Periodic planar straight-frame graph drawings with polynomial resolution

(Luca Castelli Aleardi)

(joint work with Eric Fusy and Anatolii Kostrygin)

(work supported by the french ANR Egos)
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Let’s start from planar graphs
Some facts about planar graphs

Thm (Schnyder, Trotter, Felsner)
$G$ planar if and only if $\dim(G) \leq 3$

Thm (Koebe-Andreev-Thurston)
Every planar graph with $n$ vertices is isomorphic to the intersection graph of $n$ disks in the plane.

Thm (Kuratowski, excluded minors)
$G$ planar if and only if $G$ contains neither $K_5$ nor $K_{3,3}$ as minors

Thm (Y. Colin de Verdière)
$G$ planar if and only if $\mu(G) \leq 3$ ($\mu(G) =$ multiplicity of $\lambda_2$ of a generalized laplacian)

$$L_G = \begin{bmatrix} 4 & -1 & \ldots & \ldots & 0 \\ -1 & 5 & \ldots \\ \vdots & \vdots \\ 0 & \ldots & \ldots & \ldots & 3 \end{bmatrix} \quad L_G[i, k] = \begin{cases} \deg(v_i) \\ -A_G[i, j] \end{cases}$$
Planar straight-line drawings
(of planar graphs)
Planar straight-line drawings

[Wagner’36]
[Fary’48]
Planar straight-line drawings

Classical algorithms:

- spring-embedding
- incremental (Shift-algorithm)
- face-counting principle

existence of straight-line drawing

[Wagner’36]
[Fary’48]
[Stein’51]
[Tutte’63]
[De Fraysseix, Pach, Pollack 89]
[Schnyder’90]
Periodic straight-line drawings
(statement of the problem)
Drawing graphs on surfaces

\[ g = 0 \]
Drawing graphs on surfaces \( g \geq 2 \)

Universal cover

Polygonal scheme

Periodic drawing out of circle packing

\( [\text{Mohar'99}] \)

(Palais de la Découverte, Fête de la Science, October 2013)
Drawing toroidal graphs

For the torus you can get periodic drawings

\[ g = 1 \]
Straight-line toroidal drawings

On the torus

- x-periodic and y-periodic drawing
- Drawing on the flat torus

O\((n \times n^3)\) grid

- [Castelli-Aleardi Devillers Fusy, GD'12]
- [Goncalves Lévêque, DCG'14]

O\((n^2 \times n^2)\) grid

- [Duncan, Goodrich, Kobourov, GD'09]
- [Chambers, Eppstein, Goodrich, Löffler, GD'10]
some useful previous results
(key ingredients for our work)
Incremental shift algorithm
[de Fraysseix, Pollack, Pach’89]

1. Grid size of $G_k$: $2k \times k$

2. use the canonical ordering
1) $k$-scheme triangulation is a quasi-triangulation s.t.
- $k$ marked outer vertices are called corners;
- each path of the outer face contour between two consecutive corners is chordless.

2) $G$ is a 4-scheme triangulation.
A straight-frame drawing of $G$ is
- a planar straight-line drawing of $G$;
- the outer face is an axis-aligned rectangle;
- its corners are the corners of $G$.

**Theorem** [Duncan et al., GD09]
Each 4-scheme triangulation with $n$ vertices admits a straight-frame drawing on a grid of size $O(n^2 \times n)$. 
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Our main result

(statement and the key idea)
Straight-frame periodic drawing

1) Denote the paths between consecutive corners by $S_1, \ldots, S_k$. Then a 4-scheme triangulation satisfying $|S_1| = |S_3|$ and $|S_2| = |S_4|$ is called balanced.

2) Its straight-frame drawing is periodic if
- the abscissas of vertices of the same rank along $S_1$ and $S_3$ coincide;
- the ordinates of vertices of the same rank along $S_2$ and $S_4$ coincide.

**Theorem (Castelli Aleardi, Fusy, Kostrygin)**
Each balanced 4-scheme-triangulation admits a periodic straight-frame drawing on a (regular) grid of size $O(n^4 \times n^4)$. 
Main idea: key picture

Before drawing a balanced 4-scheme triangulation . . .
Key picture: first cut

Before drawing a balanced 4-scheme triangulation . . .

compute a special partition of its internal faces
Key picture: then draw and stretch

draw each piece according to its type
stretch each piece identifying coordinates on opposite sides
Key picture: finally glue all pieces

stretch each piece identifying coordinates on opposite sides
Why does it work
(overview of the proof)
Step 1: decomposition phase

- Suppose that there is no "vertical" cord.
- Then there exists a closest to the upper-side cordless path.
- Each vertex of the path is on the dist. 1 from the upper-side.
- Let’s cut the graph along this path.
**Step 2: compute a river**

- for identification: we need to take care only about left and right sides.
- Upper side is not required to be straight.

- Find a river from upper to bottom side.
- Let’s cut along this river.
Step 3: draw left bottom piece

- Turn by 90 degrees (to help intuition).
- Find a canonical order.
Step 3: draw left bottom piece

- Turn by 90.
- Find a canonical order.
- Draw with incremental algorithm.
Step 3: draw left bottom piece

- Turn by 90.
- Find a canonical order.
- Draw with incremental algorithm.
- Remember the distance between vertices on the bottom side.
Step 3b: modified shift algorithm

- Remember the distance between vertices on the bottom side (shift vector).
- Use the shift vector to perform a second drawing pass.

First pass

[First pass diagram]

Second pass

(initial positions are given by the shift vector)
Step 4a: draw both bottom corners

- Repeat previous step 3 for right bottom piece.
- Turn the two pieces by 90 degrees
- re-draw and adjust sizes (using stored shift vectors)
Step 4b: align and add the river

- Repeat previous step 3 for right bottom piece.
- Turn the two pieces by 90 degrees
- re-draw and adjust sizes (using stored shift vectors)
- add the river
- align opposite vertices (modified shift algorithm)
Step 5: decompose the upper graph

- Cut upper corners (along largest upper chords)
Step 5: decompose the upper graph

- Cut upper corners (along largest upper chords)
- Find the edge adjacent to the river
- Decompose the rest into 3 parts
Step 6: draw upper corners
Step 7: align upper corners
Step 8: draw remaining pieces

refinement factor for the upper piece

\[ O(n \cdot \max(n, K)) \]

where \( K = O(n^2) \) is the grid width
Final step: glue all pieces together

Drawing in the naïve way $O(K^2n \times K^2n) = O(n^5 \times n^5)$ area

where $K = O(n^2)$ is the grid width
Final step: glue all pieces together

Drawing in a clever way $\implies O(n^4 \times n^4)$ area

(recall on the torus there are non contractible cycles of length $O(\sqrt{n})$) [Hutchinson, Albert '78]
applications and extensions
(geodesic spherical drawing)
**Algorithm:**
- partition the faces of the initial graph;
- dessiner draw every rectangle according to their lateral sides;
- construct a pyramid from the rectangles;
- place a small copy in the center of the sphere;
- project its edges on the sphere.
Draw arbitrary polygons

- Suppose we can draw an arbitrary quadrangle, let $P(n)$ be its grid size.
- Using divide and conquer strategy we can draw any $k$-gon.
- Grid size will be proportional to $O(P(n)^{\log k})$. 

![Diagram showing the process of drawing arbitrary polygons using divide and conquer strategy.](image-url)