Reducts of homogeneous relational structures

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Submitted to
Central European University
Department of Mathematics and its Applications

In partial fulfillment of the requirements for the degree of
Doctor of Philosophy in Mathematics and its Applications

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Eötvös Loránd University

Budapest, Hungary

2012

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Abstract

In this dissertation I summarize my work in counting numerical invariants of structures in model theory and general algebra. This involves the investigation of the generative spectra of monounary algebras and the free spectra of certain semigroups arising in automata theory. By counting the number of n-element monounary algebras in given varieties, we obtained an enumerative combinatorial result on the number of rooted trees of given depth. We determine, up to the equivalence of first-order interdefinability, all structures which are first-order definable in the random partial order. It turns out that these structures fall into precisely five equivalence classes. We achieve this result by showing that there exist exactly five closed permutation groups which contain the automorphism group of the random partial order, and thus expose all symmetries of this structure. The second major result is the characterization of the reducts of the structures obtained by adding a constant to the random K_n-free graph for any $n \geq 3$, the so-called Henson graphs. Up to first-order interdefinability, there are 13 reducts if $n = 3$, and 16 reducts if $n \geq 4$. In all these topics I have published, accepted or submitted papers in various mathematical journals.

Acknowledgment

First and foremost, I wish to express my gratitude to Prof. Csaba Szabó for his enthusiasm and endless amount of advice. His guidance was an indispensable component of my becoming a researcher.

I am also thankful to Michael Pinsker for showing the techniques in Ramsey theory, and for his useful comments.

1. Introduction

In this dissertation, we investigate some numerical invariants of structures. A structure is either a general algbebraic or a model theoretic notion. The language of algebraic structures consists of functional symbols, i.e., operations, while the language of structures we study in model theory consists of relations. There is only one exception from this rule, namely that we let constant symbols be added to the language of relational structures. In this area I have the following 9 published, accepted or submitted papers [34, 31, 30, 19, 14, 21, 16, 15, 32]. The papers [19, 14, 21] are already published, [16] is accepted for publication, [34, 31, 30, 32] are submitted, and these 8 papers are not contained in any PhD dissertation different from the present thesis.

There are several types of numerical parameters of structures. One of the basic concepts in algebra is the number of n-element models of a set of axioms or properties. Usually, these questions are interesting up to isomorphism, and one can define the labelled and unlabelled versions, as well. For example, determine the number of - labelled or unlabelled - graphs, trees or rooted trees on n vertices, the number of n-element posets, etc. Cayley has proven that the number of labelled trees on an n-element set is exatly n^{n-2}. In [29] Otter has given an asymptotic formula for the number of unlabelled trees and unlabelled rooted trees. In [32] we have given estimations for the number $t_k(n)$ of n-element rooted trees of given depth k, and determined the asymptotics of the logarithm of this function in terms of n. By using the method of generating functions we have obtained a complicated recursion, and we have shown that $t_2(n) \sim \frac{1}{4n\sqrt{3}} e^{\sqrt{\frac{2n}{3}}}$, and for $k \geq 3$ we have $\log t_k(n) \sim \frac{\pi^2}{6} \frac{n}{\log\log\cdots\log n}$, where the logarithm in the denominator is iterated $k-2$ times.

Rooted trees are closely related to monounary algebras. A monounary algebra is an algebra with a single unary operation. The theory of monounary algebras is well-developed, for a recent monograph see [18]. Let (A, f) be a monounary algebra. The function f defines a directed graph on A. Let $G_A = (A, E)$, the vertex set is A and the edges are $E = (a, f(a))|a \in A$. In G_A every vertex has outdegree 1, and every graph G with outdegree 1 defines a monounary algebra on its vertex set, where $f(a)$ is the single vertex such that $(a, f(a))$ is an edge in G. Hence, there is a bijection between monounary algebras and directed graphs, where each vertex has outdegree 1. A monounary algebra (A, f) is connected if G_A is connected. By making use of Otter's method, we have shown that there exists some $\alpha > 1$ such that $\log_\alpha C_n \sim \log_\alpha M_n \sim n$, where C_n is the number of n-element conneted monounary algebras, and M_n is the number of n-element monounary algebras [14].

The theory of formal languages goes back to natural languages. The foundation of the mathematical theory is based on Kleene's theorem: it proves that the class of recognizable languages (e.g. recognized by finite automata) coincides with the class of rational languages, which are given by rational expressions. Rational expressions are the generalization of polynomials involving three operations: union, product and star operation. A real break-through in the history of language theory is a work of Schützenberger: he established an equivalence between finite automata and finite semigroups. He showed that a finite monoid, called the syntactic monoid, can be assigned to each recognizable language; this is the smallest monoid recognizing the language. According to Eilenberg's theorem varieties of finite monoids are in a one-to-one correspondence with classes of recognizable languages closed under product and boolean operations. For example, star-free languages correspond to aperiodic monoids. For more details, see [33].

A large class of star-free languages is the family of piecewise testable languages, which has been deeply studied in formal language theory. We analyzed the word problem for the syntactic monoids of the varieties of k-piecewise testable languages [21]. We have given a normal form of the words for $k = 2$ and 3, and an asymptotic formula for the logarithm of the number of words for arbitrary k. In other words, we give an estimation for the n-ary terms over the syntactic monoids of k-piecewise testable languages. This function shows an unusual behaviour, as the result is essentially different for odd and even k.

The *random partial order* $\mathbb{P} := (P; \leq)$ is the unique countable partial order which is *universal* in the sense that it contains all countable partial orders as induced suborders and which is *homogeneous*, i.e., any isomorphism between two finite induced suborders of $\mathbb{P}$ extends to an automorphism of $\mathbb{P}$. Equivalently, $\mathbb{P}$ is the *Fraïssé limit* of the class of finite partial orders – confer the textbook [13].

As the "generic order" represents all countable partial orders, the random partial order is of both theoretical and practical interest. The latter becomes in particular evident with the recent applications of homogeneous structures in theoretical computer science; see for example [5, 6, 3, 24]. It is therefore tempting to classify all structures which are first-order definable in $\mathbb{P}$, i.e., all relational structures on domain P all of whose relations can be defined from the relation $\leq$ by a first-order formula. Such structures have been called *reducts* of $\mathbb{P}$ in the literature [37, 38]. It is one of the goals of the present dissertation to obtain such a classification *up to first-order interdefinability*, that is, we consider two reducts Γ, Γ' equivalent iff they are reducts of one another. As one of the main results of this thesis, it is shown that up to this equivalence, there are precisely five reducts of $\mathbb{P}$ [30].

This result lines up with a number of previous classifications of reducts of similar generic structures up to first-order interdefinability. The first non-trivial classification of this kind was obtained by Cameron [8] for the order of the rationals, i.e., the Fraïssé limit of the class of finite linear orders; he showed that this order has five reducts up to first-order interdefinability. Thomas [37] proved that the random graph has five reducts up to first-order interdefinability as well, and later generalized this result by showing that for all $k \geq 2$, the random hypergraph with k-hyperedges has $2^k + 1$ reducts up to first-order interdefinability [38]. Junker and Ziegler [20] showed that the structure $(\mathbb{Q}; <, 0)$, i.e., the order of the rationals with an additional constant symbol, has 116 reducts up to first-order interdefinability. Moreover, they characterized the reducts of all the structures obtained from the dense linear order by adding finitely many constants. The classifiation of the reducts of the random tournament was obtained by Bennett (5 reducts, see [1]). A negative "result" is the random graph with a fixed constant, on which the author of the present dissertation, together with two collaborators, gave up after having found 300 reducts. Obviously, the successful classifications have in common that the number of reducts is finite.

Conjecture 1 (Thomas). *Every homogeneous relational structure over a finite language has finitely many reducts up to first-order interdefinability.*

In [37], Thomas has also shown that the Henson graphs (H_n, E), i.e., the random K_n-free graphs has no proper reducts. That is, any reduct of (H_n, E) is first-order interdefinable with (H_n, E) itself or the "empty" structure $(H_n, =)$. The second major result of the present dissertation is the classification of the reducts of the structures obtained from the Henson graphs by adding a constant. It is shown that up to first-order interdefinability $(H_3, E, 0)$ has 13 reducts, and $(H_n, E, 0)$ has 16 reducts for $n \geq 4$ [34].

The mentioned classifications have all been obtained by means of the automorphism groups of the reducts, and we will proceed likewise in the present dissertation. It is clear that if Δ is a reduct of a structure Γ, then the automorphism group $\mathrm{Aut}(\Delta)$ of Δ is a permutation group containing $\mathrm{Aut}(\Gamma)$, and also is a closed set with respect to the convergence topology on the space of all permutations on the domain of Γ. If Γ is ω-categorical, i.e., if Γ is up to isomorphism the only countable model of its first-order theory, then it follows from the theorem of Ryll-Nardzewski, Engeler and Svenonius (confer [13]) that the converse is true as well: the closed permutation groups acting on the domain of Γ and containing $\mathrm{Aut}(\Gamma)$ are precisely the automorphism groups of reducts of Γ; moreover, two reducts have equal automorphism groups if and only if they are first-order interdefinable. Since homogeneous structures in a finite language are ω-categorical, it is enough for us to determine all closed permutation groups that contain $\mathrm{Aut}(\Gamma)$ for $\Gamma = \mathbb{P}$ and $\Gamma = (H_n, E, 0)$ in order to obtain our classifications.

The fact that the reducts of an ω-categorical structure Γ correspond to the closed permutation groups containing $\mathrm{Aut}(\Gamma)$ not only yields a method for classifying these reducts, but also a meaningful interpretation of such classifications: for just like $\mathrm{Aut}(\Gamma)$ is the group of all symmetries of Γ, the closed permutation groups containing $\mathrm{Aut}(\Gamma)$ stand for all symmetries of Γ if we are willing to give up some of the structure of Γ. As for an example, it is obvious that turning the random partial order upside down, one obtains again a random partial order; this symmetry is reflected by one of the closed groups containing $\mathrm{Aut}(\mathbb{P})$, namely the group of all automorphisms and antiautomorphisms of $\mathbb{P}$. It will follow from our classification that $\mathbb{P}$ has only one more symmetry of this kind – this second symmetry is much less obvious, and so we argue that the classification of the reducts of $\mathbb{P}$, or indeed of any ω-categorical structure, is much more than a mere sportive challenge – it is an essential part of understanding the structure itself.

Our approach to investigating the closed groups containing $\mathrm{Aut}(\mathbb{P})$ and $\mathrm{Aut}(H_n, E)$ is based on a Ramsey-theoretic analysis of functions, and in particular permutations, on the domain of $\mathbb{P} = (P; \leq)$ or (H_n, E); this allows us to find patterns of regular behaviour with respect to the structure $\mathbb{P}$ or (H_n, E) in any arbitrary function acting on the domain. The method as we use it has been developed in [7, 6, 4, 5] and is a general powerful technique for dealing with functions on *ordered homogeneous Ramsey structures* in a finite language. But while this machinery has previously been used, for example, to re-derive and extend Thomas' classification of the reducts of the random graph, it is only in the present dissertation that it is applied to obtain a new full classification of reducts of a homogeneous structure up to first-order interdefinability.

Before stating our result, we remark that finer classifications of reducts of homogeneous structures, for example up to existential, existential positive, or primitive positive interdefinability, have also been considered in the literature, in particular in applications – see [2, 7, 4, 5].

2. Homogeneous structures

2.1. **Fraïssé theory.** The following probabilistic construction is due to Erdős and Rényi [11]. Let X be a countably infinite set. We chose a graph on the vertex set X by selecting edges from the unordered pairs of X independently and with probability 1/2. In [11] it is shown that there exists a countable graph R - called the random graph or Rado graph - such that $P(X \cong R) = 1$. Similarly, one can define the random digraph, the random tournament and random hypergraphs, as well. Although it is not apparent, the countable endpoint-free dense linear order $(\mathbb{Q}, \leq)$ has a random construction, too. All these structures have a common property, namely that they are homogeneous.

Definition 2. *A countable structure Γ is homogeneous if every partial isomorphism $f : A \to B$ between finite induced substructures of Γ extends to an automorphism of Γ. I.e., there exists an $\alpha \in \mathrm{Aut}(\Gamma)$ such that $f = \alpha|_A$.*

Fraïssé has found a very plausible approach to dealing with homogeneous structures. According to his definition, the age of a structure Γ is the class of finite structures that can be embedded into Γ.

Theorem 1. *A class $\mathcal{C}$ of finite structures over a given relational language is the age of a homogeneous structure Γ if and only if the following hold.*

(1) *$\mathcal{C}$ is closed under isomorphism.*
(2) *If $A \leq B$ and $B \in \mathcal{C}$, then $A \in \mathcal{C}$.*
(3) *Up to isomorphism there are countably many structures in $\mathcal{C}$.*
(AP) *For any $A, B_1, B_2 \in \mathcal{C}$ and embeddings $f_1 : A \hookrightarrow B_1$, $f_2 : A \hookrightarrow B_2$ there exist a $D \in \mathcal{C}$ and embeddings $g_1 : B_1 \hookrightarrow D$, $g_2 : B_2 \hookrightarrow D$ such that $g_1 \circ f_1 = g_2 \circ f_2$.*

Moreover, if $\mathcal{C}$ satisfies the above conditions, then up to isomorphism there is a unique homogeneous structure Γ with $\mathrm{Age}(\Gamma) = \mathcal{C}$.

The abbreviation (AP) in Theorem 1 stands for amalgamation property.

Classes of finite structures $\mathcal{C}$ over a given ralational language that satisfy the conditions of Theorem 1 are called Fraïssé classes. The unique Γ with $\mathrm{Age}(\Gamma) = \mathcal{C}$ is called the Fraïssé limit of $\mathcal{C}$, and is denoted by $\mathrm{Frlim}(\mathcal{C})$. It is very easy to check that the class of all finite graphs, all finite k-uniform hypergraphs and all finite total orders are Fraïssé classes, and the Fraïssé limits are the random graph, the random hypergraphs and the dense linear order, respectively. Moreover, for any $n \geq 3$ the class of all K_n-free graphs is a Fraïssé class, and its Fraïssé limit is the so-called Henson graph (H_n, E). Except for trivial constructions, the Henson graphs, the complement of the Henson graphs and the random graph are the only homogeneous graphs [23]. The class of all finite partially ordered sets is again a Fraïssé class. Its Fraïssé limit is called the random poset or the generic poset, and is denoted by $\mathbb{P}$.

The classification programme of homogeneous structures is still a vivid area. As we have indicated before, Lachlan and Woodrow characterized the homogeneous graphs in [23]. The classification of homogeneous tournaments was given by Cherlin [10], and the case of posets was settled by Schmerl [35]. Schmerl's result shows that the random poset is the only non-trivial construction for a homogeneous poset. Recently, Nešetřil and Hubička gave several presentations for the generic poset [17]. These descriptions often reveal properties of the structure which were not obvious from the definition or the recursive construction. For more details about homogeneous structures, see [9].

2.2. **Ramsey structures.**

Definition 3. *A class $\mathcal{C}$ of finite structures over a given language is a Ramsey class, if for all $k \in \mathbb{N}$, $A, B \in \mathcal{C}$ there exists a $C \in \mathcal{C}$ such that if the copies of A in C are colored with k colors, then there is a monochromatic copy B' of B in C, i.e., a copy of B such that all copies of A in B' have the same color.*

The first prominent result concerning Ramsey structures was due to Nešetřil. He proved that if a Ramsey class satisfies items (1), (2) and (3) of Theorem 1, then it satisfies item (AP), as well. In particular, if the age of a countable structure Γ is a Ramsey class, then Γ is homogeneous. If this is the case, then Γ is called a homogeneous Ramsey structure.

Homogeneous Ramsey structures are thoroughly investigated, see for example [25, 26, 22].

2.3. **Reducts of homogeneous structures.** We have already defined the notion of a reduct in the introduction. The main goals of the present thesis is to characterize the reducts of the random partial order and the Henson graphs with a constant. Although the basic techniques used for these two results are very similar, both proofs are self-contained for the sake of transparency. Hence, the most important definitions and theorems are stated in both sections.

2.4. **Galois connection.** Homogeneous structures over a finite relational language are ω-categorical. In particular, the therorem of ??? implies that the reducts of a homogeneous structure Γ up to first-order interdefinability are in a one-to-one Galois correspondence with the closed permutation groups containing $\mathrm{Aut}(\Gamma)$. The (local) closure of a permutation group G acting on Γ consists of the permutations α such that for all $A \subseteq \Gamma$ finite there exists a permutation $g \in G$ such that $g|_A = \alpha|_A$. A permutation group G is closed if G equals to its local closure. That is, G is closed if every permutation that can be interpolated on any finite set is in G.

Throughout the dissertation we use this Galois connection.

3. Reducts of the random poset

3.1. **The reducts of the random partial order.**

3.1.1. *The group formulation.* In a first formulation of our result, we will list the closed groups containing $\mathrm{Aut}(\mathbb{P})$ by means of sets of permutations generating them: we say that a set $\mathcal{S}$ of permutations on P *generates* a permutation α on P iff α is an element of the smallest closed permutation group $\langle \mathcal{S} \rangle$ that contains $\mathcal{S}$. Equivalently, writing id for the identity function on P, for every finite set $A \subseteq P$ there exist $n \geq 0$, $\beta_1, \ldots, \beta_n \in \mathcal{S}$, and $i_1, \ldots, i_n \in \{1, -1\}$ such that $\beta_1^{i_1} \circ \cdots \circ \beta_n^{i_n} \circ \mathrm{id}$ agrees with α on A. We also say that a permutation β generates α iff $\{\beta\}$ generates α.

If for $x, y \in P$ we define $x \geq y$ iff $y \leq x$, then the structure $(P; \geq)$ is isomorphic to $\mathbb{P}$ – it is, for example, easy to verify that it contains all finite partial orders and that it is homogeneous. Hence, there exists an isomorphism between the two structures, and we fix one such isomorphism $\updownarrow\colon P \to P$; so the function $\updownarrow$ simply reverses the order $\leq$ on P. It is easy to see that any two isomorphisms of this kind generate one another, and the exact choice of the permutation is thus irrelevant for our purposes.

The class $\mathcal{C}$ of all finite structures of the form $(A; \leq', F')$, where $\leq'$ is a partial order on A, and $F' \subseteq A$ is an upward closed set with respect to $\leq'$, is an *amalgamation class* in the sense of [13]. Hence, it has a Fraïssé limit; that is, there exists an up to isomorphism unique countable structure which is homogeneous and whose *age*, i.e., the set of finite structures isomorphic with one of its induced substructures, equals $\mathcal{C}$. The partial order of this limit is just the random partial order, and thus we can write $(P; \leq, F)$ for this structure, where $F \subseteq P$ is an upward closed set with respect to $\leq$. By homogeneity and universality of $(P; \leq, F)$, F is even a *filter*, i.e., any two elements of F have a lower bound in F. We call $(P; \leq, F)$ the *random partial order with a random filter*, and any filter $W \subseteq P$ with the property that $(P; \leq, W)$ is isomorphic with $(P; \leq, F)$ *random*.

Let $F \subseteq P$ be a random filter, and let $I := P \setminus F$. Then I is downward closed, and in fact an *ideal*, i.e., any two elements of I have an upper bound in I. Define a partial order $\trianglelefteq_F$ on P by setting

$$
\begin{aligned}
x \trianglelefteq_F y \ \leftrightarrow\ & x, y \in F \text{ and } x \leq y, \text{ or} \\
& x, y \in I \text{ and } x \leq y, \text{ or} \\
& x \in F \wedge y \in I \text{ and } y \not\leq x,
\end{aligned}
$$

where $a \not\leq b$ is short for $\neg(a \leq b)$. It is easy to see that $(P; \trianglelefteq_F)$ is indeed a partial order, and we will verify in the next subsection that $(P; \trianglelefteq_F)$ and $\mathbb{P}$ are isomorphic. Pick an isomorphism $\circlearrowright_F : (P; \trianglelefteq_F) \to \mathbb{P}$. Then for $x, y \in F$, we have $f(x) \leq f(y)$ if and only if $x \leq y$, and likewise for $x, y \in I$; if $x \in F$ and $y \in I$, then $f(x) \leq f(y)$ if and only if $y \not\leq x$; and moreover, $f(x) \not\geq f(y)$ for all $x \in F$ and $y \in I$. It is not hard to see that any two permutations obtained this way generate one another, even if they were defined by different random filters. We therefore also write $\circlearrowright$ for any $\circlearrowright_F$ when the filter F is not of particular interest.

Theorem 2. *The following five groups are precisely the closed permutation groups on P which contain* $\mathrm{Aut}(\mathbb{P})$*.*

(1) $\mathrm{Aut}(\mathbb{P})$*;*
(2) $\mathrm{Rev} := \langle \{\updownarrow\} \cup \mathrm{Aut}(\mathbb{P}) \rangle$*;*
(3) $\mathrm{Turn} := \langle \{\circlearrowright\} \cup \mathrm{Aut}(\mathbb{P}) \rangle$*;*
(4) $\mathrm{Max} := \langle \{\updownarrow, \circlearrowright\} \cup \mathrm{Aut}(\mathbb{P}) \rangle$*;*
(5) *The full symmetric group* Sym_P *of all permutations on P.*

As a consequence, the only symmetries of $\mathbb{P}$ in the sense mentioned above are turning it upside down, and "turning" it around a random filter F via the function $\circlearrowright_F$. These symmetries suggest the investigation of the corresponding operations on finite posets (essentially, the restrictions of $\updownarrow$ and $\circlearrowright_F$ to finite substructures of $\mathbb{P}$). While $\updownarrow$ for finite posets is, of course, combinatorially not very exciting, the study of "turns" of finite posets seems to be quite worthwhile – we refer to the companion paper [31].

We will also obtain explicit descriptions of the elements of the groups in Theorem 2. Clearly, the group Rev contains exactly the automorphisms of $\mathbb{P}$ and the isomorphisms between $\mathbb{P}$ and $(P; \geq)$. We will show that Turn consists precisely of what we will call *rotations* in Definition 30 – these are functions of slightly more general form than the functions $\circlearrowright_F$. Moreover, Max turns out to be simply the union of Rev, Turn, and the set of all functions of the form $\updownarrow \circ f$, where f is a rotation.

3.1.2. *The reduct formulation.* We now turn to the relational formulation of our result; that is, we will specify five reducts of $\mathbb{P}$ such that any reduct of $\mathbb{P}$ is first-order interdefinable with one of the reducts of our list.

Define a binary relation $\perp$ on P by $\perp := \{(x, y) \in P^2 \mid x \not\leq y \wedge y \not\leq x\}$. We call the relation the *incomparability relation*, and refer to elements $x, y \in P$ as *incomparable* iff (x, y) is an element of $\perp$; in that case, we also write $x \perp y$. Elements $x, y \in P$ are *comparable* iff they are not incomparable.

For $x, y \in P$, write $x < y$ iff $x \leq y$ and $x \neq y$. Now define a ternary relation cycl on P by

$$\begin{aligned}\text{cycl} := \{(x,y,z) \in P^3 \mid\ & (x<y<z) \vee (y<z<x) \vee (z<x<y) \vee \\ & (x<y \wedge x \perp z \wedge y \perp z) \vee \\ & (y<z \wedge y \perp x \wedge z \perp x) \vee \\ & (z<x \wedge z \perp y \wedge x \perp y)\}.\end{aligned}$$

Finally, define a ternary relation Par on P by

$$\begin{aligned}\text{Par} := \{(x,y,z) \in P^3 \mid\ & x,y,z \text{ are distinct and the number of} \\ & \text{2-element subsets of incomparable elements of } \{x,y,z\} \text{ is odd}\}.\end{aligned}$$

Theorem 3. *Let Γ be a reduct of $\mathbb{P}$. Then Γ is first-order interdefinable with precisely one of the following structures.*

(1) $\mathbb{P} = (P; \leq)$;
(2) $(P; \perp)$;
(3) $(P; \text{cycl})$;
(4) $(P; \text{Par})$;
(5) $(P; =)$.

Moreover, for $1 \leq x \leq 5$, Γ is first-order interdefinable with structure (x) if and only if $\text{Aut}(\Gamma)$ *equals group number (x) in Theorem 2.*

3.2. **Random filters and the extension property.** Before turning to the main proof of our theorems, we verify the existence of the permutation $\circlearrowleft_F$. That is, we must show that if $F \subseteq P$ is a random filter, then $(P; \lhd_F)$ and $\mathbb{P}$ are isomorphic. The easiest way to see this is by checking that $(P; \unlhd_F)$ satisfies the following *extension property*, which determines $\mathbb{P}$ up to isomorphism and which we will use throughout the paper: for any finite set $S = \{s_1, \ldots, s_k\} \subseteq P$ and any partial order with domain $\{y\} \cup S$ extending the order induced by $\mathbb{P}$ on S, there exists $x \in P$ such that the assignment from $\{x\} \cup S$ to $\{y\} \cup S$ which sends x to y and leaves all elements of S fixed is an isomorphism. In logic terminology, the extension property says that if we fix any finite set of elements $s_1, \ldots, s_k \in P$, and express properties of another imaginary element x by means of a quantifier-free $\{\leq\}$-formula with one free variable using parameters $s_1, \ldots, s_k$, then an element enjoying these properties actually exists in $\mathbb{P}$ unless the properties are inconsistent with the theory of partial orders.

Proposition 4. *Let $F \subseteq P$ be a random filter of $\mathbb{P}$. Then $(P; \triangleleft_F)$ satisfies the extension property. Consequently, $(P; \triangleleft_F)$ and $\mathbb{P}$ are isomorphic and $\circlearrowright_F$ exists.*

Proof. Let $s_1, \ldots, s_k \in P$ and an extension of the order induced by $\triangleleft_F$ on $S = \{s_1, \ldots, s_k\}$ by an element y outside S be given. We will denote the order on $T := S \cup \{y\}$ by $\triangleleft_F$ as well. Let $I := P \setminus F$ be the ideal in $\mathbb{P}$ corresponding to the filter F, and write S as a disjoint union $S_F \cup S_I$, where $S_F := S \cap F$, and $S_I := S \cap I$. Now suppose that there exist $a \in S_I$ and $b \in S_F$ such that $a \triangleleft_F y \triangleleft_F b$. Then $a \triangleleft_F b$, which is impossible by the definition of $\triangleleft_F$, since $a \in I$ and $b \in F$. Hence, assume without loss of generality that we do not have $y \triangleleft_F b$ for any $b \in S_F$. Then $W := S_I \cup \{y\}$ is upward closed and S_F downward closed in $(T; \triangleleft_F)$. Now define an order $\leq_W$ on T by setting

$$u \leq_W v \;\leftrightarrow (u, v \in W \wedge u \triangleleft_F v) \vee (u, v \in T \setminus W \wedge u \triangleleft_F v) \vee (u \in W \wedge v \in T \setminus W \wedge \neg(v \triangleleft_F u)).$$

Note that this defines $\leq_W$ from $\triangleleft_F$ in precisely the same way as $\triangleleft_F$ is defined (though on $\mathbb{P}$) from $\leq$. Hence, $\leq_W$ is a partial order on T, and for $u, v \in S$ we have $u \leq_V v$ if and only if $u \leq v$ in $\mathbb{P}$. Now the downward closed set S_F in $(T; \triangleleft_F)$ is an upward closed set in $(T; \leq_W)$. Hence, the structure $(T; \leq_W, S_F)$ has an embedding ξ into the universal object $(P; \leq, F)$. Since $\leq_W$ agrees with $\leq$ on S, and by homogeneity, we may assume that ξ is the identity on S. Set $x := \xi(y)$. We leave the straightforward verification of the fact that the assignment from $\{x\} \cup S$ to $\{y\} \cup S$ which sends x to y and leaves all elements of S fixed is an isomorphism from $(\{x\} \cup S; \triangleleft_F)$ onto $(T; \triangleleft_F)$ to the reader. □

Let us remark that the ideal $I = P \setminus F$ corresponding to a random filter F on $\mathbb{P}$ is random in the analogous sense for ideals. Moreover, under $\circlearrowright_F$ the random filter F is sent to a random ideal, and vice-versa. One could thus assume that the image of F under $\circlearrowright_F$ equals I, in which case $\circlearrowright_F$ becomes, similarly to $\updownarrow$, its own "almost" inverse in the sense that applying it twice yields an automorphism of $\mathbb{P}$. By adjusting it with such an automorphism, one could even assume that $\circlearrowright_F = \circlearrowright_F^{-1}$.

3.3. **Ramsey theory: canonizing functions.** Our combinatorial method for proving Theorem 2 is to apply Ramsey theory in order to find patterns of regular behaviour in arbitrary functions on $\mathbb{P}$, and follows [7, 6, 4, 5]. We make this more precise.

Definition 5. *Let Δ be a structure. The* type $\mathrm{tp}(a)$ *of an n-tuple a of elements in Δ is the set of first-order formulas with free variables $x_1, \ldots, x_n$ that hold for a in Δ.*

Definition 6. *Let Δ, Λ be structures. A* type condition *between Δ and Λ is a pair (t, s), where t is a type of an n-tuple in Δ, and s is a type of an n-tuple in Λ, for some $n \geq 1$.*

A function $f : \Delta \to \Lambda$ satisfies *a type condition* (t, s) *between* Δ *and* Λ *iff for all* n*-tuples* $a = (a_1, \dots, a_n)$ *of elements of* Δ *with* $\mathrm{tp}(a) = t$ *the* n*-tuple* $f(a) := (f(a_1), \dots, f(a_n))$ *has type* s *in* Λ. *A* behaviour *is a set of type conditions between structures* Δ *and* Λ. *A function from* Δ *to* Λ has behaviour B *iff it satisfies all the type conditions of* B.

Definition 7. *Let* Δ, Λ *be structures. A function* $f : \Delta \to \Lambda$ *is* canonical *iff for all types* t *of* n*-tuples in* Δ *there exists a type* s *of an* n*-tuple in* Λ *such that* f *satisfies the type condition* (t, s). *In other words,* n*-tuples of equal type in* Δ *are sent to* n*-tuples of equal type in* Λ *under* f, *for all* $n \geq 1$.

We remark that since $\mathbb{P}$ is homogeneous, every first-order formula is over $\mathbb{P}$ equivalent to a quantifier-free formula, and so the type of an n-tuple a in $\mathbb{P}$ is determined by which of its elements are equal and between which elements the relation $\leq$ holds. In particular, the type of a only depends on its binary subtypes, i.e., the types of the pairs (a_i, a_j), where $1 \leq i, j \leq n$. Therefore, a function $f : \mathbb{P} \to \mathbb{P}$ is canonical iff it satisfies the condition of the definition for types of 2-tuples.

Roughly, our strategy is to make the functions we work with canonical, and thus easier to handle. To achieve this, we first enrich the structure $\mathbb{P}$ by a linear order in order to improve its combinatorial properties, as follows. We do not give the – in some cases fairly technical – definitions of all notions in this discourse, as they will not be needed later on; in any case, Proposition 8 that follows is used as a black box for this paper, and the reader interested in its proof is referred to [7]. The class $\mathcal{D}$ of all finite structures $(A; \leq', \prec')$ with two binary relations $\leq'$ and $\prec'$, where $\leq'$ is a partial order and $\prec'$ is a total order extending $\leq'$, is an amalgamation class, and moreover a *Ramsey class* (see for example [36, Theorem 1 (1)]). By the first property, it has a Fraïssé limit. Checking the extension property, one sees that the partial order of this limit is just the random partial order, and by uniqueness of the dense linear order without endpoints its total order is isomorphic to the order of the rationals. Hence, there exists a linear order $\prec$ on P which is isomorphic to the order of the rationals, which extends $\leq$, and such that the structure $\mathbb{P}^+ := (P; \leq, \prec)$ is precisely the Fraïssé limit of the class $\mathcal{D}$. So $\mathbb{P}^+$ is a homogeneous structure in a finite language which has a linear order among its relations and which is *Ramsey*, i.e. its age, which equals the class $\mathcal{D}$, is a Ramsey class. The following proposition is then a consequence of the results in [7, 5] about such structures. To state it, let us extend the notion "generates" to non-permutations: for a set of functions $\mathcal{F} \subseteq P^P$ and $f \in P^P$, we say that f is *M-generated* by $\mathcal{F}$ iff it is contained in the smallest transformation monoid on P which contains $\mathcal{F}$ and which is a closed set in the convergence topology on P^P. In other words, f is M-generated by $\mathcal{F}$ iff for all finite $A \subseteq P$ there exist $n \geq 0$ and $f_1, \dots, f_n \in \mathcal{F}$ such that $f_1 \circ \dots \circ f_n \circ \mathrm{id}$

agrees with f on A. For a structure Δ and elements $c_1, \dots, c_n$ of Δ, we write $(\Delta, c_1, \dots, c_n)$ for the structure obtained by adding the constant symbols $c_1, \dots, c_n$ to Δ.

Proposition 8. *Let $f : P \to P$ be a function, and let $c_1, \dots, c_n, d_1, \dots, d_m \in P$. Then $\{f\} \cup \mathrm{Aut}(\mathbb{P}^+)$ M-generates a function which is canonical as a function from $(\mathbb{P}^+, c_1, \dots, c_n)$ to $(\mathbb{P}^+, d_1, \dots, d_m)$, and which is identical with f on $\{c_1, \dots, c_n\}$.*

Any canonical function g from $(\mathbb{P}^+, c_1, \dots, c_n)$ to $(\mathbb{P}^+, d_1, \dots, d_m)$ defines a function from the set T of types of pairs of distinct elements in $(\mathbb{P}^+, c_1, \dots, c_n)$ to the set S of such types in $(\mathbb{P}^+, d_1, \dots, d_m)$ – this "type function" simply assigns to every element t of T the type s in S for which the type condition (t, s) is satisfied by g. Already when $n = m = 0$, i.e., there are no constants added to $\mathbb{P}^+$, then $|T| = |S| = 4$, so in theory there are 4^4 such type functions. The following lemma states which of them actually occur.

Lemma 9. *Let $g : \mathbb{P}^+ \to \mathbb{P}^+$ be canonical and injective. Then it has one of the following behaviours.*

- (i a) *g behaves like* id*, i.e., it preserves $\leq$ and $\perp$ (and hence also $\prec$);*
- (i b) *g behaves like $\updownarrow$, i.e., it reverses $\leq$ and preserves $\perp$ (and hence reverses $\prec$);*
- (ii a) *g sends P order preservingly onto a chain with respect to $\leq$ (and hence preserves $\prec$);*
- (ii b) *g sends P order reversingly onto a chain with respect to $\leq$ (and hence reverses $\prec$);*
- (iii a) *g sends P onto an antichain with respect to $\leq$ and preserves $\prec$;*
- (iii b) *g sends P onto an antichain with respect to $\leq$ and reverses $\prec$.*

Proof. We first prove that g either preserves or reverses the order $\prec$.

Suppose there exist $a, b \in P$ with $a \prec b$ such that $g(a) \prec g(b)$. Assume first that $a \leq b$. Then $g(c) \prec g(d)$ for all $c, d \in P$ with $c \prec d$ and $c \leq d$ because g is canonical. Now using the universality of $\mathbb{P}^+$, pick $u, v, w \in P$ with $u \prec v \prec w$, $u \leq w$, $u \perp v$, and $v \perp w$. Then $g(u) \leq g(w)$ by our observation above. If $g(v) \prec g(u)$, then also $g(w) \prec g(v)$ as g is canonical, and hence $g(w) \prec g(u)$, a contradiction. Hence, $g(u) \prec g(v)$, and so $g(c) \prec g(d)$ for all $c, d \in P$ with $c \prec d$, so g preserves $\prec$. Now suppose that $a \perp b$. Then $g(c) \prec g(d)$ for all $c, d \in P$ with $c \prec d$ and $c \perp d$, because g is canonical. Pick $u, v, w \in P$ as before. This time, $g(u) \prec g(v) \prec g(w)$, and hence $g(u) \prec g(w)$. Therefore, $g(c) \prec g(d)$ for all $c, d \in P$ with $c \prec d$, so g again preserves $\prec$.

By the dual argument, the existence of $a, b \in P$ with $a \prec b$ such that $g(b) \prec g(a)$ implies that g reverses $\prec$.

We next show that if g preserves $\prec$, then one of the situations (i a), (ii a), (iii a) occurs; then by duality, if g reverses $\prec$, one of (i b), (ii b), (iii b) hold. We distinguish two cases.

Suppose first that $g(a) \perp g(b)$ for all $a, b \in P$ with $a \leq b$. Let $c, d, e \in P$ such that $c \prec d \prec e$, $c \perp d$, $c \leq e$, and $e \perp d$. If $g(c)$ and $g(d)$ were comparable, then $g(c) \leq g(d)$ since $g(c) \prec g(d)$, and likewise $g(d) \leq g(e)$, so that $g(c) \leq g(d)$, a contradiction. Hence, $g(c)$ and $g(d)$ are incomparable, and so, since g is canonical, (iii a) holds.

Assume now that $g(a) \leq g(b)$ for all $a, b \in P$ with $a \leq b$. If $g(c) \leq g(d)$ also for all $c, d \in P$ with $c \perp d$ and $c \prec d$, then clearly we have situation (ii a). Otherwise, $g(c) \perp g(d)$ for all $c, d \in P$ with $c \perp d$ and $c \prec d$, and we have case (i a).

Since one of these two situations must be the case, we are done. □

When applying Proposition 8, we will be able to ignore most of the possible behaviours of canonical functions as a consequence of the following lemma.

Lemma 10. *Let $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ be a closed group such that for all finite $A \subseteq P$ there is a function M-generated by $\mathcal{G}$ which sends A to a chain or an antichain. Then $\mathcal{G} = \mathrm{Sym}_P$.*

Proof. Suppose first that for all finite $A \subseteq P$ there is a function M-generated by $\mathcal{G}$ which sends A to an antichain. Let s, t be injective n-tuples of elements in P, for some $n \geq 1$. Let $g : P \to P$ and $h : P \to P$ be functions M-generated by $\mathcal{G}$ such that $g(s)$ (the n-tuple obtained by applying g to every component of s) and $h(t)$ induce antichains in $\mathbb{P}$. By the homogeneity of $\mathbb{P}$, there exists an automorphism $\alpha \in \mathrm{Aut}(\mathbb{P})$ such that $\alpha(g(s)) = h(t)$. Also, since $\mathcal{G}$ contains the inverse of all of its functions, there exists a function $p : P \to P$ M-generated by $\mathcal{G}$ such that $p(h(t)) = t$, and hence $p(\alpha(g(s))) = t$. Since $p \circ \alpha \circ g$ is M-generated by $\mathcal{G}$, there exists $\beta \in \mathcal{G}$ which agrees with this function on s. Hence, $\beta(s) = t$, proving that $\mathcal{G}$ is n-transitive for all $n \geq 1$, and so $\mathcal{G} = \mathrm{Sym}_P$.

Now suppose that for all finite $A \subseteq P$ there is a function M-generated by $\mathcal{G}$ which sends A to a chain. Let any finite $A \subseteq P$ be given, and let $B \subseteq P$ be so that $|B| = |A|$ and such that B induces an independent set in $\mathbb{P}$. Let $g : P \to P$ and $h : P \to P$ be functions M-generated by $\mathcal{G}$ such that $g[A]$ and $h[B]$ induce chains in $\mathbb{P}$. There exists $\alpha \in \mathrm{Aut}(\mathbb{P})$ such that $\alpha[g[A]] = h[B]$. Let $p : P \to P$ be a function generated by $\mathcal{G}$ such that $p[h[B]] = B$. Then $p[\alpha[g[A]]] = B$, and hence we are back in the preceding case.

Finally, observe that one of the two cases must occur: for otherwise, there exist finite $A_1, A_2 \subseteq P$ such that A_1 cannot be set to an antichain, and A_2 cannot be sent to a chain by any function which is M-generated by $\mathcal{G}$. But then $A_1 \cup A_2$ can neither be sent to a chain nor to an antichain by any such function, a contradiction. □

Lemma 11. *Let $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ be a closed group which M-generates a canonical function of behaviour (ii a), (ii b), (iii a) or (iii b) in Lemma 9. Then $\mathcal{G} = \mathrm{Sym}_P$.*

Proof. This is a direct consequence of Lemma 10. □

Having enriched $\mathbb{P}$ with the linear order $\prec$ and taken advantage of Proposition 8, we pass to a suitable substructure of $(\mathbb{P}^+, c_1, \ldots, c_n)$ in order to get rid of $\prec$ – this substructure will be called a *$\prec$-clean skeleton*. Before giving the exact definition, we need more notions and notation concerning the definable subsets of $(\mathbb{P}, c_1, \ldots, c_n)$ and of $(\mathbb{P}^+, c_1, \ldots, c_n)$.

Definition 12. *Let $\mathcal{G}$ be a permutation group acting on a set D. Then for $n \geq 1$ and $a = (a_1, \ldots, a_n) \in D^n$, the set*

$$\{(\alpha(a_1), \ldots, \alpha(a_n)) : \alpha \in \mathcal{G}\} \subseteq D^n$$

is called an n-orbit of $\mathcal{G}$. The 1*-orbits are just called* orbits*. If Δ is a structure, then the n-orbits of Δ are defined as the n-orbits of* $\mathrm{Aut}(\Delta)$*.*

By the theorem of Ryll-Nardzewski, Engeler and Svenonius, two n-tuples in an ω-categorical structure belong to the same n-orbit if and only if they have the same type; in particular, this is true in the structures $(\mathbb{P}, c_1, \ldots, c_n)$ and $(\mathbb{P}^+, c_1, \ldots, c_n)$.

Notation 13. *Let $c_1, \ldots, c_n \in P$. For $R_1, \ldots, R_n \in \{=, <, \perp, >\}$ and $S_1, \ldots, S_n \in \{\prec, \succ\}$, we set*

$$X_{R_1,\ldots,R_n} := \{x \in P : c_1 R_1 x \ \wedge \ \cdots \ \wedge \ c_n R_n x\}$$

and

$$X^{S_1,\ldots,S_n}_{R_1,\ldots,R_n} := \{x \in P : (c_1 R_1 x \ \wedge c_1 S_1 x) \ \wedge \ \cdots \ \wedge \ (c_n R_n x \wedge x_n S_n x)\}.$$

The constants $c_1, \ldots, c_n$ are not specified in the notation, but will always be clear from the context.

The following is well-known and easy to verify using the homogeneity and universality of $\mathbb{P}$ and $\mathbb{P}^+$, and in particular the fact that first-order formulas over these structures are equivalent to quantifier-free formulas.

Fact 14. *Let $c_1, \ldots, c_n \in P$.*

- *The sets $X_{R_1,\ldots,R_n}$ are either empty, or equal to $\{c_i\}$ for some $1 \leq i \leq n$, or infinite and induce $\mathbb{P}$. The orbits of $(\mathbb{P}, c_1, \ldots, c_n)$ are precisely the non-empty sets of this form.*
- *The sets $X^{S_1,\ldots,S_n}_{R_1,\ldots,R_n}$ are either empty, or equal to $\{c_i\}$ for some $1 \leq i \leq n$, or infinite and induce $\mathbb{P}^+$. The orbits of $(\mathbb{P}^+, c_1, \ldots, c_n)$ are precisely the non-empty sets of this form.*

Definition 15. *Let Δ be a structure on domain D. A subset S of D is called a* skeleton *of Δ iff it induces a substructure of Δ which is isomorphic to Δ. Now let $\sqsubset$ be a linear* *order on D. Then a skeleton S is called* $\sqsubset$-clean *iff whenever $a = (a_1, a_2), b = (b_1, b_2) \in S^2$ have the same type in Δ, then either a, b or $a, \tilde{b} := (b_2, b_1)$ have the same type in $(\Delta, \sqsubset)$.*

In this paper, we only need a $\prec$-clean skeleton of $(\mathbb{P}, c_1, \ldots, c_n)$, but we stated Definition 15 generally since we believe it could be useful in other situations where a homogeneous structure is extended by a linear order with the goal of making it Ramsey.

Lemma 16. *Let $c_1, \ldots, c_n \in P$. Then $(\mathbb{P}, c_1, \ldots, c_n)$ has a skeleton which is $\prec$-clean.*

Proof. Let $O_1, \ldots, O_k$ be the orbits of $(\mathbb{P}, c_1, \ldots, c_n)$, and pick one representative element r_i of each orbit O_i. By relabelling the orbits, we may assume that $r_1 \prec \cdots \prec r_k$; pick an additional $r_0 \in P$ with $r_0 \prec r_1$. Now for all $1 \le j \le k$ for which O_j is infinite set

$$S_j := \{s \in O_j \mid r_{j-1} \prec s \prec r_j\}.$$

Let S be the union of all the S_j with $\{c_1, \ldots, c_n\}$. To see that S is a skeleton, it suffices to verify the extension property for $(S; \le)$. Let $U = \{u_1, \ldots, u_l\} \subseteq S$ induce a finite substructure of $(S; \le)$, and let $U \cup \{y\}$ be an extension of U by an element $y \notin U$. We may assume that U contains $\{c_1, \ldots, c_n\}$. By the extension property for $\mathbb{P}$, we may assume that y is an element of this structure, and so $y \in O_j$ for some $1 \le j \le k$. Since $y \notin \{c_1, \ldots, c_n\}$, O_j is infinite. We claim there exists $x \in S_j$ such that x, y have the same type in $(\mathbb{P}, u_1, \ldots, u_l)$ – then picking any such x yields the desired extension. Suppose there exists $1 \le i \le l$ such that $r_j \le u_i < y$ – we will derive a contradiction. Since $r_j \le u_i$, we have $u_i \not\prec r_j$, implying $u_i \notin S_j$; but $u_i \in S$, and so $u_i \notin O_j$. Hence, the orbits of u_i and r_j in $(\mathbb{P}, c_1, \ldots, c_n)$ are distinct. Therefore, there exists $1 \le m \le n$ such that either $u_i \ge c_m$ and $r_j \not\ge c_m$, or $r_j < c_m$ and $u_i \not< c_m$. In the first case we infer $y > c_m$, contradicting the fact that y and r_j have the same type in $(\mathbb{P}, c_1, \ldots, c_n)$. In the second case it follows that $y \not< c_m$, yielding the same contradiction. An isomorphic argument shows that there is no $1 \le i \le l$ such that $y < u_i \le r_{j-1}$, and so we henceforth assume that $y \not\le u_i$ for all $u_i \le r_{j-1}$ and $y \not\ge u_i$ for all $u_i \ge r_j$. Let x be a new object, and extend the partial order $\le$ from $W := \{u_1, \ldots, u_l, r_{j-1}, r_j\}$ to $W \cup \{x\}$ by setting $x \le u_i$ iff $y \le u_i$ and $u_i \le x$ iff $u_i \le y$ for all $1 \le i \le l$, and taking the reflexive and transitive closure of this relation on $W \cup \{x\}$. Then $x \le r_{j-1}$ iff this relation is forced by a transitivity condition, i.e., iff there exists $1 \le i \le l$ such that $x \le u_i \le r_{j-1}$; by our above observation, this is not the case. Similarly, $r_j \le x$ does not hold. Therefore extending $\prec$ from W to $W \cup \{x\}$ by setting $r_{j-1} \prec x \prec r_j$ yields a partial order with linear extension on $W \cup \{x\}$. By the extension property for $\mathbb{P}^+$ (the analogue of the extension property for $\mathbb{P}$), we may assume that $x \in \mathbb{P}^+$ and that the partial order with linear extension on W is induced by $\mathbb{P}^+$; then

x has the same type as y in $(\mathbb{P}, c_1, \ldots, c_n)$ by the definition of $\leq$ on $W \cup \{x\}$, and $x \in S_j$ since $r_{j-1} \prec x \prec r_j$.

We show that S is $\prec$-clean. Let $a = (a_1, a_2), b = (b_1, b_2) \in S^2$ have the same type in $(\mathbb{P}, c_1, \ldots, c_n)$. Then there exist $1 \leq i, j \leq k$ such that $a_1, b_1 \in O_i$ and $a_2, b_2 \in O_j$. Suppose $i = j$. If a_1, a_2 are comparable, say $a_1 \leq a_2$, then $b_1 \leq b_2$, $a_1 \prec a_2$, and $b_1 \prec b_2$, and we are done. If $a_1 \perp a_2$, then $b_1 \perp b_2$ and so either a, b or $a, \tilde{b} = (b_2, b_1)$ have the same type in $(\mathbb{P}^+, c_1, \ldots, c_n)$. Now suppose $i \neq j$, say $i < j$. Then $a_1 \prec a_2$ and $b_1 \prec b_2$, and so a, b have the same type in $(\mathbb{P}^+, c_1, \ldots, c_n)$.

□

Lemma 17. *Let $f : P \to P$ be a permutation, and let $c_1, \ldots, c_n, d_1, \ldots, d_m \in P$. Then $\{f, f^{-1}\} \cup \mathrm{Aut}(\mathbb{P})$ M-generates a function $g : P \to P$ with the following properties.*

- *g agrees with f on $\{c_1, \ldots, c_n\}$;*
- *g is canonical as a function from $(\mathbb{P}, c_1, \ldots, c_n)$ to $(\mathbb{P}, d_1, \ldots, d_m)$.*

Proof. Let h be the function guaranteed by Proposition 8. Since every infinite orbit X of $(\mathbb{P}^+, c_1, \ldots, c_n)$ induces $\mathbb{P}^+$, h must have one of the behaviours of Lemma 9 on X. By Lemma 10, we may assume that h behaves like $\updownarrow$ or like id on every infinite orbit of $(\mathbb{P}^+, c_1, \ldots, c_n)$; for otherwise, $\langle \{f\} \cup \mathrm{Aut}(\mathbb{P}) \rangle$ is the full symmetric group Sym_P, which implies that $\{f, f^{-1}\} \cup \mathrm{Aut}(\mathbb{P})$ M-generates all injective functions, and in particular a function with the desired properties.

Now let $S \subseteq P$ be a $\prec$-clean skeleton of $(\mathbb{P}, c_1, \ldots, c_n)$. We claim that h, considered as a function from $(\mathbb{P}, c_1, \ldots, c_n)$ to $(\mathbb{P}, d_1, \ldots, d_m)$, is canonical on S, that is, it satisfies the definition of canonicity for tuples in S. To see this, let $a = (a_1, a_2), b = (b_1, b_2) \in S^2$ have the same type in $(\mathbb{P}, c_1, \ldots, c_n)$. Then either a, b or $a, \tilde{b} = (b_2, b_1)$ have the same type in $(\mathbb{P}^+, c_1, \ldots, c_n)$, and so either $h(a), h(b)$ or $h(a), h(\tilde{b})$ have the same type in $(\mathbb{P}^+, d_1, \ldots, d_m)$, and hence also in $(\mathbb{P}, d_1, \ldots, d_m)$. In the first case we are done; in the second case, $\mathrm{tp}(a) = \mathrm{tp}(b) = \mathrm{tp}(\tilde{b})$ in $(\mathbb{P}, c_1, \ldots, c_n)$ implies that a_1, a_2, b_1, b_2 all belong to the same orbit in $(\mathbb{P}, c_1, \ldots, c_n)$. Since h behaves like $\updownarrow$ or like id on this orbit, we conclude that $f(a), f(b)$ have the same type in $(\mathbb{P}, d_1, \ldots, d_m)$.

Let $i : (P; \leq, c_1, \ldots, c_n) \to (S; \leq, c_1, \ldots, c_n)$ be an isomorphism, and set $g := h \circ i$. Then g is canonical as a function from $(\mathbb{P}, c_1, \ldots, c_n)$ to $(\mathbb{P}, d_1, \ldots, d_m)$, and agrees with f on $\{c_1, \ldots, c_n\}$. Since i preserves $\leq$ and its negation, it is M-generated by $\mathrm{Aut}(\mathbb{P})$. Hence so is g, proving the lemma. □

3.4. **Applying canonical functions.**

3.4.1. *Ordering orbits.*

Definition 18. *For disjoint subsets X, Y of P we write*

- $X \leq Y$ *iff there exist $x \in X$, $y \in Y$ such that $x \leq y$;*
- $X \perp Y$ *iff $x \perp y$ for all $x \in X$, $y \in Y$;*
- $X < Y$ *iff $x < y$ for all $x \in X$ and all $y \in Y$.*

We call X, Y incomparable *iff $X \perp Y$, and* comparable *otherwise (which is the case iff $X \leq Y$ or $Y \leq X$). We say that X, Y are* strictly comparable *iff $X < Y$ or $Y < X$.*

Lemma 19. *Let $c_1, \ldots, c_n \in P$. The relation $\leq$ defines a partial order on the orbits of $(\mathbb{P}, c_1, \ldots, c_n)$.*

Proof. Reflexivity is obvious. To see that $X \leq Y$ and $Y \leq X$ imply $X = Y$, observe first that it follows from Fact 14 that X is convex, i.e., if $x, z \in X$ satisfy $x \leq z$ and $y \in P$ is so that $x \leq y$ and $y \leq z$, then $y \in X$. Now there exist $x, x' \in X$ and $y, y' \in Y$ such that $x \leq y$ and $x' \geq y'$. Since y, y' belong to the same orbit, they satisfy the same first-order formulas over $(\mathbb{P}, c_1, \ldots, c_n)$, and hence there exists $z \in X$ such that $z \geq y$. Since X is convex, we have $y \in X$, which is only possible if $X = Y$ since distinct orbits are disjoint.

Suppose that $X \leq Y$ and $Y \leq Z$. Then there exist $x \in X$, $y, y' \in Y$ and $z \in Z$ such that $x \leq y$ and $y' \leq z$. Since y, y' satisfy the same first-order formulas, there exists $x' \in X$ such that $x' \leq y'$. Hence $x' \leq z$ and so $X \leq Z$, proving transitivity. □

Let X, Y be infinite orbits of $(\mathbb{P}, c_1, \ldots, c_n)$. Then precisely one of the following cases holds.

- X and Y are strictly comparable;
- X and Y are incomparable;
- X and Y are comparable, but not strictly comparable.

In the third case, if $X \leq Y$, then there exist $x, x' \in X$ and $y, y' \in Y$ such that $x < y$ and $x' \perp y'$, and there are no $x'' \in X$ and $y'' \in Y$ such that $x'' > y''$.

Definition 20. *If for two disjoint subsets X, Y of P we have $X \leq Y$, $Y \not\leq X$, and $X \not< Y$, or vice-versa, then we write $X \div Y$.*

3.4.2. *Behaviours generating* Sym_P.

Definition 21. *Let $X, Y \subseteq P$ be disjoint, and let $f : P \to P$ be a function. We say that f*

- behaves like id on X *iff $x < x'$ implies $f(x) < f(x')$ and $x \perp x'$ implies $f(x) \perp f(x')$ for all $x, x' \in X$;*

- behaves like $\updownarrow$ on X *iff* $x < x'$ *implies* $f(x) > f(x')$ *and* $x \perp x'$ *implies* $f(x) \perp f(x')$ *for all* $x, x' \in X$*;*
- behaves like id between X and Y *iff* $x < y$ *implies* $f(x) < f(y)$*,* $x > y$ *implies* $f(x) > f(y)$*, and* $x \perp y$ *implies* $f(x) \perp f(y)$ *for all* $x \in X, y \in Y$*.*

Lemma 22. *Let* $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ *be a closed group, and let* $c_1, \ldots, c_n \in P$*. Let* $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ *be a canonical function M-generated by* $\mathcal{G}$*. Then* g *behaves like* id *or like* $\updownarrow$ *on each infinite orbit* X *of* $(\mathbb{P}, c_1, \ldots, c_n)$*, or else* $\mathcal{G} = \mathrm{Sym}_P$*.*

Proof. Let X be an infinite orbit, and let $x, x' \in X$ such that $x \perp x'$. Then the type of (x, x') in $(\mathbb{P}, c_1, \ldots, c_n)$ equals the type of (x', x) in $(\mathbb{P}, c_1, \ldots, c_n)$. Hence, the type of $(g(x), g(x'))$ must equal the type of $(g(x'), g(x))$ in $\mathbb{P}$, which is only possible if $g(x) \perp g(x')$, and hence g preserves $\perp$ on X.

Now if $g(a) < g(a')$ for some $a, a' \in X$ with $a < a'$, then the same holds for all $a, a' \in X$ with $a < a'$, and g behaves like id on X. If $g(a') < g(a)$ for some $a, a' \in X$ with $a < a'$, then g behaves like $\updownarrow$ on X. Finally, if $g(a) \perp g(a')$ for some $a, a' \in X$ with $a < a'$, then g sends X to an antichain. Since X contains all finite partial orders, and by the homogeneity of $\mathbb{P}$, we can then refer to Lemma 10 to conclude that $\mathcal{G} = \mathrm{Sym}_P$. □

Lemma 23. *Let* $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ *be a closed group, and let* $c_1, \ldots, c_n \in P$*. Let* $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ *be a canonical function M-generated by* $\mathcal{G}$*. Then* $g[X] \div g[Y]$ *for all infinite orbits* X, Y *of* $(\mathbb{P}, c_1, \ldots, c_n)$ *with* $X \div Y$*, or else* $\mathcal{G} = \mathrm{Sym}_P$*.*

Proof. Suppose there are infinite orbits X, Y with $X \div Y$ but for which $g[X] \div g[Y]$ does not hold. Assume without loss of generality that $X \leq Y$. By Lemma 22, we may assume that g behaves like id or like $\updownarrow$ on X and on Y.

First consider the case where $g[X] < g[Y]$ or $g[Y] < g[X]$. Let $A \subseteq P$ be finite; we claim that $\mathcal{G}$ M-generates a function which sends A to a chain. There is nothing to show if A is itself a chain, so assume that there exist x, y in A with $x \perp y$. Then using the extension property, one readily checks that there exists $\alpha \in \mathrm{Aut}(\mathbb{P})$ which sends the principal ideal of x in A into X and all other elements of A, and in particular y, into Y. Set $h := g \circ \alpha$. Then $h(x)$ and $h(y)$ are comparable, and h does not add any incomparabilities between elements of A. Hence, repeating this procedure and composing the functions, we obtain a function which sends A to a chain. Lemma 10 then implies $\mathcal{G} = \mathrm{Sym}_P$.

The other case is where $g[X] \perp g[Y]$. Then an isomorphic argument shows that we can map any finite subset A of P to an antichain via a function which is M-generated by $\mathcal{G}$. Again, Lemma 10 yields $\mathcal{G} = \mathrm{Sym}_P$. □

Lemma 24. *Let* $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ *be a closed group, and let* $c_1, \ldots, c_n \in P$*. Let* $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ *be a canonical function M-generated by* $\mathcal{G}$*. Then* g *behaves like* id *on all infinite orbits of* $(\mathbb{P}, c_1, \ldots, c_n)$*, or it behaves like* $\updownarrow$ *on all infinite orbits of* $(\mathbb{P}, c_1, \ldots, c_n)$*, or else* $\mathcal{G} = \mathrm{Sym}_P$*.*

Proof. By Lemma 22, we may assume that g behaves like id or $\updownarrow$ on all infinite orbits. Suppose that the behaviour of g is not the same on all infinite orbits. Consider the graph H on the infinite orbits of $(\mathbb{P}, c_1, \ldots, c_n)$ in which two orbits X, Y are adjacent if and only if $X \div Y$ holds. We claim that H is connected. To see this, let X, Y be infinite orbits with $X < Y$. Pick $x, x' \in X$ and $y, y' \in Y$ such that $x < x'$ and $y' < y$. By the extension property, there exists $z \in P$ such that $x < z$, $z \perp x'$, $z \perp y'$, and $z < y$. Let Z be the orbit of z in $(\mathbb{P}, c_1, \ldots, c_n)$. Then $X \div Z$ and $Z \div Y$, and so there is a path from X to Y in H. Now if X, Y are infinite orbits which are incomparable, then there exists an infinite orbit Z with $X < Z$ and $Y < Z$, and so again there is a path from X to Y in H.

Since H is connected, there exist infinite orbits X, Y with $X \div Y$ such that g behaves like id on X and like $\updownarrow$ on Y. Assume that $X \leq Y$; the proof of the case $Y \leq X$ is dual. By Lemma 23, we may furthermore assume that $g[X] \div g[Y]$, or else we are done. This leaves us with two possibilities, $g[X] \leq g[Y]$ or $g[Y] \leq g[X]$.

The first case $g[X] \leq g[Y]$ splits into two subcases:

- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) < g(y)$ and $x \perp y$ implies $g(x) \perp g(y)$;
- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) \perp g(y)$ and $x \perp y$ implies $g(x) < g(y)$.

Let $x, x' \in X$ and $y, y' \in Y$ be so that $x < x'$, $x < y'$, $x' < y$, $y' < y$, and $x' \perp y'$. Then in the first subcase we can derive $g(x') < g(y)$, $g(y) < g(y')$, and $g(x') \perp g(y')$, a contradiction. In the second subcase, $g(x) < g(x')$, $g(x') < g(y')$, and $g(x) \perp g(y')$, again a contradiction.

In the second case $g[Y] \geq g[X]$ we have the following possibilities:

- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) > g(y)$ and $x \perp y$ implies $g(x) \perp g(y)$;
- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) \perp g(y)$ and $x \perp y$ implies $g(x) > g(y)$.

Let $x, x' \in X$ and $y, y' \in Y$ be as before. Then in the first subcase we can derive $g(x) < g(x')$, $g(y') < g(x)$, and $g(x') \perp g(y')$, a contradiction. In the second subcase, $g(y) < g(y')$, $g(y') < g(x')$, and $g(y) \perp g(x')$, again a contradiction. $\square$

3.4.3. *Behaviours generating* Rev*.*

Lemma 25. *Let* $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ *be a closed group, and let* $c_1, \ldots, c_n \in P$*. Let* $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ *be a canonical function M-generated by* $\mathcal{G}$*. If* g *behaves like* $\updownarrow$ *on some infinite orbit of* $(\mathbb{P}, c_1, \ldots, c_n)$*, then* $\mathcal{G} \supseteq \mathrm{Rev}$*.*

Proof. Let X be the infinite orbit. Pick an isomorphism $i : (P; \leq) \to (X; \leq)$. Then given any finite $A \subseteq P$, there exists $\alpha \in \mathrm{Aut}(\mathbb{P})$ such that $\alpha \circ g \circ i$ agrees with $\updownarrow$ on A. Since

g and i are generated by $\mathcal{G}$, there exists $\beta \in \mathcal{G}$ such that β agrees with $\updownarrow$ on A. Hence, $\updownarrow \in \mathcal{G}$. $\square$

3.4.4. *Behaviours generating* Turn.

Lemma 26. *Let $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ be a closed group, and let $c_1, \ldots, c_n \in P$. Let $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ be a canonical function M-generated by $\mathcal{G}$ which behaves like* id *on all of its orbits. Then g behaves like* id *between all infinite orbits of $(\mathbb{P}, c_1, \ldots, c_n)$, or else $\mathcal{G} \supseteq$* Turn.

Proof. Let infinite orbits X, Y be given.

We start with the case $X \div Y$. Say without loss of generality $X \leq Y$. By Lemma 23, we may assume that $g[X] \div g[Y]$, or else $\mathcal{G} = \mathrm{Sym}_P$. Hence $g[X] \leq g[Y]$ or $g[Y] \leq g[X]$. If $g[X] \leq g[Y]$, then either g behaves like id between X and Y and we are done, or $x < y \to g(x) \perp g(y)$ and $x \perp y \to g(x) < g(y)$ hold for all $x \in X$, $y \in Y$; the latter, however, is impossible, as for $x, x' \in X$ and $y \in Y$ with $x < x'$, $x < y$, and $x' \perp y$ we would have $g(x) < g(x') < g(y)$ and $g(x) \perp g(y)$. Now suppose $g[Y] \leq g[X]$. Then we have one of the following:

- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) > g(y)$ and $x \perp y$ implies $g(x) \perp g(y)$;
- For all $x \in X$, $y \in Y$, $x < y$ implies $g(x) \perp g(y)$ and $x \perp y$ implies $g(x) > g(y)$.

The first case is absurd since picking x, x', y as above yields $g(x) < g(x')$, $g(x) > g(y)$, and $g(x') \perp g(y)$. We claim that in the second case $\mathcal{G}$ contains $\circlearrowright$. Let $F \subseteq P$ be any random filter. Let $A \subseteq P$ be finite, and set $A_2 := A \cap F$, and $A_1 := A \setminus A_2$. Then there exists an automorphism α of $\mathbb{P}$ which sends A_2 into Y and A_1 into X. The composite $g \circ \alpha$ behaves like $\circlearrowright_F$ on A for what concerns comparabilities and incomparabilities, and hence there exists $\beta \in \mathrm{Aut}(\mathbb{P})$ such that $\beta \circ g \circ \alpha$ agrees with $\circlearrowright_F$ on A. By topological closure we infer $\circlearrowright_F \in \mathcal{G}$.

Now consider the case where X, Y are strictly comparable, say $X < Y$. Then we know from the proof of Lemma 24 that there exists an infinite orbit Z such that $X \leq Z \leq Y$, $X \div Z$ and $Z \div Y$. Let $x \in X$ and $y \in Y$ be arbitrary. There exists $z \in Z$ such that $x < z < y$. As g behaves like id between X and Z and between Z and Y, we have that $g(x) < g(z) < g(y)$, and hence g behaves like id between X and Y.

It remains to discuss the case $X \perp Y$. Suppose that $g[X]$ and $g[Y]$ are comparable, say $g[X] < g[Y]$. Then given any finite $A \subseteq P$ with incomparable elements x, y, using the extension property we can find $\alpha \in \mathrm{Aut}(\mathbb{P})$ which sends x into X, all elements of A which are incomparable with x into Y, and all other elements of A into infinite orbits which are comparable with both X and Y. Applying $g \circ \alpha$ then increases the number of comparabilities on A, and hence repeated applications of such functions will send A onto a chain, proving $\mathcal{G} = \mathrm{Sym}_P$. □

Lemma 27. *Let* $\mathcal{G} \supseteq \mathrm{Aut}(\mathbb{P})$ *be a closed group, and let* $c_1, \ldots, c_n \in P$. *Let* $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ *be a canonical function M-generated by* $\mathcal{G}$ *which behaves like* id *on all of its orbits. Then* g *behaves like* id *between all orbits of* $(\mathbb{P}, c_1, \ldots, c_n)$ *(including the finite ones), and hence is M-generated by* $\mathrm{Aut}(\mathbb{P})$, *or else* $\mathcal{G} \supseteq \mathrm{Turn}$.

Proof. Let $1 \leq i \leq n$, and let X be an infinite orbit which is incomparable with $\{c_i\}$. Suppose that $g[X]$ and $\{g(c_i)\}$ are strictly comparable, say $\{g(c_i)\} < g[X]$. Let Y be an infinite orbit such that $X \leq Y$, $X \div Y$, and $\{c_i\} < Y$. Let moreover Z be an infinite orbit such that $Z < \{c_i\}$, $Z \leq X$ and $Z \div X$. Then by the preceding lemma, we may assume that g behaves like id between X, Y and Z. We cannot have $g[Z] < \{g(c_i)\}$ as this would imply $g[Z] < g[X]$, contradicting the fact that g behaves like id between Z and X. Suppose that $g[Z] \perp \{g(c_i)\}$. Set $S := Z \cup X \cup Y \cup \{c_i\}$. Then it is easy to see that $(S; \leq)$ satisfies the extension property, and hence is isomorphic which $\mathbb{P}$; fix an isomorphism $i : (P; \leq, c_i) \to (S; \leq, c_i)$. This isomorphism is M-generated by $\mathrm{Aut}(\mathbb{P})$ since it can be approximated by automorphisms of $\mathbb{P}$ on all finite subsets of P. The restriction of g to S is canonical as a function from $(S; \leq, c_i)$ to $\mathbb{P}$. Hence, the function $h := g \circ i$ is canonical as a function from $(\mathbb{P}, c_i)$ to $\mathbb{P}$, and has the same behaviour as the restriction of g to S. Let $\alpha \in \mathrm{Aut}(\mathbb{P})$ be so that $\alpha(h(c_i)) = c_i$. Then $t := h \circ \alpha \circ h$ has the property that $t(x) > t(c_i)$ for all $x \neq c_i$, and that $t(x) \perp t(y)$ if and only if $x \perp y$, for all $x, y \in P \setminus \{c_i\}$. Hence, given any finite $A \subseteq P$ which is not a chain, we can pick $x \in A$ which is not comparable to all other elements of A, and find $\beta \in \mathrm{Aut}(\mathbb{P})$ which sends x to c_i; then $t \circ \beta$ strictly increases the number of comparabilities among the elements of A. Repeating this process and composing the functions, we find a function which is M-generated by $\mathcal{G}$ and which maps A onto a chain. Hence, $\mathcal{G} = \mathrm{Sym}_P$.

Therefore, we may henceforth assume that g behaves like id between all $\{c_i\}$ and all infinite orbits X with $\{c_i\} \perp X$. Now suppose that there exists $1 \leq i \leq n$ and an infinite orbit X with $X < \{c_i\}$ such that $\{g(c_i)\} < g[X]$. Pick an infinite orbit Y which is incomparable with c_i, and which satisfies $X \leq Y$. Then $\{g(c_i)\} < g[Y]$ since g behaves like id between X and Y, a contradiction. Next suppose there exists $1 \leq i \leq n$ and an infinite orbit X with $X < \{c_i\}$ such that $\{g(c_i)\} \perp g[X]$. Then pick an infinite orbit Y as in the preceding case, and an infinite orbit Z with $\{c_i\} < Z$. Now given any finite $A \subseteq P$ which does not induce an antichain, we can pick $y \in A$ which is not minimal in A. Taking $\alpha \in \mathrm{Aut}(\mathbb{P})$ which sends y to c_i and A into $X \cup Y \cup Z \cup \{c_i\}$, we then have that application of $g \circ \alpha$ increases the number of incomparabilites of A. Repeated composition of such functions yields a function which sends A onto an antichain. Hence, $\mathcal{G} = \mathrm{Sym}_P$. The case where there exist $1 \le i \le n$ and an infinite orbit X with $\{c_i\} < X$ such that $\{g(c_i)\} \perp g[X]$ is dual.

We turn to the case where we have two distinct finite orbits $\{c_i\}$ and $\{c_j\}$. Suppose first that they are comparable, say $c_i < c_j$. Picking an infinite orbit Z with $\{c_i\} < Z < \{c_j\}$ then yields, by what we know already, $\{g(c_i)\} < g[Z] < \{g(c_j)\}$, so we are done. Finally, suppose that $c_i \perp c_j$. Then given any finite $A \subseteq P$ which has incomparable elements x, y, we can send x to c_i, y to c_j, and the rest of A to infinite orbits via some $\alpha \in \mathrm{Aut}(\mathbb{P})$. But then application of $g \circ \alpha$ increases the number comparabilities on A, and hence repeating the process yields a function which sends A to a chain. Hence, $\mathcal{G} = \mathrm{Sym}_P$. □

3.4.5. *Climbing up the group lattice.*

Proposition 28. *Let $\mathcal{G} \supsetneq \mathrm{Aut}(\mathbb{P})$ be a closed group. Then $\mathcal{G}$ contains either* Rev *or* Turn.

Proof. There exist $\pi \in \mathcal{G} \setminus \mathrm{Aut}(\mathbb{P})$ and elements $u, v \in P$ such that $u \le v$ and $\pi(u) \not\le \pi(v)$. Let $g : (\mathbb{P}, u, v) \to \mathbb{P}$ be a canonical function M-generated by $\mathcal{G}$ which agrees with π on $\{u, v\}$. If g behaves like $\updownarrow$ on some infinite orbit of $(\mathbb{P}, u, v)$, then $\mathcal{G} \supseteq \mathrm{Rev}$ by Lemma 25. Otherwise Lemma 27 states that g is generated by $\mathrm{Aut}(\mathbb{P})$ or $\mathcal{G} \supseteq \mathrm{Turn}$. Since $g(u) \not\le g(v)$, only the latter possibility can be the case. □

Proposition 29. *Let $\mathcal{G} \supsetneq$ Rev *be a closed group. Then $\mathcal{G}$ contains* Turn.

Proof. Let $\pi \in \mathcal{G} \setminus \mathrm{Rev}$. Then there exists a finite tuple $c = (c_1, \ldots, c_n)$ of elements of P such that no function in Rev agrees with π on c. Let $g : (\mathbb{P}, c_1, \ldots, c_n) \to \mathbb{P}$ be a canonical function which is M-generated by $\mathcal{G}$ and which agrees with π on $\{c_1, \ldots, c_n\}$. By Lemma 24, we may assume that either g behaves like id on all infinite orbits, or it behaves like $\updownarrow$ on all infinite orbits of $(\mathbb{P}, c_1, \ldots, c_n)$. By composing g with $\updownarrow$, we may assume that it behaves like id on all infinite orbits. But then Lemma 27 implies that $\mathcal{G} \supseteq \mathrm{Turn}$, or that g is M-generated by $\mathrm{Aut}(\mathbb{P})$. The latter is, of course, impossible. □

3.4.6. *Relational descriptions of* Turn *and* Max. Before climbing up further, we need to describe the groups Turn and Max relationally. The componentwise action of the group Turn on triples of distinct elements of P has three orbits, namely:

Par: the orbit of the 3-element antichain, i.e., the set of all tuples $(a, b, c) \in P^3$ such that one of the following holds: $a \perp b, b \perp c, c \perp a$;

$a < b, a < c, b \perp c$; $b < a, b < c, a \perp c$; $c < a, c < b, b \perp c$;

$a > b, a > c, b \perp c$; $b > a, b > c, a \perp c$; $c > a, c > b, b \perp c$;

cycl: the orbit of the 3-element chain $a < b < c$, i.e., the set of all $(a, b, c) \in P^3$ such that one of the following holds:

$a < b < c$; $b < c < a$; $c < a < b$;

$a < b, c \perp a, c \perp b$; $b < c, a \perp b, a \perp c$; $c < a, b \perp a, b \perp c$;

cycl′: the dual of cycl; that is, the orbit of the chain $a > b > c$, or more precisely the set of all $(a, b, c) \in P^3$ such that one of the following holds:

$a > b > c$; $b > c > a$; $c > a > b$;

$a > b, c \perp a, c \perp b$; $b > c, a \perp b, a \perp c$; $c > a, b \perp a, b \perp c$.

Definition 30. *Let $\{X, Y, Z\}$ be a partition of P into disjoint subsets such that X is an ideal of $\mathbb{P}$, Z is a filter of $\mathbb{P}$, $X \leq Y$, $Y \leq Z$ and $X < Z$. A* rotation *on $\mathbb{P}$ with respect to X, Y, Z is any permutation f on P which behaves like* id *on each class of the partition, and such that for all $x \in X$, $y \in Y$, and $z \in Z$ we have*

- *$f(z) < f(x)$;*
- *$f(y) < f(x)$ iff $x \perp y$ and $f(y) \perp f(x)$ iff $x < y$;*
- *$f(z) < f(y)$ iff $y \perp z$ and $f(z) \perp f(y)$ iff $y < z$.*

Observe that if F is a random filter, then $\circlearrowright_F$ is a rotation with respect to the partition $\{\emptyset, P \setminus F, F\}$.

Proposition 31. Turn *contains all rotations on $\mathbb{P}$.*

Proof. Let f be a rotation on $\mathbb{P}$, let $\{X, Y, Z\}$ be the corresponding partition, and let $S \subseteq P$ be finite. Set $X' := X \cap S$, $Y' := Y \cap S$, and $Z' := Z \cap S$. Let $F \subseteq P$ be a random filter with $F \supseteq Z'$ and $P \setminus F \supseteq X' \cup Y'$. Since $\circlearrowright_F(u) \not< \circlearrowright_F(z)$ for all $u \in X' \cup Y'$ and all $z \in Z'$, there exists a random filter F' with $F' \supseteq \circlearrowright_F[X' \cup Y']$ and $P \setminus F' \supseteq \circlearrowright_F[Z']$. It is a straightforward verification that $\circlearrowright_{F'} \circ \circlearrowright_F$ changes the relations between elements of $X' \cup Y' \cup Z'$ in the very same way as the rotation f, and hence there exists an automorphism α of $\mathbb{P}$ such that $\alpha \circ \circlearrowright_{F'} \circ \circlearrowright_F$ agrees with f on $X' \cup Y' \cup Z'$. □

Lemma 32. $\mathrm{Turn} = \mathrm{Aut}(P; \mathrm{Par}, \mathrm{cycl}, \mathrm{cycl}')$.

Proof. To show that $\circlearrowright$ preserves Par, cycl and cycl′ is only a matter of verification of a finite number of cases. For the converse, let $f \in \mathrm{Aut}(P; \mathrm{Par}, \mathrm{cycl}, \mathrm{cycl}')$; we show it is a rotation. Define a binary relation $\sim$ on P by setting $x \sim y$ if and only if (x, y) and $(f(x), f(y))$ have the same type in $\mathbb{P}$, for all $x, y \in P$. Clearly, $\sim$ is reflexive and symmetric; we claim it is transitive, and hence an equivalence relation. To this end, let $x, y, z \in P$ such that $x \sim y$ and $y \sim z$. Now by going through all possible relations that might hold between x, y, z, using the fact that these relations remain unaltered between x and y as well as between y and z, and taking into account the fact that (x, y, z) in Par (cycl, cycl′) implies $(f(x), f(y), f(z))$ in Par (cycl, cycl′), one checks that the relation which holds between x and z has to remain unchanged as well – this is a finite case analysis which we leave to the reader.

If $\sim$ has only one equivalence class, then f it is an automorphism of $\mathbb{P}$ and there is nothing to show, so assume henceforth that this is not the case. Then there exist equivalence classes X, Y and $x \in X$, $y \in Y$ such that $x \perp y$; we may assume without loss of generality that $f(x) > f(y)$.

Let $u, v \in X \cup Y$ such that $u < v$, and suppose that $f(v) < f(u)$. Pick $r \in P$ incomparable with u, v, x, y. Then $(r, x, y) \in$ Par, so $(f(r), f(x), f(y)) \in$ Par. Consequently, $f(r) \perp f(x)$ or $f(r) \perp f(y)$, and hence $r \in X \cup Y$. Now observe that (u, v, r), and hence also its image under f, is an element of cycl. Hence $f(v) < f(u)$ yields $f(v) < f(r) < f(u)$, contradicting $r \in X \cup Y$. We conclude that comparable elements of $X \cup Y$ either belong to the same class, or they are sent to incomparable elements.

Pick any $u \in P$ such that $u < x$ and $u \perp y$. Then $(f(u), f(x), f(y)) \in$ cycl and $f(x) > f(y)$ imply $f(u) < f(x)$, and so $u \in X$. Similarly, any $v \in P$ such that $y < v$ and $v \perp x$ is an element of Y, and in particular $X \leq Y$.

We next claim that $Y \nleq X$. Suppose there exist $u \in Y$, $v \in X$ with $u < v$. If $u > x$, then $(x, u, v) \in$ cycl, and so $f(x) < f(v)$ and the fact that we cannot have $f(x) < f(u)$ yield a contradiction. Hence, $u \not> x$, and by symmetry $v \not< y$. Suppose $v > y$. If $v > x$, then $(y, x, v) \in$ Par, but $f(y) < f(x) < f(v)$, a contradiction. By the preceding paragraph, $v \perp x$ would imply $v \in Y$; so $v \perp y$, and by symmetry $u \perp x$. If $u < y$ and $x < v$, then $f(u) < f(y) < f(x) < f(v)$, contradicting the fact that u and v are elements of different classes. So assume without loss of generality that $u \not< y$; since $u > y$ would imply $v > y$, which we already excluded, we then have $u \perp y$. Since $(x, u, y) \in$ Par and $f(x) > f(y)$, we conclude $f(u) < f(x)$. Hence, if $v > x$, then $f(u) < f(x) < f(v)$, a contradiction, so we must have $v \perp x$. But then $(u, v, x) \in$ cycl, $f(u) < f(x)$, and the fact that $f(v)$ is incomparable with $f(u)$ and $f(x)$ yield the final contradiction.

Suppose there exist $u \in X$ and $y \in Y$ with $u \perp v$ and such that $f(u) < f(v)$. As above, we could then conclude that $X \nleq Y$, a contradiction.

Say that A, B are equivalence classes for which $A < B$. Picking $a \in A$, $b \in B$, and any $c \in P$ which is incomparable with a and b, we then have $(a, b, c) \in$ cycl. We cannot have $c \in A \cup B$, and so $f(c)$ must be comparable with $f(a)$ and $f(b)$. The only possibility then is that $f(b) < f(a)$.

Let Z be an equivalence class distinct from X, Y and such that $Y \leq Z$. Then $X \leq Z$. We claim that $Z > Y$ is impossible. Otherwise, there exist $x \in X$, $y \in Y$, and

$z \in Z$ such that $x < y < z$, and so $(x, y, z) \in \text{cycl}$. But $f(x) \perp f(y)$ and $f(z) < f(y)$ imply $(f(x), f(y), f(z)) \notin \text{cycl}$, a contradiction. We next claim that $X < Z$. Otherwise, pick $x \in X$ and $z \in Z$ with $x \perp z$, and an arbitrary $y \in Y$ such that $x < y$. Then $(f(x), f(y), f(z)) \in \text{cycl}$, $f(x) > f(z)$ and $f(x) \perp f(y)$ yield a contradiction. Suppose next that there exist two distinct classes Z_1, Z_2 with $Y \leq Z_1, Z_2$. We know that Z_1, Z_2 must be comparable, say $Z_1 \leq Z_2$. Pick $z_1 \in Z_1$, $z_2 \in Z_2$ with $z_1 < z_2$. Since $X < Z_1, Z_2$, we then have $f(x) > f(z_1), f(z_2)$, and $f(z_1) \not< f(z_2)$ yields a contradiction. So there is at most one class Z distinct from Y with $Z \geq Y$, and it satisfies $Z > X$ and $Z \not> Y$.

Similarly there is at most one class W distinct from X with $W \leq X$, and it satisfies $W < Y$ and $W \not< X$. By the same kind of argument that yielded uniqueness of Z above, W and Z cannot exist simultaneously, say that W does not. Let U be any other class distinct from X and Y. Then $X \leq U \leq Y$, and so $X < Y$, a contradiction.

If Z does not exist, then Y is a filter and f is of the form $\circlearrowright_Y$. If Z does exist, then f is a rotation with respect to the partition $\{X, Y, Z\}$. $\square$

Corollary 33. *The group* Turn *consists precisely of the rotations on* $\mathbb{P}$. *In particular, the composition of two rotations is again a rotation.*

Proof. By Lemma 32, if $f \in \text{Turn}$, then $f \in \text{Aut}(P; \text{Par}, \text{cycl}, \text{cycl}')$. It then follows from the proof of the other direction of same lemma that f is a rotation. $\square$

Proposition 34. $\text{Turn} = \text{Aut}(P; \text{cycl})$.

Proof. By Lemma 32, $\text{Turn} \subseteq \text{Aut}(P; \text{cycl})$. If the two groups were not equal, then $\text{Aut}(P; \text{cycl})$ would contain a function f which sends a triple $a = (a_1, a_2, a_3)$ in Par to a triple in cycl'. Moreover, by first applying a function in Turn, we could assume that a induces an antichain in $\mathbb{P}$. But then for any automorphism α of $\mathbb{P}$ sending a to (a_3, a_2, a_1) we would get that $f \circ \alpha$ sends a to a triple in cycl, a contradiction. $\square$

Lemma 35. *Let* $f \in \text{Aut}(P; \text{Par}) \setminus \text{Turn}$. *Then for all* $a \in P^3$ *we have* $a \in \text{cycl}$ *if and only if* $f(a) \in \text{cycl}'$, *i.e.,* f *switches* cycl *and* cycl'.

Proof. Suppose there exists $a = (a_1, a_2, a_3) \in \text{cycl}$ with $f(a) \in \text{cycl}$ – we will derive a contradiction, implying $f(a) \in \text{cycl}'$. By symmetry, it then follows that all tuples in cycl' are sent to cycl, and we are done.

Since $f \in \text{Aut}(P; \text{Par}) \setminus \text{Turn}$, there exists $b = (b_1, b_2, b_3)$ in cycl such that $f(b) \in \text{cycl}'$. We first claim that by replacing a and b with adequate triples, we may assume that both a and b are strictly ascending, i.e., $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$. Otherwise, either all strictly ascending triples are sent to cycl, or all strictly ascending triples are sent to cycl$'$. Assume without loss of generality the former. Let $g \in \mathrm{Turn}$ be so that it sends some strictly ascending triple $e \in P^3$ to b. Then $f \circ g$ sends e to $f(b) \in \mathrm{cycl}'$; on the other hand, since g is a rotation by Corollary 33, it sends some other strictly ascending triple $w \in P^3$ onto a strictly ascending triple, and so $f \circ g(w) \in \mathrm{cycl}$. Thus by replacing f by $f \circ g$, a by w and b by e, we may indeed henceforth assume that both a and b are strictly ascending triples.

Now let $c = (c_1, c_2, c_3)$ be a strictly ascending triple such that $a_i < c_j$ and $b_i < c_j$ for all $1 \leq i, j \leq 3$. If $f(c) \in \mathrm{cycl}$, then we replace a by c, and otherwise we replace b by c. Assume without loss of generality the former; hence, from now on we assume $b_1 < b_2 < b_3 < a_1 < a_2 < a_3$, $f(b) \in \mathrm{cycl}'$, and $f(a) \in \mathrm{cycl}$. By replacing f by $h \circ f$ for an appropriate function $h \in \mathrm{Turn}$ we may moreover assume that $f(a_i) = a_i$ for all $1 \leq i \leq 3$.

Suppose that $f(b_i) \perp a_j$ for some $1 \leq i, j \leq 3$. Then, for any $1 \leq k \leq 3$ with $k \neq j$, the fact that $(b_i, a_j, a_k) \notin \mathrm{Par}$ implies $(f(b_i), a_j, a_k) \notin \mathrm{Par}$, and consequently $f(b_i) \perp a_k$. Hence, if $f(b_i)$ is incomparable with some a_j, then it is incomparable with all a_j, and if it is comparable with some a_j, then it is comparable with all a_j. Suppose that $f(b_i) \perp a_1$ for some $1 \leq i \leq 3$, and consider $f(b_j)$, where $j \neq i$. Since $(b_i, b_j, a_1) \notin \mathrm{Par}$, we have $(f(b_i), f(b_j), a_1) \notin \mathrm{Par}$. This implies that if $f(b_j) \perp f(b_i)$, then $f(b_j)$ and a_1 are comparable. Putting this information together, we conclude that any two distinct elements $f(b_i), f(b_j)$ which are incomparable with the a_k are mutually comparable. Thus, the image of $S := \{a_1, a_2, a_2, b_1, b_2, b_3\}$ under f is the disjoint union of at most two chains; by applying $\circlearrowright_F$ for an appropriate random filter $F \subseteq P$, we may assume its image is a single chain. By the same argument, we may assume that a_3 is the largest element of this chain.

Since $f(b) \notin \mathrm{cycl}$, there exists b_i, b_j with $b_i < b_j$ such that $f(b_j) < f(b_i)$. As in the following, we will not make use of the third element of b anymore, we may assume that this is the case for b_1, b_2. Then either $f(b_2) < f(b_1) < a_2 < a_3$, or $a_1 < a_2 < f(b_2) < f(b_1)$, or $f(b_2) < a_2 < f(b_1) < a_3$. We will derive a contradiction from each of the three cases.

Pick any $u_1, u_2, u_3, u_4 \in P$ such that $u_1 < u_2$, $u_3 < u_4$, and such that any other two elements u_i, u_j are incomparable. Then there is an random filter $F \subseteq P$ containing u_1, u_2 but not u_3, u_4, and so $\circlearrowright_F(u_1) < \circlearrowright_F(u_2) < \circlearrowright_F(u_3) < \circlearrowright_F(u_4)$. Now if $f(b_2) < f(b_1) < a_2 < a_3$, then by applying an automorphism of $\mathbb{P}$, we may assume that (b_1, b_2, a_2, a_3) coincides with the ascending 4-tuple t containing the $\circlearrowright_F(u_i)$. Picking an random filter $F' \subseteq P$ containing a_2 but not $f(b_1)$ and setting $h := \circlearrowright_{F'} \circ f \circ \circlearrowright_F$, we get that $h(u_2) < h(u_1)$, $h(u_3) < h(u_4)$, and all other $h(u_i), h(u_j)$ are incomparable. Pick any $x \in P$ such that $x > u_1$, $x > u_3$, $x \perp u_3$, and $x \perp u_4$. Then $(u_4, u_3, x) \in \mathrm{Par}$ implies that $h(x) > h(u_3)$, $(x, u_1, u_2) \in \mathrm{Par}$ implies $h(x) < h(u_1)$, and hence $h(u_3) < h(u_1)$, a contradiction. If $a_1 < a_2 < f(b_2) < f(b_1)$, then by applying an automorphism of $\mathbb{P}$, we may assume that

(b_1, b_2, a_1, a_2) coincides with the tuple t. Picking an random filter $F' \subseteq P$ containing $f(b_2)$ but not a_2 and setting $h := \circlearrowright_{F'} \circ f \circ \circlearrowright_F$, we get that $h(u_2) < h(u_1)$, $h(u_3) < h(u_4)$, and all other $h(u_i), h(u_j)$ are incomparable, leading to the same contradiction as in the preceding case. Finally, assume $f(b_2) < a_2 < f(b_1) < a_3$, and assume that (b_1, b_2, a_2, a_3) coincides with t. Picking an random filter $F' \subseteq P$ containing $f(b_1)$ but not a_2 and setting $h := \circlearrowright_{F'} \circ f \circ \circlearrowright_F$, we get that $h(u_2) < h(u_3)$, $h(u_1) < h(u_4)$, and all other $h(u_i), h(u_j)$ are incomparable. Now pick $x \in P$ such that $x > u_i$ for all $1 \leq i \leq 4$. Then $(u_1, u_2, x) \notin \mathrm{Par}$ implies that $h(x) \perp h(u_1)$ or $h(x) \perp h(u_2)$. However, $(u_2, u_4, x) \in \mathrm{Par}$ implies that $h(x)$ is comparable with $h(u_2)$, and similarly $(u_1, u_3, x) \in \mathrm{Par}$ implies that $h(x)$ is comparable with $h(u_1)$, a contradiction.

□

Proposition 36. $\mathrm{Max} = \mathrm{Aut}(P; \mathrm{Par})$.

Proof. By Lemma 32, Turn is contained in $\mathrm{Aut}(P; \mathrm{Par})$. Obviously, $\updownarrow$ preserves Par, so that indeed $\mathrm{Max} \subseteq \mathrm{Aut}(P; \mathrm{Par})$.

For the other direction, let $f \in \mathrm{Aut}(P; \mathrm{Par})$. If $f \in \mathrm{Turn}$ then $f \in \mathrm{Max}$ by definition of Max, so assume $f \notin \mathrm{Turn}$. Then f switches cycl and cycl$'$ by Lemma 35. Since $\updownarrow$ switches cycl and cycl$'$ as well, $\updownarrow \circ f$ preserves Par, cycl and cycl$'$. Thus, by Lemma 32, $\updownarrow \circ f$ is an element of Turn, and so $f \in \mathrm{Max}$. □

3.4.7. *Climbing to the top.*

Proposition 37. *Let $\mathcal{G} \supsetneq \mathrm{Max}$ be a closed group. Then $\mathcal{G}$ is 3-transitive.*

Proof. Since $\mathcal{G}$ is not contained in Max, Par cannot be an orbit of its componentwise action on P^3. Since it contains Max, the orbits of this action are unions of the orbits of the corresponding action of Max. However, the latter action has only two orbits of triples of distinct elements, namely Par and $\mathrm{cycl} \cup \mathrm{cycl}'$. Hence, $\mathcal{G}$ has only one such orbit, and is 3-transitive. □

Proposition 38. *Let $\mathcal{G}$ be a* 3*-transitive closed group containing* Turn*. Then* $\mathcal{G} = \mathrm{Sym}_P$.

Proof. We prove by induction that $\mathcal{G}$ is n-transitive for all $n \geq 3$. Our claim holds for $n = 3$ by assumption. So let $n \geq 4$ and assume that $\mathcal{G}$ is $(n-1)$-transitive. We claim that every n-element subset of P can be mapped onto an antichain by a permutation in $\mathcal{G}$; n-transitivity then follows as in the proof of Lemma 10. We prove this claim in several steps, and will need the following partial orders.

For every natural number k with $1 \leq k \leq n$, let

- S_n^k be the n-element poset consisting of k independent points and a chain of $(n-k)$ elements below them;
- T_n^k be the dual of S_n^k;
- A_n^k be the n-element poset consisting of k independent points, an element below them, and an antichain of size $(n-k-1)$ independent from these points;
- B_n^k be the dual of A_n^k;
- C_k be the $k+1$-element poset consisting of k independent points and an element below them; that is, $C_k = A_{k+1}^k = S_{k+1}^k$.

Step 1: From anything to A_n^k or B_n^k for $k \geq \frac{n-1}{2}$.

We first show that any n-element set $A \subseteq P$ can me mapped to a copy of A_n^k or B_n^k, where $k \geq \frac{n-1}{2}$, via a function in $\mathcal{G}$. Let A be given, and write $A = A' \cup \{a\}$, where A' has $n-1$ elements. Then by the induction hypothesis there exists $\pi \in \mathcal{G}$ which maps A' to an antichain. Let $F \subseteq P$ be an random filter which separates $\pi(a)$ from $\pi[A']$, i.e., for all $b \in \pi[A']$ we have $b \in F$ if and only if $\pi(a) \notin F$. Then one can check that either $\pi[A]$ or $(\circlearrowright_F \circ \pi)[A]$ induce A_n^k or B_n^k in $\mathbb{P}$ for some $k \geq \frac{n-1}{2}$.

Step 2: From A_n^k (B_n^k) to S_n^k (T_n^k) for $k \geq \frac{n-1}{2}$.

We now show that any copy of A_n^k in $\mathbb{P}$ can be mapped to a copy of S_n^k via a function in $\mathcal{G}$. The dual proof then shows that any copy of B_n^k can be mapped to a copy of T_n^k.

Let $\{x_1, \ldots, x_{n-1}\}$ and $\{y_1, \ldots, y_{n-1}\}$ be disjoint subsets of P inducing an antichain and a chain, respectively. By the $(n-1)$-transitivity of $\mathcal{G}$, the map $x_i \mapsto y_i$, $1 \leq i \leq n-1$, can be extended to a permutation $\pi \in \mathcal{G}$. Let X be the orbit of $(\mathbb{P}, x_1, \ldots, x_{n-1})$ such that $x \perp x_i$ for all $x \in X$ and all $1 \leq i \leq n-1$. By Lemma 17 there exists a canonical function $g : (\mathbb{P}, x_1, \ldots, x_{n-1}) \to (\mathbb{P}, y_1, \ldots, y_{n-1})$ M-generated by $\mathcal{G}$ that agrees with π on $\{x_1, \ldots, x_{n-1}\}$. We may assume that g behaves like id or like $\updownarrow$ on X, by Lemma 22. If g behaves like $\updownarrow$ on X, then $\mathcal{G}$ contains $\updownarrow$ by Lemma 25; replacing g by $\updownarrow \circ g$ and replacing each y_i by $\updownarrow(y_i)$, we may assume that g behaves like id on X. Let $D \subseteq X$ be so that it induces C_k, and observe that $D' := D \cup \{x_1, \ldots, x_{n-k-1}\}$ induces a copy of A_n^k in $\mathbb{P}$. Since g is canonical, all elements of X, and in particular all elements of D are sent to the same orbit Y of $(\mathbb{P}, y_1, \ldots, y_{n-1})$. Thus for all $1 \leq i \leq n-1$ we have that either $g[D] < \{y_i\}$, or $g[D] \perp \{y_i\}$, or $g[D] > \{y_i\}$. Let S be the set of those y_i for which the first relation holds, and set $E := g[D] \cup (\{y_1, \ldots, y_{n-1}\} \setminus S)$. Let $F \subseteq P$ be an random filter which separates E from S, i.e., F contains S, but does not intersect E. Then $\circlearrowright_F[S] \perp \circlearrowright_F[E]$. Choose an random filter F' which contains $\circlearrowright_F[S]$ and which does not intersect $\circlearrowright_F[E]$. Then $\circlearrowright_{F'} \circ \circlearrowright_F[S] < \circlearrowright_{F'} \circ \circlearrowright_F[E]$. Set $h := \circlearrowright'_F \circ \circlearrowright_F \circ g$. Now for all $1 \leq i \leq n-1$ we have that either $h[D] > \{h(x_i)\}$ or $h[D] \perp \{h(x_i)\}$. Moreover, h behaves like id on D, and the $h(x_i)$ form a chain. Either there are at least $\frac{n-1}{2}$ elements among the $h(x_i)$ for which $h[D] > \{h(x_i)\}$, or there are at least $\frac{n-1}{2}$ of the $h(x_i)$ for which $h[D]\perp\{h(x_i)\}$. In the first case, observe that $k \geq \frac{n-1}{2}$ implies $\frac{n-1}{2} \geq n-k-1$. Hence, by relabelling the x_i, we may assume that $h[D] > \{h(x_i)\}$ for $1 \leq n-k-1$, and so h sends D' to a copy of S_n^k, finishing the proof. In the second case, pick an random filter $F'' \subseteq P$ which contains all $h(x_i)$ for which $h[D]\perp\{h(x_i)\}$, and which does not contain any element from $h[D]$. Then replacing h by $\circlearrowleft_{F''} \circ h$ brings us back to the first case.

Step 3: From S_n^k (T_n^k) to an antichain when $k > \frac{n-1}{2}$.

We show that if $k > \frac{n-1}{2}$, then any copy of S_n^k in $\mathbb{P}$ can be mapped to an antichain by a permutation in $\mathcal{G}$. Clearly, the dual argument then shows the same for T_n^k. Let $\{u_1, \ldots, u_{n-1}\} \subseteq P$ be so that it induces a chain. By the $(n-1)$-transitivity of $\mathcal{G}$, there is some $\rho \in \mathcal{G}$ that maps $\{u_1, \ldots, u_{n-1}\}$ to an antichain $\{v_1, \ldots, v_{n-1}\}$. Let Z be the orbit of $(\mathbb{P}, u_1, \ldots, u_{n-1})$ that is above all the u_j. By Lemma 17 there exists a canonical function $f : (\mathbb{P}, u_1, \ldots, u_{n-1}) \to (\mathbb{P}, v_1, \ldots, v_{n-1})$ M-generated by $\mathcal{G}$ that agrees with ρ on $\{u_1, \ldots, u_{n-1}\}$. All elements of Z are mapped to one and the same orbit O of $(\mathbb{P}, v_1, \ldots, v_{n-1})$. Now pick $z_1, \ldots, z_k \in Z$ which induce an antichain. By applying an appropriate instance of $\circlearrowleft$ in a similar fashion as in Step 2, we may assume that O is incomparable with at least $\frac{n-1}{2}$ of the singletons $\{v_i\}$. Choose $(n-k)$ out of these v_i. This is possible, as $k > \frac{n-1}{2}$ and consequently $\frac{n-1}{2} \geq n-k$. By relabelling the u_i, we may assume that the chosen elements are $v_1, \ldots, v_{n-k}$. Then $f[\{z_1, \ldots, z_k\}] \cup \{v_1, \ldots, v_{n-k}\}$ is an antichain. Since $\{z_1, \ldots, z_k, u_1, \ldots, u_{n-k}\}$ induces a copy of S_n^k, we are done.

Step 4: From A_n^k to an antichain when $k = \frac{n-1}{2}$.

Assuming that $k = \frac{n-1}{2}$, we show that any copy of A_n^k in $\mathbb{P}$ can be mapped to an antichain by a function in $\mathcal{G}$. Note that this assumption implies that n is odd, so $n \geq 5$, and thus $k = \frac{n-1}{2} \geq 2$.

Let $\{x_1, \ldots, x_{k-1}\} \subseteq P$ induce an antichain. Let $s \in P$ be a point below all the x_i, and let $\{y_1, \ldots, y_k\} \subseteq P$ induce an antichain whose elements are incomparable with all the x_i and s. The set $A := \{s, x_1, \ldots, x_{k-1}, y_1, \ldots, y_k\}$ induces a copy of A_{n-1}^{k-1}. By the $(n-1)$-transitivity of $\mathcal{G}$ there exists $\varphi \in \mathcal{G}$ which maps A to an antichain $\{z_1, \ldots, z_{n-1}\} \subseteq P$. Without loss of generality, we write $\varphi(s) = z_{n-1}$, $\varphi(x_i) = z_i$ for $1 \leq i \leq k-1$, and $\varphi(y_i) = z_{k+i}$ for $1 \leq i \leq k$. By Lemma 17 there exists a canonical function $h : (\mathbb{P}, s, x_1, \ldots, x_{k-1}, y_1, \ldots, y_k) \to (\mathbb{P}, z_1, \ldots, z_{n-1})$ M-generated by $\mathcal{G}$ which agrees with φ on A. Let U be the orbit of $(\mathbb{P}, s, x_1, \ldots, x_{k-1}, y_1, \ldots, y_k)$ whose elements are larger than s and incomparable to all other elements of A. Since h is canonical, $h[U]$ is contained in an orbit V of $(\mathbb{P}, z_1, \ldots, z_{n-1})$.

Assume that the elements of the orbit V do not satisfy the same relations with all the z_i for $1 \leq i \leq n-2$. Then there is a partition $R \cup S = \{z_1, \ldots, z_{n-2}\}$, with both R and S non-empty, such that the elements of V are incomparable with the elements of R and strictly comparable with the elements of S. By applying an appropriate instance of $\circlearrowright$ we may assume that $|R| \geq k$. Pick any $R' \subseteq R$ of size k, any $S' \subseteq S$ of size 1, and a k-element antichain $W \subseteq U$. Then $h^{-1}[R'] \cup h^{-1}[S'] \cup W$ induces an antichain of size n whose image I under h induces either A_n^k or B_n^k. In the second case, let $F \subseteq P$ be an random filter which separates the largest element of I from its other elements. Then $\circlearrowright_F$ sends I to a copy of A_n^k. Thus in either case, $\mathcal{G}$ contains a function which sends an n-element antichain to a copy of A_n^k. Since $\mathcal{G}$ contains the inverse of all of its functions, it also maps a copy of A_n^k to an antichain.

Finally, assume that V satisfies the same relations with all the z_i for $1 \leq i \leq n-2$. By applying an appropriate instance of $\circlearrowright$ we may assume that V is incomparable with all the z_i for $1 \leq i \leq n-2$. Let $W \subseteq U$ induce a $(k-1)$-element antichain, and consider $R := W \cup \{x_1, y_1, \ldots, y_k, s\}$; then R induces a copy of A_n^k. If V is incomparable with z_{n-1}, then $h[R]$ is an antichain and we are done. So assume that V and z_{n-1} are comparable. Then $h[R]$ induces A_n^{k-1} or B_n^{k-1}. Let $F \subseteq P$ be an random filter that separates $h(s)$ from the other elements of $h[R]$. Then $\circlearrowright_F \circ h[R]$ induces B_n^{n-k+1} or A_n^{n-k+1}. By Steps 2 and 3, both A_n^{n-k+1} and B_n^{n-k+1} can be mapped to an antichain by permutations from $\mathcal{G}$, finishing the proof. $\square$

Proposition 39. *Let* $\mathcal{G} \supsetneq \mathrm{Turn}$ *be a closed group. Then* $\mathcal{G}$ *contains* Max.

Proof. If $\mathcal{G} = \mathrm{Sym}_P$, then there is nothing to show, so assume this is not the case. Then $\mathcal{G}$ is not 3-transitive; since $\mathcal{G} \supseteq \mathrm{Turn}$, the orbits of its action on triples of distinct entries of P are unions of the action of Turn on such triples. Since $\mathcal{G} \neq \mathrm{Turn}$, it cannot preserve cycl or cycl$'$; thus it preserves Par. Thus $\mathcal{G} \subseteq \mathrm{Max}$ by Proposition 36. Now if $f \in \mathcal{G} \setminus \mathrm{Turn}$, then it flips cycl and cycl$'$, by Lemma 35. Hence, $\updownarrow \circ f$ preserves cycl, and so it is an element of $\mathrm{Turn} \subseteq \mathcal{G}$, by Proposition 34. But then $\updownarrow = \updownarrow \circ f \circ f^{-1} \in \mathcal{G}$, and so $\mathcal{G}$ contains Rev. $\square$

Theorem 2 now follows from Propositions 28, 29, 37, 38, and 39.

3.4.8. *Relational description of* Rev.

Proposition 40. $\mathrm{Rev} = \mathrm{Aut}(P; \perp)$.

Proof. By definition, the function $\updownarrow$ preserves the incomparability relation and its negation, so the inclusion $\supseteq$ is trivial. For the other direction, let $f \in \mathrm{Aut}(P; \perp)$. We claim that f is either an automorphism of $\mathbb{P}$, or satisfies itself the definition of $\updownarrow$ (i.e., $f(b) \leq f(a)$ iff

$a \leq b$ for all $a, b \in P$). Suppose that f is not an automorphism of $\mathbb{P}$, and pick $a \leq b$ such that $f(a) \nleq f(b)$. Since f preserves comparability, we then have $f(b) \leq f(a)$. To prove our claim, since f preserves $\perp$ it suffices to show that likewise $f(d) \leq f(c)$ for all $c \leq d$.

We first observe that if $e \leq b$ and $e \perp a$, then $f(e) \geq f(b)$. For if we had $f(e) \leq f(b)$, then it would follow that $f(e) \leq f(b) \leq f(a)$, a contradiction since f preserves $\perp$. Hence, $f(e) \nleq f(b)$, and so $f(e) \geq f(b)$ since f preserves comparability.

Next let $r, s \in P$ so that $r \leq s$, $r \leq b$, and $s \perp b$; we show $f(r) \geq f(s)$. Since $f(r)$ and $f(s)$ are comparable, it is enough to rule out $f(r) \leq f(s)$. By our previous observation, we have $f(b) \leq f(r)$, so $f(r) \leq f(s)$ would imply $f(b) \leq f(s)$, contradicting the fact that f preserves $\perp$.

Now let $u, v \in P$ be so that $u \leq v$ and such that both u and v are incomparable with both a and b. Then using the extension property, we can pick $r, s \in P$ as above and such that $u \leq s$ and $v \perp s$. By the preceding paragraph, $f(r) \geq f(s)$, and applying the above once again with (u, v) taking the role of (r, s) and (r, s) the role of (a, b), we conclude $f(v) \geq f(u)$.

Finally, given arbitrary $c, d \in P$ with $c \leq d$, we use the extension property to pick $u, v \in P$ incomparable with all of a, b, c, d, and apply the above twice to infer $f(c) \geq f(d)$. $\square$

Theorem 3 now follows from Propositions 34, 36, and 40.

4. Reducts of the Henson graphs with a constant

4.1. **Ramsey theory and other preliminaries.** The structure $\mathcal{N} = \{X, (Q_j)_{j \in J}\}$ is a reduct of the structure $\mathcal{M} = \{X, (R_i)_{i \in I}\}$ if Q_j is first-order definable from the set $\{R_i | i \in I\}$ for all $j \in J$. In this thesis, reducts are considered up to first-order interdefinability. Thus two reducts $\mathcal{A}$ and $\mathcal{B}$ are equivalent if $\mathcal{A}$ is the reduct of $\mathcal{B}$ and vice versa. The reducts of $\mathcal{M}$ up to first-order interdefinability are in a one-to-one Galois correspondence with the closed subgroups of $\mathrm{Sym}(X)$ containing $\mathrm{Aut}(\mathcal{M})$. A group C is closed if it is the automorphism group of a structure. Equivalently, a group is closed if any permutation that can be interpolated on finite substructures of $\mathcal{M}$ by C is in C. I.e., C is closed if and only if for any permutation ρ the condition that for all finite $A \leq \mathcal{M}$ there exists a $\pi \in C$ such that $\pi|_A = \rho|_A$ implies that $\rho \in C$. Instead of directly characterizing the reducts of $(H_n, E, 0)$ we determine the closed groups containing $\mathrm{Aut}(H_n, E, 0)$.

A countable relational structure $\mathcal{M}$ is homogeneous if every partial isomorphism between finite substructures of $\mathcal{M}$ is the restriction of an automorphism of $\mathcal{M}$. The complete list of homogeneous graphs was given by A. H. Lachlan and R. E. Woodrow [23]. A homogeneous graph is isomorphic to one of the following structures.

(1) The random graph (R, E).
(2) The Henson graph (H_n, E) for some $n \geq 3$.
(3) The complement of the Henson graph (H_n, E) for some $n \geq 3$.
(4) The disjoint union of complete graphs of equal size.
(5) The complement of the disjoint union of complete graphs of equal size.

The age of a structure $\mathcal{M}$ is the class of all finite structures that has an isomorphic copy in $\mathcal{M}$. The Henson graph (H_n, E) is the unique homogeneous graph with $\mathrm{Age}(H_n, E)$ being the class of finite K_n-free graphs ($n \geq 3$). In other words, (H_n, E) is the Fraïssé limit of all finite K_n-free graphs. These structures were first discovered by C. W. Henson in [12]. According to Thomas' theorem in [37] the structure (H_n, E) has no non-trivial reducts.

Theorem 4 (Thomas)**.** *There is no closed group between* $\mathrm{Aut}(H_n, E)$ *and* $\mathrm{Sym}(H_n)$*.*

We determine the reducts of the Henson graphs with a constant, that is, $(H_n, E, 0)$ up to first-order interdefinability. Usually, this notation means that the element $0 \in H_n$ is added to the language as a constant sign. Contrary to the investigation of $(\mathbb{Q}, <, 0)$ in [20] we do not allow constant signs in the language. In fact, we always work with structures that are homogeneous in a finite relational language. So in our setup $(H_n, E, 0)$ means that a unary relation 0 is added to the language, and it is interpreted as the singleton $\{0\}$. As $(H_n, E, 0)$ is not homogeneous, we need to add the unary relations to the language that are first-order definable in $(H_n, E, 0)$. Note that this does not alter the automorphism group of the structure.

Definition 41. *The set of neighbours of* 0 *in* $(H_n, E, 0)$ *is denoted by* U_1*. The set of non-neighbours of* 0 *in* $(H_n, E, 0)$ *is denoted by* U_2*. The* U_i*-part of a substructure* $A \leq (H_n, E, 0, U_1, U_2)$ *is the set of points in* $U_i \cap A$ *for* $i \in \{1, 2\}$*. The* 0*-part of a substructure* $A \leq (H_n, E, 0, U_1, U_2)$ *is* $\{0\} \cap A$*. The intermediate pairs in a substructure* $A \leq (H_n, E, 0, U_1, U_2)$ *are the* 2*-element substructures of* A *with one point in* U_1 *and one point in* U_2*. Intermediate edges and non-edges are the intermediate pairs constituting an edge and a non-edge, respectively. Throughout the dissertation the language of* $(H_n, E, 0, U_1, U_2)$ *is denoted by* τ*, and the language of* (H_n, E) *is denoted by* Δ*.*

The structure $(H_n, E, 0, U_1, U_2)$ is homogeneous. To prove this, let f be a partial τ-isomorphism. We show that if 0 is not in the domain of f, then extending f to $\bar{f}$ with $\bar{f}(0) = 0$ is still a partial τ-isomorphism. The partial map $\bar{f}$ is injective, and $\bar{f}$ does not violate the unary relations by definition. If $\bar{f}$ violates an edge e, then e must contain 0. But then the other endpoint of e is mapped from U_1 into U_2 by f, which is impossible. Similarly, $\bar{f}$ preserves non-edges, as well. Thus we may assume that 0 is in the domain of f, and then $f(0) = 0$. Hence, f is a partial Δ-isomorphism fixing 0. According to the homogeneity of (H_n, E), we have that f extends to an automorphism α of (H_n, E) fixing 0. Then α is also an isomorphism of $(H_n, E, 0, U_1, U_2)$, and α extends f.

Proposition 42. *The structures $(H_n, E, 0)$ and $(H_n, E, 0, U_1, U_2)$ have the same reducts up to first-order interdefinability.*

Proof. $(H_n, E, 0)$ and $(H_n, E, 0, U_1, U_2)$ are first-order interdefinable. □

In [27] the following Ramsey type theorem is shown.

Theorem 5 (Nešetřil, Rödl). *Let $n \geq 3$ and $r \geq 2$. Then for all K_n-free graphs A there exists a K_n-free graph B such that if the edges and non-edges of B are colored with r colors, then there exists a copy $A' \leq B$ of A that is monochromatic, i.e., edges have the same color and non-edges have the same color.*

The class of ordered K_n-free graphs has an even stronger property, namely that it is a Ramsey class [28]. A class $\mathcal{C}$ of finite structures is called a Ramsey class if for all $A, B \in \mathcal{C}$ and $r \in \mathbb{N}$ there is a $C \in \mathcal{C}$ such that if the copies of A in C are colored with r colors, then there is a copy $B' \leq C$ isomorphic to B that is monochromatic. The (homogeneous) structure $\mathcal{M}$ is a Ramsey structure if Age($\mathcal{M}$) is a Ramsey class.

Theorem 6 (Nešetřil, Rödl). *The class of finite ordered K_n-free graphs is a Ramsey class for all $n \geq 3$. In particular, the randomly ordered Henson graph $(H_n, E, <)$ is a homogeneous ordered Ramsey structure. I.e., given any $n \geq 3$, $r \geq 2$ and finite ordered K_n-free graphs A, B, there exists a finite ordered K_n-free graph C such that if the copies of A in C are colored with r colors, then there is a monochromatic copy of B in C.*

In [7] the following is shown.

Proposition 43 (Bodirsky, Pinsker, Tsankov). *Let $\mathcal{M}$ be a homogeneous ordered Ramsey structure. If we add a constant to the language of $\mathcal{M}$, then the structure obtained is an ordered Ramsey structure.*

We need a generalization of Theorem 5, when there are finitely many constants in the graph. This structure is not Ramsey, but it satisfies a weaker property that we show in Proposition 45.

For a structure $\mathcal{M}$ and a k-tuple $\underline{a} = (a_1, \ldots, a_k) \in \mathcal{M}^k$ the type $tp(\underline{a})$ is the set of first-order formulas with free variables $x_1, \ldots, x_n$ that hold for $\underline{a}$. If $\mathcal{M}$ is homogeneous in a finite relational language, then $\mathcal{M}$ is η-categorical, and thus there are finitely many k-types of $\mathcal{M}$ for all $k \in \mathbb{N}$. Moreover, the k-tuples $\underline{a}$ and $\underline{b}$ have the same type if and only if there is an automorphism $\alpha \in \operatorname{Aut} \mathcal{M}$ with $\alpha(\underline{a}) = \underline{b}$. In particular, the k-types can be identified with the orbits of $\operatorname{Aut}(\mathcal{M})$ in $\mathcal{M}^k$. This identification is used throughout the dissertation. E.g., U_1 is a one-type and the set of intermediate edges (u_1, u_2) with $u_1 \in U_1$, $u_2 \in U_2$ is a two-type of $(H_n, E, 0, U_1, U_2)$. If $c_1, \dots, c_n \in \mathcal{M}$ are given constants, then a relative k-type in $\mathcal{M}$ is a k-type in $(\mathcal{M}, c_1, \dots, c_n)$. Here, $(\mathcal{M}, c_1, \dots, c_n)$ denotes the structure $\mathcal{M}$ with the constants $c_1, \dots, c_n$ added to the language.

Definition 44. *Let $k \in \mathbb{N}$. A class $\mathcal{C}$ of finite structures is called a k-Ramsey class if for all $A, B \in \mathcal{C}$ with $|A| \leq k$ and $r \in \mathbb{N}$ there is a $C \in \mathcal{C}$ such that if the copies of A in C are colored with r colors, then there is a copy $B' \leq C$ isomorphic to B that is monochromatic. The (homogeneous) structure $\mathcal{M}$ is a k-Ramsey structure if* $\operatorname{Age}(\mathcal{M})$ *is a k-Ramsey class.*

Proposition 45. *Let $n \geq 3$, $r \geq 2$, $t \geq 0$. Let $(H_n, E, 0, 1, \dots, t-1)$ be the Henson graph with t constants. Then $(H_n, E, 0, 1, \dots, t-1)$ is 2-Ramsey.*

Proof. Let $T_1, T_2, \dots, T_k$ be the at most 2-element substructures in $(H_n, E, 0, 1, \dots, t-1)$ up to isomorphism. It is enough to prove that for any $r \geq 2$, $T \in \{T_1, T_2, \dots, T_k\}$ and $B \in \operatorname{Age}(H_n, E, 0, 1, \dots, t-1)$ there exists a $C \in \operatorname{Age}(H_n, E, 0, 1, \dots, t-1)$ such that if the copies of T in C are colored with r colors, then there is a monochromatic copy of B in C.

According to Theorem 6 and Proposition 43 we can extend the language of $(H_n, E, 0, 1, \dots, t-1)$ with a total order so that it becomes an ordered Ramsey structure. Let us denote the structure obtained in this way by $(H_n, E, <, 0, 1, \dots, t-1)$. Observe that $B \leq (H_n, E, 0, 1, \dots, t-1)$ has an ordered version $B^<$ such that all ordered versions of the copies of T in $B^<$ are isomorphic to some $T^<$. Indeed, if $|T| = 1$ or $|T| = 2$ and both vertices of T have the same relative 1-type, then T has only one ordered version up to isomorphism. If $|T| = 2$ and the two vertices of T have different 1-types Q_1 and Q_2 with Q_1 not bigger than Q_2, then we order B such that the points in Q_1 become smaller than the points in Q_2. According to the Ramsey property of $(H_n, E, <, 0, 1, \dots, t-1)$ there is a $C^<$ in $\operatorname{Age}(H_n, E, <, 0, 1, \dots, t-1)$ such that if the copies of $T^<$ in $C^<$ are colored with r colors, then there is a monochromatic copy of $B^<$ in $C^<$. We show that if we omit the total order from $C^<$, then the unordered structure C satisfies the required condition. Assume that the copies of T in C are colored with r colors. In particular, the copies of $T^<$ are colored with r colors in $C^<$. Then there is a monochromatic copy $B'^<$ of $B^<$ in $C^<$. As all ordered versions of the copies of T in $B'^<$ are isomorphic to $T^<$, we have that B' is monochromatic. □

Throughout this section we use a variant of the notion behaviour defined in [4].

Definition 46. *Let $\mathcal{M}$, $\mathcal{N}$ be countable homogeneous relational structures over finite relational languages with maximal arity of relations k. Let $T_{\mathcal{M}}$, $T_{\mathcal{N}}$ be the set of k-types, i.e. the set of orbits of the automorphism group on the k-tuples of $\mathcal{M}$ and $\mathcal{N}$, respectively. A function $b : T_{\mathcal{M}} \to T_{\mathcal{N}}$ is called a* behaviour*. A set of functions $H = \{f_i : \mathcal{M} \to \mathcal{N} | i \in I\}$* admits the behaviour b *if for all finite $A \leq \mathcal{M}$ there exists a copy $A' \leq \mathcal{M}$ and an $i \in I$ such that $f_i|A'$ has the property that the image of any k-tuple $\underline{a}$ in A' is isomorphic to $b(\underline{a})$. To* realize the behaviour b *on A means that we map A to A' with an automorphism in $\mathrm{Aut}(\mathcal{M})$, and apply f_i. We also say that f_i realizes the behaviour b on A.*

Definition 47. *Let $\mathcal{C}$ be the age of a countable relational structure over a finite language. The sequence $A_1 \subsetneq A_2 \subsetneq \cdots$ of structures in $\mathcal{C}$* tends to infinity *if for any $B \in \mathcal{C}$ there is an $i \in \mathbb{N}$ such that B embeds into A_i.*

Example 48. *For a general construction of a sequence of structures tending to infinity let $\mathcal{M}$ be a countable structure with age C. Let $m_1, m_2, \ldots$ be an enumeration of the elements of M. Then the sequence $(A_i)_{i\in\mathbb{N}}$ with $A_i = \{m_1, \ldots, m_i\}$ tends to infinity.*

Proposition 49. *Let $s, t \geq 0$. Then any set of functions $H = \{f_i : (H_n, E, 0, \ldots, s-1) \to (H_n, E, 0, \ldots, t-1) | i \in I\}$ admits some behaviour b.*

Proof. Note that in the structures $(H_n, E, 0, \ldots, s-1)$ and $(H_n, E, 0, \ldots, t-1)$ every relation is at most binary, thus a behaviour is a function between the set of two-types of the two structures.

Let $A_1 \subsetneq A_2 \subsetneq \cdots$ be a sequence of structures in $\mathrm{Age}(H_n, E, 0, \ldots, s-1)$ tending to infinity. Let $S_1, \ldots, S_q$ be the at most 2-element substructures of $(H_n, E, 0, \ldots, s-1)$, and let $T_1, \ldots, T_r$ be the at most 2-element substructures of $(H_n, E, 0, \ldots, t-1)$. According to Proposition 45 for all A_j there exists a $B_j \in \mathrm{Age}(H_n, E, 0, \ldots, s-1)$ such that if the at most 2-element substructures of B_j are colored with r colors, then there is a monochromatic copy A'_j of A_j in B_j. Let us choose an arbitrary copy of B_j in $(H_n, E, 0, \ldots, s-1)$ and color its at most 2-element substructures by the isomorphism type of their f_1-image. Then for any $1 \leq m \leq q$ we have that all copies of S_m in the monochromatic A'_j have the same f_1-image up to isomorphism. Thus we can assign a structure from $T_1, \ldots, T_r$ to any S_m. Hence, some behaviour b_j can be realized on A_j by f_1. As there are only finitely many possible behaviours, there is a behaviour b that occurs infinitely many times in the sequence $(b_j)_{j\in\mathbb{N}}$. Given a finite $A \in \mathrm{Age}(H_n, E, 0, \ldots, s-1)$ there is a j such that A_j has an induced substructure isomorphic to A and $b_j = b$. Thus the behaviour b can be realized on any $A \in \mathrm{Age}(H_n, E, 0, \ldots, s-1)$ by f_1, and consequently f_1 admits the behaviour b. $\square$

We use the terminology of [4]. Let (G, E) be a countable homogeneous graph. Then there are four possible behaviours of a function $f : (G, E) \to (G, E)$.

Definition 50. *The behaviour mapping edges to edges and non-edges to non-edges is called the identical behaviour, and it is denoted by id. The behaviour switching edges and non-edges is denoted by $-$. The constant behaviour mapping edges to both two-types is denoted by e_E. The constant behaviour mapping non-edges to both two-types is denoted by e_N.*

There are some direct connections between behaviours and closed groups. The following is proven in [4] for the random graph.

Proposition 51. *Let (G, E) be a countable homogeneous graph. If a permutation π of G admits the behaviour e_N or e_E, then* $\mathrm{Aut}(G, E)$ *and π generates the fully symmetric group* $\mathrm{Sym}(G)$ *as a closed group.*

Proof. We prove the statement for the behaviour e_N, the other case is analogous. Let C be the closed group generated by the set $\mathrm{Aut}(G, E) \cup \{\pi\}$. We have to show that for any permutation $\rho \in \mathrm{Sym}(G)$ on any finite substructure $A \leq (G, E)$ there exists a permutaion in C that agrees with ρ on A. Let $A = \{a_1, \ldots, a_n\}$. As C contains $\mathrm{Aut}(G, E)$ and π, the behaviour e_N can be realized on A and $\rho(A)$. Let $\gamma_A, \gamma_{\rho(A)} \in C$ be permutations that realize e_N on A and $\rho(A)$, respectively. As the images are empty graphs, the partial map $\gamma_A(a_i) \mapsto \gamma_{\rho(A)}(\rho(a_i))$ is a partial isomorphism. Hence, there exists an $\alpha \in \mathrm{Aut}(G, E)$ such that $\alpha(\gamma_A(a_i)) = \gamma_{\rho(A)}(\rho(a_i))$. Thus $\gamma_{\rho(A)}^{-1} \circ \alpha \circ \gamma_A \in C$ interpolates ρ on A. $\square$

Proposition 49 provides the main technique of the present thesis. Given a closed group $\mathrm{Aut}(\mathcal{M})$ we characterize the closed groups C that are minimal over $\mathrm{Aut}(\mathcal{M})$ as follows. Any $\pi \in C \setminus \mathrm{Aut}(\mathcal{M})$ together with $\mathrm{Aut}(\mathcal{M})$ generates C as a closed group. The fact that π is not an automorphism of $\mathcal{M}$ is witnessed on a finite set $\{c_1, \ldots, c_k\}$. We add $c_1, \ldots, c_k$ to the language as constants, and fix a behaviour b of π as an $(\mathcal{M}, c_1, \ldots, c_k) \to (\mathcal{M}, c_1, \ldots, c_n)$ function. As π violates a relation on $\{c_1, \ldots, c_k\}$ the behaviour b is not identical. By analyzing the possible behaviours we show that C is among finitely many closed groups.

Proposition 52. *Let* $\mathrm{Aut}(H_n, E, 0, U_1, U_2) \subseteq C$ *be a closed group. Assume that for any* $A \in \mathrm{Age}(H_n, E, 0, U_1, U_2)$ *there exists a copy A' of A in H_n and a permutation $\pi_A \in C$ such that either all intermediate pairs of A' are mapped to edges by π_A or all intermediate pairs of A' are mapped to non-edges by π_A. Then C contains* $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$. *In particular, if C admits a behaviour b that is constant on the intermediate pairs, then* $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E) \subseteq C$.

Proof. Let $A_1 \subsetneq A_2 \subsetneq \cdots$ be a sequence in $\mathrm{Age}(H_n, E, 0, U_1, U_2)$ tending to infinity. According to Proposition 45 for all A_i there is a $B_i \in \mathrm{Age}(H_n, E, 0, U_1, U_2)$ such that by coloring all the 2-element substructures $T_1, T_2, \dots, T_k$ with r colors there is a monochromatic copy $\overline{A}_i$ of A_i in B_i. Choose a copy B_i' of B_i in H_n such that all intermediate pairs in B_i' are mapped to non-edges by some π_{B_i}. Color the 2-element substructures of B_i' according to the isomorphism type of their π_{B_i}-image. Then the intermediate edges and non-edges of the monochromatic copy $\overline{A}_i$ are mapped to the same 2-type. Hence, the assigned canonical behaviour on A_i is constant on the intermediate pairs. There is a behaviour b that occurs infinitely many times as an assigned behaviour to some A_i. This behaviour b is constant on the intermediate pairs, and C admits b. To prove the assertion of the proposition we have to show that any permutation $\alpha \in \mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$ can be interpolated on the finite substuctures of H_n. Let $A = \{a_1, \dots, a_k\}$ be a finite substructure and $B = \{b_1, \dots, b_k\}$ its α-image, $b_i = \alpha(a_i)$. Then these two structures differ only in the intermediate pairs. More precisely, $a_i \mapsto b_i$ is a partial τ-isomorphism except that some intermediate edges are mapped to non-edges and vice versa. Let γ_A and γ_B be permutations in C realizing the behaviour b on A and B, respectively. Then $g : \gamma_A(a_i) \mapsto \gamma_B(b_i)$ is a partial τ-isomorphism. Thus there exists a $\beta \in \mathrm{Aut}(H_n, E, 0, U_1, U_2)$ such that $\beta(\gamma_A(a_i)) = \gamma_B(b_i)$ for all i. Hence, the composition $\gamma_B^{-1} \circ \beta \circ \gamma_A$ interpolates α on A. □

Note that in the above application of the notion behaviour, we considered permutations as functions $(H_n, E, 0, U_1, U_2) \to (H_n, E)$. In some proofs this is not enough. We will use that there is a behaviour of any permutation as a function $(H_n, E, 0, U_1, U_2) \to (H_n, E, 0, U_1, U_2)$, and sometimes several other constants are added to the language. If these more complicated structures are considered, then we use the following terminology. Let L be the language of the domain structure and L' be the language of the image structure. We say that there is a behaviour $b : L \to L'$ of the function f. E.g., there is a $\tau \to \tau$ behaviour b of any permutation π of H_n, which means that b is a behaviour of π considered as a function $(H_n, E, 0, U_1, U_2) \to (H_n, E, 0, U_1, U_2)$. For example, the behaviour of a function mapping H_n isomorphically onto U_1 is not identical as a $\tau \to \tau$ behaviour, but is identical as a $\tau \to \Delta$ behaviour.

Behaviours can be defined naturally on the one-types, as well. Thus given a $\tau \to \tau$ behaviour b it has a value at U_1, e.g., $b(U_1) = U_2$.

In order to characterize the reducts of $(H_n, E, 0)$ we often need constructions of substructures with several extra conditions, such as being connected to a particular subset. We say that a construction is realizable if there exists a substructure satisfying the given properties. In most of the cases it is straightforward to verify that the construction is realizable, and it is left to the reader. E.g., it is realizable to consider any K_n-free graph H in U_2. Indeed, the graph G that consits of H and an isolated vertex is a K_n-free graph, and thus G is a subgraph of (H_n, E). As (H_n, E) is homogeneous, there is an automorphism mapping the isolated vertex in G to 0. The image of the other $(n-1)$ vertices of G induce a subgraph isomorphic to H in U_2.

We introduce the following notations for the finite structures that are used several times in the constructions.

Definition 53. *K_n denotes the complete graph on n vertices. I_n denotes the empty graph on n vertices. K_{n-1}^* denotes the graph consisting of a K_{n-1} and an isolated vertex. K_n^- denotes the graph on n vertices with exactly one non-edge.*

4.2. **The result.** We state the second main theorem of the present dissertation and study the reducts of $(H_n, E, 0, U_1, U_2)$. Some of them occur naturally.

If we drop the constant sign 0 and the other unary relations U_1, U_2 that are definable using 0, we obtain the original Henson graph (H_n, E). This gives rise to the closed supergroup $\mathrm{Aut}(H_n, E)$.

By omitting the vertex 0, the remaining vertices constitute a graph isomorphic to the original one. Formally this is not a reduct, as the unerlying set is changed. However, this observation leads to the structure $(H_n \setminus \{0\}, E) \cup \{0\}$. From the aspect of closed groups we obtain $\mathrm{Aut}(H_n, 0, E|_{H_n \setminus \{0\}})$.

As there are intermediate edges in $(H_n, E, 0, U_1, U_2)$, an automorphism of this structure can not act independently on U_1 and U_2. If the intermediate edges are omitted, the connection disappears. This yields $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$. In the next subsection it is shown that these three closed groups are exactly the minimal ones above $\mathrm{Aut}(H_n, E, 0)$.

According to Theorem 4 the only closed group above $\mathrm{Aut}(H_n, E)$ is $\mathrm{Sym}(H_n)$. Thus the only closed group above $\mathrm{Aut}(H_n, 0, E|_{H_n \setminus \{0\}})$ that stabilizes 0 is $\mathrm{Sym}(H_n \setminus \{0\})$, the stabilizer of 0 in $\mathrm{Sym}(H_n)$. This corresponds to the reduct $(H_n, 0)$.

If $n \geq 4$ then (U_1, E) is not an empty graph. In this case $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1)$ is a proper supergroup of $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$. This is the permutation group compatible with all the unary relations and the edge relation in U_2.

The group $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2)$ is always a proper supergroup of $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$. The relations can be chosen similarly to the previous case.

For $n \geq 4$ we have $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$, the group consisting of the permutations compatible with all the unary relations $0, U_1, U_2$.

The permutations preserving the partition $(U_1 \cup \{0\}) \cup U_2$ and the graph structure on U_2 are $\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Aut}(U_2, E)$. Analogously, we have $\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Aut}(U_1, E)$.

This is strictly smaller than $\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)$ for $n \geq 4$. The group $\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)$ consists of the permutations compatible with U_1. Similarly, there is the closed group $\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Sym}(U_2)$ containing the permutations that are compatible with U_2. This is always bigger than $\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Aut}(U_2, E)$.

Given a partition $X \cup Y$ of a set Γ into countably infinite sets X and Y the group $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ is not a maximal closed group in $\mathrm{Sym}(\Gamma)$. According to Theorem 55 in Subsection 4.3 there is exactly one closed group above $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ that is not the fully symmetric group. We denote the unique proper closed supergroup of $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ by $(\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$.

Definition 54. *The group* $(\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$ *consists of the permutations that preserve* X *and* Y *or switch* X *and* Y.

As an abstract group this is indeed the semidirect product of $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ and a 2-element subgroup generated by a permutation that is given by infinitely many transpositions. This construction is the automorphism group of the complete bipartite graph with bipartition $\{X, Y\}$. Thus we have the closed group $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)) \rtimes Z_2$ compatible with the unary relation 0 and the complete bipartite graph with classes U_1 and U_2.

The same construction yields the groups $(\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Sym}(U_2)) \rtimes Z_2$ and $(\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)) \rtimes Z_2$. These are the automorphism groups of the bipartite graphs with bipartition $\{(U_1 \cup \{0\}), U_2\}$ and $\{(U_2 \cup \{0\}), U_1\}$, respectively.

Finally there is the biggest group $\mathrm{Sym}(H_n)$ corresponding to the trivial structure $(H_n, \emptyset)$.

In Subsection 4.5 it is shown that these are all closed groups containing $\mathrm{Aut}(H_n, E, 0)$. As all relations in the description of these groups were at most binary, the action of the automorphism group on the pairs determines every reduct of $(H_n, E, 0)$.

4.3. **Big closed groups.**

Proposition 55. *Let* $\Gamma := X \cup Y$ *be a partition with both classes* X *and* Y *infinite. Then* $(\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$ *is the unique proper closed group above* $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$*, in particular,* $(\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$ *is a maximal closed group in* $\mathrm{Sym}(\Gamma)$.

Proof. Assume we have a closed group $\mathrm{Sym}(X) \times \mathrm{Sym}(Y) \subsetneq H$ different from $(\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$. Note that $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ is a maximal (closed) subgroup of $(\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$, as it has index 2. Hence, there is some $\alpha \in H \setminus (\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$. We show that H is k-transitive for any fixed k. As H is a closed group this implies $H = \mathrm{Sym}(\Gamma)$.

An element of Γ is class-changing if α maps it from X to Y, or vice versa. Otherwise, it is class-preserving. A simple analysis shows that either there are infinitely many class-preserving points in both classes, or there are infinitely many class-changing points in both classes.

Assume that there are infinitely many class-preserving elements in X and in Y, as well. As $\alpha \in H \setminus (\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$, we have a class-changing element. Without loss of generality, assume that $\alpha(x) \in Y$ for some $x \in X$. Let $(a_1, \ldots, a_k) \in Y^k$ be a fixed k-tuple. Given $x_1, \ldots, x_m \in X$ and $y_{m+1}, \ldots, y_k \in Y$ it is enough to show that an appropriate permutation in H maps the k-tuple $\underline{t} = (x_1, \ldots, x_m, y_{m+1}, \ldots, y_k)$ to $(a_1, \ldots, a_k)$. By applying an appropriate permutation in $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ we first map all points to class-preserving elements, except for x_m which we map to x. By applying α the element x_m is mapped to Y and every other elements in the tuple $\underline{t}$ preserves its class. By iterating this step all the elements of $\underline{t}$ can be mapped to Y. Then a permutation in $\mathrm{Sym}(Y)$ maps them to $(a_1, \ldots, a_k)$.

Now assume that there are infinitely many class-changing points in both classes. As $\alpha \notin (\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$, we have a class-preserving element. Without loss of generality, assume that $\alpha(x) \in X$ for some $x \in X$. Let $(a_1, \ldots, a_k) \in Y^k$ be a fixed k-tuple. Let $x_1, \ldots, x_m \in X$ and $y_{m+1}, \ldots, y_k \in Y$, and let us denote the k-tuple $(x_1, \ldots, x_m, y_{m+1}, \ldots, y_k)$ by $\underline{t}$. First we map all points to class-changing elements, except for x_m which is mapped to x. By applying α the element x_m is preserved in X and every other element in $\underline{t}$ changes its class. If we map all the images to class changing elements, and apply α again, then x_m is mapped to Y and every other element of $\underline{t}$ preserves its class. By iterating this step all the elements of $\underline{t}$ can be mapped to Y. Then a permutation in $\mathrm{Sym}(Y)$ maps them to $(a_1, \ldots, a_k)$. ☐

Proposition 56. $\mathrm{Sym}(\Gamma \setminus \{c\})$ *is a maximal closed subgroup of* $\mathrm{Sym}(\Gamma)$ *for any* $c \in \Gamma$.

Proof. We prove the stronger statement that $\mathrm{Sym}(\Gamma \setminus \{c\})$ is a maximal subgroup of $\mathrm{Sym}(\Gamma)$. Assume that there is some $\mathrm{Sym}(\Gamma \setminus \{c\}) \subsetneq H \subsetneq \mathrm{Sym}(\Gamma)$. Then H is transitive. Let $\alpha \in \mathrm{Sym}(\Gamma)$ be a permutation, and assume that it maps c to some $d \in \Gamma$. As H is transitive there is a $\beta \in H$ such that $\beta(c) = d$. Then $\pi = \alpha \circ \beta^{-1}$ stabilizes c, and hence it is in $\mathrm{Sym}(\Gamma \setminus \{c\}) \leq H$. Thus $\alpha = \pi \circ \beta \in H$, as well. ☐

4.4. **Minimal closed groups.**

Lemma 57. *Assume that a closed group* $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E) \subsetneq C$ *admits the behaviour* $b : \tau \to \tau$ *which as a* $\tau \to \Delta$ *behaviour is* e_N *on* U_k *for some* $k \in \{1, 2\}$. *Then* C *contains* $\mathrm{Aut}(U_m, E) \times \mathrm{Sym}(U_k)$ *with* $\{k, m\} = \{1, 2\}$.

Proof. To prove that C contains $\mathrm{Aut}(U_m, E) \times \mathrm{Sym}(U_k)$ we need to show that any permutation $\zeta \in \mathrm{Aut}(U_m, E) \times \mathrm{Sym}(U_k)$ can be interpolated on the finite substructures. Now assume that we are given two substructures A and B of $(H_n, E, 0, U_1, U_2)$ such that $\zeta|_A(A) = B$. Then their U_m-parts are isomorphic, their 0-parts are also isomorphic, and they have the same number of vertices in U_2. Applying some permutations in $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$ we may assume that both of them have only non-edges for intermediate pairs. Choose π_A and π_B in C that realize the behaviour b on A and B, respectively. As b is e_N in U_k as a $\tau \to \Delta$ behaviour, mapping elements in $\pi_A(A)$ to the corresponding elements in $\pi_B(B)$ is a τ-isomorphism. Let $\varphi \in \mathrm{Aut}(H_n, E, 0, U_1, U_2)$ be an automorphism extending this partial isomorphism. Then $\pi_B^{-1} \circ \varphi \circ \pi_A$ interpolates ζ on A. □

Theorem 7. *Assume that a closed group* $\mathrm{Aut}(H_n, E, 0) \subsetneq C$ *admits the behaviour* $b : \tau \to \tau$ *which as a* $\tau \to \Delta$ *behaviour is not identical on the 2-types contained in* U_2. *Then* C *contains* $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2)$.

Proof. As I_n is a substructure of U_2, the behaviour b can not map edges to the non-edges in U_2. Hence, b is neither e_E nor $-$ on U_2. As b is not identical, $b = e_N$ on U_2.

By Lemma 57 it is enough to show that C contains $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2)$.

According to Proposition 52 it is enough to prove that for any finite substructure $A \leq (H_n, E, 0, U_1, U_2)$ there exists an element of C that maps the intermediate pairs of A to non-edges. Let us denote the vertices of A in U_1 by $X = \{x_1, \ldots, x_r\}$ and in U_2 by $Y = \{y_1, \ldots, y_s\}$. The intermediate edges of A are going to be deleted in r steps, i.e., with the composition of r permutations $\pi_1, \ldots, \pi_r$ in C. The i-th step is as follows. Assume that the intermediate pairs containing $x_1, \ldots, x_{i-1}$ are already mapped to non-edges by the composition $\pi_{i-1} \circ \cdots \circ \pi_1$ such that $\pi_{i-1} \circ \cdots \circ \pi_1$ maps the elements of Y to U_2. The elements of X are not necessarily mapped to U_1. Let $v_1, v_2, \ldots, v_{i-1}$ be the images of $x_1, x_2, \ldots, x_{i-1}$, let $u_i, \ldots, u_r$ be the images of $x_i, \ldots, x_r$, and let $z_1, \ldots, z_s$ be the images of $y_1, \ldots, y_s$, respectively. We shall find a permutation π_i that stabilize the vertices $v_1, v_2, \ldots, v_{i-1}$ and maps the pairs (u_i, z_j) to non-edges.

Let $(\pi_{i-1} \circ \cdots \circ \pi_1)(A) = A'$. We construct a substructure B of $(H_n, E, 0, U_1, U_2)$ using A'. At first we omit the vertices $z_1, \ldots, z_s$ and add the new elements $z_{1,1}, z_{1,2}, \ldots, z_{s,n-1}$. We keep all edges and non-edges between the remaining vertices of A'. A vertex $g \in A'$ that was not omitted is connected to some $z_{p,q}$ in B if and only if g and z_p are connected in A'. Similarly, the $z_{p,q}$ are connected according to their first index. Then we omit all vertices from A' those are not in the image of Y except for $\{v_1, \ldots, v_{i-1}\}$. We need to check that replacing the z_p by the $z_{p,q}$ is realizable.

A complete subgraph in the union of that graph and $\{0\}$ can not contain two vertices of the form $\{z_{p,q_1}, z_{p,q_2}\}$, as such vertices constitute a non-edge. Thus all the vertices of the form $z_{p,q}$ in the complete subgraph have different first indices. The map that is identical on the original vertices of A' and assigns z_p to $z_{p,q}$ for all vertices of the form $z_{p,q}$ in this complete subgraph is an embedding into A'. Thus a complete subgraph has at most $n-1$ elements.

Drop all edges from the substructure induced by $\{z_{1,1}, z_{1,2}, \ldots, z_{s,n-1}\}$ in B and put edges between $z_{t,p}z_{t,q}$ for all $1 \leq t \leq s$, $1 \leq p, q \leq n-1$. Fix an isomorphic copy in $(H_n, E, 0, U_1, U_2)$ of the τ-structure D obtained in this way. We may assume that the vertices corresponding to $\{v_1, \ldots, v_{i-1}\}$ are $\{v_1, \ldots, v_{i-1}\}$ themselves. There are permutations $\gamma_B, \gamma_D \in C$ realizing the behaviour b on the substructures B and D, respectively. The resulting structures are isomorphic, as the only difference between B and D is in their U_2-part. Hence, by using an automorphism $\alpha \in \mathrm{Aut}(H_n, E, 0, U_1, U_2)$ we have that $\delta = \gamma_D \circ \alpha \circ \gamma_B$ maps B to D. For any $1 \leq t \leq s$ the vertex $\delta(u_i)$ can not be connected to all of the $\delta(z_{t,p})$, as the $\delta(z_{t,p})$ induce a K_{n-1}. Thus for all $1 \leq t \leq s$ there exists a smallest $1 \leq p_t \leq n-1$ such that $N(\delta(u_i)\delta(z_{t,p_t}))$. By omitting all vertices of the form $z_{t,q}$ from B except for the z_{t,t_p}, and by adding the vertices in the image of A those were omitted in the construction of B, we obtain an isomorphic copy of A'. Let us denote by $\alpha_{A'} \in \mathrm{Aut}(H_n, E, 0, U_1, U_2)$ an automorphism such that the image of A' under $\alpha_{A'}$ is this structure. Observe that δ maps all pairs (u_i, z_{t,t_p}) $(1 \leq t \leq s)$ to non-edges, and fixes the vertices $\{v_1, \ldots, v_{i-1}\}$. It also maps the z_{t,t_p} $(1 \leq t \leq s)$ to U_2. Hence, $\pi_i = \delta \circ \alpha_{A'}$ is an appropriate choice.

Hence, C indeed contains $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$ by Proposition 52, and thus the statement follows from Lemma 57. $\square$

The analogue version of this theorem is somewhat more complicated to prove.

Theorem 8. *Assume that a closed group* $\mathrm{Aut}(H_n, E, 0) \subsetneq C$ *admits the behaviour* $b : \tau \to \tau$ *which as a* $\tau \to \Delta$ *behaviour is not identical on the 2-types contained in* U_1*. Then* C *contains* $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1)$.

Just as in the previous case we have that b restricted to the 2-types in U_1 can not be $-$ or e_E, thus it must be e_N. For $n = 3$ the behaviours e_N and id are the same on U_1. Thus we may assume that $n \geq 4$. According to Lemma 57 it suffices to show that C contains $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$.

Lemma 58. *Let* C *be a closed group containing* $\mathrm{Aut}(H_n, E, 0)$ *with* $n \geq 4$*. Assume that for any* K_n*-free graph* S *there exists a permutation* $\pi_S \in C$ *and a copy* S' *of* S *in* U_2 *such* *that $\pi_S(S')$ is triangle-free. Then C admits a behaviour $b : \tau \to \tau$ that is not identical on U_2 as a $\tau \to \Delta$ behaviour.*

Proof. Let $A_1 \subsetneq A_2 \subsetneq \dots$ be a sequence in $\mathrm{Age}(H_n, E, 0, U_1, U_2)$ tending to infinity. Let k be large enough so that the U_2-part of A_k contains a triangle and an I_n. Let $m \geq k$. According to Proposition 45 for every $r \leq 2$ there is a $B_m \in \mathrm{Age}(H_n, E, 0, U_1, U_2)$ such that coloring all the 2-types with r colors there is a monochromatic copy $\overline{A}_m$ of A_m in B_m. Let r be the number of 2-types in $(H_n, E, 0, U_1, U_2)$. We use the following coloring. Choose B_m in H_n such that its U_2-part S_m can be mapped to a triangle free graph by $\pi_{S_m} \in C$. Color the 2-types of B_m according to their π_{S_m}-image. Then the U_2-part Y of the monochromatic copy $\overline{A}_m$ is mapped to a triangle-free graph. As Y contains a triangle, the edges in U_2 can not be mapped to edges. As Y contains an independent set of n vertices, the non-edges in U_2 can not be mapped to edges. Hence the assigned canonical behaviour on A_m maps non-edges to both 2-types in U_2. As we assign such behaviours for large enough structures (in fact for all $m \geq k$), there is a behaviour b that occurs infinitely many times as an assigned behaviour in the sequence $(A_i)_{i \in \mathbb{N}}$. This behaviour b as a $\tau \to \Delta$ behaviour is not identical on U_2. □

Proof of Theorem 8. According to Lemma 58 we may assume that there is a K_n-free graph S such that the image of any copy of S in U_2 under any permutation in C contains a triangle. Throughout the proof we fix such a graph S.

The method is similar to that of the previous proof. Assume we are given a finite τ-substructure A. Let us denote the vertices of A in U_1 by $X = \{x_1, \dots, x_r\}$ and in U_2 by $Y = \{y_1, \dots, y_s\}$. According to Proposition 52 it is enough to show that there exists an element of C such that the intermediate edges of A are mapped to non-edges. This permutation will be constructed in s steps. In the j-th step we construct a perutation π_j. Assume we have already dealt with some vertices in Y, i.e., the intermediate pairs containing $y_1, \dots, y_{j-1}$ are mapped to non-edges by the permutation $\pi_{j-1} \circ \dots \circ \pi_1$. Assume further that the permutation $\pi_{j-1} \circ \dots \circ \pi_1$ maps X to U_1. Let us denote the images of the points $x_1, \dots, x_r, y_1, \dots, y_{j-1}, y_j$ under the permutation $\pi_{j-1} \circ \dots \circ \pi_1$ by $z_1, \dots, z_r, v_1, \dots, v_{j-1}, u_j$, respectively. Our goal is to construct a permutation π_j that maps $z_1, \dots, z_r$ to X, stabilizes $v_1, \dots, v_{j-1}$, preserves the non-edges $v_t x_p$ for all $1 \leq t \leq j-1$, $1 \leq p \leq s$, and turns all the $u_j x_p$ into non-edges.

We first show that we may assume that the U_1-part of $A' = \{z_1, \dots, z_r, v_1, \dots, v_{j-1}, u_j\}$ is an independent set. Let us consider the τ-structure B in H_n which is ismorphic to A except that all the edges in U_2 are removed. As C admits the behaviour b, there are permutations $\nu_1, \nu_2 \in C$ that realize the behaviour b on A and B, respectively. Mapping the elements in $\nu_1(A)$ to the corresponding elements in $\nu_2(B)$ is a τ-isomorphism, as A and B differ only in the U_1-part. Thus $\nu_2^{-1} \circ \mu \circ \nu_1$ maps A to B with some $\mu \in \operatorname{Aut}(H_n, E, 0, U_1, U_2)$. As $\{z_1, \ldots, z_r\}$ is in the U_1-part of A', the vertices $\{z_1, \ldots, z_r\}$ induce an empty graph. If $u_j \in U_1$ then we are done, so we may assume that $u_j \in U_2$. Let us denote by r_E and r_N the number of vertices in $\{z_1, \ldots, z_r\}$ that are connected and not connected to u_j, respectively. A is determined by the τ-structure B induced by $\{v_1, \ldots, v_{j-1}, u_j\}$ and the natural numbers r_E and r_N.

We construct a τ-structure D. Consider $|S|$ isomorphic copies of B. These are $B_1, \ldots, B_{|S|}$. In the k-th copy the points are $v_1^k, \ldots, v_{j-1}^k, u_j^k$. Between two such copies there are no edges, except that the set $\{u_j^k | k = 1, 2, \ldots, |S|\}$ induce a graph in U_2 isomorphic to S. Let $\{z_{p,q}^t | 1 \leq p \leq n-2, 1 \leq q \leq 3(2r_E - 1) + (2r_N - 1), 1 \leq t \leq \binom{|S|}{3}\}$ be $\binom{|S|}{3}(n-2)(3(2r_E - 1) + (2r_N - 1))$ independent points in U_1. All the pairs $z_{p,q}^t v_m^k$ are non-edges. To finish the construction of D it remains to define when $z_{p,q}^t$ is connected to u_j^k. Let us list the three-element subsets of S. Thus every three-element subset of S has an index between 1 and $\binom{|S|}{3}$. Each subset is ordered according to the parameter k of the u_j^k. The vertex $z_{p,q}^t$ is connected to u_j^k if and only if

(1) u_j^k is in the three-element subset of index t,
(2) $1 \leq q \leq 2r_E - 1$ and u_j^k is the second or third element inside its three-element subset,
(3) $2r_E \leq q \leq 2(2r_E - 1)$ and u_j^k is the third or first element inside its three-element subset,
(4) $2(2r_E - 1) + 1 \leq q \leq 3(2r_E - 1)$ and u_j^k is the first or second element inside its three-element subset.

The first thing to check is that this structure D is realizable. For this we have to show that considering an extra vertex 0 connected to the U_1-part of D a K_n-free is obtained. It is clear that a subgraph containing 0 can not constitute a K_n as the U_1-part of D is empty. So we can restrict ourselves to D. For the same reason, a complete subgraph of D can contain at most one point in the U_1-part.

First assume the complete subgraph contains one vertex in the U_1-part. If this vertex is v_m^k, then all the vertices of the complete subgraph are in B_k as v_m^k is not connected to anything else. However, B_k is isomorphic to B, which is a subgraph of the K_n-free graph A. If the vertex of the complete subgraph is $z_{p,q}^t$, then the complete subgraph has at most three points, as all vertices of the form $z_{p,q}^t$ have degree at most two in D.

Now assume that all vertices of the complete subraph are in U_2. If it contains a v_m^k, then, again all of its vertices are in B_k which is K_n-free. If the whole complete subgraph is in S, then it has at most $(n-1)$ vertices, as S is K_n-free.

As D is realizable, we can consider an isomorphic copy of it in H_n. Now we construct a τ-structure F. It is derived from D in the following way. Its underlying set is $D \setminus S$. The edge relation is unaltered everywhere except between the points $z_{p,q}^t$. Two vertices $z_{p_1,q_1}^{t_1}$ and $z_{p_2,q_2}^{t_2}$ are connected if and only if $q_1 = q_2$ and $t_1 = t_2$. We show that F is realizable. If a complete subgraph of $F \cup \{0\}$ does not contain any vertex of the form $z_{p,q}^t$, then it is contained in $D \cup \{0\}$, and we have already proven that $D \cup \{0\}$ is K_n-free. If the complete subgraph contains some $z_{p,q}^t$, then it can only contain vertices of the form $z_{p,q}^t$ and possibly 0. However, the clique number of the subgraph induced by the $z_{p,q}^t$ is $n-2$, thus the biggest clique obtained by the addition of 0 has cardinality $n-1$.

We fix an isomorphic copy of F in H_n. Choose $\pi_F, \pi_{D \setminus S} \in C$ that realize the behaviour on F and $D \setminus S$, respectively. As F and $D \setminus S$ differ only in edges in U_1, the natural mapping from $\pi_F(F)$ to $\pi_{D\setminus S}(D \setminus S)$ is a τ-isomorphism. Thus there exists an $\alpha \in C$ such that $\pi_F^{-1} \circ \alpha \circ \pi_{D \setminus S} = \rho \in C$ maps $D \setminus S$ to F. According to the choice of S it has a three-element subset such that its image under ρ is a triangle. Without loss of generality we may assume that it is $\{u_j^1, u_j^2, u_j^3\}$ which has index $t = 1$. The $(n-2)$ vertices $\rho(z_{p,q}^1)$ for a fixed $1 \le q \le 3(2r_E - 1) + (2r_N - 1)$ form a K_{n-2}. Hence, there are at least two points in $\{\rho(u_j^1), \rho(u_j^2), \rho(u_j^3)\}$ that are disconnected to at least one of these $(n-2)$ points for a fixed q. For all $1 \le q \le 3(2r_E - 1) + (2r_N - 1)$ let us assign two such vertices from $\{\rho(u_j^1), \rho(u_j^2), \rho(u_j^3)\}$.

By a simple pigeonhole argument, there are at least two vertices in $\{\rho(u_j^1), \rho(u_j^2), \rho(u_j^3)\}$ that are assigned at least r_N times to some $3(2r_E - 1) + 1 \le q \le 3(2r_E - 1) + (2r_N - 1)$. Without loss of generality we may assume that $\rho(u_j^1)$ and $\rho(u_j^2)$ are such. Again, by a simple pigeonhole argument, either $\rho(u_j^1)$ or $\rho(u_j^2)$ is assigned to at least r_E times to some $2(2r_E - 1) + 1 \le q \le 3(2r_E - 1)$. Without loss of generality we may assume that $\rho(u_j^1)$ is such.

Thus there exist

- r_E numbers $Q_1, \ldots, Q_{r_E}$ such that $2(2r_E - 1) + 1 \le Q_1, \ldots, Q_{r_E} \le 3(2r_E - 1)$ and for some $1 \le p(Q_i) \le n-2$ depending on Q_i we have that $\rho(z_{p(Q_i),Q_i}^1)$ is not connected to $\rho(u_j^1)$ $(1 \le i \le r_E)$,
- r_N numbers $Q_1', \ldots, Q_{r_E}'$ such that $3(2r_E - 1) + 1 \le Q_1', \ldots, Q_{r_E}' \le 3(2r_E - 1) + (2r_N - 1)$ and for some $1 \le p(Q_i') \le n-2$ depending on Q_i' we have that $\rho(z_{p(Q_i'),Q_i'}^1)$ is disconnected to $\rho(u_j^1)$ $(1 \le i \le r_N)$.

The $r_E + r_N = r$ vertices $\{\rho(z^1_{p(Q_i),Q_i})|1 \leq i \leq r_E\} \cup \{\rho(z^1_{p(Q'_i,Q'_i})|1 \leq i \leq r_N\}$ together with the v^1_m $(1 \leq m \leq j-1)$ and u^1_j constitute a τ-structure A''. The function $f : A' \to A''$ with

(1) $f(v_m) = v^1_m$ $(1 \leq m \leq j-1)$,
(2) $f(u_j) = u^1_j$,
(3) f mapping the r_E vertices in A' of the form z_m connected to u_j to the r_E vertices of the form $z^1_{p(Q_i),Q_i}$ with $2(2r_E-1)+1 \leq Q_i \leq 3(2r_E-1)$ such that $\rho(z^1_{p(Q_i),Q_i})$ is not connected to $\rho(u^1_j)$,
(4) f mapping the r_N vertices in A' of the form z_m disconnected to u_j to the r_N vertices of the form $z^1_{p(Q'_i),Q'_i}$ with $3(2r_E-1)+1 \leq Q'_i \leq 3(2r_E-1)+(2r_N-1)$ such that $\rho(z^1_{p(Q'_i),Q'_i})$ is not connected to $\rho(u^1_j)$

is a τ-isomorphism. Let $\beta \in \mathrm{Aut}(H_n, E, 0, U_1, U_2)$ be an automorphism that extends f. Then $\sigma = \rho \circ \beta|_{A'}$ preserves everything except that it deletes the edges of the form $z_m u_j$ and possibly maps u_j outside U_2. Thus $\sigma|_{A'\setminus\{u_j\}}$ is a partial τ-isomorphism. Let $\gamma \in \mathrm{Aut}(H_n, E, 0, U_1, U_2)$ be an automorphism extending $\sigma|_{A\setminus\{u_j\}}$. Then $\gamma^{-1} \circ \sigma$ stabilizes the v_m $(1 \leq m \leq j-1)$, preserves the z_m $(1 \leq m \leq r)$ in U_1, and prevesrves the τ-structure of A' except that it turns all the pairs $z_m u_j$ into non-edges. Thus $\gamma^{-1} \circ \sigma$ is an appropriate choice for π_j. □

Proposition 59. *Assume that the closed group* $\mathrm{Aut}(H_n, E, 0) \subseteq C$ *admits the behaviour* $b : \tau \to \tau$ *that is not identical either on* U_1*, or on* U_2*, or between* U_1 *and* U_2*. Then* C *contains at least one of the groups*

(1) $\mathrm{Aut}(H_n, E)$,
(2) $\mathrm{Aut}(H_n, 0, E|_{H_n\setminus\{0\}})$,
(3) $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$.

Proof. If b as a $\tau \to \Delta$ behaviour is not identical on U_2, then item (2) applies by Theorem 7. Thus we may assume that b as a $\tau \to \Delta$ behaviour is identical on U_2. As a K_{n-1} in U_2 can not be mapped identically to U_1, we have that $b(U_2) \neq U_1$. Hence, b is identical on U_2 as a $\tau \to \tau$ behaviour. Intermediate non-edges can not be turned into edges by b, otherwise b could not be realized on a K^*_{n-1} with the K_{n-1} in U_2 and the isolated vertex in U_1. According to Theorem 8 and Proposition 52 we may assume that b as a $\tau \to \Delta$ behaviour is also identical on U_1 and between U_1 and U_2, otherwise we are done by item (2). Then the condition of the statement can hold for b only if $b(U_1) = U_2$.

Note that as b is identical on U_2, the non-edges containing 0 can not be turned into edges, otherwise a K_{n-1} in U_2 together with 0 would be mapped to a K_n. So there are two cases.

First, if b maps the edges between 0 and U_1 to non-edges, then b is constant on the 2-types containing 0 as a $\tau \to \Delta$ behaviour. We show that in this case $\mathrm{Aut}(H_n, 0, E|_{H_n\setminus\{0\}})$ is contained in C. Let $\pi \in \mathrm{Aut}(H_n, 0, E|_{H_n\setminus\{0\}})$ and let $A = \{a_1, \ldots, a_k\}$ be a finite substructure of H_n. Let $\pi(a_i) = b_i$ and $\pi(A) = B$. Then $a_i \mapsto b_i$ is a graph isomorphism fixing 0 except that there might be edges containing 0 mapped to non-edges containing 0, and vice versa. Let γ_A and γ_B be permutations in C realizing the behaviour b on A and B, respectively. Then $f : \gamma_A(a_i) \mapsto \gamma_B(b_i)$ is a partial τ-isomorphism that extends to some automorphism $\alpha \in \mathrm{Aut}(H_n, E, 0)$. Thus $\gamma_B^{-1} \circ \alpha^{-1} \circ \gamma_A$ interpolates π on A.

Secondly, suppose that b maps the edges between 0 and U_1 to edges. Then b is identical everywhere, i.e.,on and between every one-type as a $\tau \to \Delta$ behaviour. We prove that C contains $\mathrm{Aut}(H_n, E)$. It is enough to find a permutation ρ in C such that any finite substructure A of H_n is mapped to U_2 by ρ and $\rho|_A$ is a graph isomorphism. We may assume that the U_1-part of A is non-empty, otherwise it can be extended by a vertex in U_1. Let γ_A be a permutation in C that realizes the behaviour b on A. If $0 \notin A$ then $\gamma_A = \rho$ is an appropriate choice. Assume that $0 \in A$. As U_1 is mapped to U_2 by b, and the edges containing 0 are mapped to edges, 0 can not be fixed. So 0 is either mapped to U_1 or to U_2. If 0 is mapped to U_2, then $\gamma_A = \rho$ is an appropriate choice, again. If 0 is mapped to U_1, then let us apply a permutation that realizes b on the image. The composition of these two permutations in C maps A into U_2, and the restriction of this map to A is a graph isomorphism. □

Corollary 60. *Let* $\mathrm{Aut}(H_n, E, 0) \subseteq C$ *be a closed group. Assume that for all* $A \in \mathrm{Age}(H_n, E, 0, U_1, U_2)$ *there exists a permutation* $\pi_A \in C$ *such that either the* U_1*-part of* A *is mapped to* U_2 *or the* U_2*-part of* A *is mapped to* U_1*. Then* C *contains at least one of the groups*

(1) $\mathrm{Aut}(H_n, E)$,
(2) $\mathrm{Aut}(H_n, 0, E|_{H_n\setminus\{0\}})$,
(3) $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$.

Proof. Let $A_1 \subsetneq A_2 \subsetneq \cdots$ be a sequence of τ-structures tending to infinity. Let r be the number of two-element substructures up to isomorphism in $(H_n, E, 0)$. According to Proposition 4.1, for all A_m there is a τ-structure B_m such that coloring all the two-types with r colors there is a monochromatic copy $\overline{A}_m$ of A_m in B_m. Choose the copy of B_m in H_n such that its U_1-part is mapped to U_2 or its U_2-part is mapped to U_1 by π_{B_m}. Color the two-element substructures of B_m according to their π_{B_m}-image. Then we have a canonical behaviour b_i on the monochromatic copy $\overline{A}_m$ such that $b_i(U_1) = U_2$ or $b_i(U_2) = U_1$. Let us assign this behaviour to A_i. Then there is a behaviour b that occurs infinitely many times as an assigned behaviour in the sequence $(A_i)_{i\in\mathbb{N}}$. Thus C admits the behaviour b, and $b(U_1) = U_2$ or $b(U_2) = U_1$. Either way, C satisfies the conditions of Proposition 59. ☐

In the upcoming proofs we use the followinng notations.

Definition 61. *Let $c_1, \dots, c_k \in H_n$. We denote by $U_{i_0,i_1,\dots,i_k}$ with $i_0 \in \{1, 2\}$ and $i_j \in \{c_j, \not{c}_j\}$ the subset of H_n that consists of the vertices w such that*

- *w is connected to 0 if and only if $i_0 = 1$,*
- *for $j = 1, \dots, k$ we have that w is connected to c_j if and only if $i_j = c_j$.*

For $k \geq 1$ we will always denote the language of $(H_n, E, 0, c_1, \dots, c_k, \{U_{i_0,\dots,i_k}\})$ by η. The sets $U_{i_0,\dots,i_k}$ are either empty or infinite, and induce Henson graphs (of different degree). The sets $U_{i_0,\dots,i_k}$ are exactly the 1-types relative to the constants $0, c_1, \dots, c_k \in H_n$ that are different from these constants. After showing three technical lemmata, we are ready to determine the minimal closed supergroups of $\mathrm{Aut}(H_n, E, 0)$.

Lemma 62. *Let $\mathrm{Aut}(H_n, E, 0) \subseteq C$ be a closed group such that $\{0\}$ and U_i is in the same orbit for some $i \in \{1, 2\}$. Then there exist a $u \in U_i$ and a permutation $\pi \in C$ such that π switches 0 and u.*

Proof. Let U_j denote the infinite 1-type different form U_i. Let $\rho \in C$ be such that $\rho(0) = u$. If the ρ-preimage v of 0 is also in U_i, then there is a permutation $\alpha \in \mathrm{Aut}(H_n, E, 0)$ such that $\alpha(u) = v$. Thus $\pi = \rho \circ \alpha$ switches 0 and u. Assume that $v \in U_j$. Let $w \neq v$ be a vertex in U_j, and let $\rho(w) = z$. If $z \in U_j$ then there are $\beta, \gamma \in \mathrm{Aut}(H_n, E, 0)$ with $\beta(v) = z$ and γ switching v and w. Hence, $\pi = \rho \circ \beta^{-1} \circ \rho \circ \gamma \circ \rho^{-1} \circ \beta \circ \rho^{-1}$ switches 0 and u. Finally, if $z \in U_i$ then there exist $\mu, \nu \in \mathrm{Aut}(H_n, E, 0)$ with $\mu(v) = w$ and ν switching u and z. Thus $\pi = \rho \circ \mu^{-1} \circ \rho^{-1} \circ \nu \circ \rho \circ \mu \circ \rho^{-1}$ switches 0 and u. ☐

Lemma 63. *Assume that the vertex $u \neq 0$ in H_n is added to the language of $(H_n, E, 0)$ as a constant. Let us denote the language of the structure obtained by η. Assume further that a closed group $\mathrm{Aut}(H_n, E, 0) \subseteq C$ admits the behaviour $b : \eta \to \eta$ such that $b(U_{ij})$ is in U_i for $i, j \in \{1, 2\}$.*

1. *Then either C contains $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$, or b as an $\eta \to \Delta$ behaviour is identical on and between U_{1k} and U_{2m} for any $k, m \in \{u, \not{u}\}$.*
2. *Moreover, for any $i \in \{1, 2\}$ we have that C contains $\mathrm{Sym}(U_i) \times \mathrm{Aut}(U_t, E)$ with $\{i, t\} = \{1, 2\}$ or b as an $\eta \to \Delta$ behaviour is identical on the U_{ij} for $j \in \{u, \not{u}\}$.*

Proof. At first item (2) is shown. Any τ-structure has a copy such that its U_i-part is in $U_{i\not{u}}$. Thus b is identical on $U_{i\not{u}}$ as an $\eta \to \Delta$ behaviour, or otherwise C admits a behaviour that is e_N on U_i, and we are done by Theorem 7 or Theorem 8. It is clear that b can not map the non-edges to edges on any of the U_{ij} ($j \in \{u, \not{u}\}$), as both contain an I_n.

It is realizable to put a K_{n+i-3} into $U_{i\not{u}}$ and an isolated vertex into U_{iu}. Thus b maps non-edges with one endpoint in $U_{i\not{u}}$ to non-edges, otherwise b could not be realized on this $(n+i-2)$-element structure. Given any finite τ-structure A, its U_i-part can be put into $U_{i\not{u}}$ except for one vertex which is put into U_{iu} (provided that U_{iu} is non-empty). Thus edges between U_{iu} and $U_{i\not{u}}$ are mapped to edges by b, otherwise we can kill all edges in the U_i-part while preserving the non-edges there, and obtain e_N on U_i. If U_{iu} contains an edge then the following construction is realizable. Given a finite τ-structure A such that its U_i-part contains an edge. Then this edge can be put into U_{iu}, while all other vertices of the U_i-part is put into $U_{i\not{u}}$. This shows that if b deletes the edges in U_{iu}, then C admits a behaviour that is e_N on U_i, as we can delete the edges in U_i one-by-one while preserving the non-edges there. Thus b is identical on U_{iu}, as well, otherwise C contains $\mathrm{Sym}(U_i) \times \mathrm{Aut}(U_t, E)$.

To show item (1) observe that for any $i \in \{1, 2\}$ we have that b is identical on and between U_{ij} for all $j \in \{u, \not{u}\}$, otherwise item (2) can be applied. It is realizable to put a K_{n-1} into $U_{2\not{u}}$ and an isolated vertex in any of the U_{1j} with $j \in \{u, \not{u}\}$. Thus intermediate non-edges between U_{1j} and U_{22} are mapped to non-edges for $j \in \{u, \not{u}\}$, otherwise b could not be realized on this structure. As any τ-structure A has a copy such that the U_1-part of A is in $U_{1\not{u}}$ and the U_2-part of A is in $U_{2\not{u}}$, we have that edges between $U_{1\not{u}}$ and $U_{2\not{u}}$ can not be mapped to non-edges, otherwise C would admit a behaviour that is e_N between U_1 and U_2. For the same reason edges between U_{1u} and $U_{2\not{u}}$ are preserved, as any finite τ-structure A with a fixed intermediate edge has a copy such that the U_2-part of A is in $U_{2\not{u}}$, and the U_1-part is in $U_{1\not{u}}$ except for the endpoint of the edge which is in U_{1u}. Thus b is identical between U_{1j} and U_{22} ($j \in \{1, 2\}$) as an $\eta \to \Delta$ behaviour. We show that non-edges between U_{1j} and U_{2u} ($j \in \{u, \not{u}\}$) are mapped to non-edges by b. It is well defined to extend such a non-edge to a K_n^- such that all other vertices of it are in $U_{2\not{u}}$. If the non-edge is mapped to an edge by b, then b could not be realized on this structure. It remains to show that edges between U_{1j} and U_{2u} ($j \in \{1, 2\}$) are mapped to edges by b. Assume there is at least one of these 2-types is mapped to a non-edge by b. Any τ-structure containing an intermediate edge has a copy such that at least one edge is moved into the violated two-type. Then realizing the behaviour b would delete this edge, while intermediate non-edges are preserved. By iterating this step we can delete all intermediate edges while preserving the intermediate non-edges, so C would admit a behaviour that is e_N on the intermediate pairs. $\square$

Lemma 64. *Let the vertices $u, v \neq 0$ in H_n be added to the language of $(H_n, E, 0)$ as constants. Let us denote the language of the structure obtained by η. Assume further that a closed group* $\mathrm{Aut}(H_n, E, 0) \subseteq C$ *admits the behaviour $b : \eta \to \eta$ such that $b(U_{ijk})$ is in U_i for $i \in \{1, 2\}$, $j \in \{u, \not{u}\}$, $k \in \{v, \not{v}\}$.*

(1) • *Either C contains* $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$*,*
 • *or b as an $\eta \to \Delta$ behaviour is identical on and between the U_{1jk} and U_{2mp} for $j, m \in \{u, \not{u}\}$ and $k, p \in \{v, \not{v}\}$.*

(2) *Moreover, for any $i \in \{1, 2\}$ we have that*
 • *either C contains* $\mathrm{Sym}(U_i) \times \mathrm{Aut}(U_t, E)$ *with $\{i, t\} = \{1, 2\}$,*
 • *or b as an $\eta \to \Delta$ behaviour is identical on the U_{ijk} for $j \in \{u, \not{u}\}$, $k \in \{v, \not{v}\}$.*

Proof. At first item (2) is shown. Any τ-structure has a copy such that its U_i-part is in $U_{i\not{u}\not{v}}$. Thus b is identical on $U_{i\not{u}\not{v}}$ as an $\eta \to \Delta$ behaviour, or otherwise C admits a behaviour that is e_N on U_i, and we are done by Theorem 7 or Theorem 8. It is clear that b can not kill non-edges on any of the U_{ijk}, as all of these contain an I_n.

It is realizable to put a K_{n+i-3} into $U_{i\not{u}\not{v}}$ and an isolated vertex to any other 1-type of η in U_i. Thus b maps non-edges with one endpoint in $U_{i\not{u}\not{v}}$ to non-edges, otherwise b could not be realized on this $(n+i-2)$-element structure. Given any finite τ-structure A, its U_i-part can be put into $U_{i\not{u}\not{v}}$ except for one vertex which is put into U_{ijk} with $(j, k) \neq (\not{u}, \not{v})$ (provided that U_{ijk} is non-empty). Thus edges between U_{ijk} and $U_{i\not{u}\not{v}}$ are mapped to edges by b, otherwise we can kill all edges in the U_i-part while preserving the non-edges there, and obtain e_N on U_i. If U_{ijk} $((j, k) \neq (\not{u}, \not{v}))$ contains an edge then the following construction is well defined. Given a finite τ-structure A such that its U_i-part contains an edge. Then this edge can be put in U_{ijk}, while all other vertices of the U_i-part is put into $U_{i\not{u}\not{v}}$. This shows that if b deletes the edges in U_{ijk}, then C admits a behaviour that is e_N on U_i, as we can delete the edges in U_i one-by-one while preserving the non-edges there. Thus b is identical on U_{ijk}, otherwise we are ready either by Theorem 7 or Theorem 8. To obtain that b is identical as an $\eta \to \Delta$ behaviour between $U_{iu\not{v}}$ and $U_{i\not{u}v}$ we first show that non-edges are preserved between these one-types. If this is not the case, then let us consider a copy of K^-_{n+i-2} in $U_{iu\not{v}} \cup U_{i\not{u}v} \cup U_{i\not{u}\not{v}}$ such that the endpoints of the non-edge are moved to $U_{iu\not{v}}$ and $U_{i\not{u}v}$ and all other points are moved to $U_{i\not{u}\not{v}}$. Then b cannot be realized on this structure. If there are edges between $U_{iu\not{v}}$ and $U_{i\not{u}v}$, then the U_i-part of any finite τ-structure A containing an edge in U_i could be put into $U_{iu\not{v}} \cup U_{i\not{u}v} \cup U_{i\not{u}\not{v}}$ such that one edge is moved to a $(U_{iu\not{v}}, U_{i\not{u}v})$-edge. The realization of b on this structure delets an edge in the U_i-part of A and preserves the non-edges. Thus C would admit a behaviour that is e_N on U_i. Hence, we have that b as an $\eta \to \Delta$ behaviour is identical on and between $U_{iu\not{v}}$, $U_{i\not{u}v}$ and $U_{i\not{u}\not{v}}$. We have also shown that b as an $\eta \to \Delta$ behaviour is identical on and between U_{iuv} and $U_{i\not{u}\not{v}}$. The only thing left to prove is that b as an $\eta \to \Delta$ behaviour is identical between U_{iuv} and $U_{iu\not{v}}$ and between U_{iuv} and $U_{i\not{u}v}$. We prove the former one, the latter can be shown similarly. A K^-_{n+i-3} can be put in U_i such that the non-edge is a pair with one vertex in U_{iuv} and the other one in $U_{i\not{u}v}$, and all other points are in $U_{i\not{u}\not{v}}$. Thus non-edges between U_{iuv} and $U_{i\not{u}v}$ are mapped to edges, otherwise b cannot be realized on this structure. If there are edges between U_{iuv} and $U_{i\not{u}v}$, then the U_i-part of any finite τ-structure A can be put in U_i such that an edge is moved to an edge between U_{iuv} and $U_{i\not{u}v}$, and all other vertices are in $U_{i\not{u}\not{v}}$. If the edges between U_{iuv} and $U_{i\not{u}v}$ are deleted, then one-by-one we could delete all edges from the U_i-part and obtain e_N on U_i.

To show item (1) observe that for any $i \in \{1, 2\}$ we have that b is identical on and between U_{ijk} for all j, k, otherwise we are done by item (2). It is realizable to consider a K^*_{n-1} with a K_{n-1} in $U_{2\not{u}\not{v}}$ and an isolated vertex in some U_{1jk}. Thus intermediate non-edges between any U_{1jk} and $U_{2\not{u}\not{v}}$ are mapped to non-edges, otherwise b could not be realized on this structure. As any τ-structure A has a copy such that the U_1-part of A is in $U_{1\not{u}\not{v}}$ and the U_2-part of A is in $U_{2\not{u}\not{v}}$, we have that edges between $U_{1\not{u}\not{v}}$ and $U_{2\not{u}\not{v}}$ can not be mapped to non-edges, otherwise C would admit a behaviour that is e_N on the intermediate edges. For the same reason edges between U_{1jk} and $U_{2\not{u}\not{v}}$ are preserved for all j, k. Indeed, any finite τ-structure A with a fixed intermediate edge has a copy such that the U_2-part of A is in $U_{2\not{u}\not{v}}$, and the U_1-part is in $U_{1\not{u}\not{v}}$ except for one endpoint of the edge which is in U_{1jk}. Thus b is identical between U_{1jk} and $U_{2\not{u}\not{v}}$ as an $\eta \to \Delta$ behaviour. We show that non-edges between U_{1jk} and U_{2mp} for $(m, p) \neq (\not{u}, \not{v})$ are mapped to non-edges by b. It is realizable to extend such a non-edge to a K^-_n such that all other vertices of it are in $U_{2\not{u}\not{v}}$. If the non-edge is mapped to an edge by b, then b could not be realized on this structure. To finish the proof we show that edges between U_{1jk} and U_{2mp} for $(m, p) \neq (\not{u}, \not{v})$ are mapped to edges by b. Assume that there is at least one such two-type that is mapped to a non-edge by b. Any τ-structure containing an intermediate edge has a copy such that at least one edge is put in the violated 2-type. Then realizing the behaviour b would kill this edge, while intermediate non-edges are preserved. By iterating this step we can delete all intermediate edges while preserving the intermediate non-edges, so C would admit a behaviour that is e_N on the intermediate pairs. □

Theorem 9. *Let* $\mathrm{Aut}(H_n, E, 0) \subsetneq C$ *be a closed group. Then* C *contains one of the following groups*

(1) $\mathrm{Aut}(H_n, E)$,
(2) $\mathrm{Aut}(H_n, 0, E|_{H_n\setminus\{0\}})$,
(3) $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$.

Proof. The proof is a case-by-case analysis of the orbit system of C. This system can be refined to $\{0, U_1, U_2\}$, thus the following cases cover everyhing.

- Either U_1 and U_2 are in one C-orbit,
- or $\{0\}$ is not a C-orbit, and U_1 and U_2 are not in the same C-orbit,
- or the orbit system is $\{0, U_1, U_2\}$.

*C*ase 1. U_1 and U_2 are in one C-orbit.

Then there exists a $u \in U_1$ that is mapped to U_2 by some $\pi \in C$. Let us add u to the language as a constant. Then we obtain the structure $(H_n, E, 0, u, U_{1u}, U_{1\not u}, U_{2u}, U_{2\not u})$. Let η denote the language of this structure.

Let b be a behaviour $\eta \to \tau$.

Let $A \in \mathrm{Age}(H_n, E, 0)$. It is realizable to extend A by a vertex in U_1 that is disconnected to all vertices of A (except for 0.) Hence, every τ-structure A has a copy in H_n such that the U_1-part of A is in $U_{1\not u}$ and the U_2-part of A is in $U_{2\not u}$. Thus we have that b is identical on and between $U_{1\not u}$ and $U_{2\not u}$, otherwise we are done by to Proposition 59. Another realizable way to distribute the nonzero vertices of any $A \in \mathrm{Age}(H_n, E, 0, U_1, U_2)$ is that we put the U_1-part into $U_{1\not u}$, and all but one points in the U_2 part into $U_{2\not u}$, while the last point is put in U_{2u}. Thus if U_{2u} is mapped to U_1, then by using this construction and by applying a permutation in C that realizes the behaviour b on A, we can map the U_2-part of A element-by-element to U_1. According to Corollary 60 this would imply that C contains one of the minimal groups. Thus we may assume that U_{2u} is preserved in U_2 by b. Given any $A \in \mathrm{Age}(H_n, E, 0, U_1, U_2)$ we can map an arbitrary point from the U_1-part of A to u by a permutation in $\mathrm{Aut}(H_n, E, 0)$. By applying a permutation that realizes the behaviour b on this substructure, we map at least one vertex in the U_1-part of A to U_2, while all its vertices in U_2 are preserved there. Thus after finitely many steps we can collect the U_1-part of A in U_2. Hence, we are ready by Corollary 60.

*C*ase 2. $\{0\}$ is not a C-orbit, and U_1 and U_2 are not in the same C-orbit.

The orbit system is either $\{U_1 \cup \{0\}, U_2\}$ or $\{U_2 \cup \{0\}, U_1\}$. We denote by U_i the set that is in the same orbit as 0. According to Lemma 62 there is a $u \in U_i$ and a $\pi \in C$ such that π switches 0 and u. Again, we add u to the language as a constant Then we obtain the structure $(H_n, E, 0, u, U_{1u}, U_{1\not u}, U_{2u}, U_{2\not u})$ with language denoted by η. Let $b : \eta \to \tau$ be a behaviour that is admitted by π.

According to Lemma 63, we have that $b(U_{i\not{u}}) = U_{i\not{u}}$ ($i \in \{1,2\}$), as a K_{n+i-3} in $U_{i\not{u}}$ can not be identically mapped into U_{iu}. Assume that the orbit system of C is $\{U_p \cup \{0\}, U_q\}$. Let $A \in \mathrm{Age}(H_n, E, 0, U_1, U_2)$.

If $p = 1$, that is, if $u \in U_1$, then we put a vertex in the U_1-part of A into u. By applying a permutation in C that realizes the behaviour b on A, this vertex goes to 0, and the τ-structure induced by the remaining vertices in $U_1 \cup U_2$ is preserved. It is realizable to put the U_1-part of the image into $U_{1\not{u}}$ and the U_2-part to $U_{2\not{u}}$. By applying a permutation in C that realizes the behaviour b on the substructure obtained, the original intermediate pairs containing the vertex put into u turn to non-edges, while all other intermediate pairs are preserved. By iterating this step, we can turn all intermediate pairs of A into non-edges, and then we are ready by Proposition 52.

If $p = 2$, i.e., if $u \in U_2$, then we put a vertex in the U_2-part into u. By applying a permutation in C that realizes the behaviour b on A, this vertex goes to 0. It is realizable to move all vertices in the U_2-part of the image into $U_{2\not{u}}$. By applying a permutation in C that realizes the behaviour b on the image, the original pairs in U_2 containing the vertex put into u turn to non-edges, while all other edges and non-edges in U_2 are preserved. By iterating this step, we can turn all pairs of A in U_2 into non-edges, and then we are done by Theorem 7.

*C*ase 3. The orbit system is $\{0, U_1, U_2\}$.

Then C does not violate any of the unary relations in $(H_n, E, 0, U_1, U_2)$, thus it violates E. As edges and non-edges containing 0 are preserved, an edge or a non-edge in $H_n \setminus \{0\}$ is violated. Considering the inverse of the permutation if necessary, we may assume that an edge is mapped to a non-edge by some $\pi \in C$. Assume that the edge relation is violated only on U_1 by π and nowhere else. Then there is a non-edge in U_1 that is mapped to an edge in U_1. Let us extend the violated non-edge with $(n-2)$ points in U_2. It is realizable that these n vertices constitute a K_n^-. Then π maps this substructure to a K_n, which is a contradiction.

So there are two cases left; the violated edge is either an intermediate one or an edge in U_2.

*C*ase 3.a. The edge uv that we map to a non-edge by $\pi \in C$ is an intermediate one.

Let us add the vertices $u \in U_1$ and $v \in U_2$ to the language as constants. Consider the structure $(H_n, E, 0, U_{1uv}, U_{1u\not{v}}, U_{1\not{u}v}, U_{1\not{u}\not{v}}, U_{2uv}, U_{2u\not{v}}, U_{2\not{u}v}, U_{2\not{u}\not{v}})$ the language of which is denoted by η.

According to item (1) of Lemma 64 we may assume that b is identical on and between the infinite 1-types U_{ijk}. Let $w \in \{u, v\}$. The behaviour b can't map non-edges between w and $U_{2\not{u}\not{v}}$ to edges, otherwise b could not be realized on a K_{n-1}^* with a K_{n-1} in $U_{2\not{u}\not{v}}$ and an isolated vertex in w. Thus b is identical between w and $U_{2\not u\not v}$. Assume that b violates the edge relation between u and $U_{2u\not v}$. Let A be a finite τ-structure which has an edge in U_2. It is realizable to put an endpoint of this edge into u, the vertices in the U_2-part connected to u into $U_{2u\not v}$ and the vertices in the U_2-part not connected to u into $U_{2\not u\not v}$. If we realize the behaviour on this copy of A, then the vertex mapped to u becomes isolated, but otherwise the graph structure of the U_2-part of A is preserved. Hence, by iterating this procedure, we could map the U_2-part to an independent set of vertices. Then C admits a behaviour that is e_N on U_2, and then we are done by Theorem 7. Thus we may assume that edges between u and $U_{2u\not v}$ are mapped to edges. Similarly, we may assume that edges between v and $U_{2\not u v}$ are mapped to edges. Let $U_{ijk} \neq U_{2\not u\not v}$ such that there are non-edges between u and U_{ijk}, and assume that b maps this non-edge to an edge. It is realizable to put a copy of K_n^- into H_n such that the non-edge is moved to a non-edge between u and U_{ijk}, and all other vertices are put into $U_{2u\not v}$. As b can not be realized on this structure, this is a contradiction. Hence, b maps non-edges to all types of non-edges between u and some U_{ijk}. Similarly, b maps non-edges to all types of non-edges between v and some U_{ijk}. Assume that an edge relation between u and some U_{ijk} is mapped to a non-edge by b. The $U_1 \cup U_2$-part of a given finite τ-structure has a copy such that an edge is put into this violated two-type (if there is such an edge in the structure). If this edge is an intermediate one, then we can map all intermediate pairs to non-edges and obtain a behaviour that is e_N between U_1 and U_2. If this edge is in U_2, then we can map all pairs in U_2 to non-edges and obtain a behaviour that is e_N on U_2. Thus we may assume that b is identical on and between all 1-types of η in $U_1 \cup U_2$ except that the edge uv is mapped to a non-edge. Then we can delete the U_2-edges of a given finite τ-structure one-by-one, while the graph relation is preserved everywhere else in the $U_1 \cup U_2$-part. This yields a behaviour that is e_N on U_2.

*C*ase 3.b. The edge uv that is mapped to a non-edge by $\pi \in C$ is in U_2.

With the same argument as in *C*ase 3.a. we may assume that b is identical on and between all one-types of η in $U_1 \cup U_2$ except that the edge uv is mapped to a non-edge. Then we can delete the intermediate edges of a given $A \in \mathrm{Age}(H_n, E, 0, U_1, U_2)$ one-by-one, while the graph relation is preserved everywhere else in the $U_1 \cup U_2$-part. Thus C admits a behaviour that is e_N between U_1 and U_2. □

4.5. **Characterization of the reducts.** In this subsection we finish the characterization of the reducts of $(H_n, E, 0, U_1, U_2)$. It is shown that the reducts of $(H_n, E, 0, U_1, U_2)$ up to first order interdefinability are those which were described in Subsection 4.2. We continue the proof of this result by determining the minimal closed groups above $\mathrm{Aut}(H_n, 0, E|_{H_n \setminus \{0\}})$ and $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$.

Theorem 10. *Every closed group C above* $\mathrm{Aut}(H_n, 0, E|_{H_n\setminus\{0\}})$ *contains* $\mathrm{Sym}(H_n \setminus \{0\})$.

Proof. If C stabilizes 0 then the restriction of all permutations in C to $H_n \setminus \{0\}$ is a closed group on $H_n \setminus \{0\}$ containing $\mathrm{Aut}(H_n, 0, E|_{H_n\setminus\{0\}})$. According to Theorem 4 the only such groups are $\mathrm{Aut}(H_n, 0, E|_{H_n\setminus\{0\}})$ and $\mathrm{Sym}(H_n \setminus \{0\})$. Thus all we have to show is that the stabilizer of 0 in C is strictly bigger than $\mathrm{Aut}(H_n, 0, E|_{H_n\setminus\{0\}})$.

By Lemma 62 we may assume that there exist $\pi \in C$ and $u \in U_1$ such that π swithes 0 and u. We add u to the language as a constant, and obtain the structure $(H_n, E, 0, u, U_{1u}, U_{1\not u}, U_{2u}, U_{2\not u})$ with language η. Let $b : \eta \to \eta$ be a behaviour admitted by π.

If b as an $\eta \to \Delta$ behaviour is not identical on $U_{2\not u}$ then C admits a behaviour that as a $\tau \to \Delta$ behaviour is e_N on U_2. If this is the case, then C contains $\mathrm{Sym}(U_2)$, which implies that there are additional permutations in the stabilizer of 0. Thus assume that b is identical on $U_{2\not u}$ as an $\eta \to \Delta$ behaviour. Then b is also identical on $U_{2\not u}$ as an $\eta \to \tau$ behaviour, as not all finite subgraphs of U_2 can be identically mapped to U_1. Moreover, b is identical on $U_{2\not u}$ as an $\eta \to \eta$ behaviour, as well. The only thing left to check is that $b(U_{2\not u}) \neq b(U_{2u})$, which holds as a K_{n-1} in $U_{2\not u}$ cannot be identically mapped to U_{2u}. Thus $b(U_{2\not u}) = U_{2\not u}$.

Assume that the behaviour b can turn non-edges between U_{2u} and $U_{2\not u}$ into edges. It is realizable to put an isolated vertex into U_{2u} and a K_{n-1} into $U_{2\not u}$, but b can not be realized on this copy of K^*_{n-1}. Thus b maps non-edges to the non-edges between U_{2u} and $U_{2\not u}$. If the edges between these two two-types are deleted by b, then we show that C contains $\mathrm{Sym}(U_2)$. Let A be any finite τ-structure, and chose a vertex in the U_2-part of A. It is realizable to consider a copy of A such that the U_1-part of A is put in $U_{1\not u}$, and the U_2-part of A is put in $U_{2\not u}$ except for the chosen vertex which is put in U_{2u}. As C contains $\mathrm{Aut}(H_n, 0, E|_{H_n\setminus\{0\}})$ we can map $A \setminus \{0\}$ to this copy. If we apply π to realize b on this copy of A, then the composition of these permutations turn all pairs containing the fixed vertex into non-edges and preserve all other edges and non-edges in $A \setminus \{0\}$. By iterating this process we map $A \setminus \{0\}$ to an independent set, in particular, e_N can be obtained in U_2. Thus we may assume that b as an $\eta \to \Delta$ behaviour is identical between U_{2u} and $U_{2\not u}$.

Given any $A \in \mathrm{Age}(H_n, E, 0, U_1, U_2)$ with U_2-part Y it is realizable to put an edge in Y into U_{2u} and all other vertices of Y into $U_{2\not u}$. Using this construction and a similar argument as above we may assume that $b : \eta \to \Delta$ is not e_N on U_{2u}. As U_{2u} contains an I_n we have that $b : \eta \to \Delta$ does not map edges to non-edges in U_{2u}. Thus we may assume that b as an $\eta \to \Delta$ behaviour is identical on and between U_{2u} and $U_{2\not u}$.

Let A be a finite τ-structure. We show that the U_2-part Y of A can be mapped to an independent set of vertices by C, and thus C contains $\mathrm{Sym}(U_2)$. We obtain a K_n-free graph by adding a vertex (0) connected to a fixed $v \in Y$ and disconnected to all other vertices in Y. Thus by an appropriate permutation in $\mathrm{Aut}(H_n, 0, E|_{H_n \setminus \{0\}})$ we can map Y to a substructure of H_n such that v is mapped to u, neighbours of v are mapped to U_{2u} and non-neighbours of v are mapped to $U_{2\not u}$. By applying π to realize the behaviour b the vertex u is mapped to 0 and the subgraph induced by the other vertices is preserved. By some permutation in $\mathrm{Aut}(H_n, 0, E|_{H_n \setminus \{0\}})$ this subgraph can be moved into $U_{2\not u}$. If we realize b again, then 0 is mapped to u while $U_{2\not u}$ is preserved. Hence, the composition of the applied permutations deletes all edges from v but preserves the graph structure of Y, otherwise. By iterating this step we can map Y to an independent set of vertices. Thus C contains $\mathrm{Sym}(U_2)$, and the stabilizer of 0 in C is strictly bigger than $\mathrm{Aut}(H_n, 0, E|_{H_n \setminus \{0\}})$. $\square$

Theorem 11. *Every closed group C above $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$ contains either $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2)$ or $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1)$.*

Proof. First assume that 0 is not an orbit of C. Then according to Lemma 62 we have a $u \in U_i$ for some $i \in \{1, 2\}$ such that there exists a $\pi \in C$ switching 0 and u. There are two cases. Either there exists a $u \in U_2$ that can be switched with $\{0\}$ by some permutation in C, or all such elements u are in U_1.

*C*ase 1. Assume that we have such a u in U_2.

We add u to the language as a constant and denote the extended language of $(H_n, E, 0, u, U_{1u}, U_{1\not u}, U_{2u}, U_{2\not u})$ by η. Let $b : \eta \to \eta$ be a behaviour admitted by π. As before, b as an $\eta \to \Delta$ behaviour is identical on $U_{1\not u}$ and on $U_{2\not u}$, otherwise C contains $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1)$ or $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2)$, respectively. In particular, $b(U_{2\not u}) = U_{2\not u}$, as $U_2 \not u$ can not be identically mapped to any other one-type of η. We can consider a K^*_{n-1} with an isolated vertex put in $U_{1\not u}$ and a K_{n-1} put in $U_{2\not u}$. Thus we have that b maps non-edges between $U_{1\not u}$ and $U_{2\not u}$ to non-edges. We show that if $b(U_{2u}) \in U_2$ then C admits a behaviour that is e_N on U_2. Let A be a finite τ-structure with U_2-part Y. Let $v \in Y$. We put v into u and realize the behaviour b. Then u maps to 0, and the τ-structure of $Y \setminus v$ is preserved. With an appropriate automorphism in $\mathrm{Aut}(H_n, E, 0)$ we can move the image of $Y \setminus v$ into $U_{2\not u}$. If we realize b, then the U_2-part is preserved except that v becomes isolated. By iterating this step Y can be mapped to an independent set of vertices.

Assume that $b(U_{2u}) \in U_1$. Again, we show that C admits a behaviour that is e_N on U_2. Let A be a finite τ-structure with U_1-part x and U_2-part y. Let $v \in Y$. If $b(U_1 \not u) \in U_1$, then we can collect $X \cup Y$ in U_1. Indeed, by applying an appropriate automorphism in $\mathrm{Aut}(H_n, E, 0)$ we may assume that $X \in U_{1u}$, $v \in U_{2u}$ and $Y \setminus v \subseteq U_{2\not u}$, as this is a realizable distribution of the vertices. By realizing b on this structure we map v in U_1, while X is preserved in U_1 and $Y \setminus \{v\}$ is preserved in U_2. Thus one-by-one all vertices in Y can be mapped to U_1. It remains to prove that C admits a behaviour that is e_N on U_2 if $b(U_{1\not u}) \in U_2$. At first we move X to $U_{1\not u}$ and Y to $U_{2\not u}$, and realize b. Thus we may assume that all non-zero vertices of A are in U_2, that is $X = \emptyset$. If we put v into U_{2u} and $Y \setminus \{v\}$ into $U_{2\not u}$, and realize b, then v is moved to U_1 and $Y \setminus \{v\}$ is preserved in U_2. By an appropriate permutation in $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$ we can map this image to a substructure such that everything is preserved except that all intermediate pairs are turned to non-edges. We may assume that the image of v is still in $U_{1\not u}$ and the image of $Y \setminus \{v\}$ is still in $U_{2\not u}$. By realizing b again Y is mapped into U_2, and the graph structure of it is preserved except that v has become an isolated vertex. After finitely many steps Y can be mapped to an independent set of vertices, and thus C admits a behaviour that is e_N on U_2.

*C*ase 2. Assume that all such u are in U_1.

In particular, $U_1 \cup \{0\}$ is a C-orbit according to Lemma 62. Let us fix a $u \in U_1$ and a $\pi \in C$ such that π switches 0 and u. We add u to the language as a constant and denote the language of $(H_n, E, 0, u, U_{1u}, U_{1\not u}, U_{2u}, U_{2\not u})$ by η.

According to Lemma 63 item (2) we may assume that for any $i \in \{1, 2\}$ the behaviour b is identical on and between U_{iu} and $U_{i\not u}$. Note that $b(U_{1\not u}) = U_{1\not u}$ as $U_{1\not u}$ can not be identically mapped to U_{1u}. We show that C admits a behaviour that is e_N on U_1. Let A be a finite τ-structure with U_1-part X. Let $v \in X$. We put v into u and realize b. Then u is mapped to 0 and the graph structure of $X \setminus \{v\}$ is preserved. By applying a permutation in $\mathrm{Aut}(H_n, E, 0)$ we can move the image of $X \setminus \{v\}$ into $U_{1\not u}$. By realizing b again the graph structure of X is preserved, except that u has become an isolated vertex. By iterating this step we can map X to an independent set of vertices, and obtain that C admits a behaviour that is e_N on U_1.

In both cases it turned out that the group C contains either $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2)$ or $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1)$. Thus we may assume that $\{0\}$ is an orbit.

Assume that C maps edges in U_2 to edges. As any non-edge in U_2 can be extended to a K_n^- in U_2, we have that non-edges is U_2 are also mapped to non-edges. Thus $C|_{U_2}$ consists of graph isomorphisms. As U_1 does not contain a K_{n-1}, U_2 can not be embedded into U_1. Thus for all $\pi \in C$ there exists a vertex $v \in U_2$ such that $\pi(v) \in U_2$. Assume that there is a $w \in U_2$ such that $\pi(w) \in U_1$ and vw is an edge. Then the edge vw is mapped to an intermediate pair by π, and the image can be mapped to a non-edge by some permutation in $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$. Hence, the neighbourhood of v is also mapped into U_2 by π. The graph (U_2, E) has diameter 2, thus $\pi(U_2) \subseteq U_2$. As π was an arbitrary element in C, we have that $\pi^{-1}(U_2) \subseteq U_2$, as well. We obtain that $\pi(U_2) = U_2$ for all $\pi \in C$, and $\pi|_{U_2} \in \mathrm{Aut}(U_2, E)$. For a $\pi \in C$ let us denote by $\bar{\pi}$ the permutation that is identical on $U_1 \cup \{0\}$ and acts as $\pi|_{U_2}$ on U_2. Then $\bar{\pi} \in \mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E) \subseteq C$ for all π. Let $\pi \in C \setminus \mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$. Then $\rho = \bar{\pi}^{-1} \circ \pi \in C \setminus \mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$ stabilizes every vertex in $U_2 \cup \{0\}$. Thus ρ violates the edge relation in U_1. As C is closed and it contains $\mathrm{Aut}(U_1, E) \cup \{\rho\}$, we have that it contains the local closure of $\mathrm{Aut}(U_1, E) \cup \{\rho\}$. According to Theorem 4 this local closure is $\mathrm{Sym}(U_1)$, thus C contains $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1)$.

Thus we may assume that C stabilizes 0 and some $\pi \in C$ violates an edge uv in U_2. We add u and v to the language as constants and obtain the structure $(H_n, E, 0, u, v, \{U_{ijk} | 1 \leq i \leq 2, j \in \{u, \not{u}\}, k \in \{v, \not{v}\}\})$ with language η. Let b be a behaviour admitted by π. We may assume that b as an $\eta \to \Delta$ behaviour is identical on $U_{1\not{u}\not{v}}$ and $U_{2\not{u}\not{v}}$, otherwise we are done by Theorem 7 or Theorem 8. In particular, $b(U_{2\not{u}\not{v}}) = U_{2\not{u}\not{v}}$ as $U_{2\not{u}\not{v}}$ can not be identically mapped to any other one-type. Given any one-type of η different from $U_{2\not{u}\not{v}}$ we can put an isolated vertex into that one-type and a K_{n-1} into $U_{2\not{u}\not{v}}$. Thus non-edges with one endpoint in $U_{2\not{u}\not{v}}$ are mapped to non-edges by b, otherwise the behaviour could not be realized on this structure. Our goal is to show that b maps the one-types of η in U_2 to U_2.

At first assume that $b(U_{1\not{u}\not{v}}) \in U_1$. Let A be a finite τ-structure with U_1-part X and U_2-part Y. By applying a permutation in $\mathrm{Aut}(H_n, E, 0)$ any vertex of Y can be put into any one-type U_{2jk} with $(j, k) \neq (\not{u}, \not{v})$, while all other vertices of Y are moved into $U_{2\not{u}\not{v}}$, and all vertices of X are moved into $U_{1\not{u}\not{v}}$. Thus if $b(U_{2jk}) \in U_1$ for some $(j, k) \neq (\not{u}, \not{v})$, then we can realize b on this structure and move an arbitrary vertex of Y to U_1, while all other vertices in Y are preserved in U_2, and all vertices in X are preserved in U_1. Hence, vertices of Y can be moved to X. Thus C admits a behaviour that maps U_2 to U_1, and consequently this behaviour is e_N on U_2. Thus $b(U_{2jk}) \in U_2$. If u (or v) is mapped to U_1, then first we map all intermediate pairs of A to intermediate non-edges by $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$. Then we can put any vertex of Y into u (or v). Then by realizing b we can move a vertex of Y to U_1, while all other vertices in Y are preserved in U_2, and all vertices in X are preserved in U_1. Thus with the same argument as above we are done. So we may assume that b maps the one-types of η in U_2 to U_2.

Assume that $b(U_{1\not{u}\not{v}}) \in U_2$. Let A be a finite τ-structure with U_1-part X and U_2-part Y. By applying a permutation in $\mathrm{Aut}(H_n, E, 0)$ any vertex of Y can be put into any one-type U_{2jk} with $(j, k) \neq (\not{u}, \not{v})$, while all other vertices of Y are moved into $U_{2\not{u}\not{v}}$, and all vertices of X are moved into $U_{1\not{u}\not{v}}$. Thus if $b(U_{2jk}) \in U_1$ for some $(j, k) \neq (\not{u}, \not{v})$, then we can realize b on this structure and move an arbitrary vertex of Y to U_1, while all other vertices in $X \cup Y$ are moved into U_2. All intermediate pairs of the image can be mapped to intermediate non-edges by $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$. We may assume that the U_1-part of this image is in $U_{1\not u \not v}$ and the U_2-part of this image is in $U_{2\not u \not v}$. Realizing b again yields that all vertices in $X \cup Y$ are moved to U_2 and a vertex in Y becomes isolated, while non-edges in $X \cup Y$ are preserved. Hence if $b(U_{2jk}) \in U_1$ for some $(j, k) \neq (\not u, \not v)$, then C admits a behaviour that is e_N on U_2, and we are done by Theorem 7. The same argument works if u (or v) is mapped to U_1. So in this case we may also assume that b maps the one-types of η in U_2 to U_2.

Acording to Lemma 63 we may assume that b as an $\eta \to \Delta$ behaviour is identical on and between the U_{2jk}. Assume that b violates the edge relation between u and $U_{2u\not v}$. Let A be a finite τ-structure which has an edge in U_2. It is well defined to put an endpoint of this edge into u, the vertices in the U_2-part connected to u into $U_{2u\not v}$ and the vertices in the U_2-part disconnected to u into $U_{2\not u \not v}$. If we realize the behaviour b on this copy of A, then the vertex mapped to u becomes isolated, but otherwise the graph structure of the U_2-part of A is preserved. Thus we could map the U_2-part to an independent set of vertices. Hence, C admits a behaviour that is e_N on U_2, and then we are done by Theorem 7. Thus we may assume that edges between u and $U_{2u\not v}$ are mapped to edges. Similarly, we may assume that edges between v and $U_{2\not u v}$ are mapped to edges. Assume that b maps the non-edge between u and $U_{2\not u v}$ to an edge. It is realizable to put a copy of K_n^- into H_n such that the non-edge is moved to a non-edge between u and $U_{2\not u v}$, and all other vertices are put into $U_{2u\not v}$. As b can not be realized on this structure, this is a contradiction. Hence, b maps non-edges to all types of non-edges between u and some U_{2jk}. Similarly, b maps non-edges to all types of non-edges between v and some U_{2jk}. Thus non-edges in the U_2-part are preserved by b. Let A be a τ-structure with at least one edge in the U_2-part. If this edge is moved into uv and b is realized, then at least one edge in the U_2-part is deleted, while the U_2-part is mapped to U_2 and non-edges are preserved there. Thus edges in the U_2-part can be successively deleted, and C admits a behaviour that is e_N on U_2. $\square$

Lemma 65. *Let C be a closed group with $\mathrm{Aut}(H_n, E, 0) \subseteq C$. Assume that $U_1 \cup U_2$ is contained in a C-orbit. Then there exist a permutation $\pi \in C$ and two vertices in U_1 such that π maps both of these vertices into U_2.*

Proof. As $U_1 \cup U_2$ is contained in a C-orbit there exist $\rho \in C$ and $x_1 \in U_1$ such that $y_1 = \rho(x_1) \in U_2$. We may assume that no vertex in U_1 other than x_1 is mapped into U_2 by ρ. There is at most one vertex in U_2 that is mapped into U_1 by ρ, otherwise ρ^{-1} satisfies the condition of the lemma. Hence, there exist $x_2 \in U_1$, $y_2 \in U_2$ such that the pair $x_2 y_1$ has the same 2-type in $(H_n, E, 0)$ as $x_1 y_2$, $\rho(y_2) \in U_2$ and $\rho^{-1}(x_2) \in U_1$. As

$(H_n, E, 0)$ is homogeneous, there is a permutation $\alpha \in \mathrm{Aut}(H_n, E, 0)$ such that $\alpha(x_2) = x_1$ and $\alpha(y_1) = y_2$. Then $\pi = \rho \circ \alpha \circ \rho \in C$ moves x_1 and $\rho^{-1}(x_2)$ into U_2. $\square$

Theorem 12. *Let C be a closed group.*

(1) *If C is above* $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2)$*, then C contains either* $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$*, or* $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2 \cup \{0\})$*.*

(2) *If C is above* $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1)$*, then C contains either* $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$*, or* $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1 \cup \{0\})$*.*

Proof. We show item (1), item (2) is analogous.

If the orbit system of C is $\{U_1, U_2, \{0\}\}$, then some $\pi \in C$ violates E on U_1. As $\pi|_{U_2} \in \mathrm{Sym}(U_2)$, we have that $\rho = \pi|_{U_2}^{-1} \circ \pi \in C$ also violates E on U_1, and ρ is identical on $U_2 \cup \{0\}$. Hence, C contains the closed group generated by $\mathrm{Aut}(U_1, E)$ and ρ_{U_1}. According to Theorem 4 this closed group is $\mathrm{Sym}(U_1)$, thus C contains $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$.

If the orbit system of C is $\{U_1, U_2 \cup \{0\}\}$, then some $\pi \in C$ moves 0 into U_2. Let $\pi(0) = u$ and $\pi^{-1}(0) = v$. Let $\alpha \in \mathrm{Sym}(U_2)$ such that $\alpha(u) = v$. Then $\rho_1 = \pi \circ \alpha \circ \pi$ stabilizes 0. Let $\beta, \gamma \in C$ such that $\beta(u) \neq v$ and $\gamma(\pi(\beta(u))) = v$. Then $\rho_2 = \pi \circ \gamma \circ \pi \circ \beta \circ \pi$ stabilizes 0. Observe that $\rho_1|_{U_1} = (\pi|_{U_1})^2$ and $\rho_2|_{U_1} = (\pi|_{U_1})^3$. If any of the $\rho_i|_{U_1}$ is not in $\mathrm{Aut}(U_1, E)$, then we are ready by the previous case. If $\rho_1|_{U_1}, \rho_2|_{U_1} \in \mathrm{Aut}(U_1, E)$, then $\pi|_{U_1}$ is also in $\mathrm{Aut}(U_1, E)$. As $\mathrm{Aut}(U_1, E) \subseteq C$, by multiplying by $(\pi|_{U_1})^{-1}$ if necessary we may assume that π is identical on U_1. According to Proposition 56 we have that C contains $\mathrm{Sym}(U_2 \cup \{0\})$, and consequently $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2 \cup \{0\}) \subseteq C$.

If the orbit system of C is $\{U_2, U_1 \cup \{0\}\}$, then according to Lemma 62 there exist $\pi \in C$ and $u \in U_1$ such that π switches 0 and u. We may assume that π is identical on U_2. We add u to the language as a constant and denote the language of the structure $(H_n, E, 0, u, U_{1u}, U_{1\not u}, U_{2u}, U_{2\not u})$ by η. Let b be a behaviour $\eta \to \eta$ admitted by π. According to Lemma 63 we may assume that b is identical on and between U_{1u} and $U_{1\not u}$. In particular, $b(U_{1\not u}) = U_{1\not u}$, as $U_{1\not u}$ can not be identically mapped to U_{11}. Let A be a finite τ-structure with U_1-part X. Assume that X contains an edge. It is realizable to put an endpoint of this edge into u, neighbours of u in X into U_{1u} and non-neighbours into $U_{1\not u}$. We realize b on this structure, and move the vertices mapped to U_1 into $U_{1\not u}$ and the vertices mapped to U_2 into $U_{2\not u}$. If we realize b then u becomes an isolated vertex in X and the graph structure of X is otherwise preserved. By applying this step to all vertices in X we can move X to an independent set of vertices in U_1. Hence, C admits a behaviour that is e_N on U_1.

Finally, we assume that $U_1 \cup U_2$ is contained in a C-orbit. According to Lemma 65 there exist $\pi \in C$ and $u, v \in U_1$ such that $\pi(u), \pi(v) \in U_2$. The transposition $t_{\pi(u)\pi(v)}$ switching $\pi(u)$ and $\pi(v)$ is in C. Thus $t_{uv} = \pi^{-1} \circ t_{\pi(u)\pi(v)} \circ \pi \in C$. Using the automorphisms in

$\mathrm{Aut}(H_n, E, 0)$ we can switch any pair in U_1 with the same τ-type as (u, v). Note that the Henson graphs and the complement of the Henson graphs are connected, except for (H_2, E), which is empty. Moreover, the non-empty graphs obtained this way have diameter 2. Hence, if the 2-type of (x, y) is different from the 2-type of (u, v) for some $x, y \in U_1$, then there is some $z \in U_1$ such that the 2-type of (x, z) and (y, z) are the same as that of (u, v). Thus $t_{xy} = t_{xz} \circ t_{zy} \circ t_{xz} \in C$. Hence, all transpositions of elements in U_1 are in C, and thus $\mathrm{Sym}(U_1) \subseteq C$. $\square$

We prove the main theorem of the section.

Theorem 13. (i) $(H_3, E, 0)$ *has 13 reducts. The closed groups* C *such that* $\mathrm{Aut}(H_3, E, 0) \subseteq C$ *are*

- $\mathrm{Aut}(H_3, E, 0)$,
- $\mathrm{Aut}(H_3, E)$,
- $\mathrm{Aut}(H_3, 0, E|_{H_3 \setminus \{0\}})$,
- $\mathrm{Sym}(U_1) \times \mathrm{Aut}(U_2, E)$,
- $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$,
- $\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Aut}(U_2, E)$,
- $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)) \rtimes Z_2$,
- $\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Sym}(U_2)$,
- $\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)$,
- $\mathrm{Stab}_0(\mathrm{Sym}(H_3))$,
- $(\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Sym}(U_2)) \rtimes Z_2$,
- $(\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)) \rtimes Z_2$,
- $\mathrm{Sym}(H_3)$.

(ii) $(H_n, E, 0)$ *has 16 reducts for* $n \geq 4$. *The closed groups* C *such that* $\mathrm{Aut}(H_n, E, 0) \subseteq C$ *are*

- $\mathrm{Aut}(H_n, E, 0)$,
- $\mathrm{Aut}(H_n, E)$,
- $\mathrm{Aut}(H_n, 0, E|_{H_n \setminus \{0\}})$,
- $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$,
- $\mathrm{Sym}(U_1) \times \mathrm{Aut}(U_2, E)$,
- $\mathrm{Sym}(U_2) \times \mathrm{Aut}(U_1, E)$,
- $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$,
- $\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Aut}(U_2, E)$,
- $\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Aut}(U_1, E)$,
- $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)) \rtimes Z_2$,

- $\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Sym}(U_2)$,
- $\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)$,
- $\mathrm{Stab}_0(\mathrm{Sym}(H_n))$,
- $(\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Sym}(U_2)) \rtimes Z_2$,
- $(\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)) \rtimes Z_2$,
- $\mathrm{Sym}(H_n)$.

Proof. Let C be a closed group above $(H_n, E, 0)$. Then C contains at least one of the closed groups $\mathrm{Aut}(H_n, E)$, $\mathrm{Aut}(H_n, 0, E|_{H_n \setminus \{0\}})$ and $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$ by Theorem 9. For $n = 3$ the group $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$ is $\mathrm{Sym}(U_1) \times \mathrm{Aut}(U_2, E)$, as (U_1, E) is an empty graph. According to Theorem 4 the group $\mathrm{Aut}(H_n, E)$ is a maximal closed subgroup of $\mathrm{Sym}(H_n)$. By Theorem 10 we have that if C is above $\mathrm{Aut}(H_n, 0, E|_{H_n \setminus \{0\}})$, then C contains $\mathrm{Stab}_0(\mathrm{Sym}(H_n))$. As this stabilizer is a maximal subgroup of $\mathrm{Sym}(H_n)$ by Proposition 56, the list contains all the groups containing $\mathrm{Aut}(H_n, E)$ or $\mathrm{Aut}(H_n, 0, E|_{H_n \setminus \{0\}})$. Assume that C is above $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$. Then according to Theorem 11 we have that C contains either $\mathrm{Sym}(U_1) \times \mathrm{Aut}(U_2, E)$ or $\mathrm{Sym}(U_2) \times \mathrm{Aut}(U_1, E)$. The latter one is $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$ if $n = 3$. By Theorem 12 either $C = \mathrm{Sym}(U_1) \times \mathrm{Aut}(U_2, E)$, or $C = \mathrm{Sym}(U_2) \times \mathrm{Aut}(U_1, E)\}$, or C contains at least one of the closed groups $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$, $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2 \cup \{0\})$ and $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1 \cup \{0\})$. If $n = 3$ then $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2 \cup \{0\}) = \mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2 \cup \{0\})$, thus these closed groups are in the provided list of groups. Hence, we may assume that C is strictly above either $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$, or $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2 \cup \{0\})$ or $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1 \cup \{0\})$. We claim that C contains at least one of the groups $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2 \cup \{0\})$, $\mathrm{Sym}(U_2) \times \mathrm{Sym}(U_1 \cup \{0\})$ and $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)) \rtimes Z_2$.

At first we show that every closed group C above $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2 \cup \{0\})$ contains $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2 \cup \{0\})$. (The argument that closed groups above $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1 \cup \{0\})$ contain $\mathrm{Sym}(U_2) \times \mathrm{Sym}(U_1 \cup \{0\})$ is analogous.) If the orbit system of C is $\{U_1, U_2 \cup \{0\}\}$, then some $\pi \in C$ violates E on U_1. We may assume that π is identical on U_2. Hence, according to Theorem 4 we have that C contains $\mathrm{Sym}(U_1)$. If C is transitive, than according to Lemma 65 there exist $\pi \in C$ and $u, v \in U_1$ such that $\pi(u), \pi(v) \in U_2$. As the transposition $t_{\pi(u)\pi(v)} \in C$, we have that $t_{uv} = \pi^{-1} \circ t_{\pi(u)\pi(v)} \circ \pi \in C$. Using the automorphisms in $\mathrm{Aut}(H_n, E, 0)$ we can switch any pair in U_1 with the same τ-type as (u, v). If the two-type of (x, y) is different from the two-type of (u, v) for some $x, y \in U_1$, then there exists a $z \in U_1$ such that the two-type of (x, z) and (y, z) are the same as that of (u, v). Thus $t_{xy} = t_{xz} \circ t_{zy} \circ t_{xz} \in C$. Hence, all transpositions of elements in U_1 are in C, and thus $\mathrm{Sym}(U_1) \subseteq C$.

Secondly, assume that C is above $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$. If $\{0\}$ is an orbit of C, then C is either $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)) \rtimes Z_2$ or $\mathrm{Stab}_0(\mathrm{Sym}(H_n))$ by Proposition 55. Assume that the orbit system of C is $\{U_1, U_2 \cup \{0\}\}$. Then some $\pi \in C$ maps 0 into U_2. We may assume that π is identical on U_1. Hence, $\mathrm{Sym}(U_2 \cup \{0\})$ is contained in C as $\mathrm{Sym}(U_2)$ is a maximal closed subgroup of $\mathrm{Sym}(U_2 \cup \{0\})$ by Proposition 56. Thus C contains $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2 \cup \{0\})$. Similarly, if the orbit system of C is $\{U_2, U_1 \cup \{0\}\}$, then C contains $\mathrm{Sym}(U_2) \times \mathrm{Sym}(U_1 \cup \{0\})$. Finally, assume that C is transitive. Then according to Lemma 62 there exist $u \in U_1$, $v \in U_2$ and $\pi_u, \pi_v \in C$ such that π_u switches 0 and u, and π_v switches 0 and v. Assume that there is an $x \in U_1 \setminus \{u\}$ such that $\pi_u(x) \in U_1$. If some $w \in U_1$ is mapped to $z \in U_2$ by π_u, then $\rho = \pi_u \circ t_{xw} \circ \pi_u^{-1} \in C$ is a permutation fixing 0, and moving $z \in U_2$ to U_1. Then $\mathrm{Stab}_0(C)$ is transitive on $U_1 \cup U_2$, and we are done by Proposition 55. Hence, $U_1 \setminus \{u\}$ is mapped into $U_1 \setminus \{u\}$ by π_u. By applying the same argument for π_u^{-1}, we may assume that π_u permutes $U_1 \setminus \{u\}$. Consequently, π_u preserves U_2, and we are done by a previous case. Thus we may assume that π_u maps $U_1 \setminus \{u\}$ into U_2. By applying the same argument for π_u^{-1}, we may assume that π_u switches $U_1 \setminus \{u\}$ and U_2. Similarly, we may assume that π_v switches $U_2 \setminus \{v\}$ and U_1. Let $p = \pi_u^{-1}(v)$. Then $\pi_v \circ \pi_u \circ t_{up} \circ \pi_u$ stabilizes 0 and maps $u \in U_1$ into U_2. Thus we have a proper supergroup of $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$ in C stabilizing 0, and we are done by Proposition 55.

To finish the proof, observe that the closed groups above $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2 \cup \{0\})$ are $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2 \cup \{0\})) \rtimes Z_2$ and $\mathrm{Sym}(H_n)$ by Proposition 55, and these are in the list. Similarly, the closed groups above $\mathrm{Sym}(U_2) \times \mathrm{Sym}(U_1 \cup \{0\})$ are $(\mathrm{Sym}(U_2) \times \mathrm{Sym}(U_1 \cup \{0\})) \rtimes Z_2$ and $\mathrm{Sym}(H_n)$ by Proposition 55, and these are also contained in the list. Finally, assume that C is above $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)) \rtimes Z_2$. If 0 is an orbit of C, then by Proposition 55 we have that C contains $\mathrm{Sym}(U_1 \cup U_2) = \mathrm{Stab}_0(\mathrm{Sym}(H_n))$, and consequently C is either $\mathrm{Stab}_0(\mathrm{Sym}(H_n))$ or $\mathrm{Sym}(H_n)\}$ by Proposition 56. Hence, assume that C is transitive. We show that either C contains an extra permutation fixing 0, or C contains $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2 \cup \{0\}))$ or $(\mathrm{Sym}(U_2) \times \mathrm{Sym}(U_1 \cup \{0\}))$, and then we are done by Proposition 55. According to Lemma 62 there exist $u \in U_1$, $v \in U_2$ and $\pi_u, \pi_v \in C$ such that π_u switches 0 and u, and π_v switches 0 and v. Assume that there is an $x \in U_1 \setminus \{u\}$ such that $\pi_u(x) \in U_1$. If some $w \in U_1$ is mapped to $z \in U_2$ by π_u, then $\rho = \pi_u \circ t_{xw} \circ \pi_u^{-1} \in C$ is a permutation fixing 0 and u, and moving $z \in U_2$ to U_1. This permutation fixes 0 and is not contained in $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)) \rtimes Z_2$, thus we have an extra permutation fixing 0. Hence, we may assume that $U_1 \setminus \{u\}$ is mapped into $U_1 \setminus \{u\}$ by π_u. By applying the same argument for π_u^{-1} we have that π_u preserves $U_1 \setminus \{u\}$, and consequently, π_u preserves U_2. Such a permutation together with $\mathrm{Sym}(U_1)$ generate $\mathrm{Sym}(U_1 \cup \{0\})$, and then C contains $(\mathrm{Sym}(U_2) \times \mathrm{Sym}(U_1 \cup \{0\}))$.

Thus we may assume that π_u maps $U_1\setminus\{u\}$ into U_2, and similarly π_v maps $U_2\setminus\{v\}$ into U_1. By applying the same argument for π_u^{-1} and π_v^{-1} we have that π_u switches $U_1\setminus\{u\}$ and U_2, and π_v switches $U_2\setminus\{v\}$ and U_1. Let $q=\pi_v(u)\in U_2$. Then $\pi_v^{-1}\circ t_{qv}\circ\pi_v\in C$ switches 0 and u while fixing all other elements of H_n. Thus C contains $(\mathrm{Sym}(U_2)\times\mathrm{Sym}(U_1\cup\{0\}))$. □

5. Additional remarks, open problems

5.1. **Rotations.** Two posets A and B of equal size are rotation equivalent if there exists a rotation $f:A\to B$. Some combinatorial, enumerative and computational complexity questions arise naturally (similar questions were investigated for the switch operation of the random graph).

Problem 66. *What is the cardinality of the smallest and the biggest rotation equivalence class of posets on n vertices?*

Problem 67. *How many rotaion equivalence classes are there (of posets on n vertices)?*

Problem 68. *Is there a well-understood subclass $\mathcal{C}$ of finite posets such that every rotation equivalence class contains at least one poset in $\mathcal{C}$?*

Problem 69. *Let X be a finite set and let R_1, R_2 and R_3 be disjoint ternary relations on X such that their union consists of all the triples with pairwise different entries. What is the complexity of deciding whether there exists a poset with underlying set X such that the* Par*,* cycl *and* cycl′ *relations defined from the poset are R_1, R_2 and R_3, respectively?*

The analoguous question corresponding to the closed group Max is as follows.

Problem 70. *Let X be a finite set and let R_1 be a ternary relations on X that consists of triples with pairwise different entries. What is the complexity of deciding whether there exists a poset with underlying set X such that the* Par *relation defined from the poset is R_1?*

5.2. **Reducts.**

Problem 71. *Give a bound for the number of the reducts of a homogeneous structure in terms of the number and arity of its relations.*

Problem 72. *Is it true that if $\mathcal{M}$ has finitely many reducts, then $(\mathcal{M},0)$ has finitely many reducts, as well?*

Problem 73. *Is it decidable about a homogeneous structure whether or not it has finitely many reducts (from the computational complexity perspective)?*

Note that if Thomas' conjecture is true, then the last two problems are trivial.

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