcal{K}}$ acts naturally on orders and one gets a $G_{\mathcal{K}}$-flow $X_{\mathcal{K}^*}$ as the orbit closure $\overline{G_{\mathcal{K}} \cdot <^*}$. Again, as shown in [14], minimality of this flow corresponds to a combinatorial property of $\mathcal{K}^*$ called the ordering property (cf. [17]), and indeed $X_{\mathcal{K}^*}$ is $M(G_{\mathcal{K}})$ if and only if $\mathcal{K}^*$

additionally is Ramsey.[1] Moreover, the Ramsey degree of $A \in \mathcal{K}$ equals the number of non-isomorphic order expansions it has in $\mathcal{K}^*$([14, §10],[18, §4]).

For example, the universal minimal $\mathrm{GL}(V_{\infty,F})$-flow is the orbit closure of the canonical order. This canonical order is forgetful in the sense that any finite dimensional $F$-vector space gets up to isomorphism only one order expansion, so Ramsey degrees are 1 in this case. The Ramsey degrees *for embeddings* on the other hand are unbounded (cf. Corollary 3.11). In general, the relationship between the two degrees is not trivial. We show that if a Ramsey class in a relational language has finite Ramsey degree for embeddings, then this degree must be a power of 2 (Theorem 3.12).

Recently, Angel, Kechris and Lyons [1] extended the duality theory to other important properties of $M(G_{\mathcal{K}})$, namely whether or not there is a (unique) $G_{\mathcal{K}}$-invariant Borel probability measure on $M(G_{\mathcal{K}})$. In this case, the group $G_{\mathcal{K}}$ is called amenable (uniquely ergodic), and this happens if and only if all minimal $G_{\mathcal{K}}$-flows admit such a (unique) measure ([1, P8.1]). For example, $\mathrm{GL}(V_{\infty,F})$ is uniquely ergodic.

The $G_{\mathcal{K}}$-flows $X_{\mathcal{K}^*}$ have a generic (i.e. comeager) orbit $G_{\mathcal{K}} \cdot <^*$ which is in fact dense $G_\delta$ [1, P14.3]. In many examples, a $G_{\mathcal{K}}$-invariant measure on $M(G_{\mathcal{K}})$, if exists, turns out to be concentrated on this generic orbit. However, answering a question in [1, Q15.3], Zucker [23, T1.2] showed that the measure on $M(\mathrm{GL}(V_{\infty,F}))$ is *not* concentrated on the generic orbit.

We show that such counterexamples rely on the language containing function symbols. More precisely, we show that if $\mathcal{K}$ is Ramsey over a relational language and $G_{\mathcal{K}}$ is amenable, then $G_{\mathcal{K}}$ is uniquely ergodic and the unique $G_{\mathcal{K}}$-invariant Borel probability measure on $M(G_{\mathcal{K}})$ is indeed concentrated on a dense $G_\delta$ orbit (Theorem 4.1).

# 2 Preliminaries

## 2.1 Notation

For $k \in \mathbb{N}$ we let $[k]$ denote $\{0, \dots, k-1\}$ and understand $[0] = \emptyset$. If $X, Y$ are sets, $f$ a function from $X$ to $Y$, $n \in \mathbb{N}$ and $Z \subseteq X^n$ we write $f(Z)$ for the set $\{f(\bar{x}) \mid \bar{x} \in Z\}$ where $f(\bar{x})$ denotes the tuple $(f(x_0), \dots, f(x_{n-1}))$ for $\bar{x} = (x_0, \dots, x_{n-1}) \in X^n$. For $X_0 \subseteq X$ we let $f \restriction X_0$ denote the restriction of $f$ to $X_0$; for a relation $Z$ as above, $Z \restriction X_0$ denotes $Z \cap (X_0^n)$. The identity on $X$ is denoted by $\mathrm{id}_X$.

[1] See [19] for a discussion of how to characterise universality alone.

## 2.2 Fraïssé theory

Fix a countable language $L$. We let $A, B, \ldots$ range over ($L$-)structures. The distinction between structures and their universes are blurred notationally. We speak of *relational* structures and classes of structures if the underlying language $L$ is relational. We write $A \leq B$ to indicate that there exists an embedding from $A$ into $B$, and we let $B^A$ denote the set of embeddings from $A$ into $B$.

The *age* $\mathrm{Age}(F)$ of a structure $F$ is the class of finitely generated structures which embed into $F$. A structure $F$ is *locally finite* if its finitely generated substructures are finite. For $A \in \mathrm{Age}(F)$ we call $F$ *$A$-homogeneous* if for all $a, a' \in F^A$ there is $g \in \mathrm{Aut}(F)$ such that $g \circ a = a'$. If $F$ is $A$-homogeneous for all $A \in \mathrm{Age}(F)$, it is *(ultra-)homogeneous.*

A structure $F$ is *Fraïssé* if it is countably infinite, locally finite and homogeneous. The age $\mathcal{K} := \mathrm{Age}(F)$ of a Fraïssé structure $F$

- is *hereditary:* for all $A, B$, if $A \leq B$ and $B \in \mathcal{K}$, then $A \in \mathcal{K}$;
- has *joint embedding:* for all $A, B \in \mathcal{K}$ there is $C \in \mathcal{K}$ such that both $A \leq C$ and $B \leq C$;
- has *amalgamation:* for all $A, B_0, B_1 \in \mathcal{K}$ and $a_0 \in B_0^A, a_1 \in B_1^A$ there are $C \in \mathcal{K}$ and $b_0 \in C^{B_0}, b_1 \in C^{B_1}$ such that $b_0 \circ a_0 = b_1 \circ a_1$.

A class $\mathcal{K}$ of finite structures that has these three properties, contains countably many structures up to isomorphism, and for every $n \in \mathbb{N}$ contains a structure (with universe) of size at least $n$, is a *Fraïssé class*. The following is well-known [21, T4.4.4]:

**Theorem 2.1** (Fraïssé 1954)**.** *For every Fraïssé class $\mathcal{K}$ there exists a Fraïssé structure $F$ with age $\mathcal{K}$.*

A standard back-and-forth argument shows that the structure $F$ in Theorem 2.1 is unique up isomorphism; it is called the *Fraïssé limit of* $\mathcal{K}$ and denoted by $\mathrm{Flim}(\mathcal{K})$.

We mention some standard examples:

**Examples 2.2.** The Fraïssé limit of the class of linear orderings is the rational order $(\mathbb{Q}, <)$. The Fraïssé limit of the class of finite Boolean algebras is the countable atomless Boolean algebra $B_\infty$. The Fraïssé limit of the class of finite graphs is the random graph $R$. The Fraïssé limit of the class of finite vector spaces over a fixed finite field $F$ is the vector space $V_{\infty,F}$ of dimension $\aleph_0$ over $F$.

We refer to [6, 7, 15] as surveys on homogeneous structures.

## 2.3 Ramsey degrees

Write $\binom{B}{A}$ for the set of substructures of $B$ which are isomorphic to $A$. Note that $\binom{B'}{A} \subseteq \binom{C}{A}$ whenever $B' \in \binom{C}{B}$. If $k, d \in \mathbb{N}$ then $C \to (B)^A_{k,d}$ means that for every colouring $\chi : \binom{C}{A} \to [k]$ there exists $B' \in \binom{C}{B}$ such that $|\chi(\binom{B'}{A})| \leq d$. The *Ramsey degree* of $A$ in a class of structures $\mathcal{K}$ is the least $d \in \mathbb{N}$ such that for all $B \in \mathcal{K}$ and $k \geq 2$ there is $C \in \mathcal{K}$ such that $C \to (B)^A_{k,d}$ – provided that such a $d$ exists; otherwise it is $\infty$. Taking the supremum over $A \in \mathcal{K}$ gives the Ramsey degree of $\mathcal{K}$, and the Ramsey degree of a structure $F$ is understood to be the Ramsey degree of $\mathrm{Age}(F)$; if this degree is 1, then $\mathcal{K}$ resp. $F$ are simply called *Ramsey*.

**Examples 2.3.** $(\mathbb{Q}, <)$, $B_\infty$ and $V_{\infty,F}$ are Ramsey [14]. The random graph $R$ has Ramsey degree $\infty$; indeed, a finite graph $G$ has Ramsey degree $|G|!/|\mathrm{Aut}(G)|$ in the class of finite graphs [14, §10].

Ramsey degrees have been introduced by Fouché in [8]. We refer to the surveys [11, 16] on Ramsey theory.

## 2.4 Topological dynamics

With a Fraïssé class $\mathcal{K}$ we associate the topological group

$$G_{\mathcal{K}} := \mathrm{Aut}(\mathrm{Flim}(\mathcal{K})),$$

the identity having basic neighbourhoods

$$G_{(A)} := \{g \in G_{\mathcal{K}} \mid g \restriction A = \mathrm{id}_A\}$$

for all finite substructures $A$ of $\mathrm{Flim}(\mathcal{K})$. For any topological group $G$ a *$G$-flow* is a continuous action $a : G \times X \to X$ of $G$ on a compact Hausdorff space $X$. When the action is understood we shall refer to $X$ as a $G$-flow and write $g{\cdot}x$ or $gx$ for $a(g, x)$. For $Y \subseteq X$ we write $G{\cdot}Y := \bigcup_{g \in G} gY = \bigcup_{y \in Y} Gy$ where $Gy := \{gy \mid g \in G\}$ denotes the *orbit of* $y$ and $gY := \{gy \mid y \in Y\}$.

**Example 2.4.** Let $G = \mathrm{Aut}(F)$ for a countable structure $F$. The *space of linear orders (on $F$)* is $LO := \{R \subseteq F^2 \mid R \text{ is a linear order on } F\}$ with topology given by basic open sets $\{R \mid R_0 \subseteq R\}$ for $R_0$ a linear order on a finite subset $A$ of $F$. This space is compact and Hausdorff, and a $G$-flow with respect to $(g, R) \mapsto g(R)$, the *logic action* of $G$ on $LO$.

A subset $Y \subseteq X$ is *$G$-invariant* if $G \cdot Y \subseteq Y$. Closed $G$-invariant subsets $Y$ are $G$-flows with respect to the restriction of the action. Such $G$-flows are *subflows* of $X$. The flow $X$ is *minimal* if $X$ and $\emptyset$ are its only subflows, that is, if and only if every orbit is dense. By Zorn's lemma, every $G$-flow contains a minimal subflow. A *homomorphism (isomorphism)* of a $G$-flow $X$ into another $Y$ is a continuous (bijective) $G$-map $\pi : X \to Y$; being a $G$-map means that $\pi(g \cdot x) = g \cdot \pi(x)$ for all $g \in G, x \in X$.

The following is well-known (cf. [22, §3]).

**Theorem 2.5.** *For every Hausdorff topological group $G$ there exists a minimal $G$-flow $M(G)$ which is* universal *in the sense that for every minimal $G$-flow $Y$ there is a homomorphism from $X$ into $Y$. Any two universal minimal $G$-flows are isomorphic.*

An interesting case is that $|M(G)| = 1$, equivalently, every $G$-flow $X$ has a fixed point, i.e. an $x \in X$ such that $G \cdot x = \{x\}$. In this case $G$ is called *extremely amenable*. Being *amenable* means that there exists a (Borel probability) measure $\mu$ on $M(G)$ which is $G$-*invariant* (i.e. $\mu(X) = \mu(g \cdot X)$ for every Borel $X \subset M(G)$ and $g \in G$). If there is exactly one such measure then $G$ is *uniquely ergodic*. It is shown in [1, P8.1] that for a uniquely ergodic $G$ in fact every minimal $G$-flow has a unique $G$-invariant measure.

We refer to [14, §1] for a survey on universal minimal flows.

## 2.5 Duality theory

Let $<$ be a binary relation symbol. A class $\mathcal{K}^*$ of finite $L \cup \{<\}$-structures is *ordered* if each of its members has the form $(A, <^A)$ for a (linear) order $<^A$ (on $A$) and some finite $L$-structure $A$; the order $<^A$ is called a $\mathcal{K}^*$-*admissible* one (cf. [17]).

The following is [14, T4.8].

**Theorem 2.6.** *Assume that $\mathcal{K}^*$ is an ordered Fraïssé class. Then $G_{\mathcal{K}^*}$ is extremely amenable if and only if $\mathcal{K}^*$ is Ramsey.*

Let $\mathcal{K} := \{A \mid (A, <^A) \in \mathcal{K}^*\}$ be the *$L$-reduct* of $\mathcal{K}^*$; $\mathcal{K}^*$ is *reasonable* if for all $A, B \in \mathcal{K}$, all $a \in B^A$ and all $\mathcal{K}^*$-admissible orders $<^A$ on $A$ there is a $\mathcal{K}^*$-admissible order $<^B$ on $B$ such that $a(<^A) \subseteq <^B$, i.e. $a \in (B, <^B)^{(A,<^A)}$.

**Lemma 2.7.** *Let $\mathcal{K}$ be a Fraïssé class and let $F = \mathrm{Flim}(\mathcal{K})$. Then $\mathcal{K}^* = \mathrm{Age}(F, R)$ is reasonable for every order $R$ on $F$.*

*Proof.* Let $A, B \in \mathcal{K}$, $a \in B^A$ and $<^A$ be a $\mathcal{K}^*$-admissible order on $A$. Let $a_0 \in (F, R)^{(A,<^A)}$ and $b \in F^B$. In particular, $a_0 \in F^A$ and $b \circ a \in F^A$, and then by homogeneity of $F$ there exists an $\alpha \in \mathrm{Aut}(F)$ such that $\alpha \circ b \circ a = a_0$. We define

$$<^B := (b^{-1} \circ \alpha^{-1})(R \restriction (\alpha \circ b)(B))$$

We need to show that $a^{-1}(<^B \restriction a(A)) = <^A$. We have that

$$a^{-1}(<^B \restriction a(A)) = a^{-1}((b^{-1} \circ \alpha^{-1})(R \restriction (\alpha \circ b)(B)) \restriction a(A)) =$$
$$a^{-1}((b^{-1} \circ \alpha^{-1})(R \restriction (\alpha \circ b)(a(A)))) = a_0^{-1}(R \restriction a_0(A)) = <^A$$

The last equality holds as $a_0 \in (F, R)^{(A,<^A)}$. □

The following is [14, P5.2, T10.8]. Recall that $LO$ denotes the space of orders (Example 2.4).

**Theorem 2.8.** *Let $\mathcal{K}^*$ be a reasonable ordered Fraïssé class in the language $L \cup \{<\}$ and $\mathcal{K}$ its L-reduct.*

1. *Then $\mathcal{K}$ is Fraïssé and $\mathrm{Flim}(\mathcal{K}^*) = (\mathrm{Flim}(\mathcal{K}), <^*)$ for some linear order $<^*$.*
2. *Let $X_{\mathcal{K}^*} := \overline{G_{\mathcal{K}} \cdot <^*}$ be the orbit closure of $<^*$ in the logic action of $G_{\mathcal{K}}$ on $LO$. Then $X_{\mathcal{K}^*}$ is the universal minimal $G_{\mathcal{K}}$-flow if and only if $\mathcal{K}^*$ is Ramsey and has the ordering property.*

That $\mathcal{K}^*$ has the *ordering property* means that for all $A \in \mathcal{K}$ there is a $B \in \mathcal{K}$ such that $(A, <^A) \leq (B, <^B)$ for all $\mathcal{K}^*$-admissible orders $<^A$ on $A$ and $<^B$ on $B$.

In [1] Kechris et al. showed that a certain quantitative version of the ordering property characterises unique ergodicity for so-called Hrushovski classes. Here, we shall only need the following [1, P9.2].

**Proposition 2.9.** *Let $\mathcal{K}^*$ be a reasonable ordered Fraïssé class which is Ramsey and satisfies the ordering property, and let $\mathcal{K}$ be its L-reduct. Then $G_{\mathcal{K}}$ is amenable (uniquely ergodic) if and only if there exists a consistent random $\mathcal{K}^*$-admissible ordering $(R_A)_{A\in\mathcal{K}}$ (and for every other consistent random $\mathcal{K}^*$-admissible ordering $(R'_A)_{A\in\mathcal{K}}$ we have that $R_A$ and $R'_A$ have the same distribution for every $A \in \mathcal{K}$).*

*Indeed, if $(R_A)_{A\in\mathcal{K}}$ is a random $\mathcal{K}^*$-admissible ordering, then there is a $G_{\mathcal{K}}$-invariant Borel probability measure $\mu$ on $X_{\mathcal{K}^*}$ such that for every $A \in \mathcal{K}$ and every $\mathcal{K}^*$-admissible ordering $<$ on $A$ we have*[2] $\mu(U(<)) = \Pr[R_A = <]$ *where*

$$U(<) := \{R \in X_{\mathcal{K}^*} \mid R \restriction A = <\}.$$

[2]Given a random variable we always use Pr to denote the probability measure of its underlying probability space.

A *random $\mathcal{K}^*$-admissible ordering* is a family $(R_A)_{A \in \mathcal{K}}$ of random variables such that each $R_A$ takes values in the set of $\mathcal{K}^*$-admissible orders on $A$. It is *consistent* if for all $A, B \in \mathcal{K}$ and $a \in B^A$ the random variables $a^{-1}(R_B \restriction \mathrm{im}(a))$ and $R_A$ have the same distribution.

**Examples 2.10.** In [14, §6] the reader can find constructions of reasonable ordered Fraïssé classes $\mathcal{K}^*$ whose reduct $\mathcal{K}$ is any of the classes mentioned in Example 2.2; in all these cases $\mathcal{K}^*$ is Ramsey and has the ordering property. By Theorem 2.6 one sees that the automorphism groups of $(\mathbb{Q}, <)$ and of certain ordered versions of $B_\infty, R, V_{\infty,F}$ are extremely amenable [14]. Theorem 2.8 allows us to calculate the universal minimal flows of the automorphism groups of $B_\infty, R$ and $V_{\infty,F}$. $\mathrm{Aut}(B_\infty)$ is not amenable, while $\mathrm{Aut}(R)$ and $\mathrm{Aut}(V_{\infty,F})$ are uniquely ergodic [1].

# 3 Automorphism groups with finite universal minimal flows

Theorem 2.6 characterises the condition that the universal minimal flow has size 1. In this section we provide a similar characterisation for the condition that it has an arbitrary finite size. To this end we consider Ramsey degrees *for embeddings*. The main result in this section reads:

**Theorem 3.1.** *Let $d \in \mathbb{N}$ and $\mathcal{K}$ be a Fraïssé class. The following are equivalent.*

1. *$M(G_{\mathcal{K}})$ has size at most $d$;*
2. *the Ramsey degree for embeddings of $\mathcal{K}$ is at most $d$.*

We start with some preliminary observations concerning finite universal minimal flows in Section 3.1. In Section 3.2 we define Ramsey degrees for embeddings and discuss their relationship to Ramsey degrees. The results proved in Sections 3.1 and 3.2 are mainly folklore. In Section 3.3 we prove the result above and in Section 3.4 we note some corollaries.

## 3.1 Finite universal minimal flows

**Lemma 3.2.** *Let $G$ be a topological Hausdorff group and $d \in \mathbb{N}$. Then $M(G)$ has size at most $d$ if and only if every nonempty $G$-flow has an orbit of size at most $d$.*

*Proof.* Assume that $|M(G)| \leq d$, and let $X$ be a nonempty $G$-flow. Then there is a minimal subflow $X'$ of $X$ and a homomorphism $\pi$ of $M(G)$ onto $X'$. Thus $|X'| \leq d$.

Conversely, if every nonempty $G$-flow has an orbit of size at most $d$, then so does $M(G)$. Since $M(G)$ is minimal, this orbit is dense in $M(G)$, so it is equal to $M(G)$ by finiteness. □

**Lemma 3.3.** *Let $G$ be a topological Hausdorff group and $H$ an extremely amenable closed subgroup of $G$ with finite index. Then $H$ is a normal clopen subgroup of $G$ and $M(G)$ is isomorphic to the action of $G$ on $G/H$ by left multiplication.*

*Proof.* Clearly, a closed subgroup of finite index is open. We first show that $G/H$ is the universal minimal $G$-flow. Since $H$ is open $G/H$ is discrete, and as $|G : H|$ is finite, $G/H$ is compact. Hence, $G/H$ is a $G$-flow. It is minimal, because $G$ acts transitively on $G/H$. If $Y$ is an arbitrary $G$-flow, then its restriction to $H$ is an $H$-flow, so it has a fixed point $y \in Y$. Then $gH \mapsto gy$ is a homomorphism from $G/H$ into $Y$.

As $gHg^{-1}$ is a closed subgroup of finite index for every $g \in G$, so is $H' = H \cap gHg^{-1}$. As above, we see that $G/H'$ is a minimal $G$-flow. By universality of $G/H$ there exists a surjection from $G/H$ onto $G/H'$, so $|G : H'| \leq |G : H|$. Thus $H = gHg^{-1}$ for every $g \in G$, that is, $H$ is normal. □

**Proposition 3.4.** *Let $G$ be a topological Hausdorff group and $d \in \mathbb{N}$. Then $M(G)$ has size $d$ if and only if $G$ has an extremely amenable, open, normal subgroup of index $d$.*

*Proof.* The backward direction follows from Lemma 3.3. Conversely, assume that $X := M(G)$ has size $d$. For $x \in X$ let $H_x \leq G$ be the stabiliser of $x$. Then there is a bijection between the set of left cosets of $H_x$ and the orbit $G \cdot x$. Since $G \cdot x$ is finite, $G \cdot x = \overline{G \cdot x}$, so $G \cdot x = X$ by minimality. Hence, $|G : H_x| = |X| = d$. As $H_x$ is closed and of finite index, so is $N := \bigcap_{x \in X} H_x$, and hence $N$ is clopen. Since $N$ is the pointwise stabiliser of $X$, it is normal. Let $Y$ be a minimal $N$-flow. Let $\tau : G/N \to G$ be a function with $\tau(hN) \in hN$. Define $a : G \times G/N \to N$ by setting

$$a(g, hN) := \tau(hN)^{-1} \cdot g^{-1} \cdot \tau(ghN).$$

A straightforward calculation shows that $a$ satisfies the so-called cocycle identity, that is, for all $g_1, g_2, h \in G$

$$a(g_1 g_2, hN) = a(g_2, hN) \cdot a(g_1, g_2 hN). \tag{1}$$

We can construct an action of $G$ on $(G/N \times Y)$ by

$$(g, (hN, y)) \mapsto (ghN, a(g, hN)^{-1} \cdot y).$$

That this indeed defines a group action follows directly from (1). The action is continuous and $(G/N \times Y)$ is compact, so $(G/N \times Y)$ is a $G$-flow.

Let $h \in G, y \in Y$ be arbitrary. We show that

$$Y(h, y) := \{a(n, hN)^{-1} \cdot y \mid n \in N\} \text{ is dense in } Y. \tag{2}$$

Indeed, as $N$ is normal, we have $Y(h, y) = \tau(hN)^{-1} \cdot N \cdot \tau(hN) \cdot y = N \cdot y$. Since $Y$ is a minimal $N$-flow, the orbit $N \cdot y$ is dense in $Y$.

The orbit $G \cdot (hN, y)$ contains $N \cdot (ghN, y')$ for every $g \in G$ and $y' := a(g, hN)^{-1} \cdot y$. But $N \cdot (ghN, y') = \{(nghN, a(n, hN)^{-1} \cdot y') \mid n \in N\} = \{ghN\} \times Y(h, y')$, where the last equality holds because $N$ is normal. So the orbit $G \cdot (hN, y)$ contains $\bigcup_{g \in G}(\{gN\} \times Y(h, y_g))$ for certain $y_g$'s, and this set is dense in $(G/N \times Y)$ by (2). Thus $(G/N \times Y)$ is a minimal $G$-flow.

By the universality of $X$ there exists a surjection from $X$ onto $(G/N \times Y)$. In particular, $|G/N \times Y| \leq d$. By definition of $N$ we have $|G : N| \geq d$, so $|Y| = 1$, $|G/N| = d$. This means $N$ is extremely amenable and has index $d$ in $G$. □

**Example 3.5.** For $d \in \mathbb{N}$ let $G^*$ be the automorphism group of $(\mathbb{Q}, <, 0, 1, \ldots, d-1)$, the structure with universe $\mathbb{Q}$ that interprets for all $i \in [d]$ a constant by $i$ and a binary relation symbol $<$ by the rational order. Let $G$ be the group generated by $G^*$ and the permutation $\alpha = (0\ 1 \ldots\ d-1)$. This is a closed subgroup of the group of all permutations of $\mathbb{Q}$, so $G = G_{\mathcal{K}}$ for some Fraïssé class $\mathcal{K}$ (see e.g. [2]). Since $\alpha$ commutes with $G$, $G^*$ is normal in $G$. Moreover, $G^*$ has index $d$ in $G$, and it follows from [5, L13] (see also [4, P24]) that $G^*$ is extremely amenable. By Lemma 3.3, $|M(G)| = |G/G^*| = d$.

**Example 3.6.** Let $G$ be the automorphism group of $(\mathbb{Q}, E_d, <)$ where $<$ is the rational order and $E_d$ is an equivalence relation with $d$ classes each of which is dense in $(\mathbb{Q}, <)$. Let $H$ be the subgroup of $G$ consisting of those automorphisms that preserve each of the classes. It is shown in [14, T8.4] that $H$ is extremely amenable and of index $d!$ in $G$. By Lemma 3.3, $|M(G)| = |G/H| = d!$.

## 3.2 Ramsey degrees for embeddings

Let $k, d \in \mathbb{N}$ and $\mathcal{K}$ be a class of finite structures. Then $C \hookrightarrow (B)^A_{k,d}$ means that for every colouring $\chi : C^A \to [k]$ there exists a $b \in C^B$ such that

$|\chi(b \circ B^A)| \leq d$. Naturally here, $b \circ B^A$ denotes $\{b \circ a \mid a \in B^A\}$. The *Ramsey degree for embeddings* of $A$ in $\mathcal{K}$ is the least $d \in \mathbb{N}$ such that for all $B \in \mathcal{K}$ and $k \geq 2$ there is a $C \in \mathcal{K}$ such that $C \hookrightarrow (B)^A_{k,d}$ – provided that such a $d$ exists; otherwise it is $\infty$. Taking the supremum over $A \in \mathcal{K}$ gives the Ramsey degree for embeddings of $\mathcal{K}$. If this degree is 1 we call $\mathcal{K}$ *Ramsey for embeddings.*

**Lemma 3.7.** *Let $d \in \mathbb{N}$, $\mathcal{K}$ be a Fraïssé class, $F = \mathrm{Flim}(\mathcal{K})$ and $A \in \mathcal{K}$. The Ramsey degree for embeddings of $A$ in $\mathcal{K}$ is at most $d$ if and only if $F \hookrightarrow (B)^A_{k,d}$ for all $B \in \mathcal{K}$ and $k \geq 2$.*

*Proof.* Assume that the Ramsey degree for embeddings of $A$ in $\mathcal{K}$ is at most $d$. Let $B \in \mathcal{K}, k \geq 2$ and $\chi : F^A \to [k]$. We are looking for $b' \in F^B$ such that $|\chi(b' \circ B^A)| \leq d$. Choose $C \in \mathcal{K}$ such that $C \hookrightarrow (B)^A_{k,d}$. Choose a $c \in F^C$ and let $\chi' : C^A \to [k]$ map $a \in C^A$ to $\chi(c \circ a)$. By $C \hookrightarrow (B)^A_{k,d}$ there is a $b \in C^B$ such that $|\chi'(b \circ B^A)| \leq d$, i.e. $|\chi(c \circ b \circ B^A)| \leq d$. Then $b' := c \circ b \in F^B$ is as desired.

Assume that there is an $A \in \mathcal{K}$ whose Ramsey degree for embeddings is bigger than $d$. Choose $B \in \mathcal{K}, k \geq 2$ such that for every finite substructure $C$ of $F$ there is a colouring $\chi : C^A \to [k]$ which is *good for* $C$, i.e. $|\chi(b \circ B^A)| > d$ for all $b \in C^B$. The set $G(C) := \{\chi \in [k]^{F^A} \mid \chi \restriction C^A \text{ is good for } C\}$ is nonempty and closed in $[k]^{F^A}$ carrying the product topology with $[k]$ being discrete. Given finitely many such sets $G(C_1), \ldots, G(C_n)$ their intersection contains the nonempty set $G(C)$ where $C$ is the substructure generated by $C_1 \cup \ldots \cup C_n$ in $F$ (note that $C$ is finite by local finiteness of $F$). Since $[k]^{F^A}$ is compact, $\bigcap_C G(C) \neq \emptyset$ where $C$ ranges over the finite substructures of $F$. Any $\chi \in \bigcap_C G(C)$ is good for $F$, so $F \not\hookrightarrow (B)^A_{k,d}$. □

We shall need the following result of Nešetřil [17, T3.2]. We include the short proof.

**Lemma 3.8.** *Let $\mathcal{K}$ be a hereditary class of finite structures with joint embedding. If $\mathcal{K}$ is Ramsey for embeddings, then it has amalgamation.*

*Proof.* Let $A, B_0, B_1 \in \mathcal{K}$ and $a_0 \in B_0^A, a_1 \in B_1^A$. Let $B \in \mathcal{K}$ and $b_0 \in B^{B_0}, b_1 \in B^{B_1}$. Choose $C \in \mathcal{K}$ with $C \hookrightarrow (B)^A_{4,1}$. We claim that there exist $e_0 \in C^{B_0}, e_1 \in C^{B_1}$ such that $e_0 \circ a_0 = e_1 \circ a_1$. Consider the following colouring $\chi : C^A \to P(\{0,1\})$: for $a \in C^A$ the colour $\chi(a) \subseteq \{0,1\}$ contains $i \in \{0,1\}$ if and only if there exists an $e \in C^{B_i}$ such that $e \circ a_i = a$. Choose $b \in C^B$ such that $\chi(b \circ B^A)$ contains precisely one colour. Then this colour is $\{0,1\}$, because for $i \in \{0,1\}$ we have $i \in \chi(b \circ b_i \circ a_i)$ and $b \circ b_i \circ a_i \in b \circ B^A$.

Let $a \in B^A$. Then $\chi(b \circ a) = \{0,1\}$, thus there are $e_0 \in C^{B_0}, e_1 \in C^{B_1}$ such that $e_0 \circ a_0 = a = e_1 \circ a_1$. $\square$

**Remark 3.9.** Clearly, $C \hookrightarrow (B)^A_{k,d}$ is equivalent to $C \to (B)^A_{k,d}$ when $A$ is rigid (i.e. $\mathrm{Aut}(A) = \{\mathrm{id}_A\}$). In particular, the Ramsey degree and the Ramsey degree for embeddings coincide for rigid structures. The following proposition generalises this observation.

**Proposition 3.10.** *Let $d \in \mathbb{N}$, and let $\mathcal{K}$ be a class of finite structures. Let $A \in \mathcal{K}$ and $\ell = |\,\mathrm{Aut}(A)|$. The Ramsey degree for embeddings of $A$ in $\mathcal{K}$ is at most $d \cdot \ell$ if and only if the Ramsey degree of $A$ in $\mathcal{K}$ is at most $d$.*

*Proof.* First assume that the Ramsey degree for embeddings of $A$ in $\mathcal{K}$ is at most $d \cdot \ell$. Let $B \in \mathcal{K}$ and $k \geq 2$. We are looking for a $C \in \mathcal{K}$ such that $C \to (B)^A_{k,d}$. By assumption we find some $C \in \mathcal{K}$ with $C \hookrightarrow (B)^A_{k,d\cdot\ell}$ and we claim that this $C$ is as desired. Let a colouring $\chi : \binom{C}{A} \to [k]$ be given. For every $A' \in \binom{C}{A}$ there are precisely $\ell$ embeddings $a_0^{A'}, \dots, a_{\ell-1}^{A'} \in C^A$ with image $A'$. Define $\chi' : C^A \to [k] \times [\ell]$ to map $a \in C^A$ to $(i,j)$ for $i := \chi(\mathrm{im}(a))$ and $j$ such that $a = a_j^{\mathrm{im}(a)}$. Since $C \hookrightarrow (B)^A_{k,d\cdot\ell}$ there is $b \in C^B$ such that $|\chi'(b \circ B^A)| \leq d \cdot \ell$. Observe that $(i,j) \in \chi'(b \circ B^A)$ implies $\{i\} \times [\ell] \subseteq \chi'(b \circ B^A)$. Hence, there are (not necessarily distinct) $i_0, \dots, i_{d-1} \in [k]$ such that $\chi'(b \circ B^A) = \{i_0, \dots, i_{d-1}\} \times [\ell]$. Clearly, $\mathrm{im}(b) \in \binom{C}{B}$ and we claim that $\chi(\binom{\mathrm{im}(b)}{A}) \subseteq \{i_0, \dots, i_{d-1}\}$. Indeed, for $A' \in \binom{\mathrm{im}(b)}{A}$ there is an $a \in B^A$ such that $\mathrm{im}(b \circ a) = A'$, namely $a := b^{-1} \circ a'$ for some isomorphism $a' : A \to A'$. As $\binom{\mathrm{im}(b)}{A} \subseteq \binom{C}{A}$ we find $j \in [\ell]$ such that $a_j^{A'} = b \circ a$. Then $\chi'(b \circ a) = (\chi(A'), j)$, and in particular $\chi(A') \in \{i_0, \dots, i_{d-1}\}$.

Conversely, assume that the Ramsey degree of $A$ in $\mathcal{K}$ is at most $d$. Let $B \in \mathcal{K}$ and $k \geq 2$ be given. By assumption there exists a $C \in \mathcal{K}$ such that $C \to (B)^A_{k^\ell,d}$. We claim that $C \hookrightarrow (B)^A_{k,d\cdot\ell}$. Let $\chi : C^A \to [k]$ be a colouring and define $\chi' : \binom{C}{A} \to [k]^\ell$ by setting $\chi'(A') := (\chi(a_0^{A'}), \dots, \chi(a_{\ell-1}^{A'}))$ for $A' \in \binom{C}{A}$; here, for $A' \in \binom{C}{A}$ we let $a_0^{A'}, \dots, a_{\ell-1}^{A'}$ enumerate the embeddings in $C^A$ with image $A'$. Since $C \to (B)^A_{k^\ell,d}$ there exists $B' \in \binom{C}{B}$ and $(i_0^0, \dots, i_{\ell-1}^0), \dots, (i_0^{d-1}, \dots, i_{\ell-1}^{d-1}) \in [k]^\ell$ such that $\chi'(\binom{B'}{A}) \subseteq \{(i_0^\nu, \dots, i_{\ell-1}^\nu) \mid \nu \in [d]\}$. Choose $b \in C^B$ with image $B'$. We claim that $\chi(b \circ B^A) \subseteq \{i_j^\nu \mid \nu \in [d], j \in [\ell]\}$. Let $a \in B^A$. Then $b \circ a \in C^A$ and $\mathrm{im}(b \circ a) \in \binom{B'}{A} \subseteq \binom{C}{A}$. Choose $j \in [\ell]$ such that $b \circ a = a_j^{\mathrm{im}(b\circ a)}$. Let $\nu \in [d]$ be such that $\chi'(\mathrm{im}(b \circ a)) = (\chi(a_0^{\mathrm{im}(b\circ a)}), \dots, \chi(a_{\ell-1}^{\mathrm{im}(b\circ a)})) = (i_0^\nu, \dots, i_{\ell-1}^\nu)$. Hence, $\chi(b \circ a) = \chi(a_j^{\mathrm{im}(b\circ a)}) = i_j^\nu$. $\square$

**Corollary 3.11.** *Let $\mathcal{K}$ be a class of finite structures and $A \in \mathcal{K}$. Then the Ramsey degree of $A$ in $\mathcal{K}$ is 1 if and only if the Ramsey degree for embeddings of $A$ in $\mathcal{K}$ is $|\operatorname{Aut}(A)|$.*

*Proof.* By Proposition 3.10 is suffices to show that the Ramsey degree for embeddings of $A$ in $\mathcal{K}$ is at least $\ell := |\operatorname{Aut}(A)|$. Let $C \in \mathcal{K}$ be arbitrary. Using the notation from the previous proof, let $\chi : C^A \to [\ell]$ map $a \in C^A$ to the $j < \ell$ such that $a = a_j^{\operatorname{im}(a)}$. Then for $B := A$ and every $b \in C^B$ we have $\chi(b \circ C^B) = [\ell]$. □

Our main result concerning the relationship of Ramsey degrees and Ramsey degrees for embeddings is the following.

**Theorem 3.12.** *Let $\mathcal{K}$ be a relational Fraïssé class which is Ramsey. Then the Ramsey degree for embeddings of $\mathcal{K}$ is infinite or a finite power of 2.*

We refer to Examples 4.5 for some natural examples of relational Fraïssé classes which are Ramsey and have infinite Ramsey degree for embeddings. We prove Theorem 3.12 in Section 4.3.

## 3.3 Proof of Theorem 3.1

Theorem 3.1 is a consequence of the following two propositions which in fact establish something stronger.

We say that a class of finite structures $\mathcal{D}$ is *cofinal in* another such class $\mathcal{K}$ if for all $A \in \mathcal{K}$ there exists $B \in \mathcal{D}$ such that $A \leq B$.

**Proposition 3.13.** *Let $d \in \mathbb{N}$ and $\mathcal{K}$ be a Fraïssé class. Assume that the class of structures with Ramsey degree for embeddings at most $d$ in $\mathcal{K}$ is cofinal in $\mathcal{K}$. Then $M(G_{\mathcal{K}})$ has size at most $d$.*

*Proof.* Write $G := G_{\mathcal{K}}$ and $F := \operatorname{Flim}(\mathcal{K})$. Let $A \in \mathcal{K}, a_0 \in F^A$ and write $A_0 := \operatorname{im}(a_0)$. Consider the map $\Phi : G \to F^A$, $g \mapsto g \circ a_0$ . By homogeneity of $F$, $\Phi$ is surjective. We have for all $g, h \in G$

$$g \circ a_0 = h \circ a_0 \iff gG_{(A_0)} = hG_{(A_0)}.$$

Hence, $\Phi$ induces a bijection e from $G/G_{(A_0)}$ onto $F^A$. Observe that

$$g \circ \mathrm{e}(hG_{(A_0)}) = g \circ (h \circ a_0) = (gh) \circ a_0 = \mathrm{e}((gh)G_{(A_0)}). \tag{3}$$

*Claim 1.* Assume that $A$ has Ramsey degree for embeddings at most $d$ in $\mathcal{K}$. Let $k \in \mathbb{N}$ and $f : G \to [k]$ be constant on each $gG_{(A_0)} \subseteq G$ for $g \in G$. Then, for every finite $H \subseteq G$ there exists $g \in G$ such that $|f(gH)| \leq d$.

*Proof of Claim 1:* The function $f$ induces a function $\tilde{f}$ from $G/G_{(A_0)}$ to $[k]$. Note that $\tilde{f} \circ \mathrm{e}^{-1} : F^A \to [k]$. There is a finite substructure $B \subseteq F$ such that

$$\{\mathrm{e}(hG_{(A_0)}) \mid h \in H\} \subseteq B^A. \tag{4}$$

By Lemma 3.7 there is $b \in F^B$ such that $|(\tilde{f} \circ \mathrm{e}^{-1})(b \circ B^A)| \leq d$. By homogeneity of $F$ there is a $g \in G$ such that $g \circ \mathrm{id}_B = b$. We show that $g$ is as desired, namely $f(gh) \in (\tilde{f} \circ \mathrm{e}^{-1})(b \circ B^A)$ for every $h \in H$:

$$f(gh) = \tilde{f}((gh)G_{(A_0)}) = \tilde{f} \circ \mathrm{e}^{-1}(\mathrm{e}((gh)G_{(A_0)})) = \tilde{f} \circ \mathrm{e}^{-1}(g \circ \mathrm{e}(hG_{(A_0)}))$$

where the last equality follows from (3). By (4) we have $g \circ \mathrm{e}(hG_{(A_0)}) \in g \circ B^A = b \circ B^A$, and our claim follows. $\dashv$

For $n \in \mathbb{N}, n \geq 1$, consider $\mathbb{R}^n$ with the Euclidian norm $\|\cdot\|$. For $\varepsilon > 0$ and $x \in \mathbb{R}^n$ let

$$B_\varepsilon(x) := \{y \in \mathbb{R}^n \mid \|x - y\| < \varepsilon\}.$$

As a topological group $G$ carries its *left* uniformity, that is, the uniformity with basic entourages $\{(g, h) \mid g^{-1}h \in G_{(A)}\}$ for $A \in \mathrm{Age}(M), A \subseteq M$.

*Claim 2.* Let $n$ be a positive integer, $f : G \to \mathbb{R}^n$ be left uniformly continuous and bounded, $H \subseteq G$ be finite and $\varepsilon$ be a positive real. Then there are $g \in G$ and $h_0, \ldots, h_{d-1} \in H$ such that

$$f(gH) \subseteq \textstyle\bigcup_{\nu < d} B_\varepsilon(f(gh_\nu)). \tag{5}$$

*Proof of Claim 2:* By left uniform continuity of $f$ there is a finite substructure $A' \subseteq F$ such that $\|f(g) - f(g')\| < \varepsilon/6$ for all $g, g' \in G$ with $gG_{(A')} = g'G_{(A')}$. By our cofinality assumption, there exist $A'' \in \mathcal{K}$ and $a' \in (A'')^{A'}$ such that $A''$ has Ramsey degree for embeddings at most $d$ in $\mathcal{K}$. Since $F$ is homogeneous, there is an embedding $a'' \in F^{A''}$ such that $a'' \circ a' = \mathrm{id}_{A'}$. Hence, the image $A$ of $a''$ has Ramsey degree for embeddings at most $d$ in $\mathcal{K}$, and $A' \subseteq A \subseteq F$. Thus $G_{(A)} \subseteq G_{(A')}$, so for all $g, g' \in G$ with $gG_{(A)} = g'G_{(A)}$

$$\|f(g) - f(g')\| < \varepsilon/6. \tag{6}$$

We claim that there exists a function $\tilde{f} : G \to \mathbb{R}^n$ such that

(a) $\mathrm{im}(\tilde{f})$ is finite;

(b) $\tilde{f}$ is constant on $gG_{(A)}$ for every $g \in G$;

(c) $\|f(g) - \tilde{f}(g)\| < \varepsilon/2$ for every $g \in G$.

By (a) and (b) we can apply Claim 1 and obtain some $g \in G$ such that $|\tilde{f}(gH)| \leq d$. Choose $h_0, \ldots, h_{d-1} \in H$ such that $\tilde{f}(gH) = \{\tilde{f}(gh_\nu) \mid \nu < d\}$. To verify (5), let $h \in H$ be given. We have to show that there exists $\nu < d$ such that $\|f(gh) - f(gh_\nu)\| < \varepsilon$. Indeed, this holds for $\nu < d$ such that $\tilde{f}(gh) = \tilde{f}(gh_\nu)$, because by (c) we have both $\|f(gh) - \tilde{f}(gh_\nu)\| = \|f(gh) - \tilde{f}(gh)\| < \varepsilon/2$ and $\|\tilde{f}(gh_\nu) - f(gh_\nu)\| < \varepsilon/2$.

Thus, we are left to find $\tilde{f}$ with properties (a)-(c).

As $f$ is bounded, its image is contained in a compact subset of $\mathbb{R}^n$. Choose finitely many points $y_\nu \in \mathbb{R}^n, \nu < k'$, such that this compact set is covered by $\bigcup_{\nu<k'} B_{\varepsilon/6}(y_\nu)$. Assume that precisely the first $k \leq k'$ balls $B_{\varepsilon/6}(y_\nu)$ contain a point from the image of $f$. For $\nu < k$ choose $\widehat{\nu} \in G$ such that $f(\widehat{\nu}) \in B_{\varepsilon/6}(y_\nu)$. Then $\bigcup_{\nu<k} B_{\varepsilon/3}(f(\widehat{\nu}))$ covers the image of $f$. Hence, for every $g \in G$ we can choose $\nu_g < k$ such that

$$\|f(g) - f(\widehat{\nu_g})\| < \varepsilon/3. \tag{7}$$

Let $c : G \to G$ be a selector for the partition $\{gG_{(A)} \mid g \in G\}$ of $G$, that is, for all $g, g' \in G$ we have $c(g) \in gG_{(A)}$, and $c(g) = c(g')$ if and only if $gG_{(A)} = g'G_{(A)}$. Define

$$\tilde{f}(g) := f(\widehat{\nu_{c(g)}}).$$

Then $\tilde{f}$ satisfies (a) and (b). For all $g \in G$ we have $c(g) \in gG_{(A)}$, so $gG_{(A)} = c(g)G_{(A)}$ and thus $\|f(g) - f(c(g))\| < \varepsilon/6$ by (6). As $\|f(c(g)) - f(\widehat{\nu_{c(g)}})\| < \varepsilon/3$ by (7), we conclude that $\tilde{f}$ satisfies (c). ⊣

We aim to show that every $G$-flow has an orbit of size at most $d$ (Lemma 3.2). So let $X$ be a $G$-flow. We are looking for some $x_0 \in X$ such that

$$|G \cdot x_0| \leq d. \tag{8}$$

Recall that the compact Hausdorff space $X$ carries a unique uniformity compatible with its topology. Suppose $f$ is a uniformly continuous function from $X$ into $\mathbb{R}^n$ for some $n \geq 1$. For each $x \in X$ define the function $f_x : G \to \mathbb{R}^n$ by

$$f_x(g) := f(g^{-1} \cdot x).$$

Then $f_x$ is left uniformly continuous. This follows from the well-known fact that for every $x \in X$ the map $g \mapsto g^{-1} \cdot x$ is left uniformly continuous (see e.g. [20, L2.1.5]).

With a triple $(H, f, \varepsilon)$ for a finite subset $H \subseteq G$, and a bounded, uniformly continuous function $f : X \to \mathbb{R}^n$, and a real $\varepsilon > 0$ we associate the set

$$Y(H, f, \varepsilon) := \left\{x \in X \mid \exists h_0, \dots, h_{d-1} \in H : f_x(H) \subseteq \bigcup_{\nu<d} \overline{B_\varepsilon(f_x(h_\nu))}\right\}.$$

Since $H$ is finite, $Y(H, f, \varepsilon)$ is a finite union of closed sets of the form $\{x \in X \mid f_x(H) \subseteq C\}$ for $C \subseteq \mathbb{R}^n$ closed, and consequently, $Y(H, f, \varepsilon)$ is closed.

*Claim 3.* The family of closed sets $Y(H, f, \varepsilon)$ with $H, f, \varepsilon$ as above has the finite intersection property.

*Proof of Claim 3:* For $j < \ell$ let $H_j \subseteq G$ be finite, $\varepsilon_j > 0$ and $f^j : X \to \mathbb{R}^{n_j}$ for $n_j \geq 1$. Set $H := \bigcup_{j<\ell} H_j, \varepsilon := \min_{j<\ell} \varepsilon_j, n := \sum_{j<\ell} n_j$ and define $f : X \to \mathbb{R}^n$ by $f(x) := f^0(x) * \cdots * f^{\ell-1}(x)$ where $*$ denotes concatenation. Then $f$ is uniformly continuous and bounded.

Let $x \in X$ be arbitrary. Since $f_x : G \to \mathbb{R}^n$ is left uniformly continuous, Claim 2 applies, and there exist $g \in G$ and $h_0, \dots, h_{d-1} \in H$ such that $f_x(gH) \subseteq \bigcup_{\nu<d} B_\varepsilon(f_x(gh_\nu))$. In other words,

$$\forall h \in H \; \exists \nu < d : \; f(h^{-1}g^{-1}x) \in B_\varepsilon(f(h_\nu^{-1}g^{-1}x)). \tag{9}$$

Any $y \in \mathbb{R}^n$ can be written as $y[0] * \cdots * y[\ell-1]$, where $y[j] \in \mathbb{R}^{n_j}$ for all $j < \ell$. In this notation, $f_x(g)[j] = f^j_x(g)$ for all $g \in G, x \in X, j < \ell$. Clearly, $f_x(g) \in B_\varepsilon(y)$ implies $f_x(g)[j] \in B_\varepsilon(y[j])$ for all $y \in \mathbb{R}^n, j < \ell$. Writing $x_0 := g^{-1}x$, (9) yields:

$$\forall j < \ell \; \forall h \in H_j \; \exists \nu < d : \; f(h^{-1}g^{-1}x)[j] = f^j_{x_0}(h) \in B_\varepsilon(f^j_{x_0}(h_\nu)).$$

Since $\varepsilon \leq \varepsilon_j$ we obtain

$$\forall j < \ell : \; f^j_{x_0}(H_j) \subseteq \bigcup_{\nu<d} B_{\varepsilon_j}(f^j_{x_0}(h_\nu)).$$

Thus, $x_0 \in \bigcap_{j<\ell} Y(H_j, f^j, \varepsilon_j) \neq \emptyset$. ⊣

By Claim 3 and since $X$ is compact, there exists an $x_0$ in the intersection of all the sets $Y(H, f, \varepsilon)$, $(H, f, \varepsilon)$ a triple as above. We claim that $x_0$ satisfies (8). Assume otherwise that there are $g_0, \dots, g_d \in G$ such that $g_0x_0, \dots, g_dx_0$ are pairwise distinct. Choose $f : X \to [0, 1] \subseteq \mathbb{R}^1$ uniformly continuous such that $f(g_\nu x_0) = \nu/d$ for all $\nu \leq d$. Then $x_0 \notin Y(\{g_\nu^{-1} \mid \nu \leq d\}, f, \varepsilon)$ for a small enough $\varepsilon > 0$, a contradiction. □

**Proposition 3.14.** *Let $d \in \mathbb{N}$, $F$ be countable and locally finite, $G := \mathrm{Aut}(F)$ and $A \in \mathrm{Age}(F)$ such that $F$ is $A$-homogeneous. If $M(G)$ has size at most $d$, then $F \hookrightarrow (B)^A_{k,d}$ for all $B \in \mathrm{Age}(F)$ and $k \geq 2$.*

*Proof.* Assume that $|M(G)| \leq d$, and let $B \in \mathrm{Age}(F), k \geq 2$ and $\chi_0 : F^A \to [k]$ be a colouring. Note that $[k]^{F^A}$ is compact Hausdorff in the product topology with $[k]$ being discrete. The group $G$ acts continuously on $[k]^{F^A}$ by shift $(g, \chi) \mapsto g \cdot \chi$, where $g \cdot \chi$ colours $a \in F^A$ by $\chi(g^{-1} \circ a)$. Consider the orbit closure $\overline{G \cdot \chi_0}$ of $\chi_0$. By Lemma 3.2, the induced action of $G$ on $\overline{G \cdot \chi_0}$ has an orbit of size at most $d$, that is, there exist $\chi_1 \in \overline{G \cdot \chi_0}$ and $\psi_0, \ldots, \psi_{d-1} \in \overline{G \cdot \chi_0}$ such that $G \cdot \chi_1 = \{\psi_i \mid i < d\}$.

Let $b \in F^B$. Observe that $b \circ B^A$ is a finite subset of $F^A$. Since $\chi_1 \in \overline{G \cdot \chi_0}$, there exists a $g \in G$ such that $g \cdot \chi_0$ and $\chi_1$ agree on $b \circ B^A$. Note that $g^{-1} \circ b \in F^B$, so we are left to show that $|\chi_0(g^{-1} \circ b \circ B^A)| \leq d$. We fix some $a_0 \in F^A$, and claim that for all $a \in g^{-1} \circ b \circ B^A$ there exists a $\nu < d$ such that $\chi_0(a) = \psi_\nu(a_0)$. To see this, let $a \in g^{-1} \circ b \circ B^A \subseteq F^A$ and choose $h \in G$ such that $h \circ a_0 = a$. Such an $h$ exists since $F$ is $A$-homogeneous. Then

$$\chi_0(a) = (g \cdot \chi_0)(g \circ a) = \chi_1(g \circ a) = \chi_1((gh) \circ a_0) = ((gh)^{-1} \cdot \chi_1)(a_0),$$

where the second equality follows from $g \circ a \in b \circ B^A$ and the choice of $g$. As $(gh)^{-1} \cdot \chi_1 \in G \cdot \chi_1$, and by choice of $\chi_1$, there exists $\nu < d$ such that $(gh)^{-1} \cdot \chi_1 = \psi_\nu$. Thus $\chi_0(a) = \psi_\nu(a_0)$ as claimed. □

*Proof of Theorem 3.1.* (1) ⇒ (2). Write $F = \mathrm{Flim}(\mathcal{K})$ and let $A \in \mathcal{K} = \mathrm{Age}(F)$. Then $F$ and $A$ satisfy the assumptions of Proposition 3.14, so $F \hookrightarrow (B)^A_{k,d}$ for all $B \in \mathcal{K}$ and $k \geq 2$. Now apply Lemma 3.7.

(2) ⇒ (1). By Proposition 3.13. □

## 3.4 Corollaries

**Corollary 3.15.** *Let $d \in \mathbb{N}$ and $\mathcal{K}$ be a Fraïssé class. The following are equivalent.*

1. *The class of structures with Ramsey degree for embeddings at most $d$ in $\mathcal{K}$ is cofinal in $\mathcal{K}$.*
2. *$\mathcal{K}$ has Ramsey degree for embeddings at most $d$.*

*Proof.* Assume (1). By Proposition 3.13 we have $|M(G_{\mathcal{K}})| \leq d$. As $F := \mathrm{Flim}(\mathcal{K})$ is Fraïssé, Proposition 3.14 implies $F \hookrightarrow (B)^A_{k,d}$ for all $A, B \in \mathcal{K}$. Then Lemma 3.7 implies (2). □

It is noted in [14, §1(D)] that a separable metrizable group $G$ is extremely amenable, i.e. $M(G)$ has size 1, if and only if every metrizable $G$-flow has a fixed point. In this context it might be of interest to note:

**Corollary 3.16.** *Let $d \in \mathbb{N}$ and $\mathcal{K}$ be a Fraïssé class. The following are equivalent.*

1. *$M(G_\mathcal{K})$ has size at most $d$.*
2. *Every continuous action of $G_\mathcal{K}$ on the Cantor space has an orbit of size at most $d$.*

*Proof.* (1) implies (2) by Lemma 3.2. Conversely, assume (2). Let $A \in \mathcal{K}$ be arbitrary and write $F := \mathrm{Flim}(\mathcal{K})$. Then $F$ and $A$ satisfy the assumptions of Proposition 3.14. In the proof of this proposition we only require the following for $G_\mathcal{K}$: for all $k \geq 2$ and all $\chi_0 \in [k]^{FA}$, the shift action of $G_\mathcal{K}$ restricted to $\overline{G_\mathcal{K} \cdot \chi_0}$ has a small orbit. But $[k]^{FA}$ is homeomorphic to the Cantor space and the restricted shift is a continuous action on this space. Thus (2) suffices to carry out this proof and we conclude that $F \hookrightarrow (B)^A_{k,d}$ for all $B \in \mathcal{K} = \mathrm{Age}(F)$. By Lemma 3.7 every $A \in \mathcal{K}$ has Ramsey degree for embeddings at most $d$ in $\mathcal{K}$. Then Proposition 3.13 implies (1). $\square$

# 4 Measure concentration

We say that a probability measure is *concentrated on* any set of measure 1. In this section we prove the following.

**Theorem 4.1.** *Let $\mathcal{K}$ be a relational Fraïssé class which is Ramsey. If $G_\mathcal{K}$ is amenable, then it is uniquely ergodic and the (unique) $G_\mathcal{K}$-invariant Borel probability measure on $M(G_\mathcal{K})$ is concentrated on a (unique) dense $G_\delta$ orbit.*

In Section 4.1 we construct a forgetful order expansion using the Ramsey property, in Section 4.2 we prove Theorem 4.1, and the final Section 4.3 contains some observations concerning the $\omega$-categorical case and a proof of (a stronger version of) Theorem 3.12.

## 4.1 Forgetful order expansions

An ordered class $\mathcal{K}^*$ of finite structures in the language $L \cup \{<\}$ is called *forgetful* if for all $A, B \in \mathcal{K}$ and $\mathcal{K}^*$-admissible orderings $<^A, <^B$ on $A, B$ respectively, we have $(A, <^A) \cong (B, <^B)$ whenever $A \cong B$; here $\mathcal{K}$ denotes the $L$-reduct of $\mathcal{K}^*$.

For example, the orderings of $B_\infty$ and $V_{\infty,F}$ mentioned in Example 2.10 have forgetful ages (see [14, §6] for details). The following is easy to see (cf. [14, P5.6]).

**Lemma 4.2.** *Let $\mathcal{K}^*$ be a forgetful ordered class of finite structures in the language $L \cup \{<\}$ and $\mathcal{K}$ its $L$-reduct. Then $\mathcal{K}^*$ has the ordering property, and $\mathcal{K}^*$ is Ramsey if and only if so is $\mathcal{K}$.*

Before showing that the Ramsey property ensures the existence of reasonable forgetful expansions, we present a well-known technical lemma. Informally, this technical lemma guarantees a monochromatic copy of a given $B$ when copies of several different $A_i$ are coloured simultaneously.

**Lemma 4.3.** *Let $\mathcal{K}$ be a Ramsey class. Let $n \in \mathbb{N}$, $k_0, \dots, k_{n-1} \in \mathbb{N}$, $A_0, \dots, A_{n-1}, B \in \mathcal{K}$. Then there exists a $C \in \mathcal{K}$ with the following property: for any family of colourings $\chi_i : \binom{C}{A_i} \to [k_i]$, $i \in [n]$, there exists a $B' \in \binom{C}{B}$ such that $\chi_i \restriction_{B'}$ is constant for all $i \in [n]$.*

*Proof.* Let $C_0 := B$, and for every $0 < i \le n$ choose $C_i \in \mathcal{K}$ such that $C_i \to (C_{i-1})^{A_{i-1}}_{k_{i-1},1}$. Let $C := C_n$. Then by using the definition of the $C_i$ and a straightforward induction on $j \in [n]$ we obtain that there is a $C'_{n-1-j} \in \binom{C}{C_{n-1-j}}$ such that $\chi_i \restriction_{C'_{n-1-j}}$ is constant for all $i \in [n] \setminus [n-1-j]$. Setting $j = n-1$ yields $B'$ as in the statement. □

**Lemma 4.4.** *Let $\mathcal{K}$ be a Fraïssé class in the language $L$. If $\mathcal{K}$ is Ramsey, then there exists a reasonable, forgetful ordered Fraïssé class $\mathcal{K}^*$ in the language $L \cup \{<\}$ with $L$-reduct $\mathcal{K}$.*

*Proof.* Let $F := \mathrm{Flim}(\mathcal{K})$ and consider the space $LO$ of linear orders on $F$ (cf. Example 2.4). Let $(A, B)$ range over pairs with $A \in \mathcal{K}$ and $B \subseteq F$. Call $R \in LO$ order forgetful for $(A, B)$ if $(A', R \restriction A') \cong (A'', R \restriction A'')$ for all $A', A'' \in \binom{B}{A}$.

*Claim.* If $n \ge 1$ and $(A_0, B_0), \dots, (A_{n-1}, B_{n-1})$ are pairs as above with all $B_i \subseteq F$ finite, then there exists $R \in LO$ that is order forgetful for every $(A_i, B_i), i \in [n]$.

*Proof of Claim:* Choose $B \subseteq F$ finite such that $\bigcup_{i\in[n]} B_i \subseteq B$. It suffices to find an order which is order forgetful for every $(A_i, B), i \in [n]$. Fix an arbitrary order $R \in LO$. For $i \in [n]$ let $\chi_i$ colour each $A'_i \in \binom{F}{A_i}$ by the isomorphism type of $(A'_i, R \restriction A'_i)$, and let $k_i \in \mathbb{N}$ be the number of colours of $\chi_i$. By Lemma 4.3 and homogeneity of $F$ there exist $B' \subseteq F$ and $g \in \mathrm{Aut}(F)$ such that $B'$ is monochromatic and $g(B') = B$. By definition of the $\chi_i$ this means that $R$ is order forgetful for $(A_i, B')$ for all $i \in [n]$. Hence, $g(R)$ is order forgetful for all $(A_i, B), i \in [n]$. $\dashv$

For every $A \in \mathcal{K}$ and $B \subseteq F$ finite, the set of orders that are order forgetful for $(A, B)$ is closed in $LO$. By the claim and compactness, there exists $R \in LO$ which is order forgetful for all pairs $(A, B)$ such that $A \in \mathcal{K}$ and $B \subseteq F$ is finite. Then $R$ is order forgetful for $(A, F)$ for every $A \in \mathcal{K}$. Equivalently, $\mathcal{K}^* := \mathrm{Age}(F, R)$ is forgetful. To see that $\mathcal{K}^*$ is Fraïssé, observe that $\mathcal{K}^*$ is hereditary and has joint embedding. As $\mathcal{K}^*$ is Ramsey by Lemma 4.2, it has amalgamation by Lemma 3.8 (and Remark 3.9; note that $\mathcal{K}^*$ is rigid because it is ordered). According to Lemma 2.7, $\mathcal{K}^*$ is reasonable. $\square$

**Examples 4.5.** The structures $F_1 := (\mathbb{Q}, \mathrm{Betw}), F_2 := (\mathbb{Q}, \mathrm{Cycl}), F_3 := (\mathbb{Q}, \mathrm{Sep})$ and $F_4 := (\mathbb{Q}, =)$ are Ramsey (see [12] for definitions). If $<$ is the rational order, then $\mathcal{K}_i^* := \mathrm{Age}((F_i, <))$ is forgetful with reduct $\mathcal{K}_i := \mathrm{Age}(F_i)$. By Lemmas 2.7, 4.2 and Theorem 2.8, $M(G_{\mathcal{K}_i})$ is $\overline{G_{\mathcal{K}_i} \cdot <}$. Then $M(G_{\mathcal{K}_1})$ is the 2-element discrete space. Hence, by Theorem 3.1, $\mathcal{K}_1$ has Ramsey degree for embeddings 2. Theorem 2.8 also allows us to explicitly describe $M(G_{\mathcal{K}_i})$ for $i = 2, 3, 4$, and these have the size of the continuum. Hence, $\mathcal{K}_2, \mathcal{K}_3$ and $\mathcal{K}_4$ have infinite Ramsey degree for embeddings.

**Examples 4.6.** Let $\mathcal{K}$ be a Fraïssé class of digraphs such that there is a directed cycle in $\mathcal{K}$. Then there does not exist a forgetful ordered Fraïssé class with $L$-reduct $\mathcal{K}$: by forgetfulness, every directed edge in any $A \in \mathcal{K}$ would be ordered in the same way and then a directed cycle contradicts transitivity of the order. For example, this applies to the age of the universal homogeneous digraph, the random tournament and the local order (see [15]).

## 4.2 Proof of Theorem 4.1

Let $F := \mathrm{Flim}(\mathcal{K})$ and $L$ denote the relational language of $\mathcal{K}$. Since $\mathcal{K}$ is assumed to be Ramsey, Lemma 4.4 applies and there is a reasonable forgetful ordered Fraïssé class $\mathcal{K}^*$ in the language $L \cup \{<\}$ with $L$-reduct $\mathcal{K}$. By Lemma 4.2 and Theorem 2.8, $\mathrm{Flim}(\mathcal{K}^*) = (F, <^*)$ for some order $<^*$, and $X_{\mathcal{K}^*} = \overline{G_{\mathcal{K}} \cdot <^*}$ is the universal minimal flow of $G_{\mathcal{K}}$.

Assume that $G_{\mathcal{K}}$ is amenable. In order to verify that $G_{\mathcal{K}}$ is uniquely ergodic, it suffices by Proposition 2.9 to show that for every consistent random ordering $(R_A)_{A \in \mathcal{K}}$ we have that each random variable $R_A$ is uniformly distributed. By forgetfulness, for any two $\mathcal{K}^*$-admissible orderings $<, <'$ on $A$ there is an $\alpha \in \mathrm{Aut}(A)$ such that $\alpha(<) =<'$, and then $\Pr[R_A =<'] =$

$\Pr[\alpha^{-1} \circ R_A =<] = \Pr[R_A =<]$ where the latter equality follows from $(R_A)_A$ being consistent.

Let $\mu$ denote the unique $G_{\mathcal{K}}$-invariant Borel probability measure on $X_{\mathcal{K}^*}$. Recall the notation $U(<)$ from Proposition 2.9. By this result, $U(<)$ and $U(<')$ have the same $\mu$-measure whenever $<$ and $<'$ are $\mathcal{K}^*$-admissible orderings of the same finite subset of $F$.

An order $R \in X_{\mathcal{K}^*}$ is outside $G_{\mathcal{K}} \cdot <^*$ if and only if $(F, R) \not\cong (F, <^*)$, if and only if $(F, R)$ is not homogeneous (cf. Section 2.2), if and only if there exist a finite $A \subseteq F$, some $(B, <^B) \in \mathcal{K}^*$ and $a \in (B, <^B)^{(A, <^* \restriction A)}$ such that $R$ is *bad for* $(B, <^B, a)$, meaning that there is no $b \in (F, R)^{(B, <^B)}$ with $b \circ a = \mathrm{id}_A$. As the language of $F$ is relational, we may assume that $B = \mathrm{im}(a) \cup \{p\}$ with $p \in F \setminus \mathrm{im}(a)$.

Observe that the set of orders $R \in X_{\mathcal{K}^*}$ which are bad for $(B, <^B, a)$ is closed. Hence, $X_{\mathcal{K}^*} \setminus G_{\mathcal{K}} \cdot <^*$ is $F_\sigma$, so $G_{\mathcal{K}} \cdot <^*$ is a dense $G_\delta$ orbit in $X_{\mathcal{K}^*}$ (see also [1, 14.3]). Since $X_{\mathcal{K}^*}$ is a Baire space, $G_{\mathcal{K}} \cdot <^*$ is clearly unique with this property. We prove that $\mu(G_{\mathcal{K}} \cdot <^*) = 1$. It suffices to show that for each $(B, <^B, a)$ with $B = \mathrm{im}(a) \dot\cup \{p\}$ as above, the set $\mathcal{B} := \{R \in X_{\mathcal{K}^*} \mid R \text{ is bad for } (B, <^B, a)\}$ has $\mu$-measure 0.

We construct a sequence $(\mathcal{U}_n)_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$

(a) $\mathcal{U}_n$ is a cover of $\mathcal{B}$, i.e. $\mathcal{B} \subseteq \bigcup \mathcal{U}_n$;

(b) every $U \in \mathcal{U}_n$ equals some $U(<')$ such that $<' \supseteq <^* \restriction A$ is a $\mathcal{K}^*$-admissible order with $|\mathrm{dom}(<')| = |A| + n$;

(c) $\mu(\bigcup \mathcal{U}_{n+1}) \leq \frac{|A|+n}{|A|+n+1} \cdot \mu(\bigcup \mathcal{U}_n)$.

Here, $\mathrm{dom}(<')$ is the set linearly orderd by $<'$; note that (b) implies that $\mathrm{dom}(<') \supseteq A$.

This finishes the proof: by (a) and (c) we have for all $n \in \mathbb{N}$

$$\mu(\mathcal{B}) \leq \mu(\bigcup \mathcal{U}_n) \leq \prod_{m<n} \tfrac{|A|+m}{|A|+m+1} \cdot \mu(\bigcup \mathcal{U}_0) = \mu(\bigcup \mathcal{U}_0) \cdot \tfrac{|A|}{|A|+n} \to_n 0.$$

Set $\mathcal{U}_0 := \{U(<^* \restriction A)\}$ and assume that $\mathcal{U}_n$ is already defined. It suffices to find for every $U(<') \in \mathcal{U}_n$ some $p' \notin \mathrm{dom}(<')$ and a family $(<_i)_{i \in I}$ such that

(a') $\bigcup_{i \in I} U(<_i) \cap \mathcal{B} = U(<') \cap \mathcal{B}$;

(b') for every $i \in I$, $<_i \supseteq <'$ is a $\mathcal{K}^*$-admissible order with $\mathrm{dom}(<_i) = \mathrm{dom}(<') \cup \{p'\}$;

(c') $\mu(\bigcup_{i \in I} U(<_i)) \leq \frac{|A|+n}{|A|+n+1} \cdot \mu(U(<'))$.

Write $A' := \mathrm{dom}(<')$, and choose $R \in \mathcal{B} \cap U(<')$. Since $R \in \overline{G_{\mathcal{K}} \cdot <^*}$ there is a $g \in G_{\mathcal{K}}$ such that

$$g(<^*) \upharpoonright A' = R \upharpoonright A' =<' . \tag{10}$$

In particular, $g(<^*) \upharpoonright A =<' \upharpoonright A =<^* \upharpoonright A$ and $(A, <^* \upharpoonright A)$ is a substructure of $(F, g(<^*))$. Since $(F, g(<^*))$ is isomorphic to $(F, <^*)$, it is homogeneous, so there exists an embedding $b \in (F, g(<^*))^{(B,<^B)}$ with $b \circ a = \mathrm{id}_A$. We set $p' := b(p)$ and claim that $p' \notin A'$. Otherwise, $\mathrm{im}(b) \subseteq A'$, so $b \in (F, R)^{(B,<^B)}$ by (10), and this contradicts $R$ being bad for $(B, <^*, a)$.

Let $<_0, \ldots, <_{s-1}$ list the $\mathcal{K}^*$-admissible orders on $A' \cup \{p'\}$ extending $<'$, and note that $s \leq |A'| + 1$. Let $I \subseteq [s]$ consist of those $i < s$ such that $U(<_i) \cap \mathcal{B} \neq \emptyset$. Then (a') and (b') follow, and we are left to verify (c'). The sets $U(<_i), i < s$, partition $U(<')$ and, as already noted, have pairwise equal $\mu$-probability, so $\mu(U(<_i)) = \mu(U(<'))/s$. Thus

$$\mu(\textstyle\bigcup_{i \in I} U(<_i)) = |I|/s \cdot \mu(U(<')). \tag{11}$$

There exists $i_0 < s$ such that $<_{i_0}= g(<^*) \upharpoonright (A' \cup \{p'\})$. Since $b \in (F, g(<^*))^{(B,<^B)}$ has $\mathrm{im}(b) \subseteq A' \cup \{p'\}$, we have that $b \in (F, S)^{(B,<^B)}$ for every $S \in U(<_{i_0})$. Hence, no such $S$ is bad for $(B, <^B, a)$, that is, $U(<_{i_0}) \cap \mathcal{B} = \emptyset$, so $i_0 \notin I$. Thus $|I| < s$. Since $|A'| = |A| + n$, we have $s \leq |A| + n + 1$, so $|I|/s \leq (s-1)/s \leq (|A| + n)/(|A| + n + 1)$. Hence, (c') follows from (11).

## 4.3 The $\omega$-categorical case

Of particular interest are Fraïssé classes $\mathcal{K}$ which have an $\omega$-categorical Fraïssé limit $F := \mathrm{Flim}(\mathcal{K})$. By the theorem of Ryll-Nardzewski (see e.g. [21, T4.3.1]) this happens e.g. if the language $L$ of $\mathcal{K}$ is finite and relational (cf. [21, T4.4.7]), and is equivalent to $G_{\mathcal{K}}$ being *oligomorphic*: for every $n \in \mathbb{N}$, $G_{\mathcal{K}}$ has only finitely many $n$-orbits. An *$n$-orbit of $G_{\mathcal{K}}$* is an orbit of the *diagonal action* of $G_{\mathcal{K}}$ on $F^n$ given by $g \cdot \bar{a} = g \cdot (a_0, \ldots, a_{n-1}) := g(\bar{a}) = (g(a_0), \ldots, g(a_{n-1}))$.

**Lemma 4.7.** *Let $\mathcal{K}^*$ be a reasonable ordered Fraïssé class in the language $L \cup \{<\}$ with $L$-reduct $\mathcal{K}$. Then $G_{\mathcal{K}^*}$ is oligomorphic if and only if so is $G_{\mathcal{K}}$.*

*Proof.* Let $F = \mathrm{Flim}(\mathcal{K})$. By Theorem 2.8 we have that $\mathrm{Flim}(\mathcal{K}^*) = (F, <^*)$ for some order $<^*$ on $F$. As $G_{\mathcal{K}^*}$ is a subgroup of $G_{\mathcal{K}}$, it suffices to show that every orbit $T \subseteq F^n$ of $G_{\mathcal{K}}$ that consists of tuples with all different entries is the union of finitely many $n$-orbits of $G_{\mathcal{K}^*}$. Let $\bar{s} = (s_1, \ldots, s_n)$ and

$\bar{t} = (t_1, \ldots, t_n)$ be tuples in $T$ such that the unique extension of the partial isomorphism $s_1 \mapsto t_1, \ldots, s_n \mapsto t_n$ to the substructures in $F$ generated by $\bar{s}$ and $\bar{t}$ is a partial isomorphism of $F^*$. Then by homogeneity of $F^*$ we have that $\bar{s}$ and $\bar{t}$ are in the same $n$-orbit of $G_{\mathcal{K}^*}$. As there are finitely many ways to define a ($\mathcal{K}^*$-admissible) order on the structure generated by a tuple in $T$, the claim follows. □

**Lemma 4.8.** *Let $\mathcal{K}^*$ be a reasonable ordered Fraïssé class in the language $L \cup \{<\}$ with $L$-reduct $\mathcal{K}$. Assume that $G_{\mathcal{K}^*}$ is oligomorphic. If $G_{\mathcal{K}^*}$ is normal in $G_{\mathcal{K}}$, then it has finite index in $G_{\mathcal{K}}$.*

*Proof.* By reasonability $\mathrm{Flim}(\mathcal{K}^*) = (\mathrm{Flim}(\mathcal{K}), <^*)$ for some order $<^*$. Consider the logic action of $G_{\mathcal{K}}$ on $LO$ (Example 2.4). Then $G_{\mathcal{K}^*}$ is the stabiliser of $<^*$. Hence, $|G_{\mathcal{K}} : G_{\mathcal{K}^*}| = |G_{\mathcal{K}} \cdot <^*|$ and it suffices to show that $G_{\mathcal{K}} \cdot <^*$ is finite. If $G_{\mathcal{K}^*}$ is normal, then it fixes every $R \in G_{\mathcal{K}} \cdot <^*$. Thus every such $R$ is a union of 2-orbits. As $G_{\mathcal{K}^*}$ is oligomorphic, there are only finitely many such $R$. □

We use the following mode of speech from [1]: let $\mathcal{K}$ be a Fraïssé class in the language $L$; a *companion of* $\mathcal{K}$ is a reasonable ordered Fraïssé class $\mathcal{K}^*$ in the language $L \cup \{<\}$ which is Ramsey, has the ordering property and has $L$-reduct $\mathcal{K}$. Note:

**Proposition 4.9.** *If a Fraïssé class is Ramsey, then it has a companion.*

*Proof.* By Lemmas 4.4 and 4.2. □

**Proposition 4.10.** *Let $\mathcal{K}$ be a relational Fraïssé class that has a companion. If $M(G_{\mathcal{K}})$ is finite, then $|M(G_{\mathcal{K}})|$ is a power of 2.*

*Proof.* Let $L$ denote the relational language of $\mathcal{K}$ and let $\mathcal{K}^*$ be a companion of $\mathcal{K}$. By Theorem 2.8 we have that $F^* := \mathrm{Flim}(\mathcal{K}^*) = (F, <^*)$ for $F := \mathrm{Flim}(\mathcal{K})$, and that $M(G_{\mathcal{K}})$ is $\overline{G_{\mathcal{K}} \cdot <^*}$. Assume that $M(G_{\mathcal{K}})$ is finite. Then $G_{\mathcal{K}} \cdot <^*$ is finite, and since $G_{\mathcal{K}^*}$ is the stabiliser of $<^*$ in the logic action of $G_{\mathcal{K}}$ on $LO$, $G_{\mathcal{K}^*}$ has finite index in $G_{\mathcal{K}}$. By Theorem 2.6, $G_{\mathcal{K}^*}$ is extremely amenable. By Lemma 3.3, $G_{\mathcal{K}^*}$ is normal in $G_{\mathcal{K}}$ and $|M(G_{\mathcal{K}})| = |G_{\mathcal{K}} : G_{\mathcal{K}^*}|$.

Consider the diagonal actions of $G_{\mathcal{K}}$ and $G_{\mathcal{K}^*}$ on $F^2$. We claim that for every $g \in G_{\mathcal{K}}$ and every 2-orbit $S$ of $G_{\mathcal{K}^*}$ the set $g \cdot S \subseteq F^2$ is also a 2-orbit of $G_{\mathcal{K}^*}$. Indeed, normality implies that two pairs in the same 2-orbit of $G_{\mathcal{K}^*}$ are mapped by $g$ to two pairs which are also in the same 2-orbit of $G_{\mathcal{K}^*}$, so there exists a 2-orbit $T$ with $g \cdot S \subseteq T$. Reasoning analogously for $g^{-1}$ and $T$ we obtain $g^{-1} \cdot T \subseteq S$, so $g \cdot S = T$.

Call a 2-orbit $S$ of $G_{\mathcal{K}^*}$ *black* if $a <^* b$ for all $(a,b) \in S$, and *white* if $b <^* a$ for all $(a,b) \in S$; orbits which are neither black nor white contain only pairs $(a,b)$ with $a = b$. Let $S$ be black or white. For every $g \in G_{\mathcal{K}}$, also $g(S)$ is black or white, and if $g(S)$ has the same colour as $S$, then $g(S) = S$. Indeed, as $g \in G_{\mathcal{K}}$, $g \restriction \{a,b\}$ preserves all relations from $L$, and as $g(S)$ has the same colour as $S$, it also preserves $<^*$. Hence, for every $(a,b) \in S$, $g \restriction \{a,b\}$ is a partial isomorphism of $F^*$, so it extends to some $h \in G_{\mathcal{K}^*}$ by homogeneity. Thus $g \cdot (a,b) = h \cdot (a,b)$, so $g \cdot (a,b) \in S$ and $g(S) = S$ follows.

We claim that $g^2 \in G_{\mathcal{K}^*}$ for every $g \in G_{\mathcal{K}}$. Seeking for contradiction, assume that there is an $(a,b) \in F^2$ such that $a <^* b$ is not equivalent to $g^2(a) <^* g^2(b)$. Then there is a black or white 2-orbit $S$ of $G_{\mathcal{K}^*}$ such that $g^2(S)$ has a different colour. The colour of $g(S)$ equals that of $S$ or $g^2(S)$, and consequently, $S = g(S)$ or $g(S) = g^2(S)$. The first case $S = g(S)$ is impossible, because it implies $S = g^2(S)$. The second case $g(S) = g^2(S)$ is also impossible, because it implies the first via $g(S) = g^{-1}(g^2(S)) = g^{-1}(g(S)) = S$.

It follows that $G_{\mathcal{K}}/G_{\mathcal{K}^*}$ is an elementary abelian 2-group, i.e., it is the direct product of copies of the 2-element group. □

**Theorem 4.11.** *Let $\mathcal{K}$ be a relational Fraïssé class with companion $\mathcal{K}^*$. Assume that $G_{\mathcal{K}}$ is oligomorphic. Then the following are equivalent.*

1. *$|G_{\mathcal{K}} : G_{\mathcal{K}^*}|$ is finite.*
2. *$|G_{\mathcal{K}} : G_{\mathcal{K}^*}|$ is a finite power of 2.*
3. *$M(G_{\mathcal{K}})$ is finite.*
4. *$|M(G_{\mathcal{K}})|$ is a finite power of 2.*
5. *$G_{\mathcal{K}^*}$ is normal in $G_{\mathcal{K}}$.*

*Proof.* By Theorem 2.8 we have that $F^* := \mathrm{Flim}(\mathcal{K}^*) = (F, <^*)$ for $F := \mathrm{Flim}(\mathcal{K})$, and that $M(G_{\mathcal{K}})$ is $\overline{G_{\mathcal{K}} \cdot <^*}$. Then $G_{\mathcal{K}^*}$ is oligomorphic by Lemma 4.7, and extremely amenable by Theorem 2.6. In a Hausdorff space a finite set equals its closure. As the elements of $G_{\mathcal{K}} \cdot <^*$ are in a one-to- one correspondence with $G_{\mathcal{K}}/G_{\mathcal{K}^*}$, we obtain (1) ⇔ (3) and (2) ⇔ (4). Proposition 4.10 implies (3) ⇔ (4), thus the first four items are equivalent. (5) ⇒ (1) follows from Lemma 4.8, and Lemma 3.3 implies (1) ⇒ (5). □

**Corollary 4.12.** *Let $\mathcal{K}$ be a relational Fraïssé class that has a companion. Then the Ramsey degree for embeddings of $\mathcal{K}$ is either infinite or a finite power of 2.*

*Proof.* By Proposition 4.10 and Theorem 3.1. □

*Proof of Theorem 3.12.* By Proposition 4.9 and Corollary 4.12. □

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