Periodic planar straight-frame graph drawings with polynomial resolution

(Torres García, Inverted map of America, 1936)

Latin 2014, Montevideo

Luca Castelli Aleardi

(joint work with Eric Fusy and Anatolii Kostrygin)

(work supported by the french ANR Egos)

ÉCOLE POLYTECHNIQUE

EGOS Embedded Graphs and their Oriented Structures ANR Project 2012-2015

Periodic planar straight-frame graph drawings with polynomial resolution

(Torres García, Inverted map of America, 1936)

Latin 2014, Montevideo

Luca Castelli Aleardi

(joint work with Eric Fusy and Anatolii Kostrygin)

(work supported by the french ANR Egos)

ÉCOLE POLYTECHNIQUE

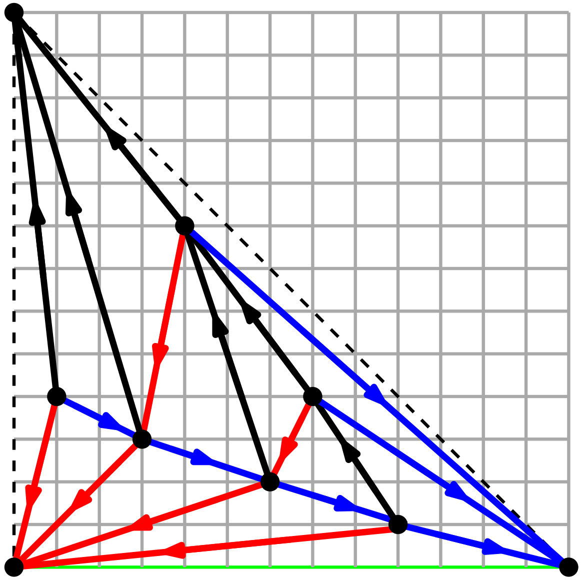
EGOS Embedded Graphs and their Oriented Structures ANR Project 2012-2015

Let’s start from planar graphs

Some facts about planar graphs

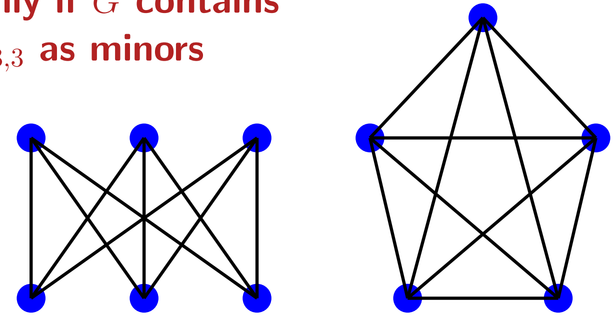
Thm (Schnyder, Trotter, Felsner)

G planar if and only if $dim(G) \leq 3$



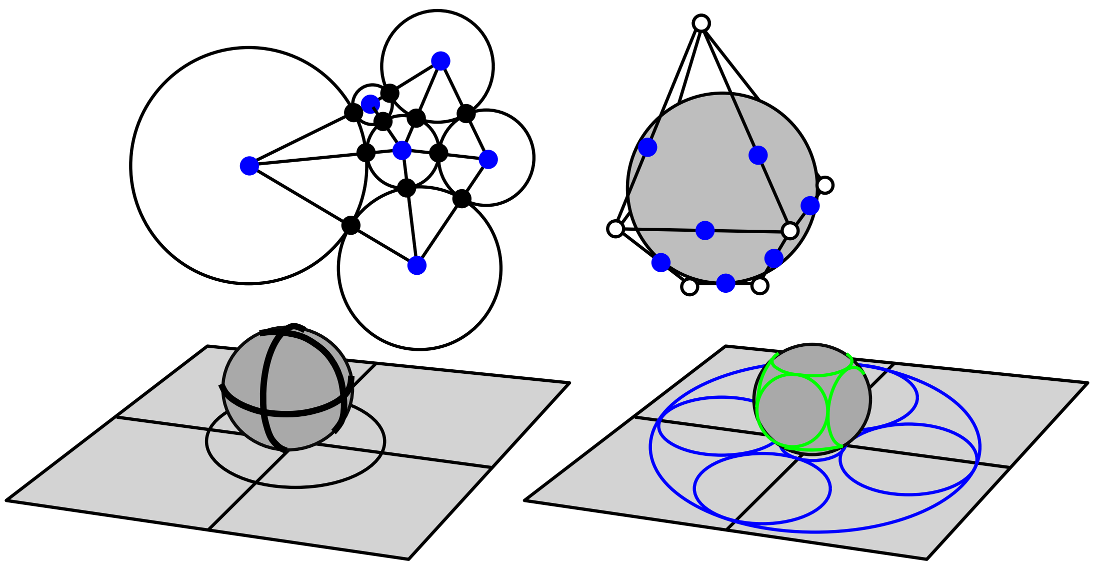
Thm (Kuratowski, excluded minors)

G planar if and only if G contains neither K_5 nor $K_{3,3}$ as minors



Thm (Koebe-Andreev-Thurston)

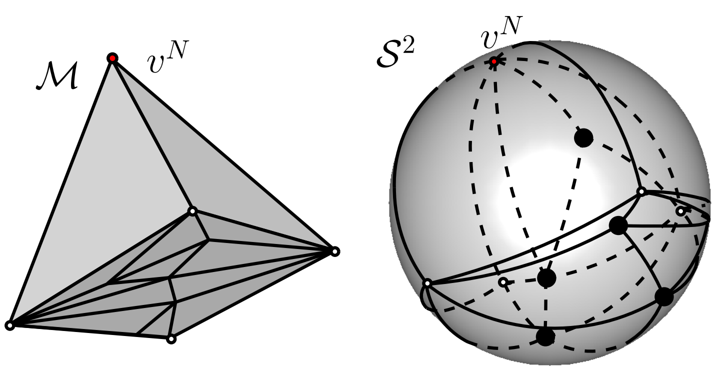
Every planar graph with n vertices is isomorphic to the intersection graph of n disks in the plane.



Thm (Y. Colin de Verdière)

G planar if and only if $\mu(G) \leq 3$
($\mu(G) =$ multiplicity of λ_2 of a generalized laplacian)

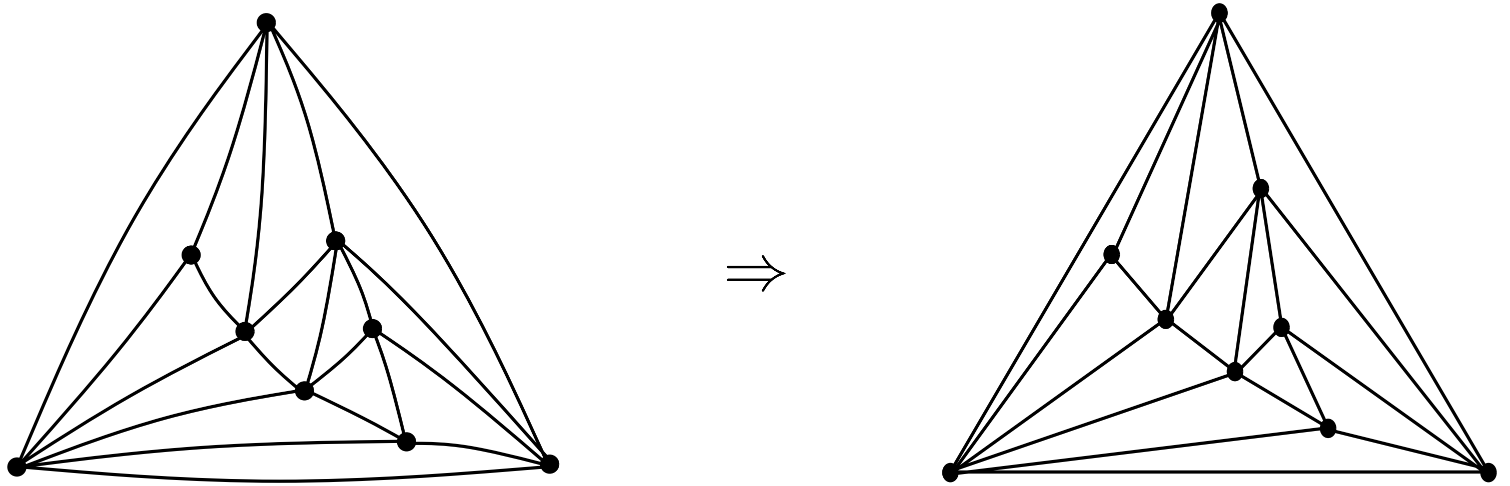
$$L_G = \begin{bmatrix} 4 & -1 & \dots & \dots & 0 \\ -1 & 5 & \dots & & \\ \dots & & \dots & & \\ \dots & & & & \dots \\ 0 & \dots & & & 3 \end{bmatrix} \qquad L_G[i,k] = \begin{cases} deg(v_i) \\ -A_G[i,j] \end{cases}$$



Planar straight-line drawings

(of planar graphs)

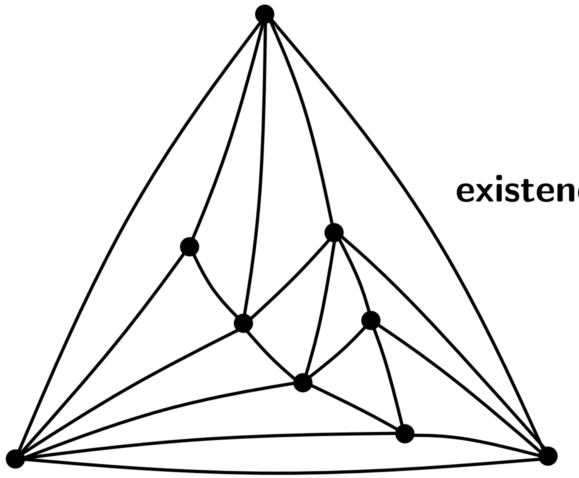
Planar straight-line drawings



[Wagner'36]

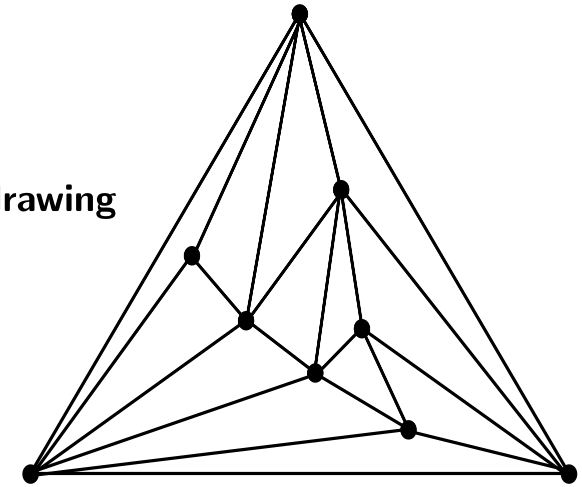
[Fary'48]

Planar straight-line drawings



existence of straight-line drawing

$\Rightarrow$

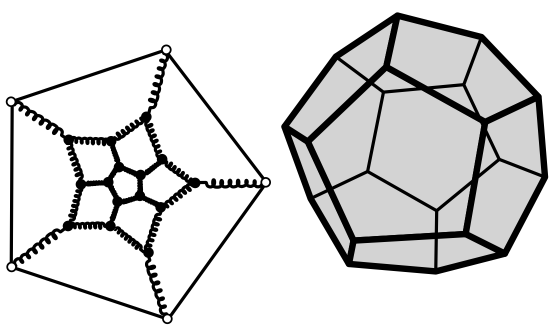


[Wagner'36]

[Fary'48]

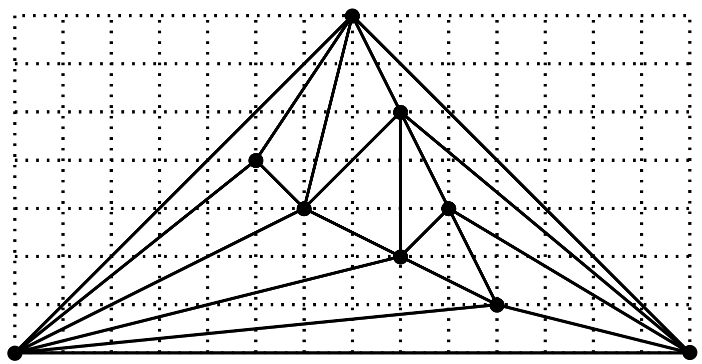
[Stein'51]

Classical algorithms:



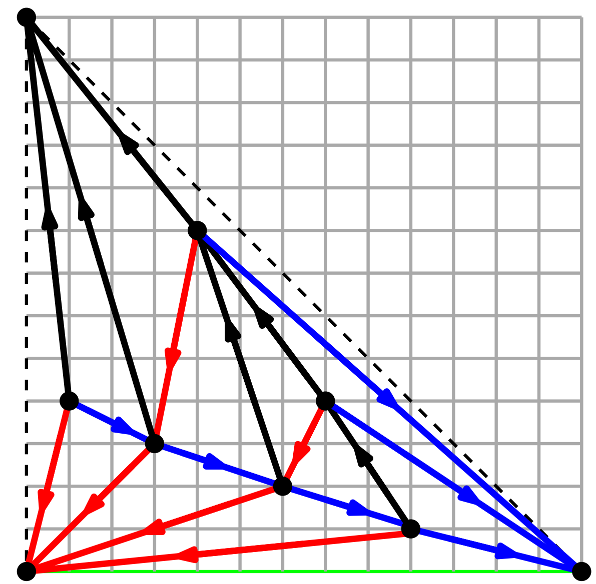
[Tutte'63]

spring-embedding



[De Fraysseix, Pach, Pollack 89]

incremental (**Shift-algorithm**)



[Schnyder'90]

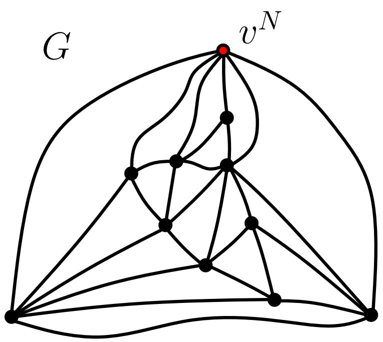
face-counting principle

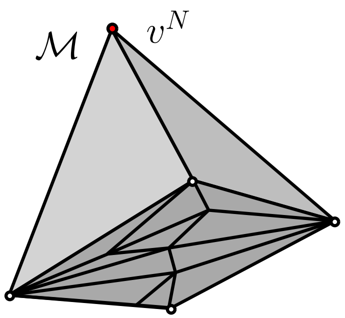
Periodic straight-line drawings

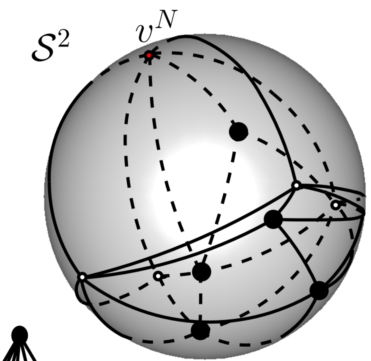
(statement of the problem)

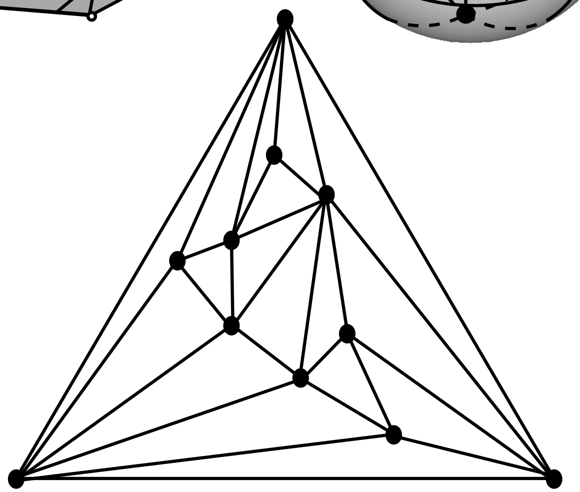
Drawing graphs on surfaces

$g = 0$



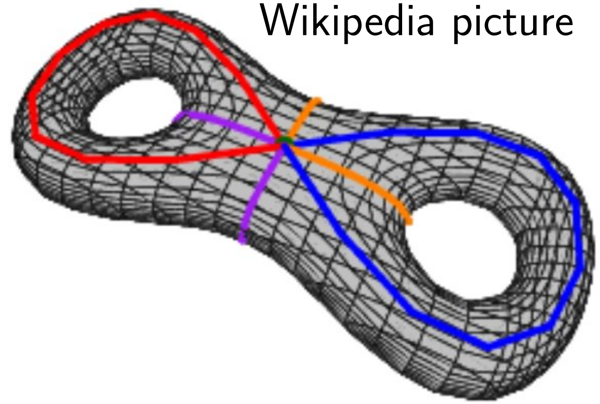






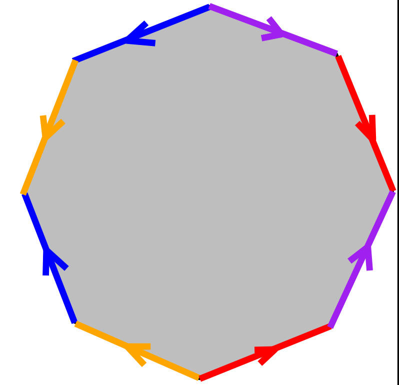
Drawing graphs on surfaces $g \geq 2$

(Palais de la Découverte,Fête de la Science, October 2013)

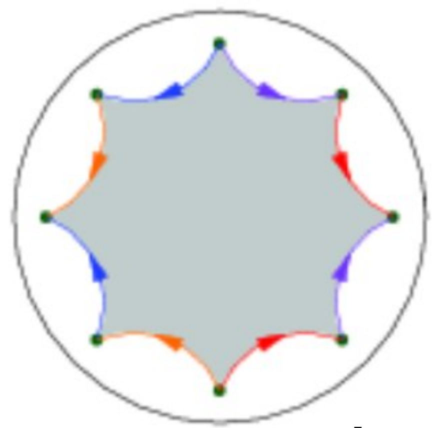


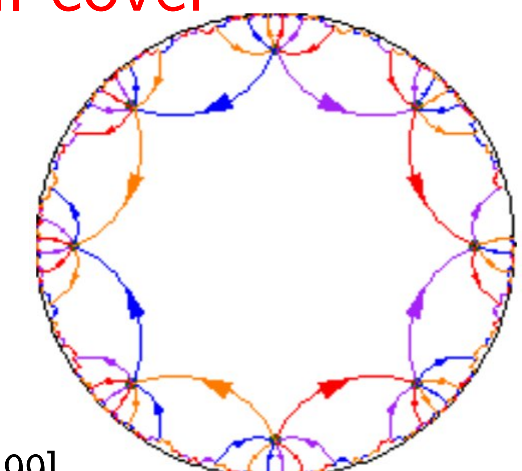
Wikipedia picture

Polygonal scheme



Universal cover



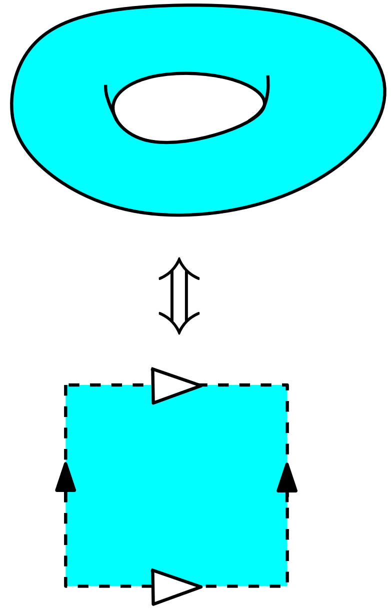


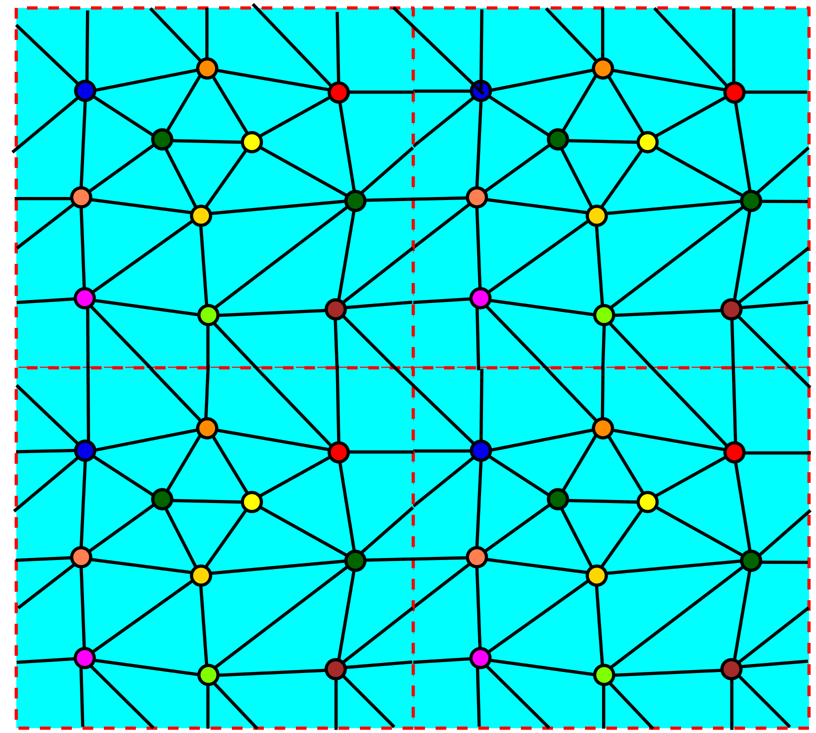
[Mohar'99]

periodic drawing   out of circle packing

Drawing toroidal graphs

$g = 1$

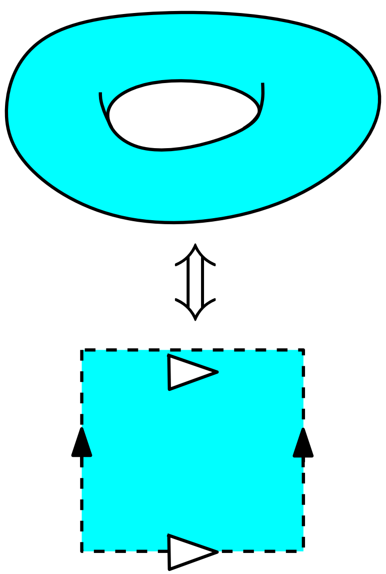


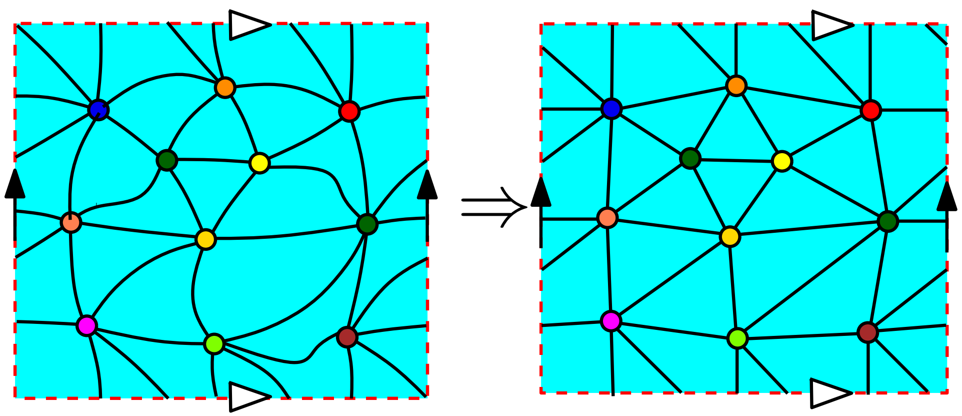


For the torus you can get periodic drawings

Straight-line toroidal drawings (existing works)

On the torus



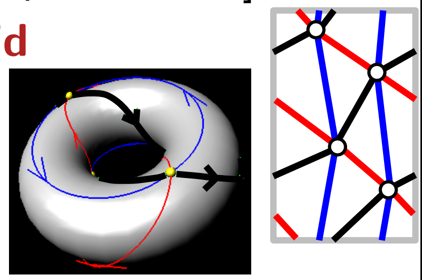


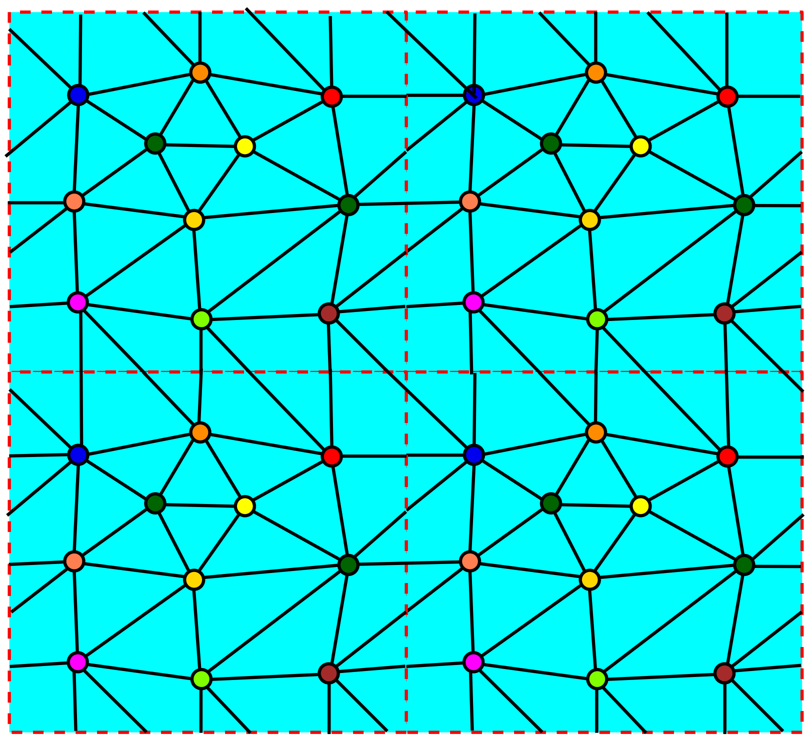
drawing on the flat torus

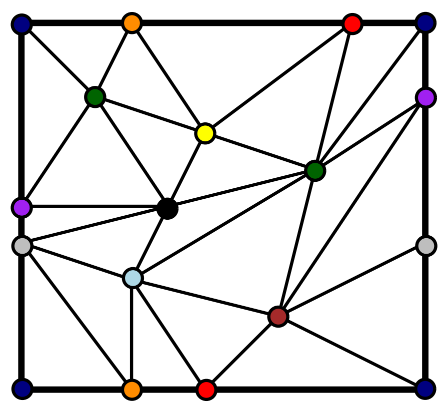
straight-line drawing
x-periodic and
y-periodic drawing

[Castelli-Aleardi Devillers Fusy, GD'12]
$O(n \times n^{\frac{3}{2}})$ **grid**

[Goncalves Lévêque, DCG'14]
$O(n^2 \times n^2)$ **grid**







straight-line frame
not x-periodic
not y-periodic

$O(n \times n^2)$ **grid**

[Duncan, Goodrich, Kobourov, GD'09]

[Chambers, Eppstein, Goodrich, Löffler, GD'10]

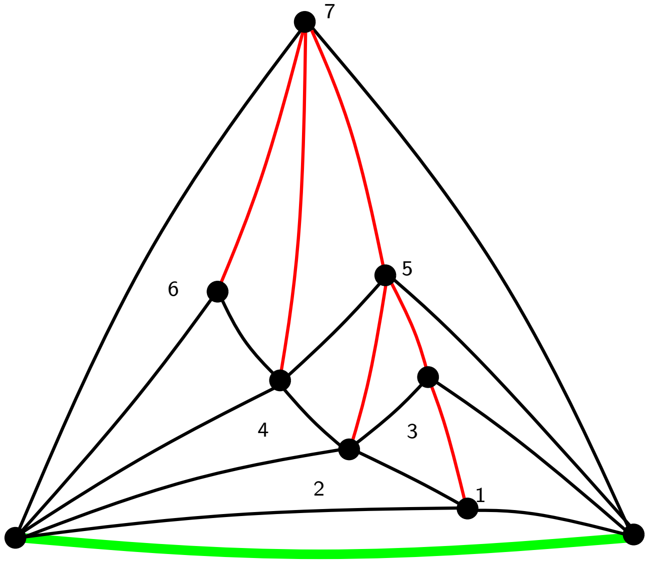
some useful previous results

(key ingredients for our work)

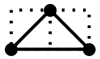
Incremental shift algorithm

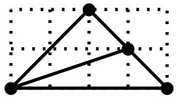
[de Fraysseix, Pollack, Pach'89]

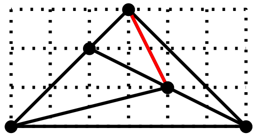
use the canonical ordering

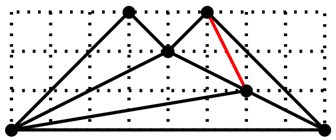


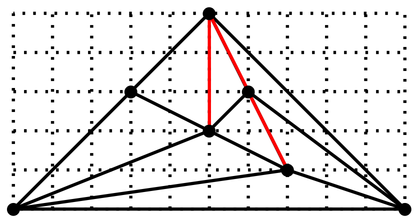
Grid size of G_k: $2k \times k$

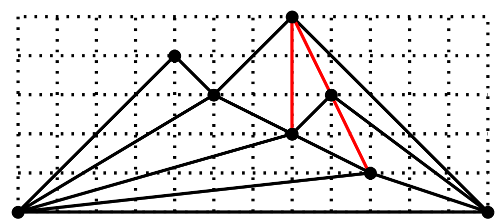
1. 

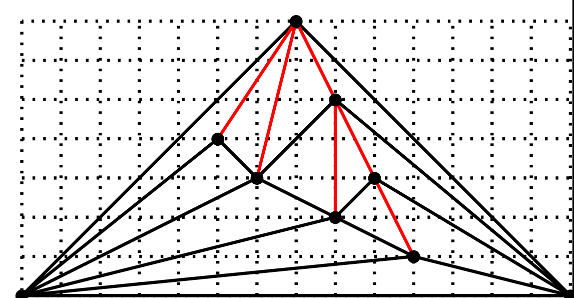
2. 

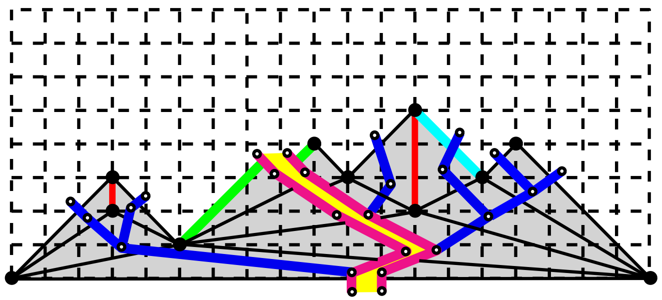
3. 

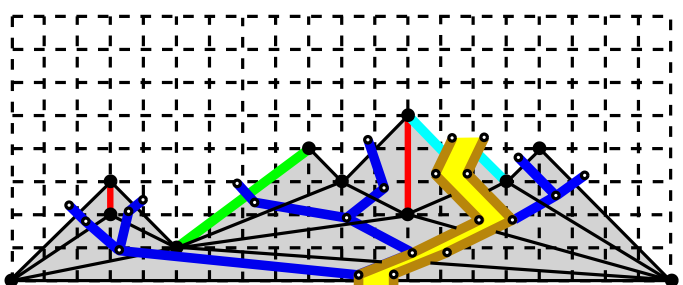
4. 

5. 

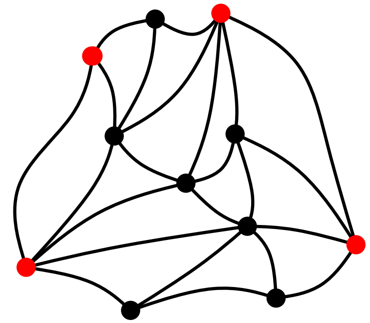
6. 

7. 

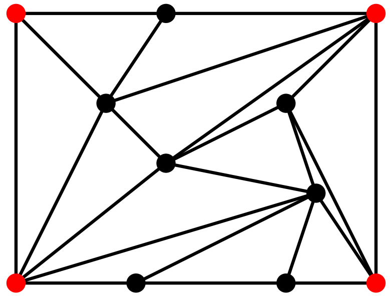




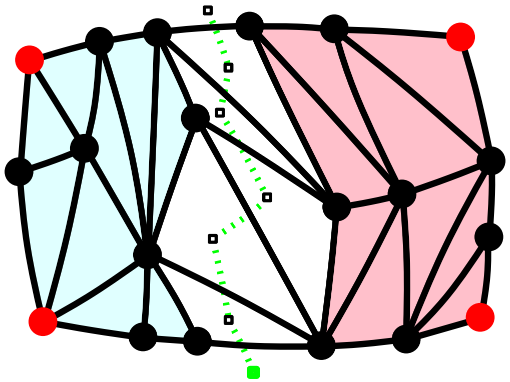
Straight-frame drawing



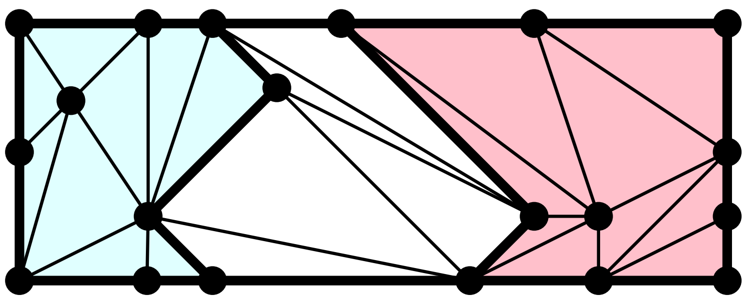
1) **k-scheme triangulation** is a quasi-triangulation s.t.
   - k marked outer vertices are called **corners**;
   - each path of the outer face contour between two consecutive corners is chordless.



2) G is a 4-scheme triangulation.
   A **straight-frame drawing** of G is
   - a planar straight-line drawing of G;
   - the outer face is an axis-aligned rectangle;
   - its corners are the corners of G.

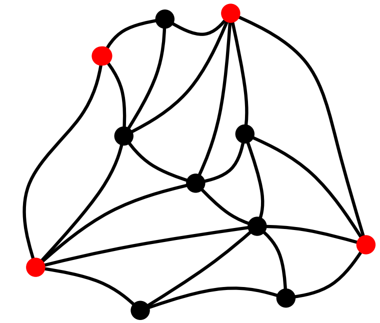


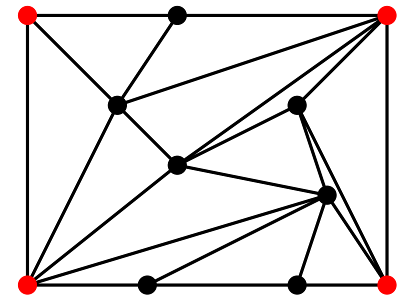
Theorem **[Duncan et al., GD09]**
Each 4-scheme triangulation with n vertices admits a straight-frame drawing on a grid of size $O(n^2 \times n)$.



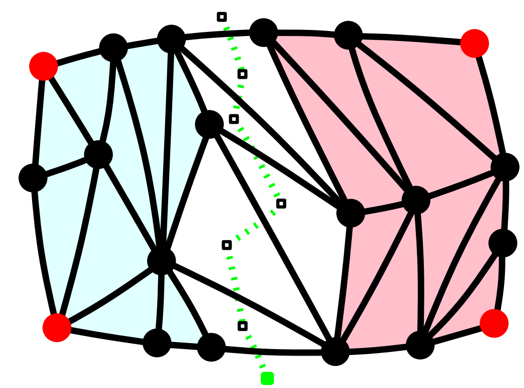
Straight-frame drawing

1) ***k*-scheme triangulation** is a quasi-triangulation s.t.
   - k marked outer vertices are called **corners**;
   - each path of the outer face contour between two consecutive corners is chordless.

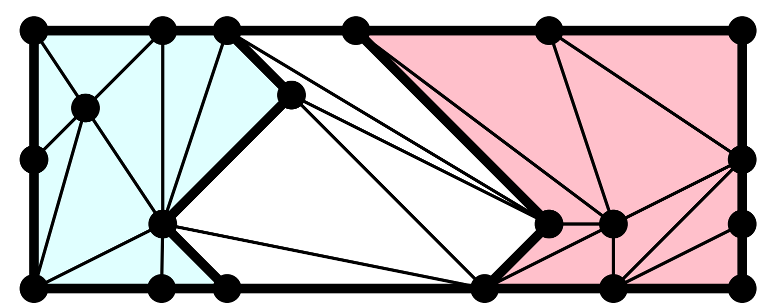




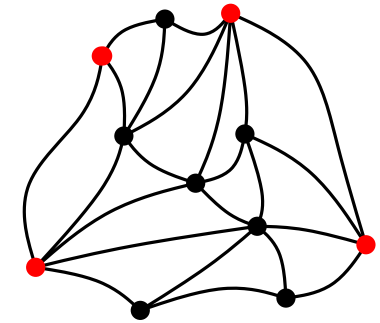
2) G is a 4-scheme triangulation.
   A **straight-frame drawing** of G is
   - a planar straight-line drawing of G;
   - the outer face is an axis-aligned rectangle;
   - its corners are the corners of G.



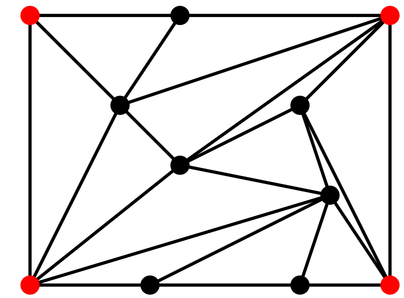
Theorem **[Duncan et al., GD09]**
Each 4-scheme triangulation with n vertices admits a straight-frame drawing on a grid of size $O(n^2 \times n)$.



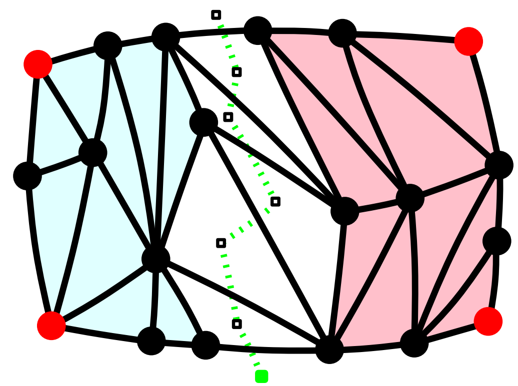
Straight-frame drawing



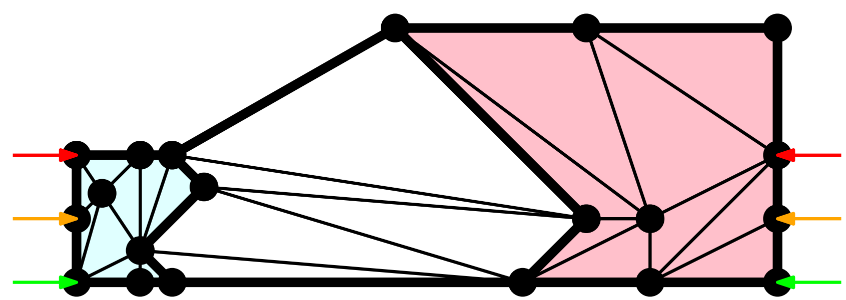
1) ***k*-scheme triangulation** is a quasi-triangulation s.t.
   - k marked outer vertices are called **corners**;
   - each path of the outer face contour between two consecutive corners is chordless.



2) G is a 4-scheme triangulation.
   A **straight-frame drawing** of G is
   - a planar straight-line drawing of G;
   - the outer face is an axis-aligned rectangle;
   - its corners are the corners of G.



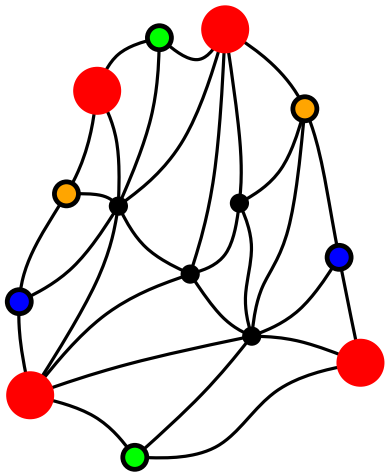
Theorem **[Duncan et al., GD09]**
Each 4-scheme triangulation with n vertices admits a straight-frame drawing on a grid of size $O(n^2 \times n)$.



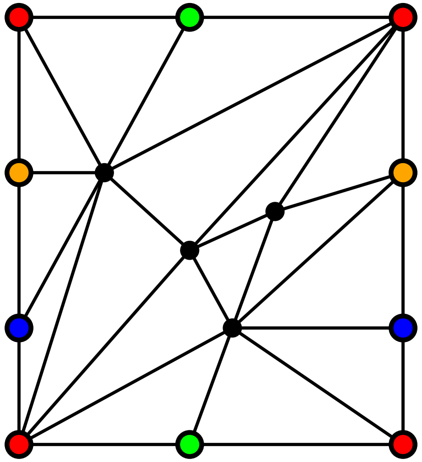
Our main result

(statement and the key idea)

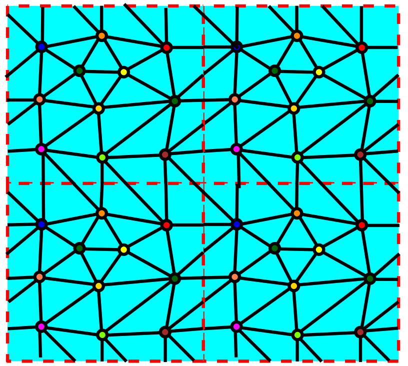
Straight-frame periodic drawing



1) Denote the paths between consecutive corners by $S_1, \ldots, S_k$. Then a 4-scheme triangulation satisfying $|S_1| = |S_3|$ and $|S_2| = |S_4|$ is called **balanced**.



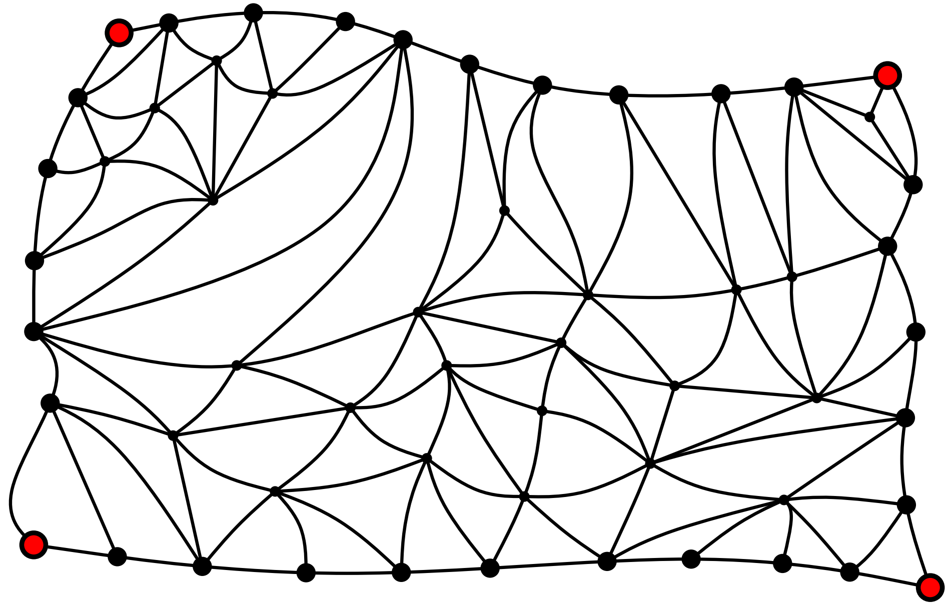
2) Its straight-frame drawing is **periodic** if
   - the abscissas of vertices of the same rank along S_1 and S_3 coincide;
   - the ordinates of vertices of the same rank along S_2 and S_4 coincide.



Theorem (Castelli Aleardi, Fusy, Kostrygin)
Each balanced 4-scheme-triangulation admits a periodic straight- frame drawing on a (regular) grid of size $O(n^4 \times n^4)$.

Main idea: key picture

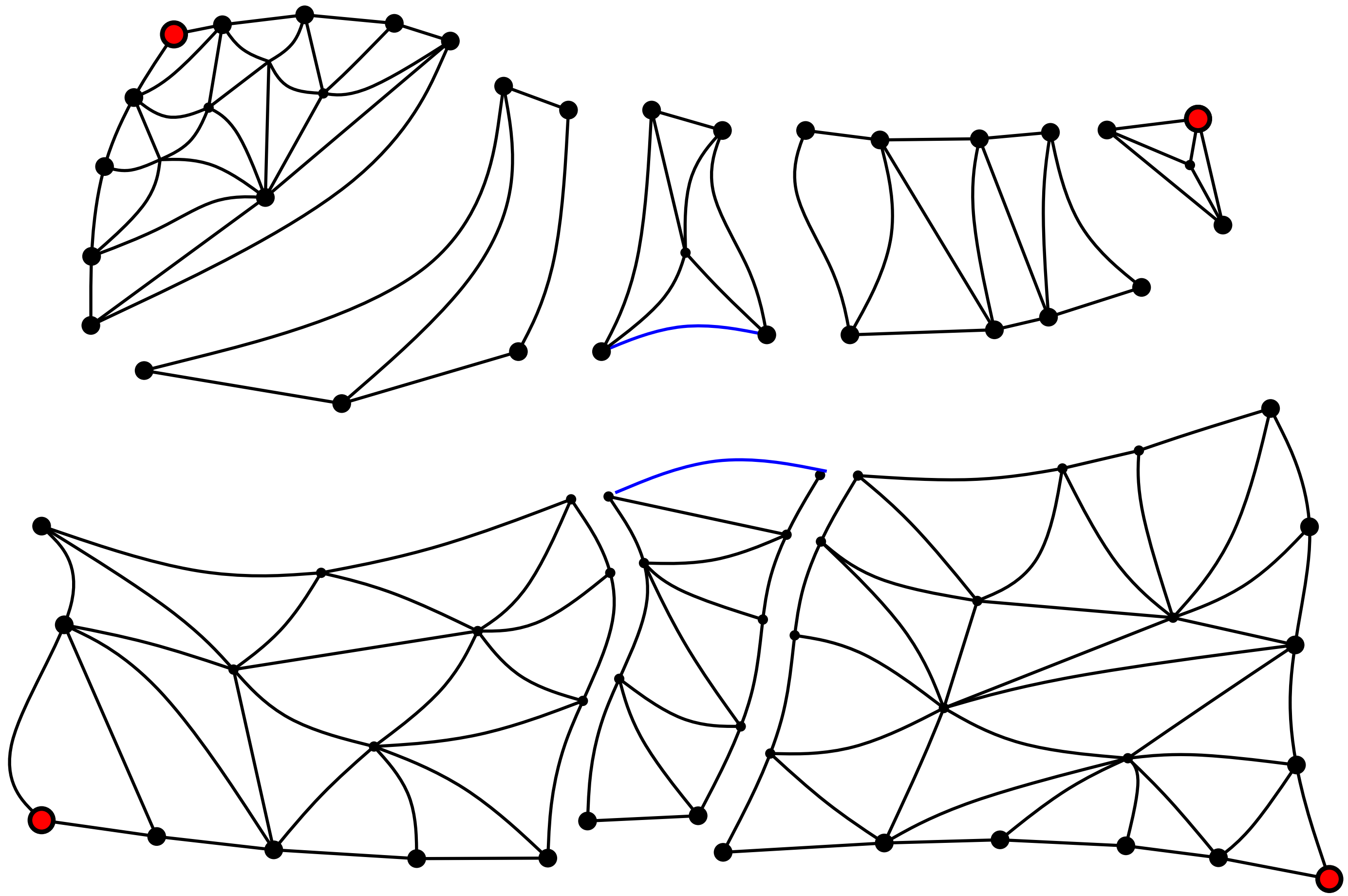
Before drawing a balanced 4-scheme triangulation . . .



Key picture: first cut

Before drawing a balanced 4-scheme triangulation . . .

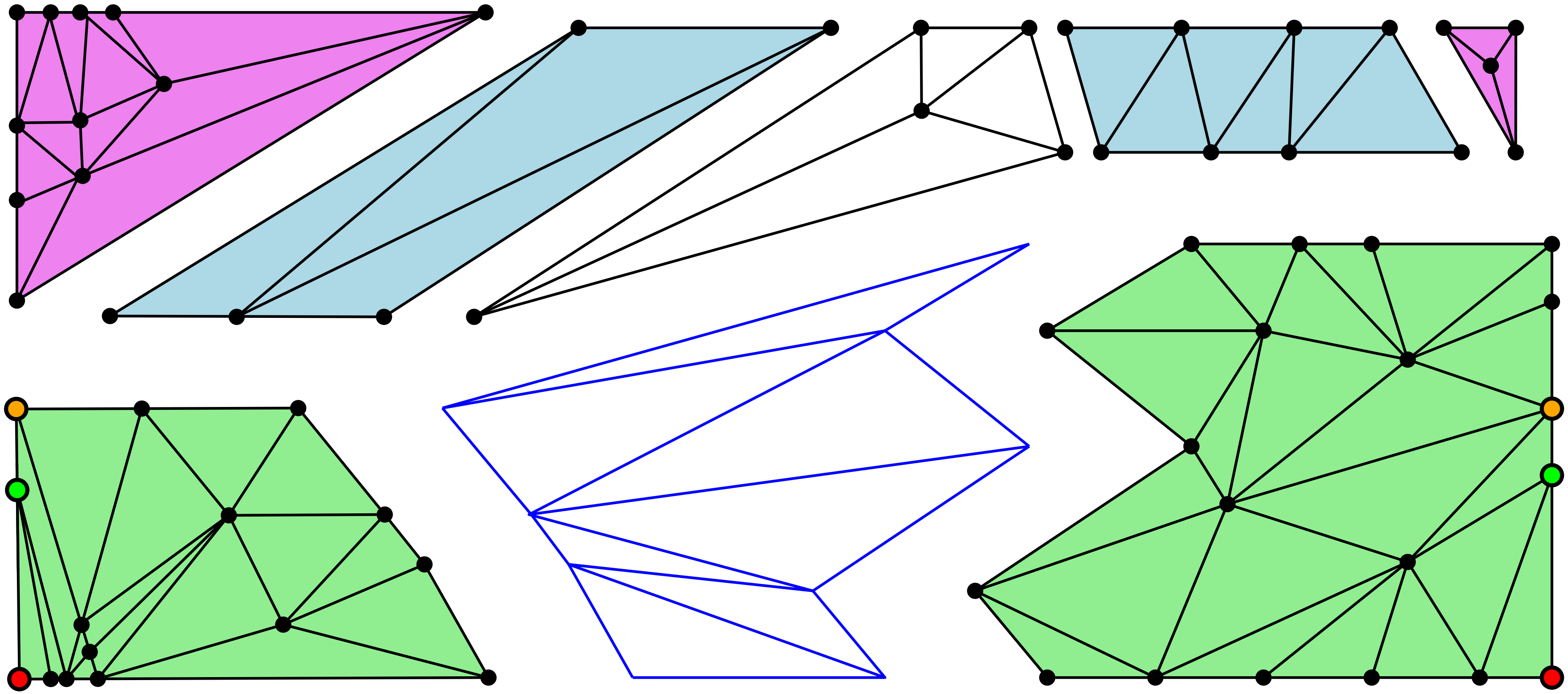
compute a special partition of its internal faces



Key picture: then draw and stretch

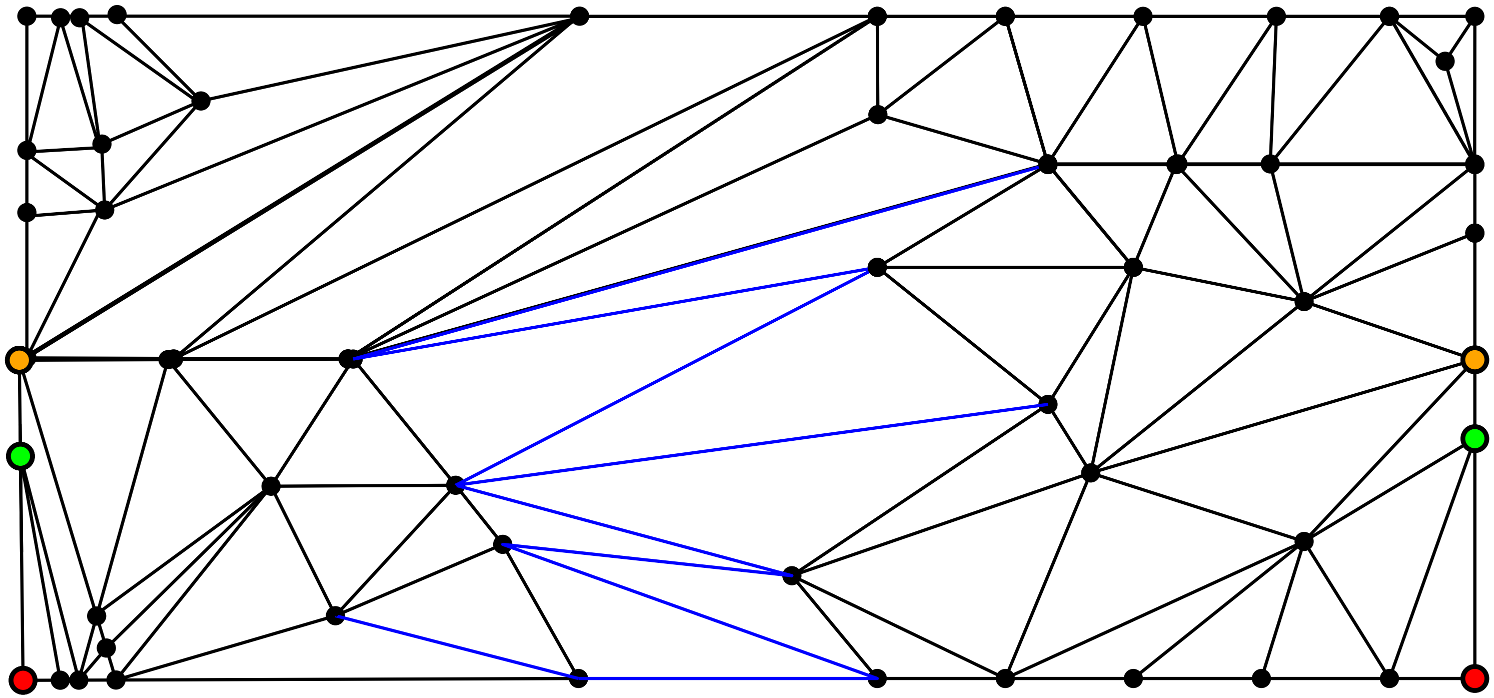
draw each piece according to its type

stretch each piece identifying coordinates on opposite sides



Key picture: finally glue all pieces

stretch each piece identifying coordinates on opposite sides

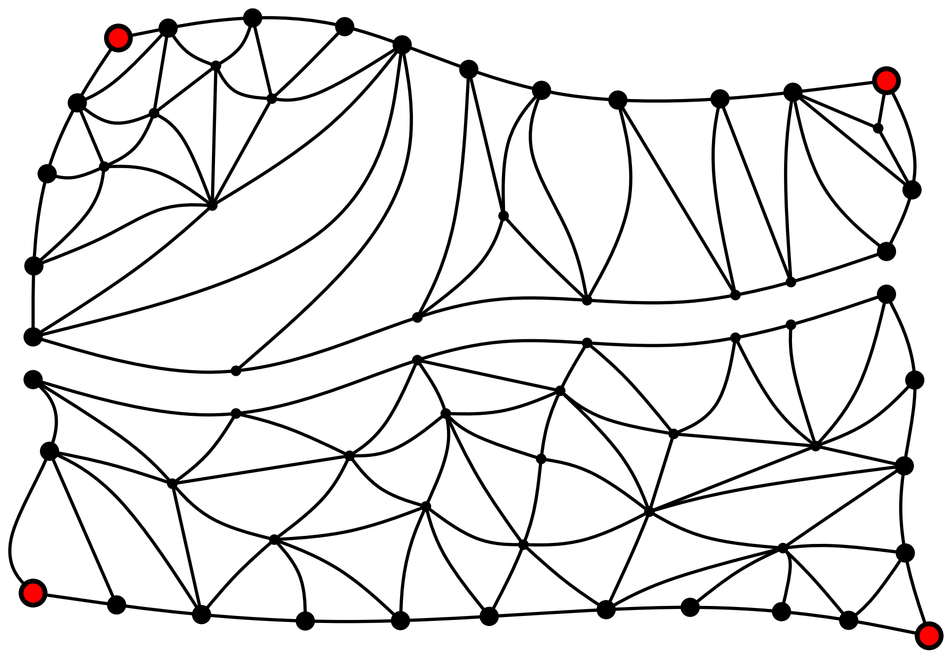


Why does it work

(overview of the proof)

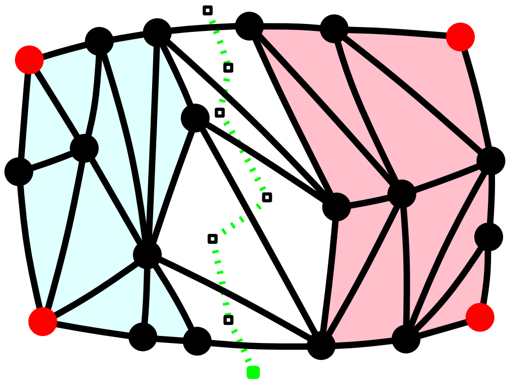
Step 1: decomposition phase

- Suppose that there is no "vertical" cord.
- Then there exists a closest to the upper-side cordless path.
- Each vertex of the path is on the dist. 1 from the upper-side.
- Let's cut the graph along this path.

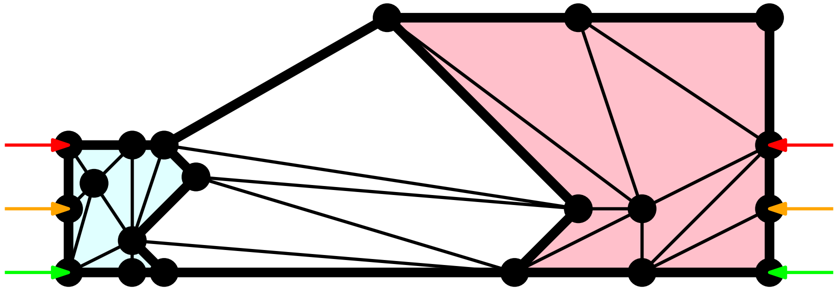


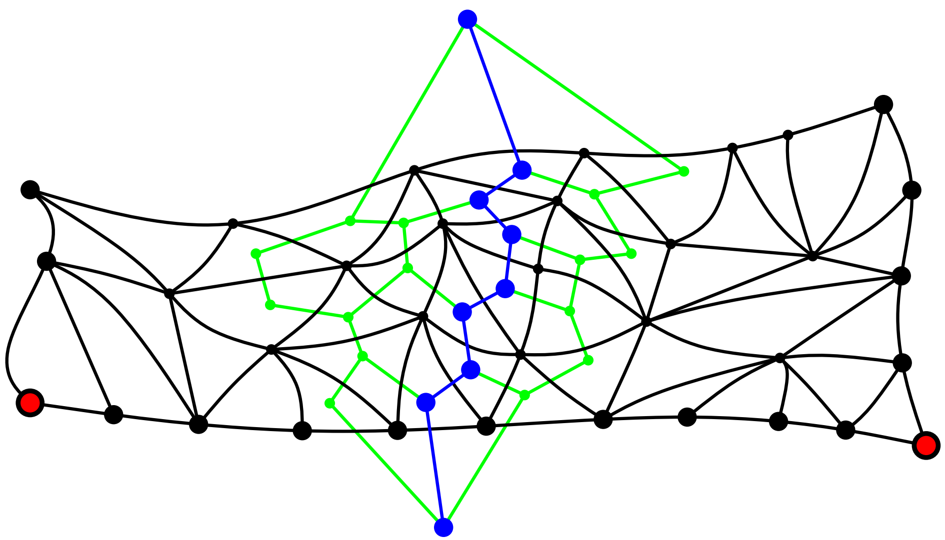
Step 2: compute a river

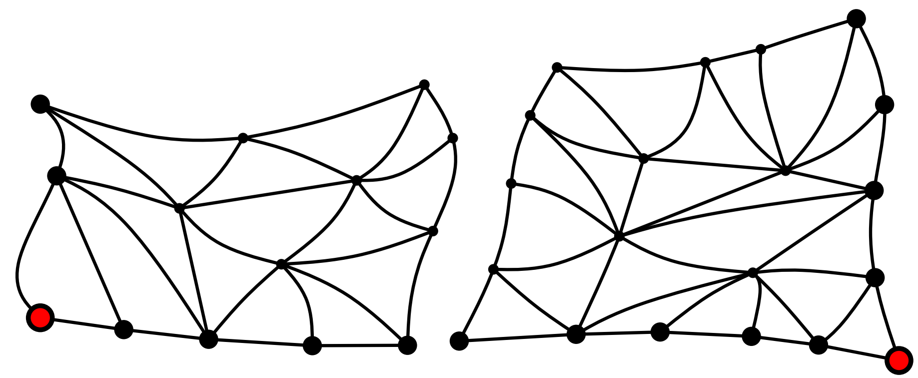
- for identification: we need to take care only about left and right sides.
- Upper side is not required to be straight.



- Find a **river** from upper to bottom side.
- Let's cut along this river.



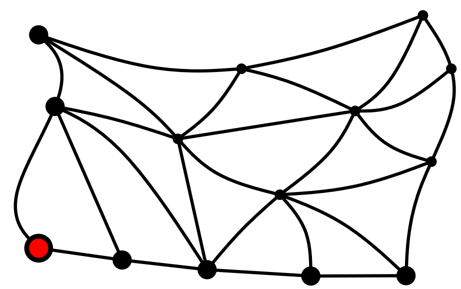




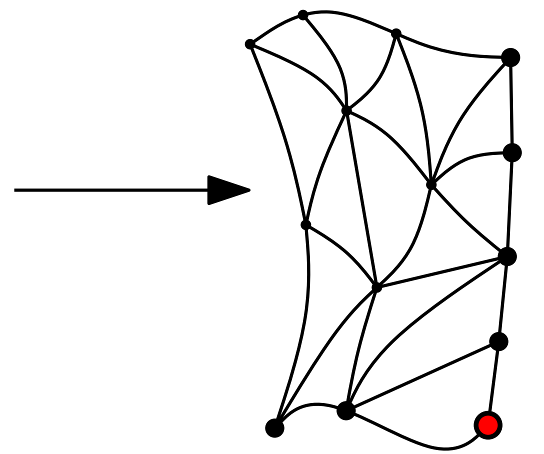
Bottom-left Bottom-right

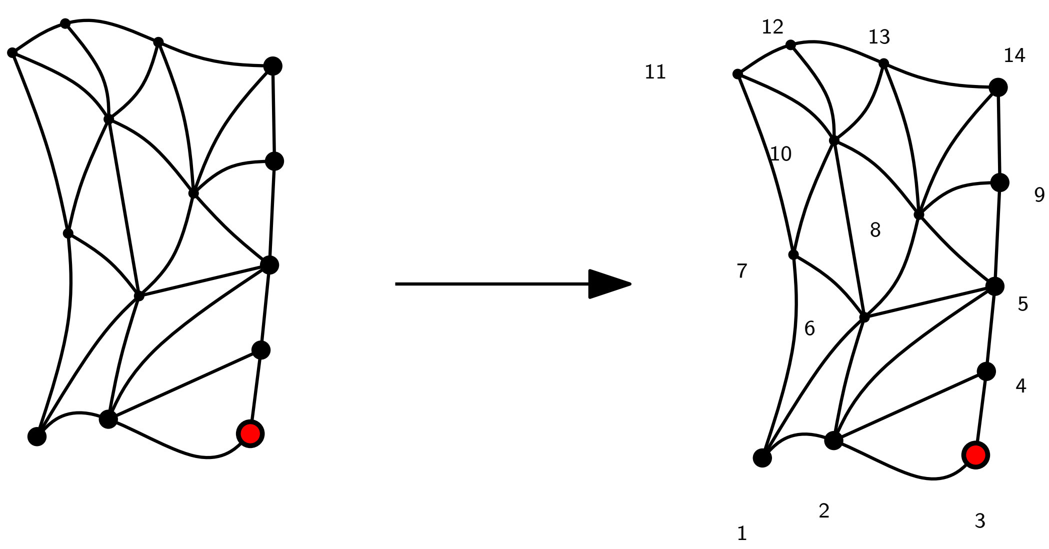
Step 3: draw left bottom piece

- Turn by 90 degrees (to help intuition).
- Find a canonical order.



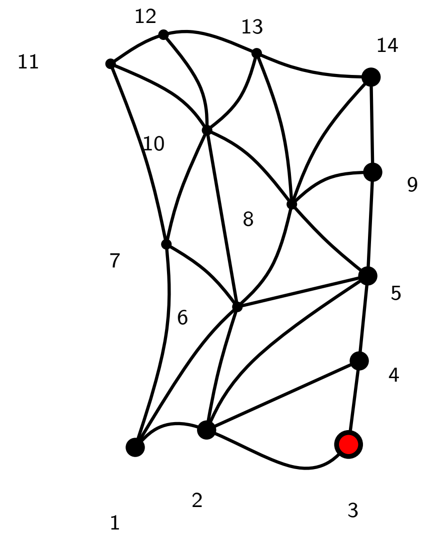
Bottom-left

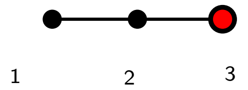


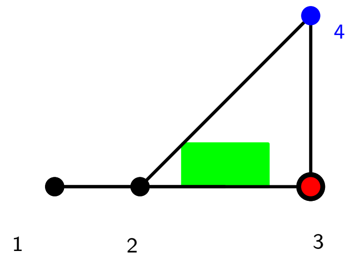


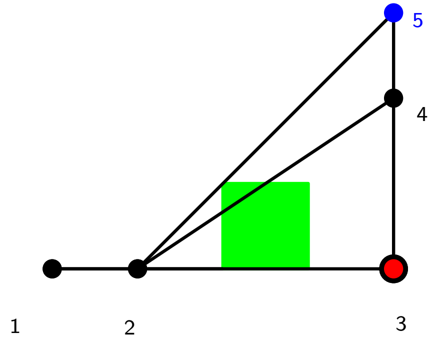
Step 3: draw left bottom piece

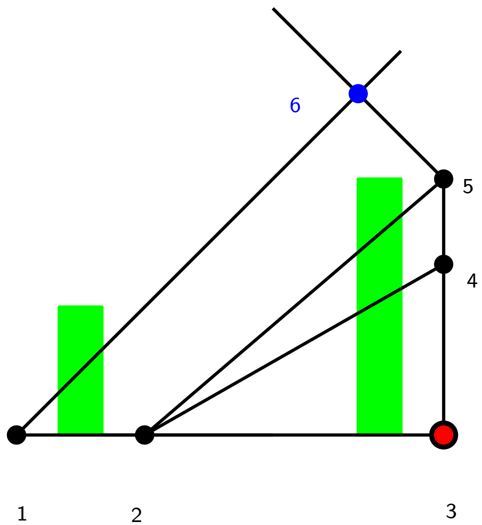
- Turn by 90.
- Find a canonical order.
- Draw with incremental algorithm.

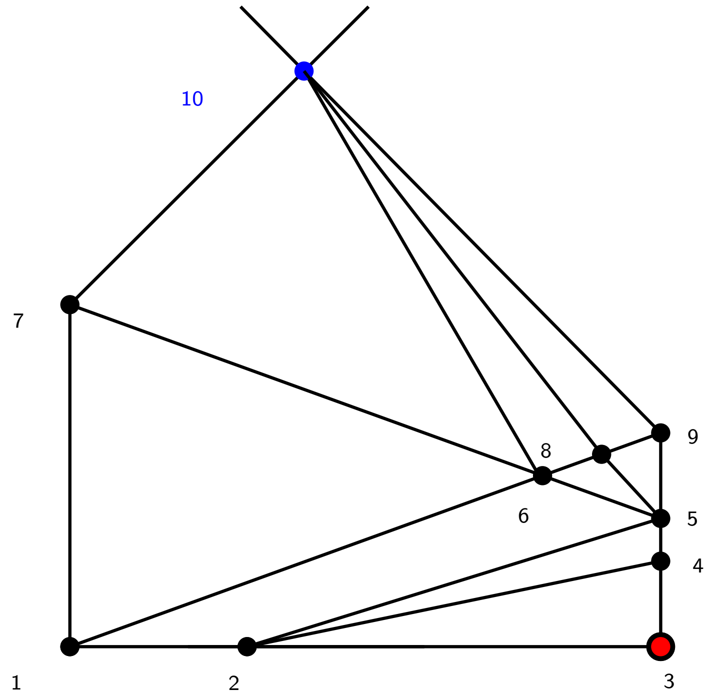






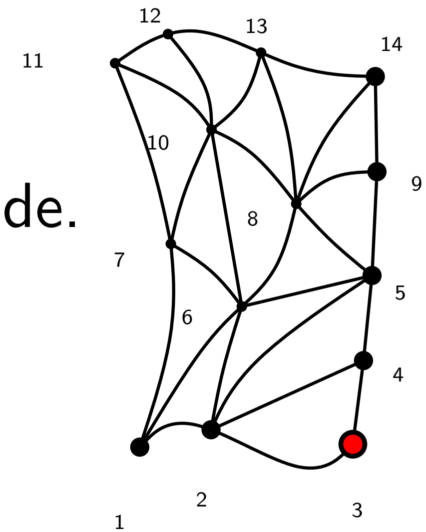


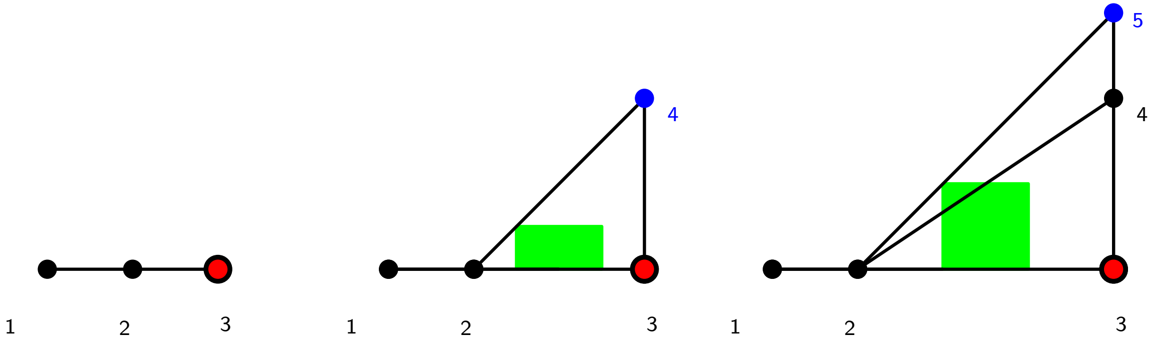


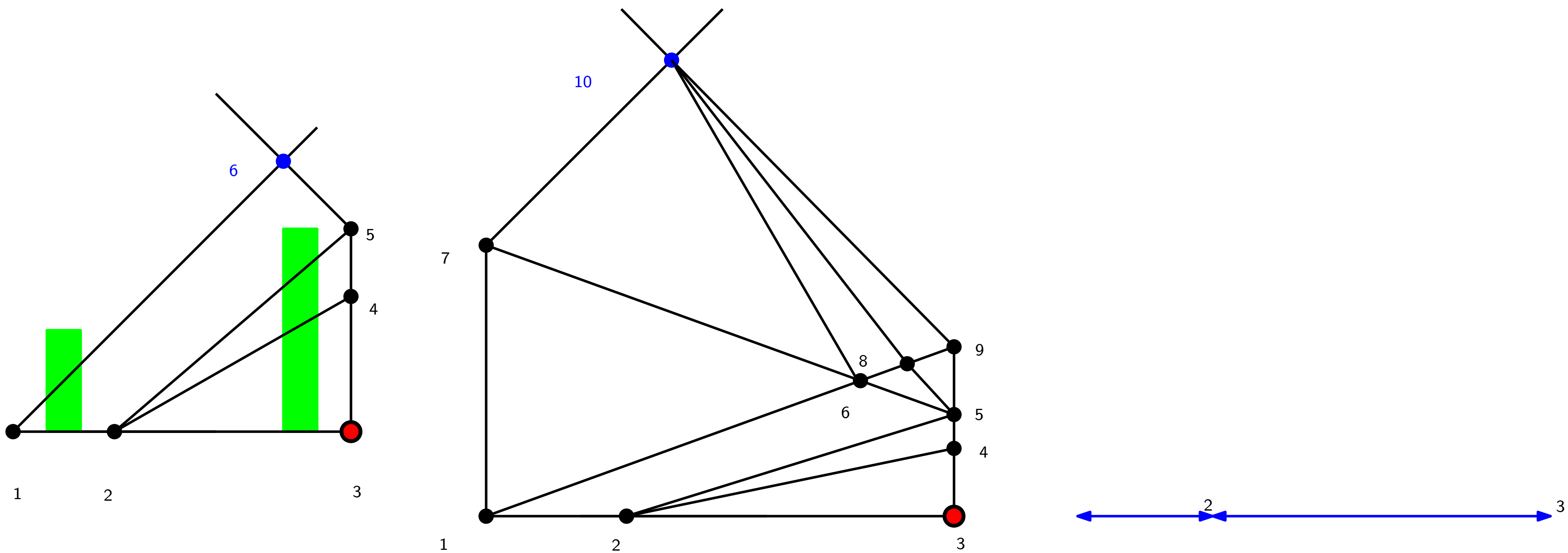


Step 3: draw left bottom piece

- Turn by 90.
- Find a canonical order.
- Draw with incremental algorithm.
- Remember the distanced between vertices on the bottom side.





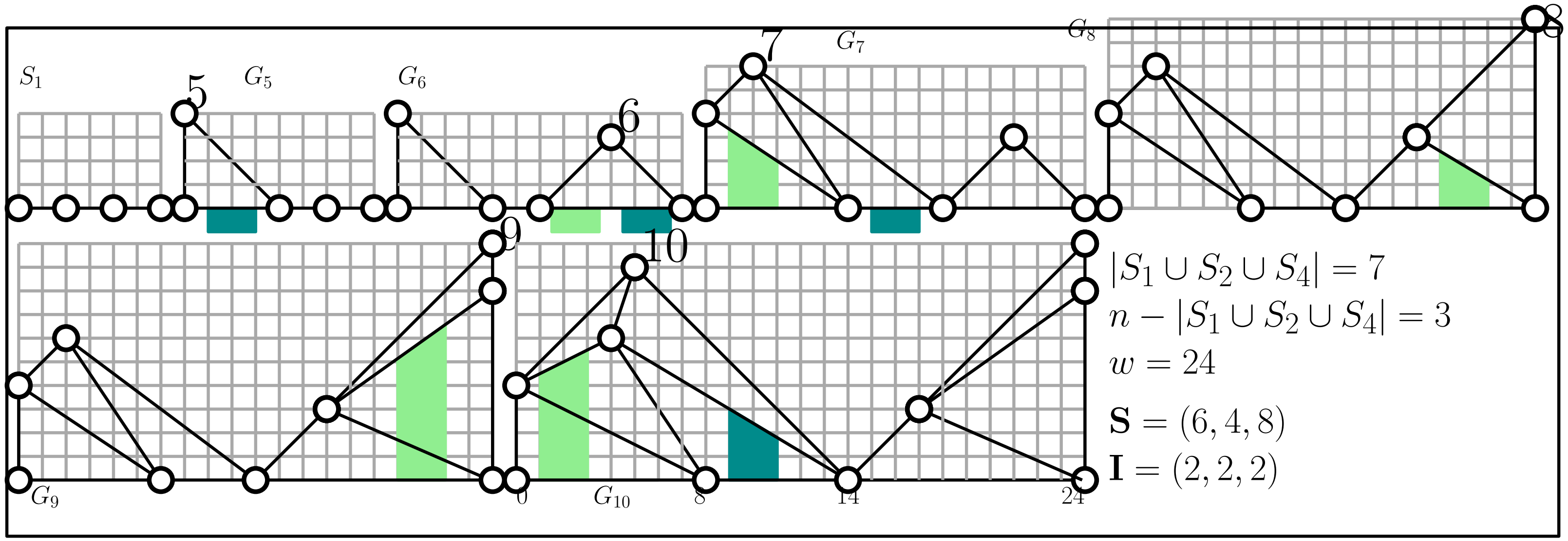


Step 3b: modified shift algorithm

- Remember the distanced between vertices on the bottom side (shift vector).
- use the shift vector to perform a second drawing pass

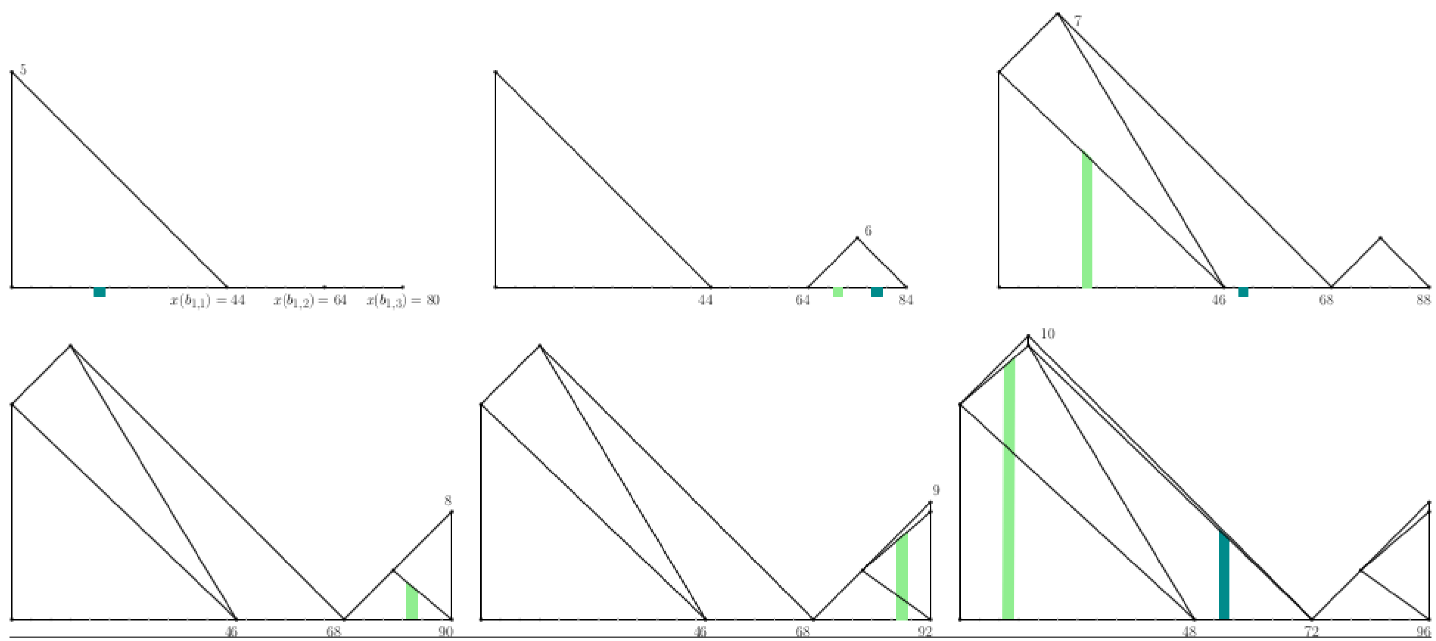
first pass

[Duncan et al., GD09]



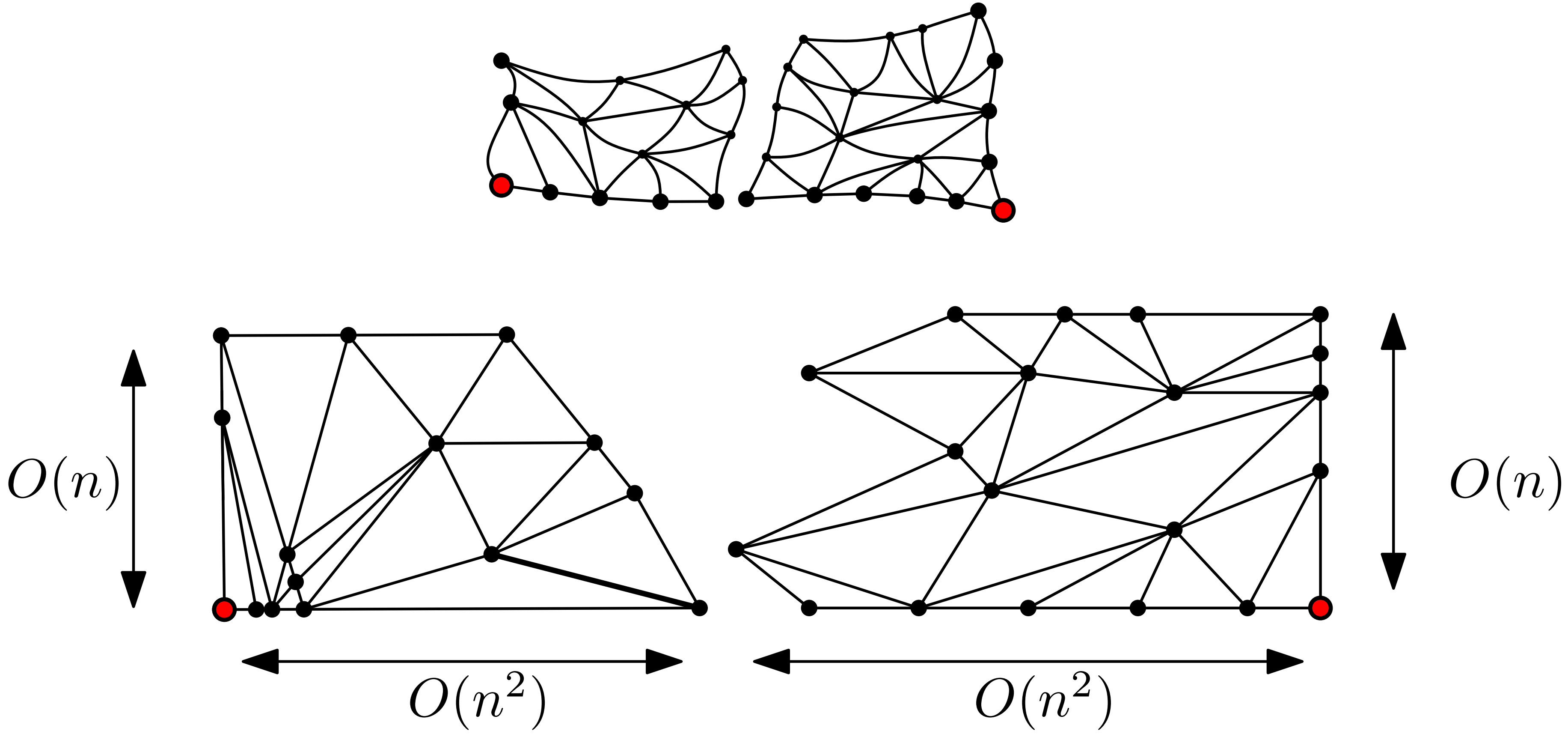
second pass

(initial positions are given by the shift vector)



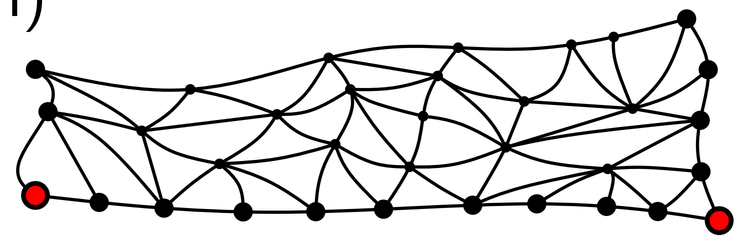
Step 4a: draw both bottom corners

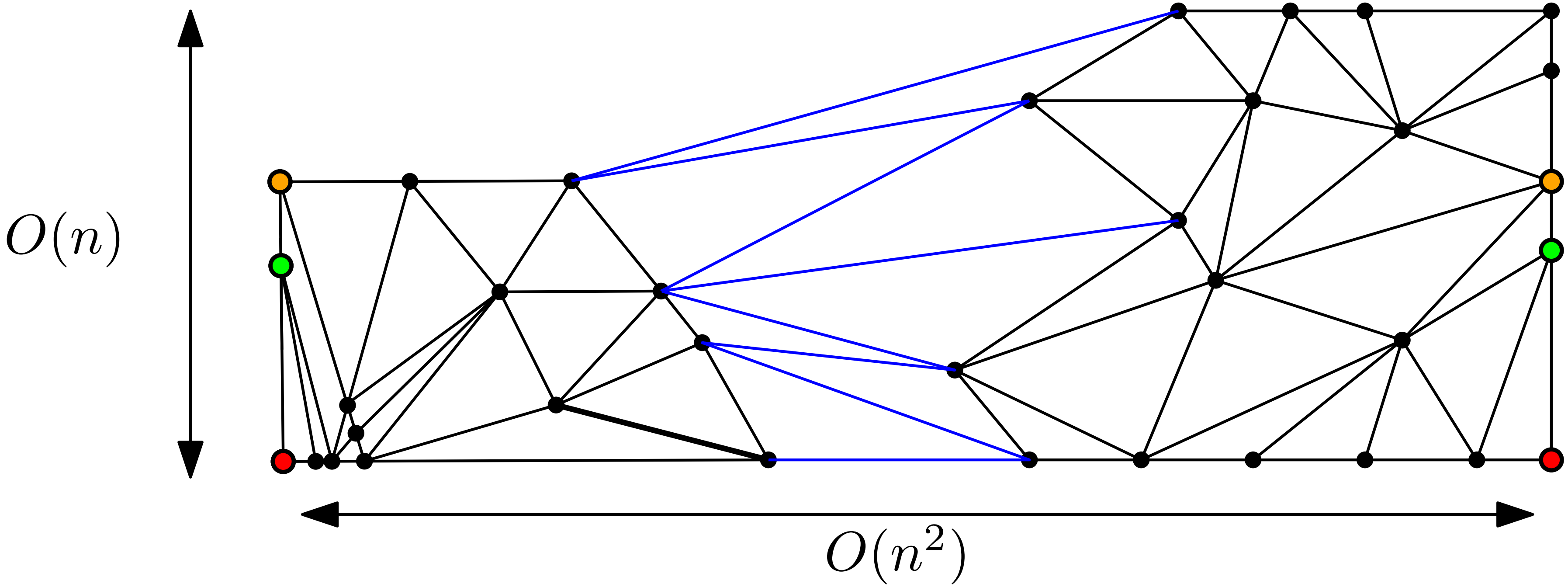
- Repeat previous step 3 for right bottom piece.
- Turn the two pieces by 90 degrees
- re-draw and adjust sizes (using stored shift vectors)



Step 4b: align and add the river

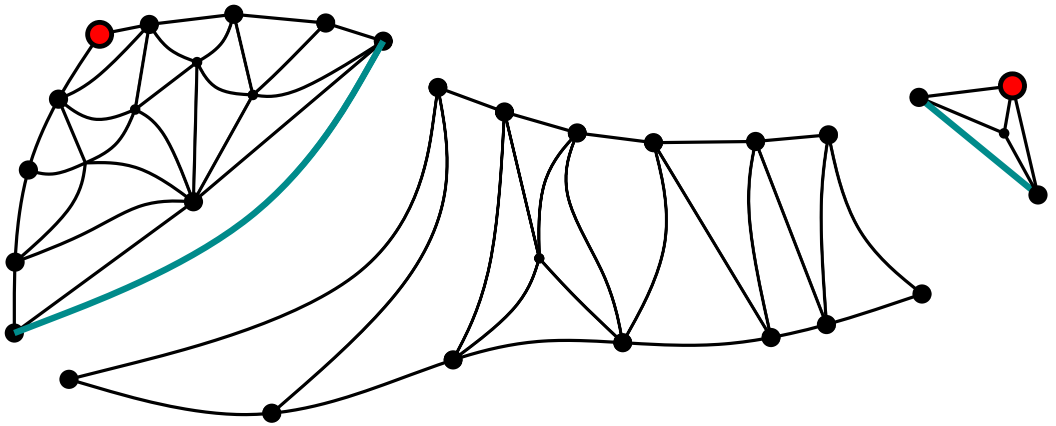
- Repeat previous step 3 for right bottom piece.
- Turn the two pieces by 90 degrees
- re-draw and adjust sizes (using stored shift vectors)
- add the river
- align opposite vertices (modified shift algorithm)





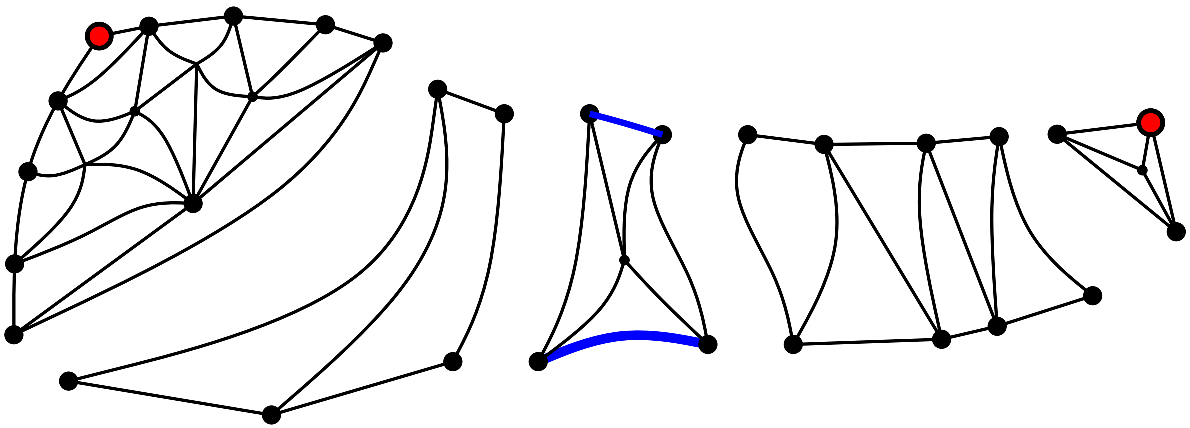
Step 5: decompose the upper graph

- Cut upper corners (along largest upper chords)

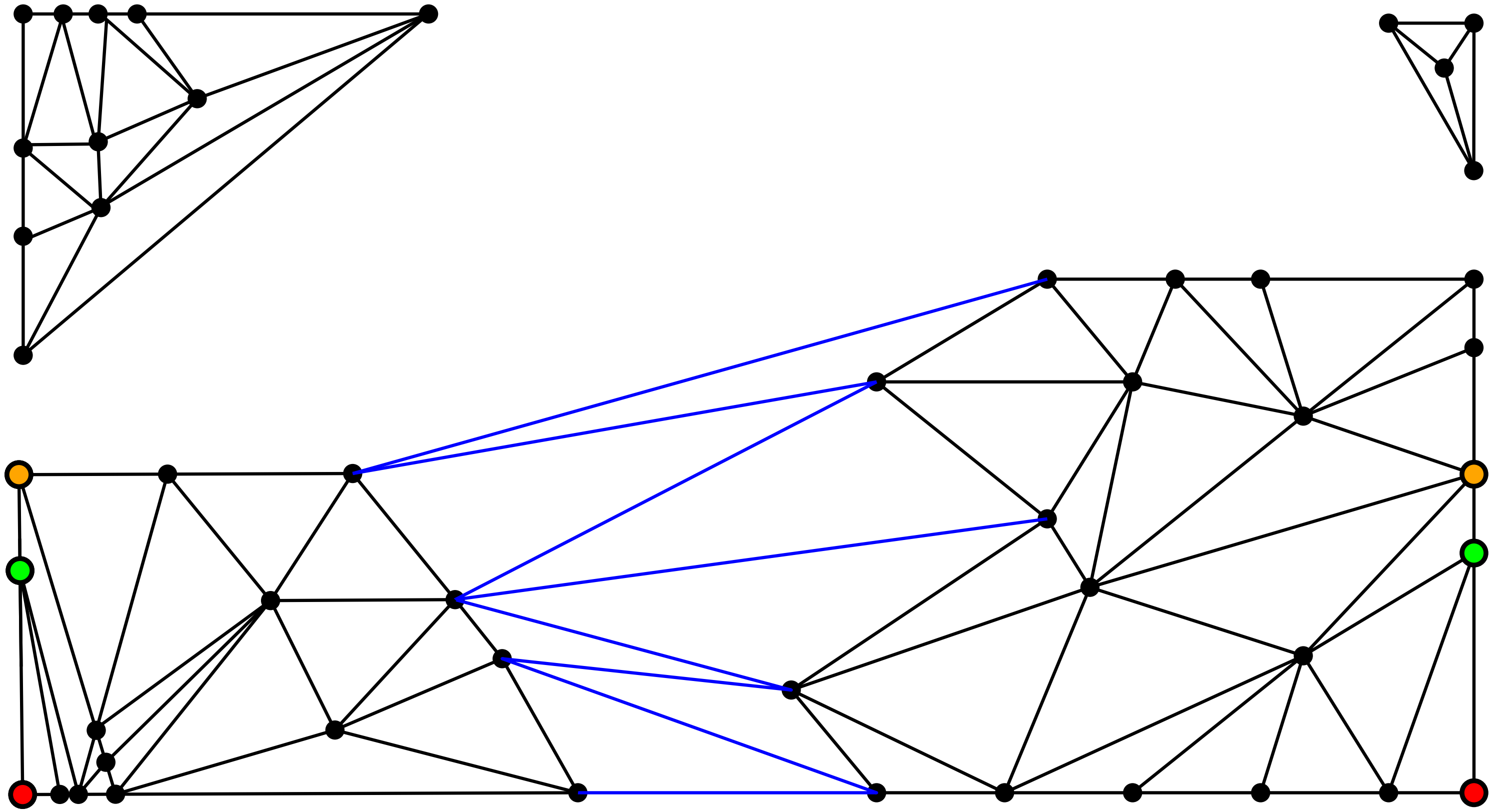


Step 5: decompose the upper graph

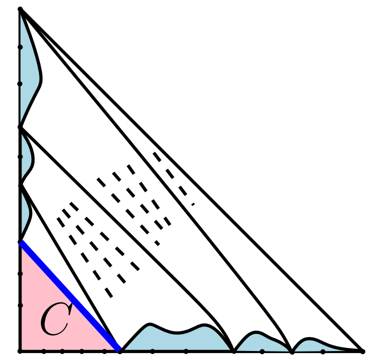
- Cut upper corners (along largest upper chords)
- Find the edge adjacent to the river
- Decompose the rest into 3 parts

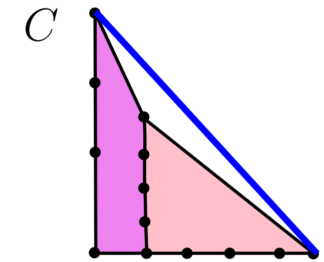


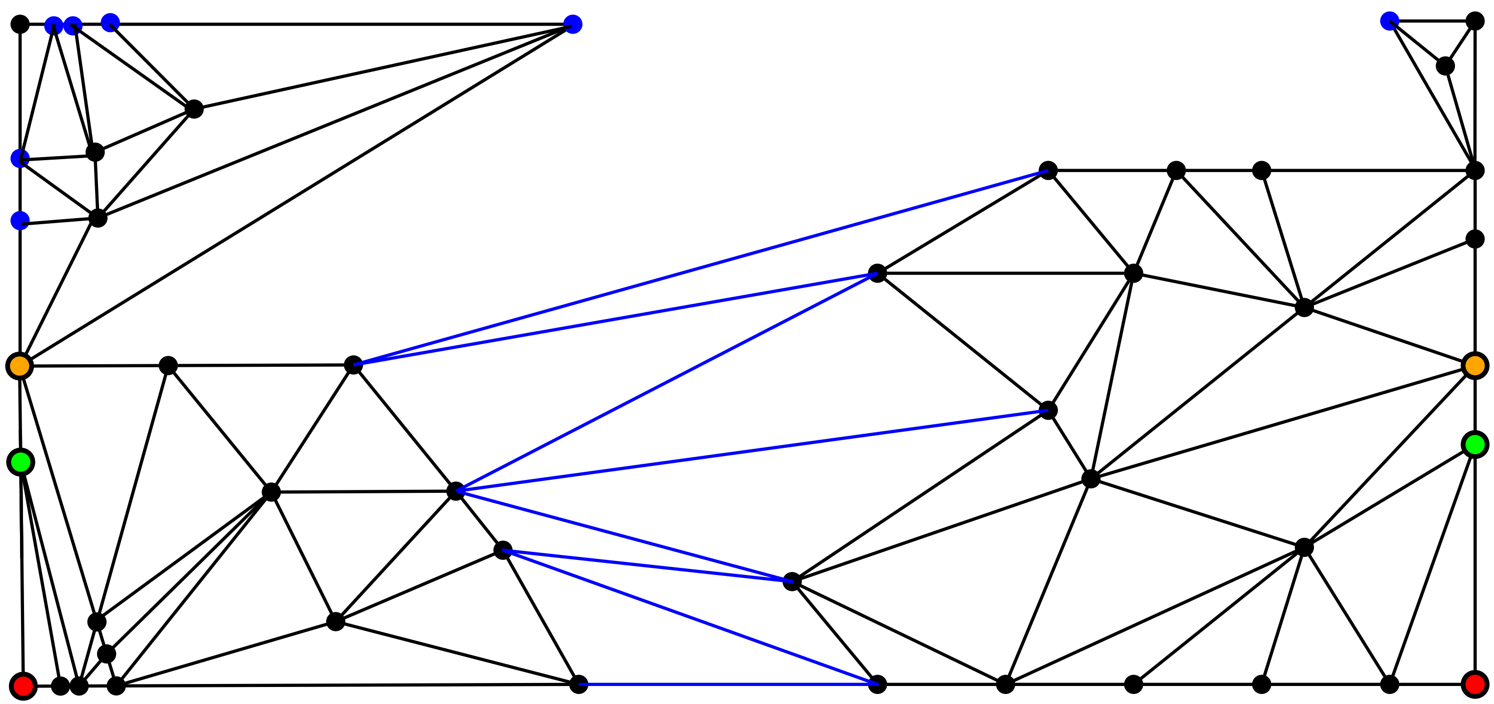
Step 6: draw upper corners



Step 7: align upper corners





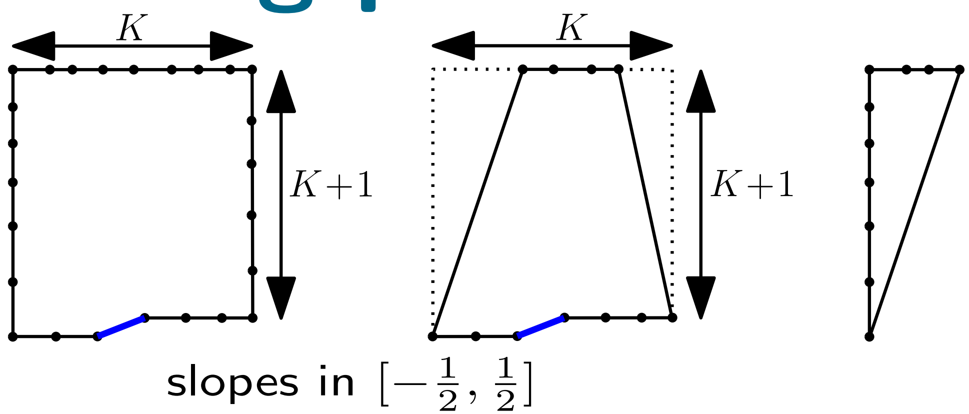


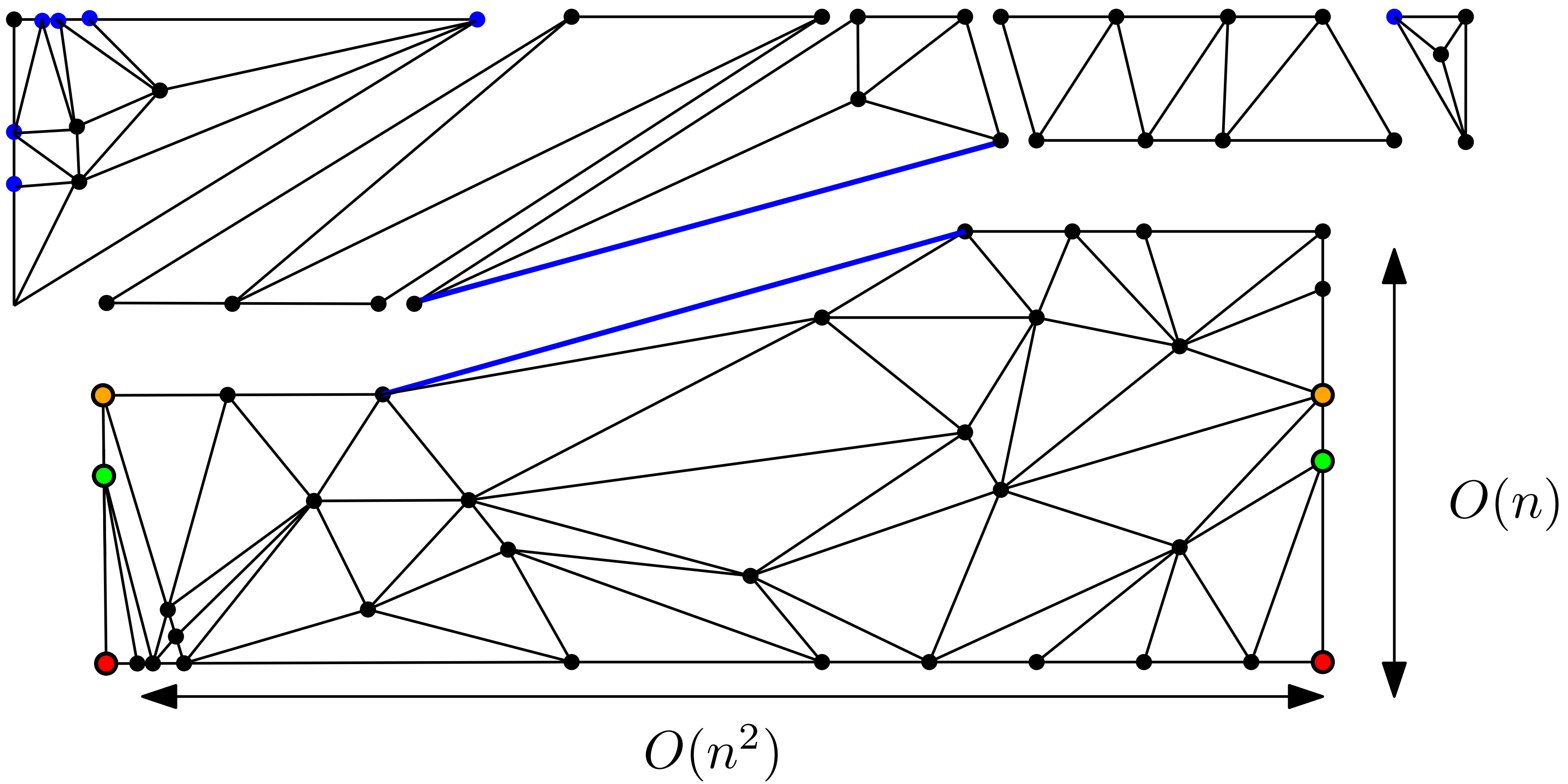
Step 8: draw remaining pieces

refinement factor for the upper piece

$$O(n \cdot \max(n, K))$$

where $K = O(n^2)$ is the grid width

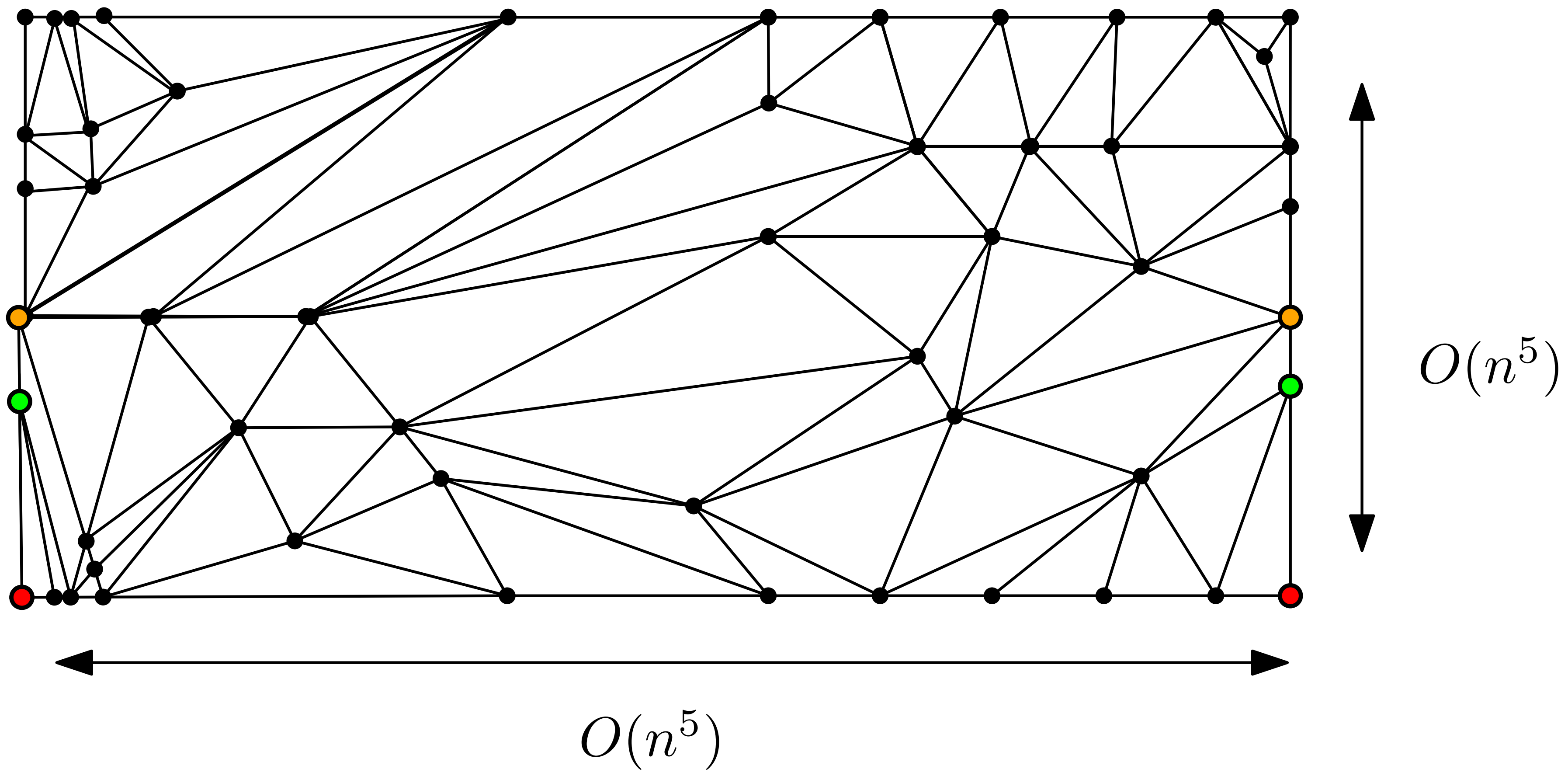




Final step: glue all pieces together

Drawing in the naïve way ⟶ $O(K^2 n \times K^2 n) = O(n^5 \times n^5)$ area

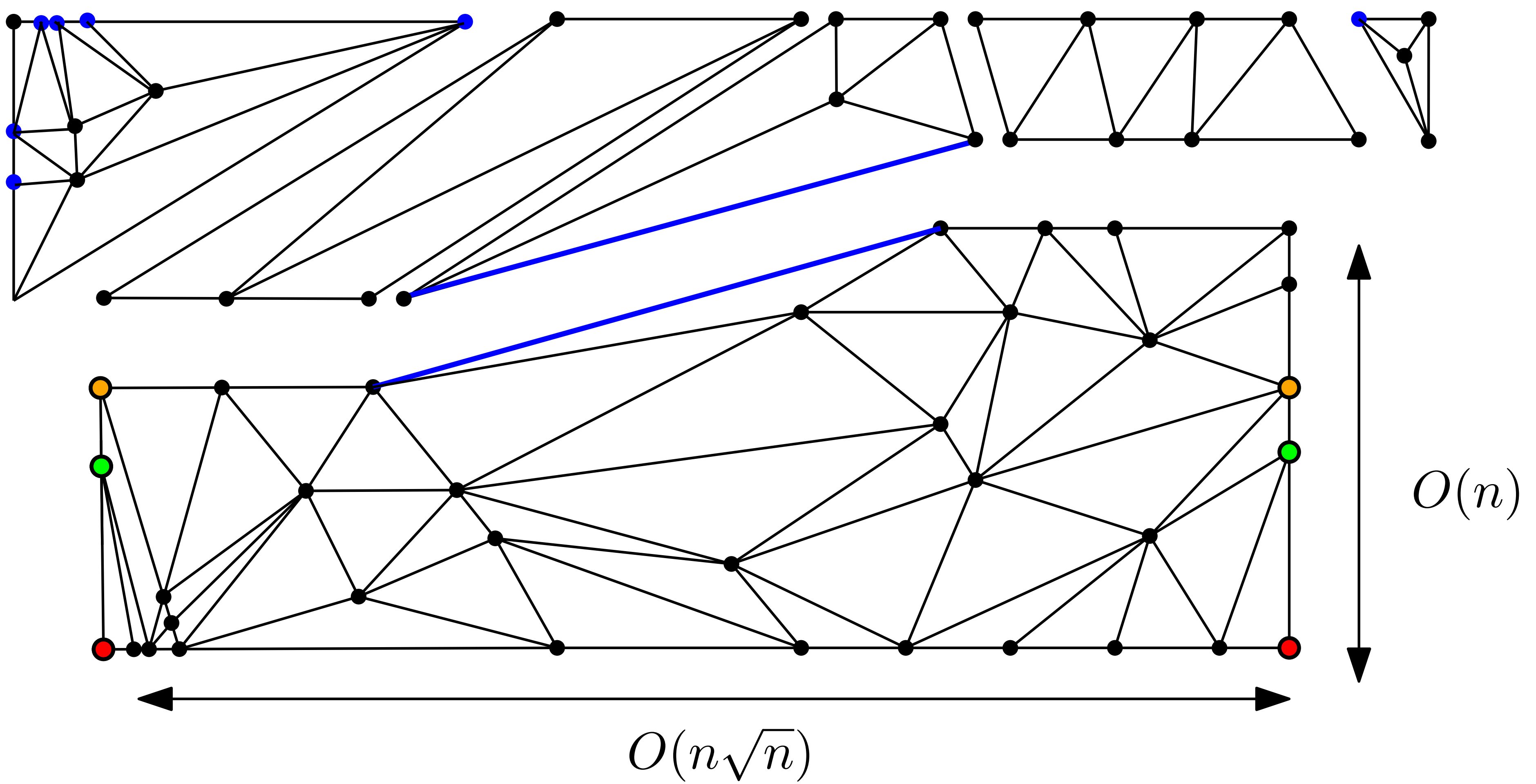
where $K = O(n^2)$ is the grid width



Final step: glue all pieces together

Drawing in a clever way ⟶ $O(n^4 \times n^4)$ area

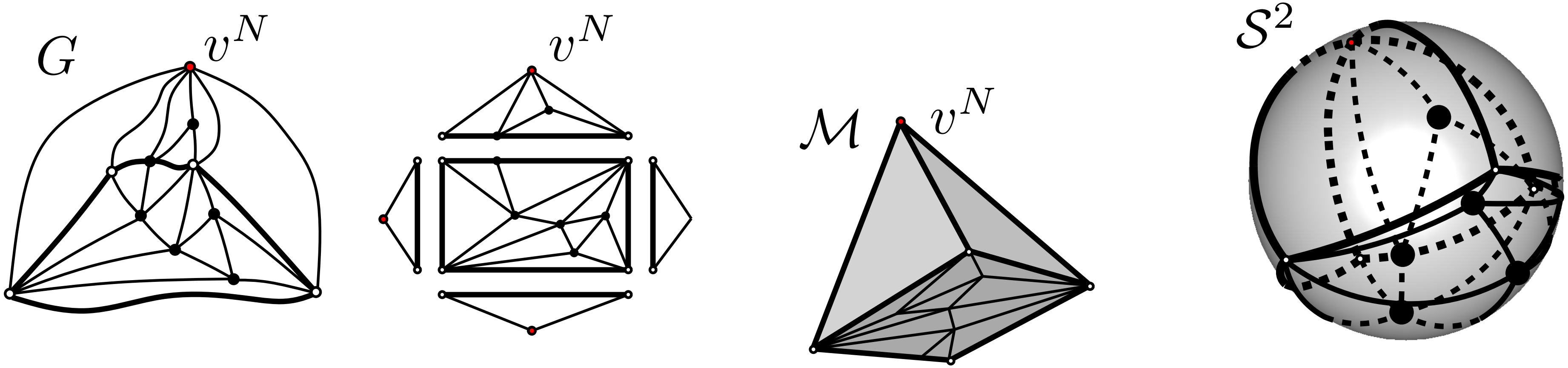
(recall on the torus there are non contractible cycles of length $O(\sqrt{n})$) [Hutchinson, Albert '78]



applications and extensions

(geodesic spherical drawing)

Geodesic spherical drawing



Algorithm:
- partition the faces of the initial graph;
- dessiner draw every rectangle according their lateral sides;
- construct a pyramid from the rectangles;
- place a small copy in the center of sphere;
- project its edges on the sphere.

Draw arbitrary polygons

- Suppose we can draw an arbitrary quadrangle, let $P(n)$ be its grid size
- Using divide and conquer strategy we can draw any k-gon
- Grid size will be proportional to $O(P(n)^{\log k})$.