Schnyder woods and graph encoding:
compression and compact data structures

journees GRATOS,
10 decembre 2010

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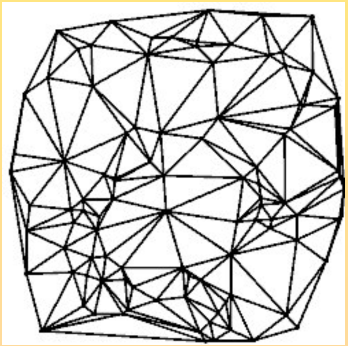
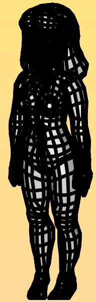
ÉCOLE POLYTECHNIQUE

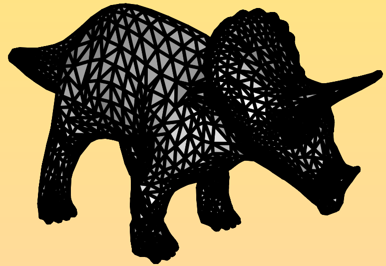
Geometric data

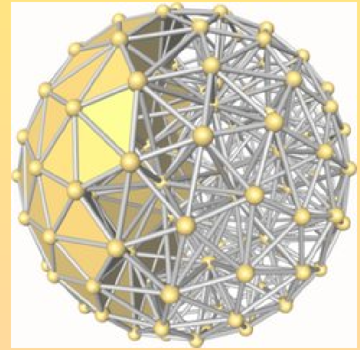
Triangulations and graphs



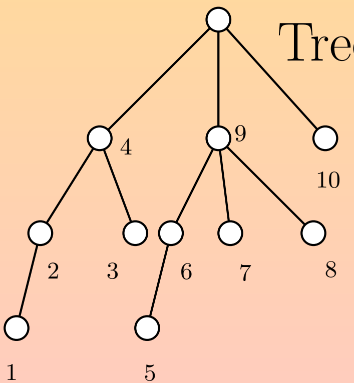
3D meshes

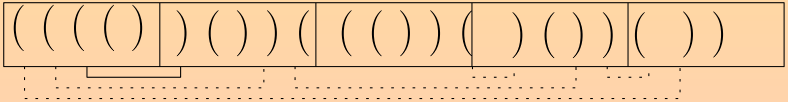






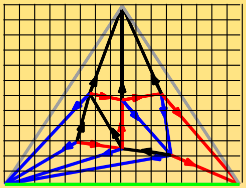
Trees





1 1 0 0 0	0 0 0 0 1	0 0 0 0 1	0 0 0 0 1	0 0

Graph planarity characterizations

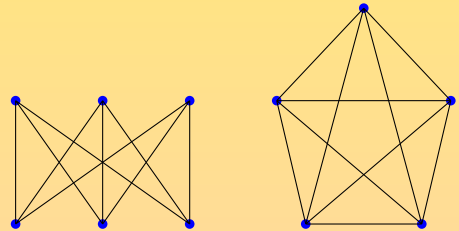


Schnyder woods (via dimension of partial orders)

- $dim(G) \leq 3$

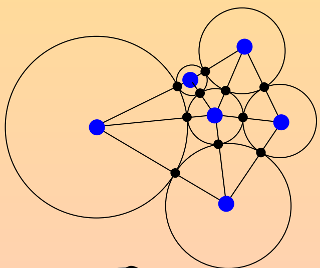
Kuratowski theorem (excluded minors)

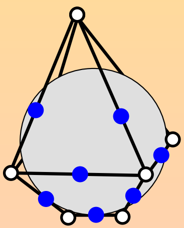
- G contains neither K_5 nor $K_{3,3}$ as minors

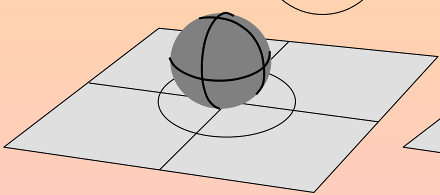


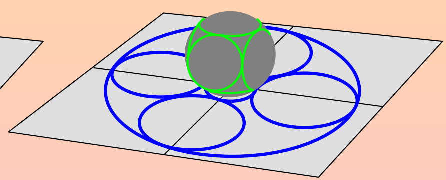
Thm (Koebe-Andreev-Thurston)

Every planar graph with n vertices is isomorphic to the intersection graph of n disks in the plane.

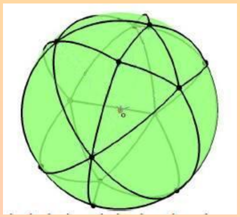








$$M = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \qquad \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{matrix} \quad \xi_x = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \xi_y = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \xi_z = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

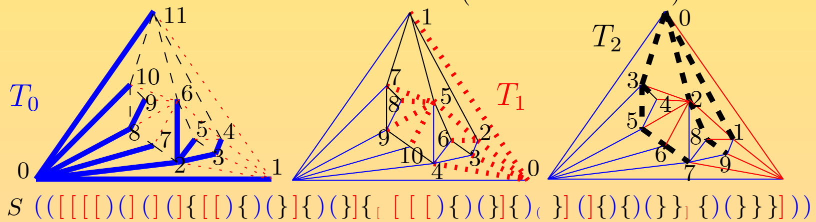


Colin de Verdiere invariant (multiplicity of λ_2 eigenvalue of a generalized laplacian)

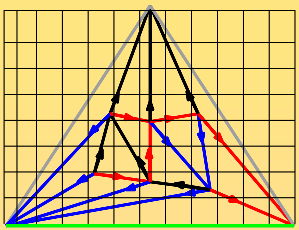
- $\mu(G) \leq 3$

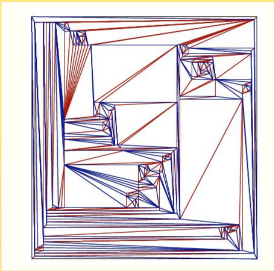
Schnyder woods: applications

Graph encoding (Chuang-Garg-He-Kao-Lu Icalp '98)
(He-Kao-Lu '99)
(Chiang et al. Soda'01)
(Barbay-Castelli Aleardi-He-Munro Isaac'07)
(Castelli Aleardi-Fusy-Lewiner SoCG08)
(Castelli Aleardi-Fusy-Lewiner CCCG'10)
(Nakano et al. '08)

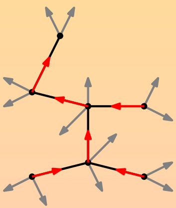


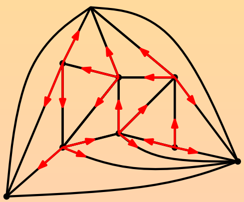
grid drawing





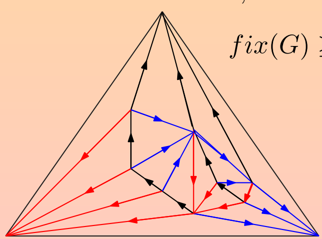
graph counting, random generation
(Poulalhon-Schaeffer, Icalp 03)



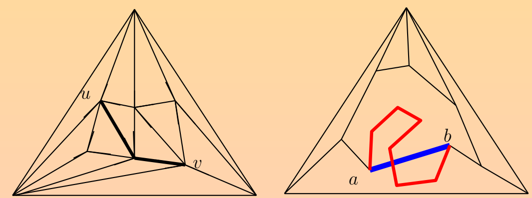


Untangling geometric graphs
Bose, Dujmovic, Hurtado, Langerman, Morin, Wood (DCG 2009)

$fix(G) \geq (\frac{n}{3})^{\frac{1}{4}}$



Greedy routing

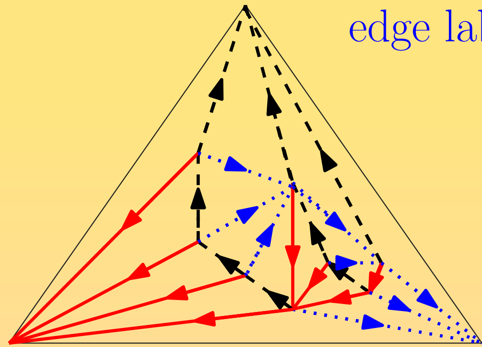


Every planar triangulation admits a *greedy drawing* (Dhandapani, Soda08)

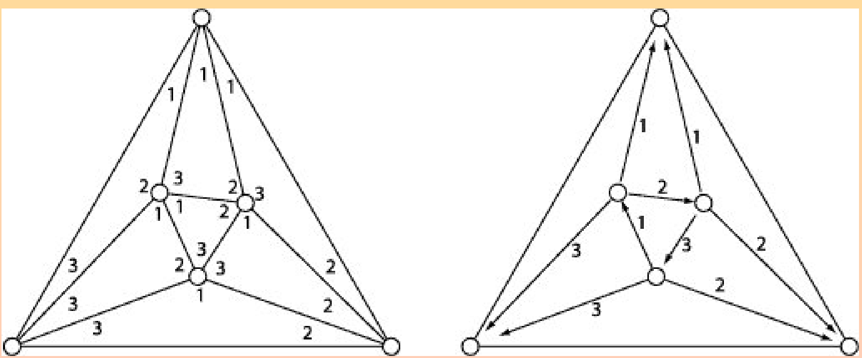
Conjectured by Papadimitriou and Ratajczak for 3-connected planar graphs

Schnyder woods: three formulations

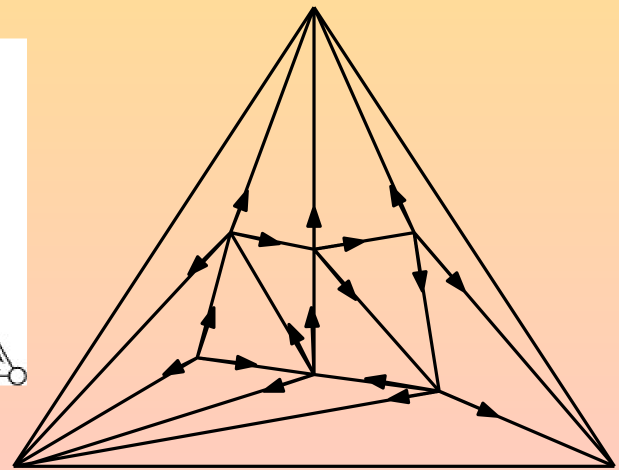
edge labelings (edge coloration)



angle labelings

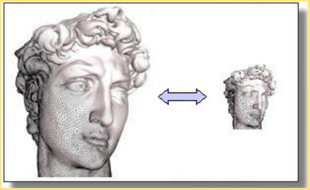


edge orientations



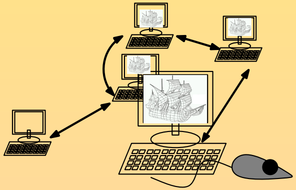
Several kinds of encodings

Mesh compression schemes

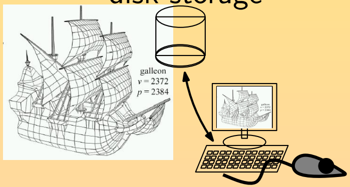


$\alpha n + O(\log n)$ bits

Transmission

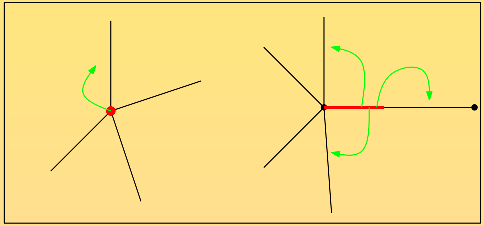


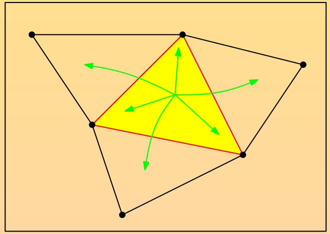
disk storage



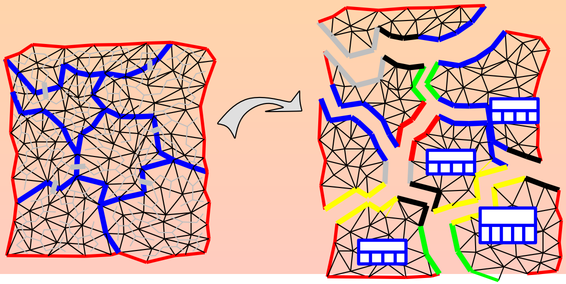
(Explicit) Geometric data structures

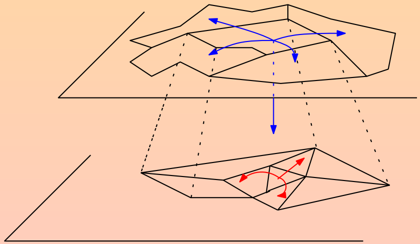
$\beta n + O(1)$ references (pointers)

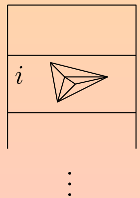




$\alpha n + O(n\frac{n \log\log n}{\log n}) = \alpha n + o(n)$ bits Succinct representations

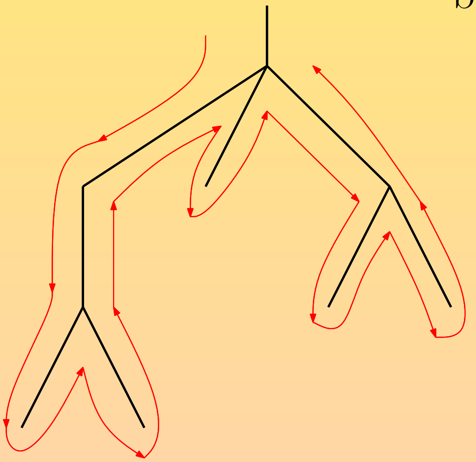




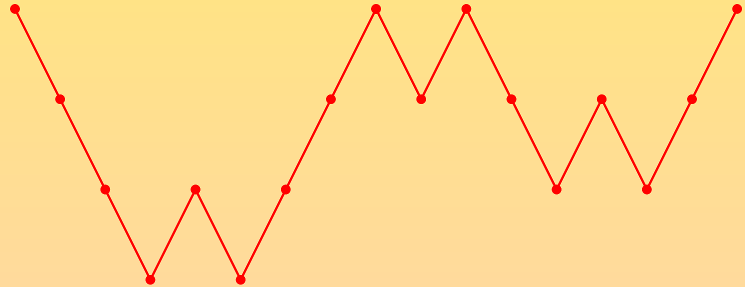


An example: plane trees

plane tree with n edges



balanced parenthesis word



1 1 1 0 1 0 0 0 1 0 1 1 0 1 0 0

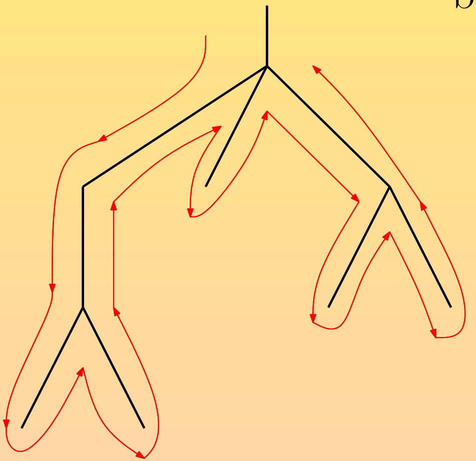
$\Rightarrow 2n$ bits for encoding a tree with n edges

Enumeration of plane trees with n edges

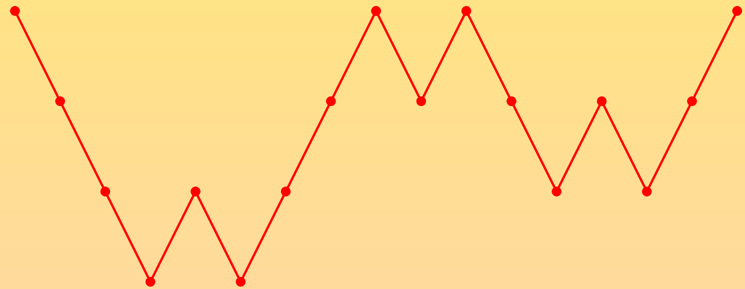
$$\|\mathcal{B}_n\| = \frac{1}{n+1}\binom{2n}{n} \approx 2^{2n} n^{-\frac{3}{2}}$$

An example: plane trees

plane tree with n edges



balanced parenthesis word



1 1 1 0 1 0 0 0 1 0 1 1 0 1 0 0

$\Rightarrow$ $2n$ bits for encoding a tree with n edges

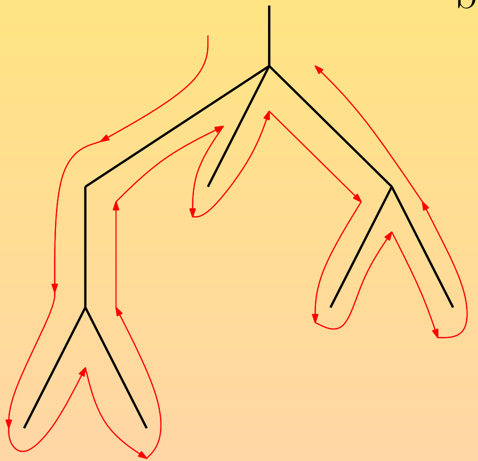
Asymptotic optimal encoding

- the cost of an object matches asymptotically the entropy

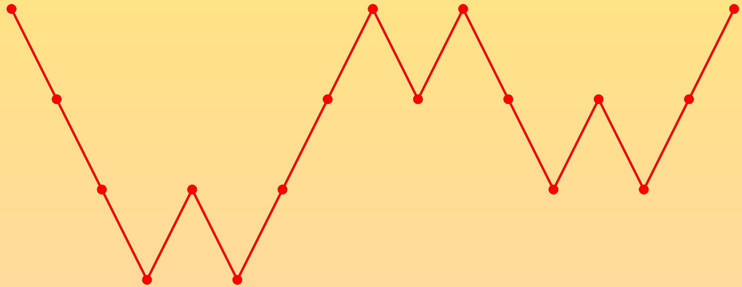
$$log_2 \|\mathcal{B}_n\| = 2n + O(\lg n)$$

An example: plane trees

plane tree with n edges



balanced parenthesis word



1 1 1 0 1 0 0 0 1 0 1 1 0 1 0 0

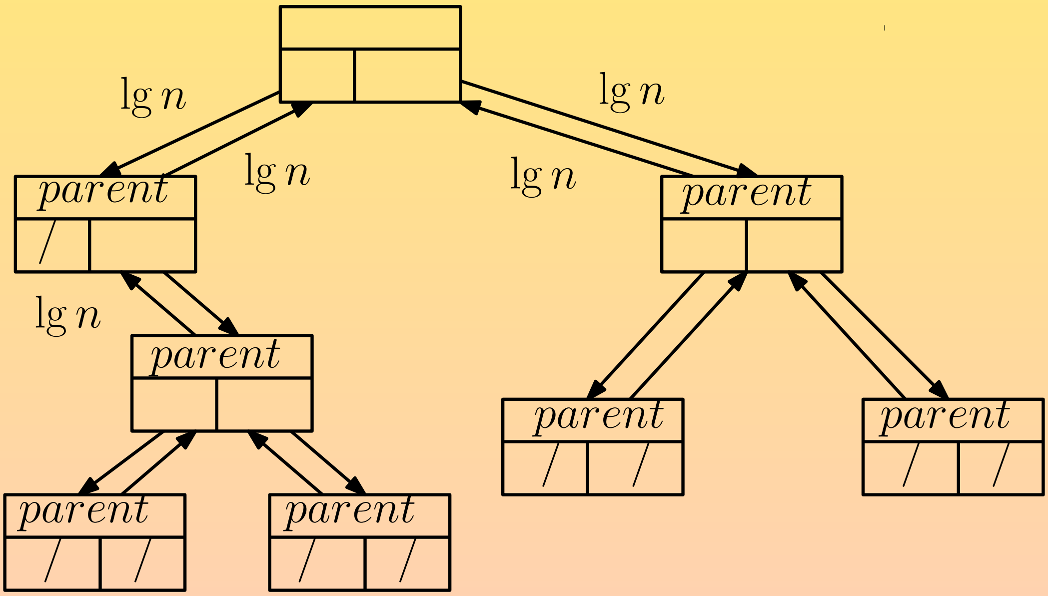
$\Rightarrow 2n$ bits for encoding a tree with n edges

Asymptotic optimal encoding

No efficient implementation of local adjacency queries

An example: plane trees

Explicit pointers based representation

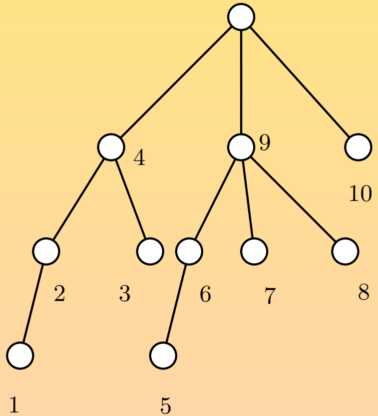


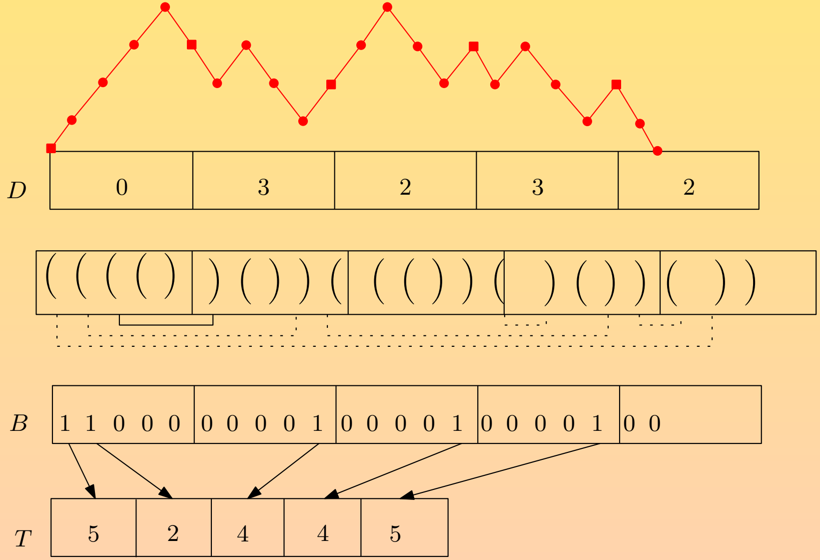
adjacency queries between vertices in $O(1)$ time

not optimal encoding: we need $\Theta(n \lg n)$ bits

An example: plane trees

(Jacobson, Focs89, Munro et Raman Focs97)



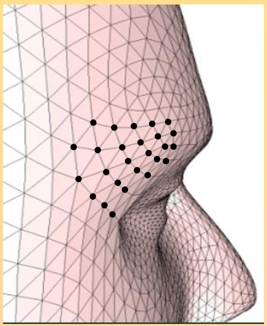


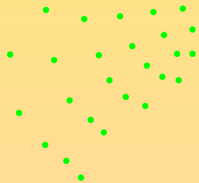
it is possible to test adjacency between vertices in $O(1)$ time with the guarantee the the encoding is still asymptotically optimal

$2n + o(n)$ bits are sufficient

Geometric information

Geometric object





$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

between 30 et 96 bits/vertex

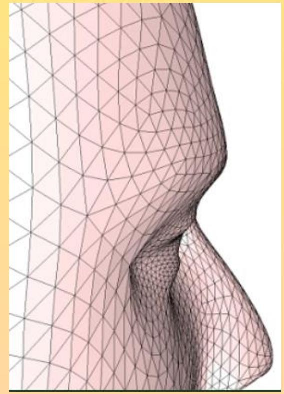
Connectivity information

vertex
triangle

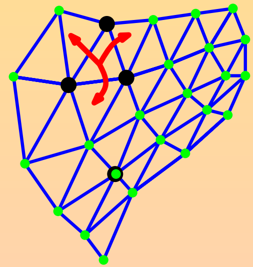
1 reference to a triangle

3 references to vertices

3 references to triangles



$\log n$ ou 32 bits



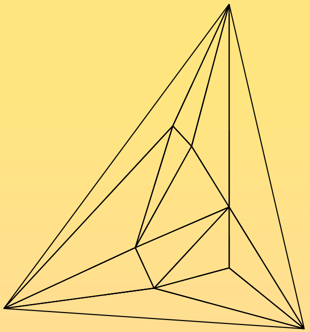
$$2 \times n \times 6 \times \log n$$

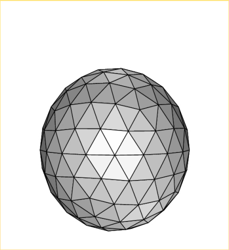
$$n \times 1 \times \log n$$

$$13n \log n$$

$416n$ bits
connectivity

Enumeration and entropy of planar triangulations





Enumeration of planar triangulations (Tutte, 1962)

$$\Psi_n = \frac{2(4n+1)!}{(3n+2)!(n+1)!} \approx \frac{16}{27}\sqrt{\frac{3}{2\pi}}n^{-5/2}\left(\frac{256}{27}\right)^n$$

Entropy

$$\frac{1}{n}\log_2 \Psi_n \approx \log_2\left(\frac{256}{27}\right) \approx 3.2451 \text{ bits/vertex}$$

Mesh compression

Computer graphics

Edgebreaker

Rossignac ('99)

Valence (degree)

Touma and Gotsman ('98)

Alliez and Debrun

Isenburg

Khodakovsky

Other approaches

Graph encoding

Graph theory / combinatorics

Turan ('84)

Keeler Westbrook ('95)

He et al. ('99)

Poulalhon Schaeffer (Icalp03)

Fusy, Poulalhon, Schaeffer (Soda05)

C-A, Fusy Lewiner (SoCG08)

C-A, Fusy Lewiner (CCCG10)

Succinct representations

algorithms and DS

Jacobson (Focs89)

Munro and Raman (Focs97)

Chuang et al. (Icalp98)

Chiang et al. (Soda01)

C-A, Devillers and Schaeffer (Wads05, CCCG05)

C-A, Devillers and Schaeffer (SoCG06)

Barbay, C-A, He, Munro (Isaac07)

Nakano et al. (2008)

Blandford Blelloch (Soda03)

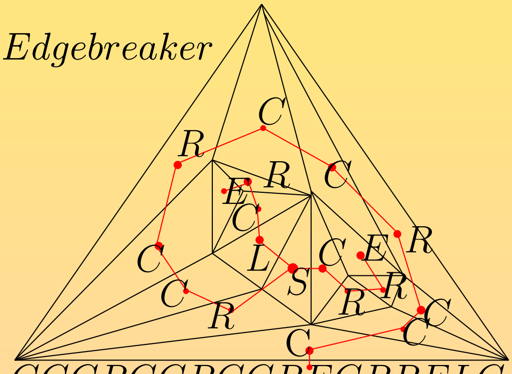
Farzan Munro (09)

Blelloch Farzan (10)

Graph encodings and spanning trees

General visual framework (Isenburg Snoeyink)

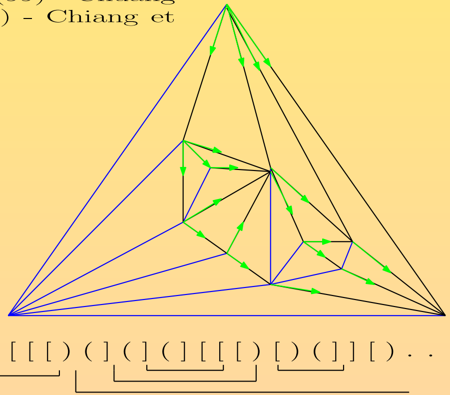
Edgebreaker



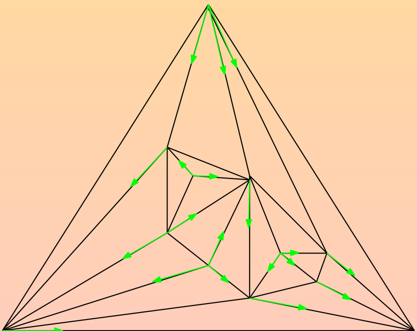
$CCCRCCRCCRECRRELCRE$

Canonical orderings, Scnhyder woods
(multiple parenthesis words)

He et al. (99) - Chuang et al. (98) - Chiang et al. (01)

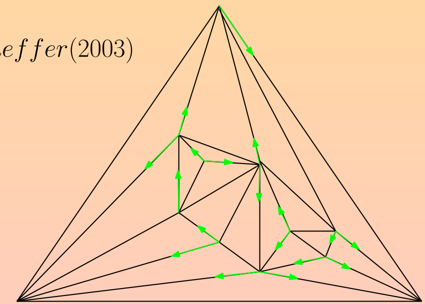


Touma Gotsman('98)



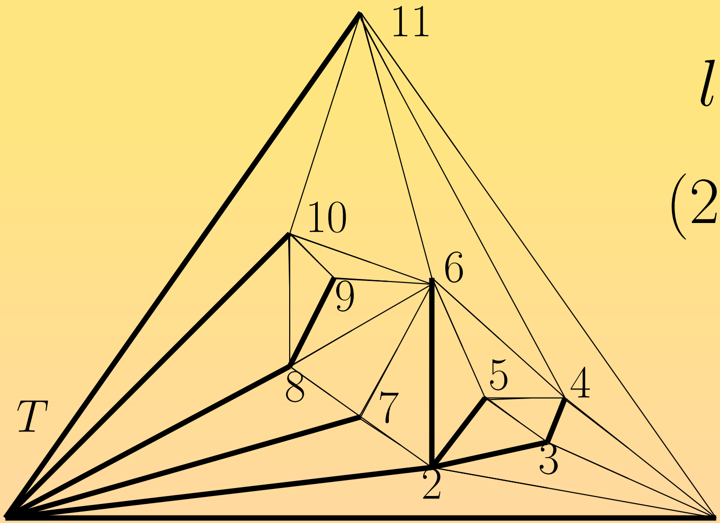
$V_5V_5V_6V_5V_4V_5V_8V_5V_5V_4S_4V_3V_4$

Poulalhon Schaeffer(2003)



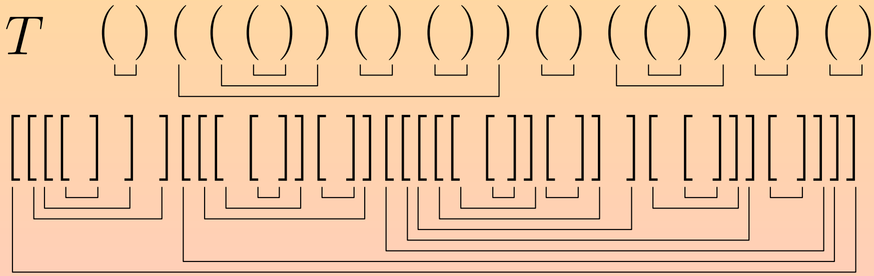
1101000110000010010000011001000000000

Turan encoding of planar map (1984)



$$length(S) = 2e \text{ symbols}$$

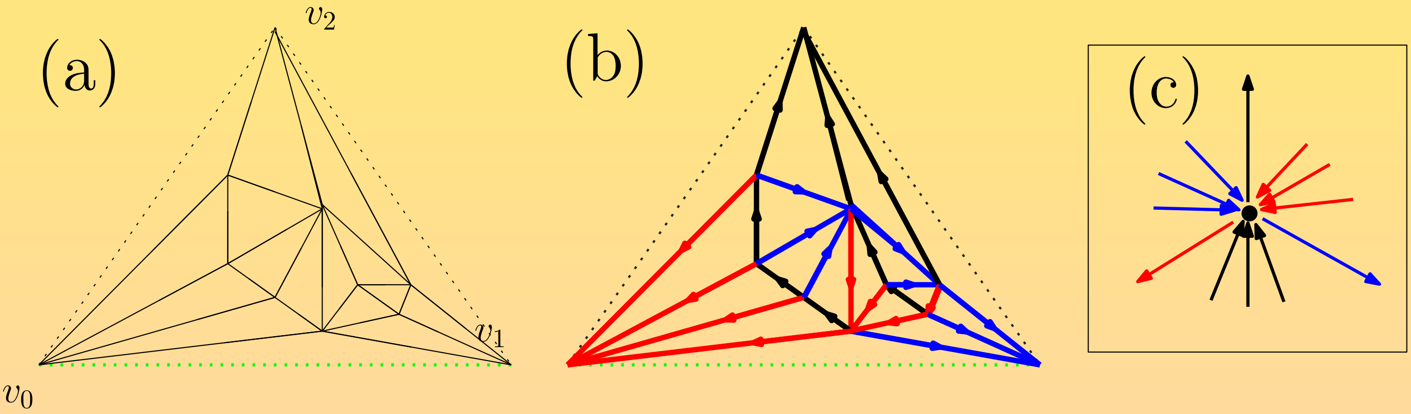
$$(2\log_2 4)e = 4e = 12n \text{ bits}$$



$S(G)$ ([[[ ) ( ] ( ] [[[ )]) ( ]] [ ) . . .

Schnyder woods: the definition

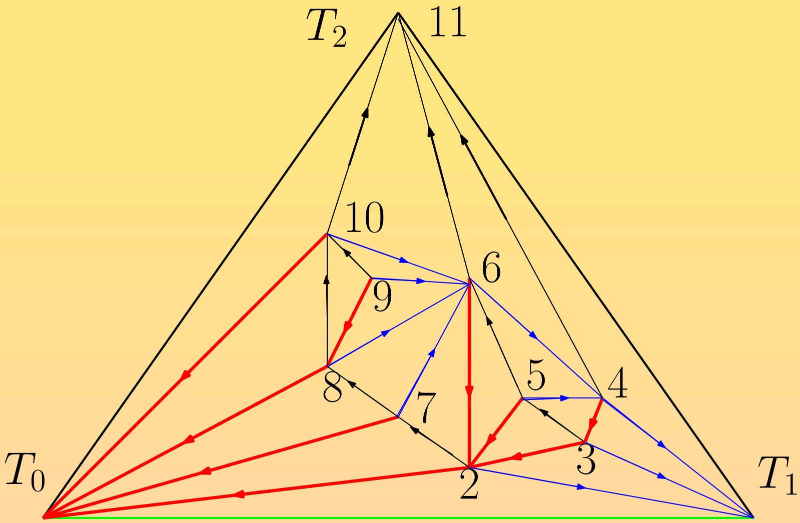
v_0 v_1 v_2 outer face



i) edge are colored and oriented in such a way that each inner node has exactly one outgoing edge of each color

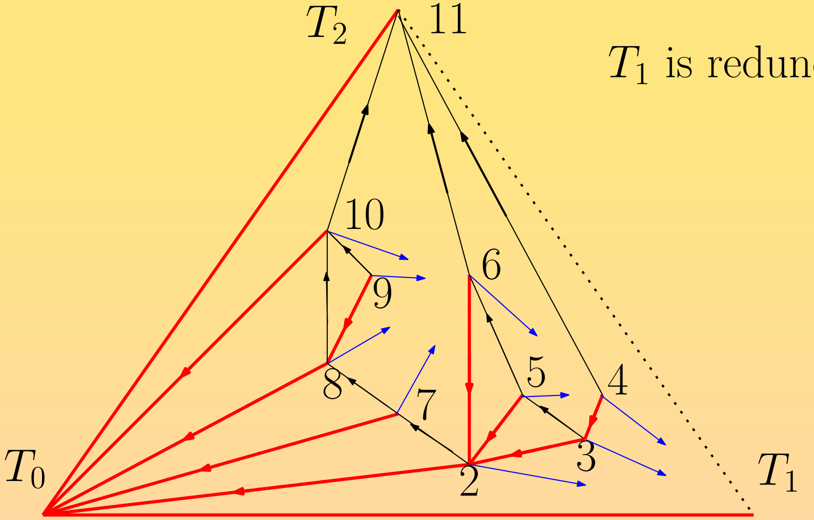
ii) colors and orientations around each inner node must respect the local Schnyder condition

Canonical orderings - Schnyder woods (He, Kao, Lu '99)

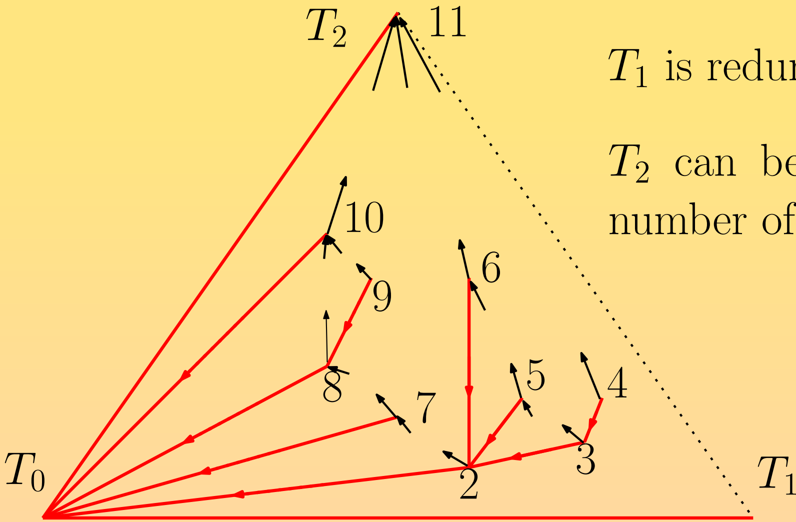


Canonical orderings - Schnyder woods (He, Kao, Lu '99)

T_1 is redundant: reconstructed from T_0, T_2



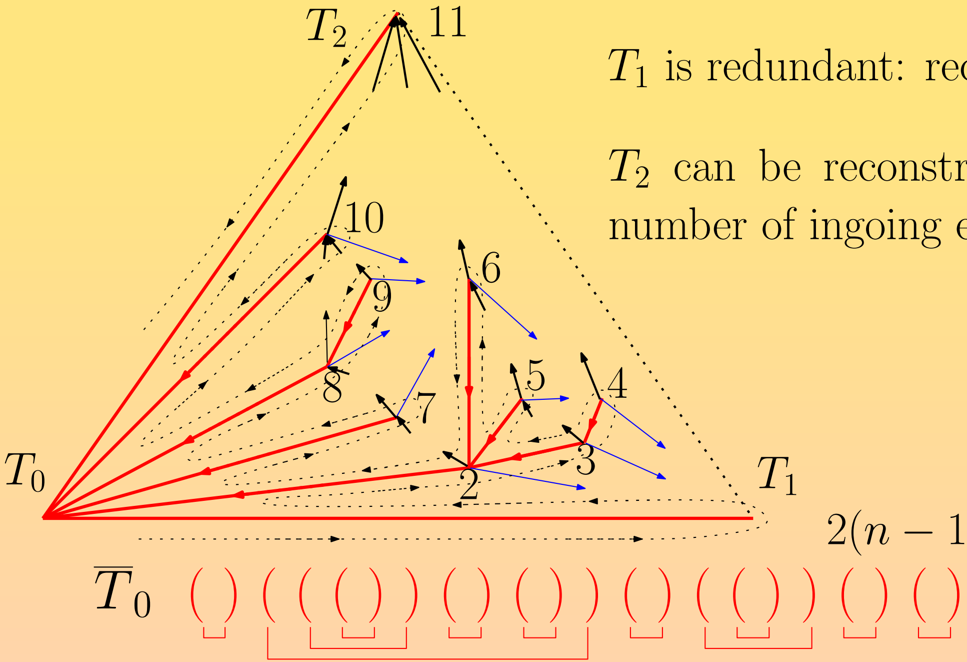
Canonical orderings - Schnyder woods (He, Kao, Lu '99)



T_1 is redundant: reconstructed from T_0, T_2

T_2 can be reconstructed from T_0 and the number of ingoing edges (for each node)

Canonical orderings - Schnyder woods (He, Kao, Lu '99)

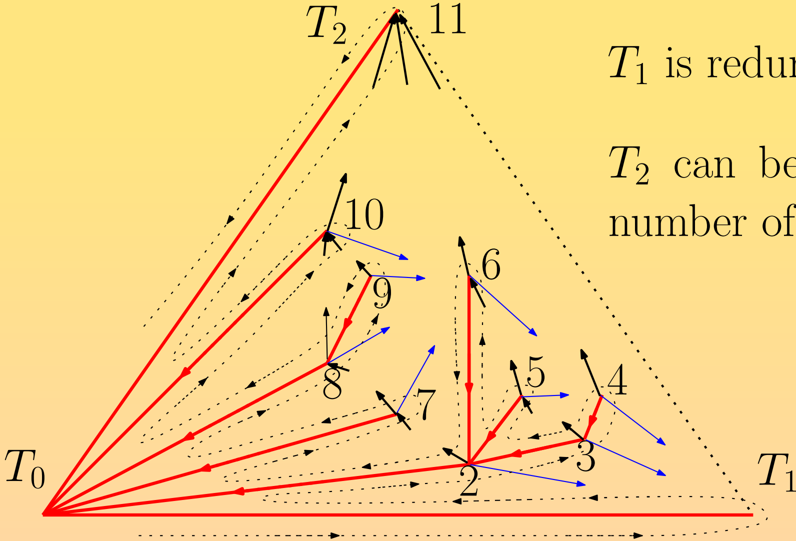


T_1 is redundant: reconstructed from $\overline{T}_0$, T_2

T_2 can be reconstructed from $\overline{T}_0$ and the number of ingoing edges (for each node)

$2(n-1)$ symbols $= 2(n-1)$ bits

Canonical orderings - Schnyder woods (He, Kao, Lu '99)



T_1 is redundant: reconstructed from $\overline{T}_0$, T_2

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$2(n-1)$ symbols$= 2(n-1)$ bits

$\overline{T}_0$ ( ) ((( )) ( ) ( )) ( ) (( )) ( ) ( )

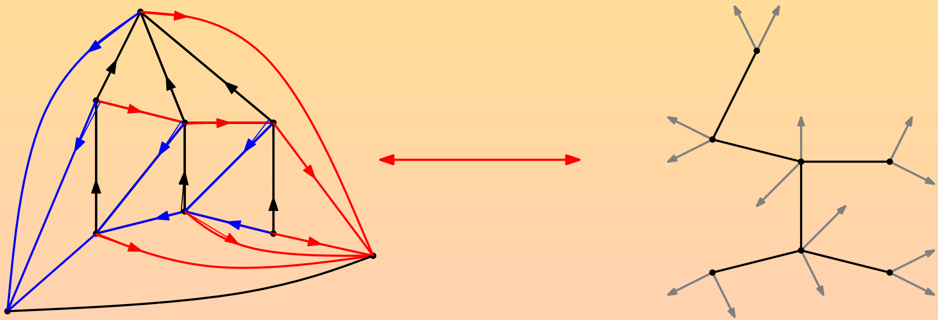
$\overline{T}_2$ 00000101010100110111 $\quad (n-1)+(n-3) = 2n-4$ bits

Opimal encoding (planar)

Theorem. (Tutte 62) The number of planar triangulations with $n+2$ vertices is

$$\frac{2(4n-3)!}{(3n-1)!n!} \asymp \left(\frac{256}{27}\right)^n .$$

Théorème. (Poulalhon–Schaeffer Icalp 03)
Bijection between plane trees of size n, having two stems per node, and the class of rooted planar triangulations with $n+2$ vertices.

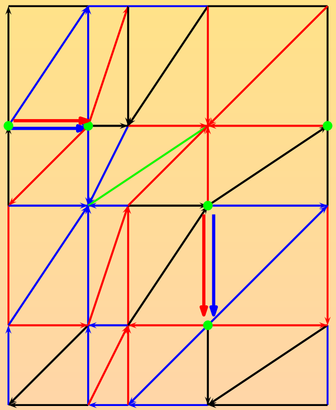


a new nice interpretation of Tutte's formula:

$$|\mathcal{T}_n| \ = \ \frac{2}{2n} \cdot |\mathcal{A}_n^{(2)}|.$$

Opimal encoding (genus g)

C-A, Fusy, Lewiner (SoCG08)



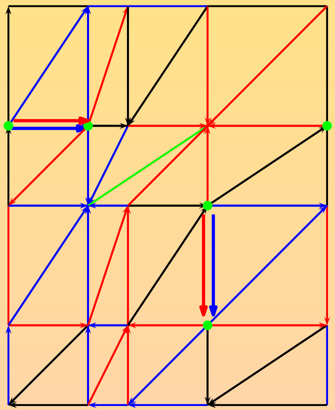
$\mathcal{S}$

triangulated graph of genus g

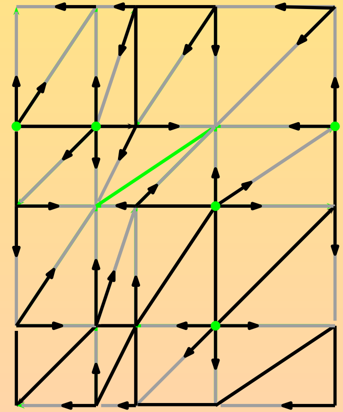
endowed with a g-Schnyder wood

Opimal encoding (genus g)

C-A, Fusy, Lewiner (SoCG08)



?



$\mathcal{S}$

$\mathcal{T}^g$

triangulated graph of genus g

endowed with a g-Schnyder wood

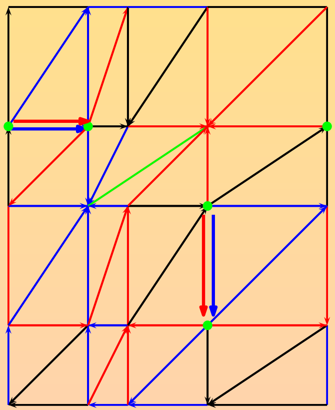
one face map of genu g

Opimal encoding (genus g)

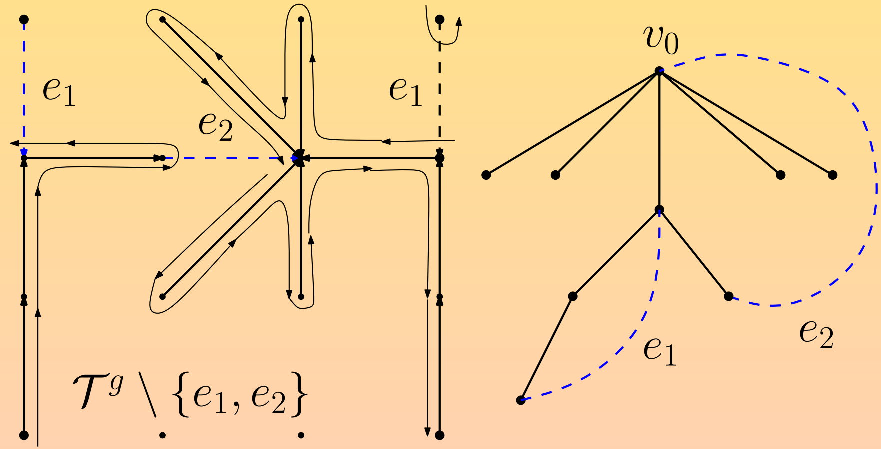
C-A, Fusy, Lewiner (SoCG08)

Corollary

A triangulation of genu g having n vertices can be encoded with $4n + O(g \log n)$ bits



$\mathcal{S}$



triangulated graph of genus g

endowed with a g-Schnyder wood

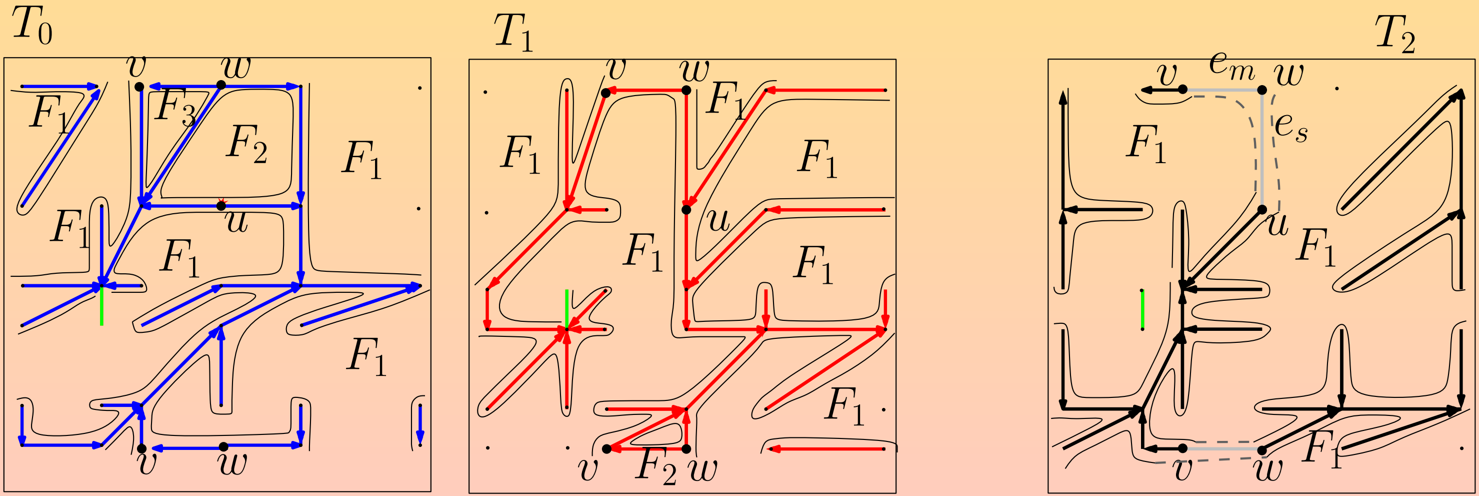
Opimal encoding (genus g)

C-A, Fusy, Lewiner (SoCG08)

Theorem

The three sets of edges T_0 and T_1 (red and blue edges), as well as the set $T_2 \cup \mathcal{E}$ (black edges and special edges) are maps of genus g satisfying:

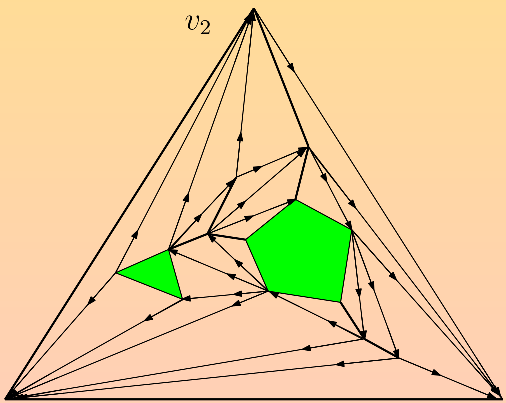
- T_0, T_1 are maps with at most $1 + 2g$ faces;
- $T_2 \cup \mathcal{E}$ is a 1 face map (a g-tree)



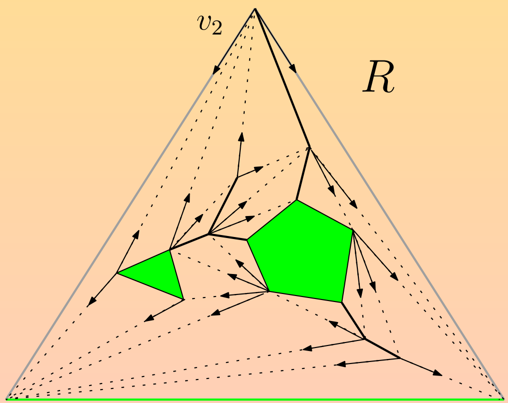
Opimal encoding (planar with b boundaries)

Théorème. (C-A, Fusy, Lewiner '10)
Optimal encoding of planar triangulations with n, b boundaries, k boundary vertices, with

$$\log_2 |T^b_{n,k}| = 2k + \log \binom{4n+2k}{n}$$

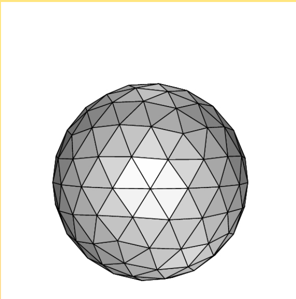


minimal 3-orientation



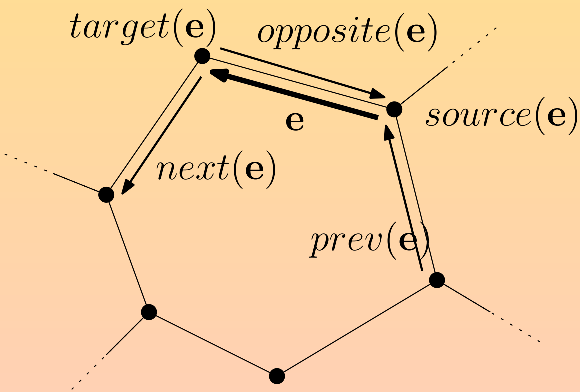
decorated tree with b boundaries

Structures de données géométriques



```
class Point{
  double x;
  double y;
}
```

information géométrique

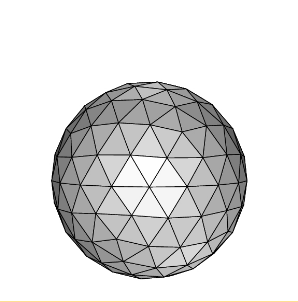


demi-arêtes

```
class Halfedge{
  Halfedge  prev, next, opposite;
  Vertex source, target;
  Face f;
}
class Vertex{
  Halfedge  e;
  Point p;
}
class Face{
  Halfedge  e;
}
```

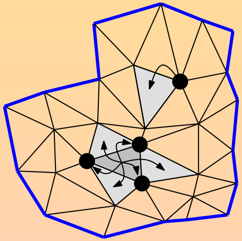
information combinatoire

A base de triangles



```
class Point{
  double x;
  double y;
}
```

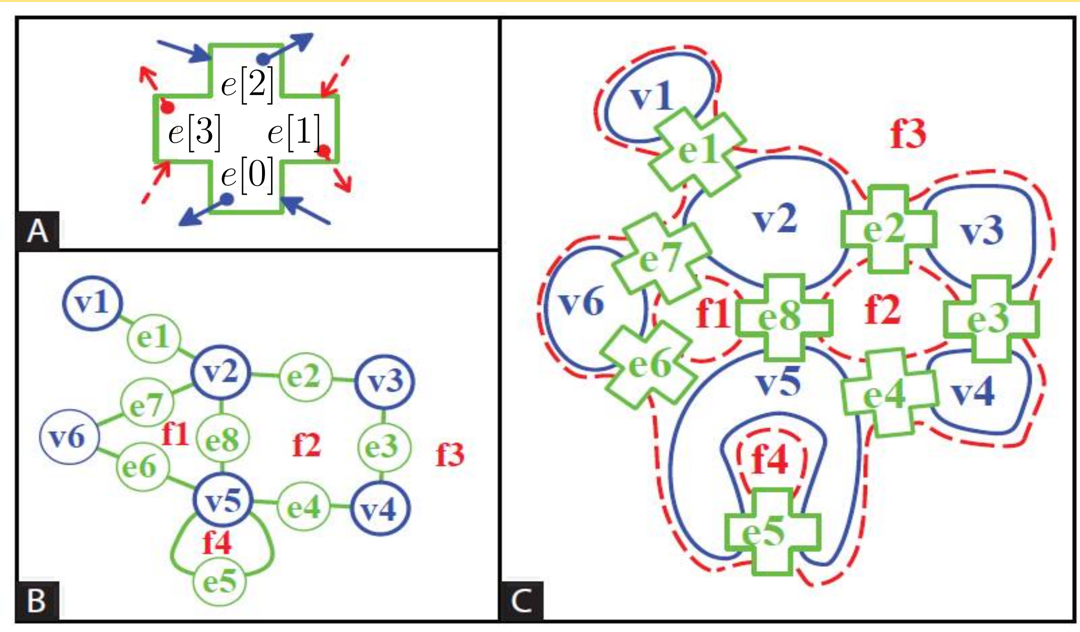
information géométrique

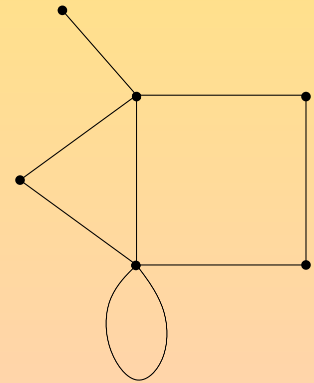


```
class Triangle{
  Triangle t1, t2, t3;
  Vertex v1, v2, v3;
}class Vertex{
  Triangle  root;
  Point p;
}
```

information combinatoire

Quad-edge

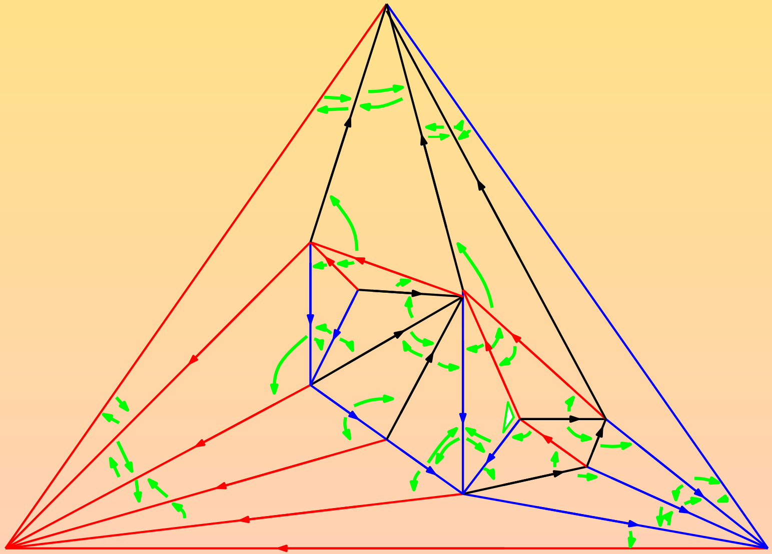




Representation	Graph encoding	Succinct representations	
	Size (references/pointers)	Navigation/access	
Halfedge	$(18+1)n$	$O(1)$	$O(1)$
Winged edge			
QuadEdge			
Triangle DS	$(12+1)n$	$O(1)$	$O(1)$
Triangle catalogs	$(9+1)n$, $(7.2+1)n$	$O(1)$	$O(1)$
Star Vertices	$7n$	$O(d)$	$O(1)$
Tripod (Snoeyink, Speckmann)	$(6+1)n$	$O(1)$	$O(d)$
SOT (Rossignac, Gurung 2010)	$6n$	$O(1)$	$O(d)$
Sorted Schnyder (C-A 2010)	$4n$	$O(1)$	$O(d)$

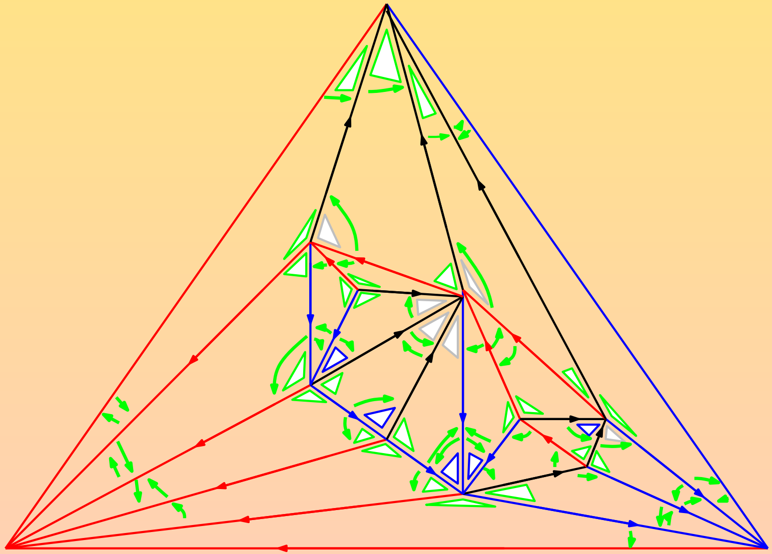
Sorted Schnyder woods representation (4n references)

Simple solution: $6n = 2e$ references



Sorted Schnyder woods representation (4n references)

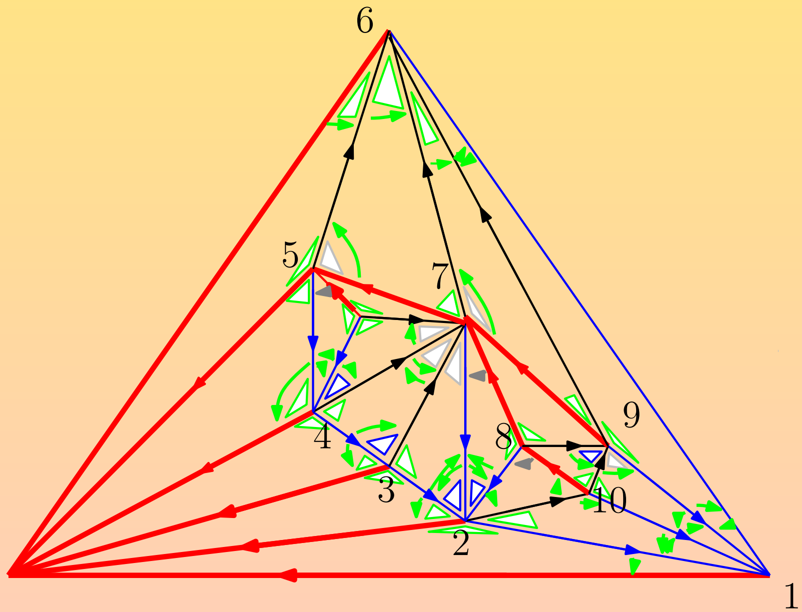
Simple solution: $5n$



Sorted Schnyder woods representation (4n references)

solution plus compacte: $4n$

DFUDS order on $\overline{T}_0$



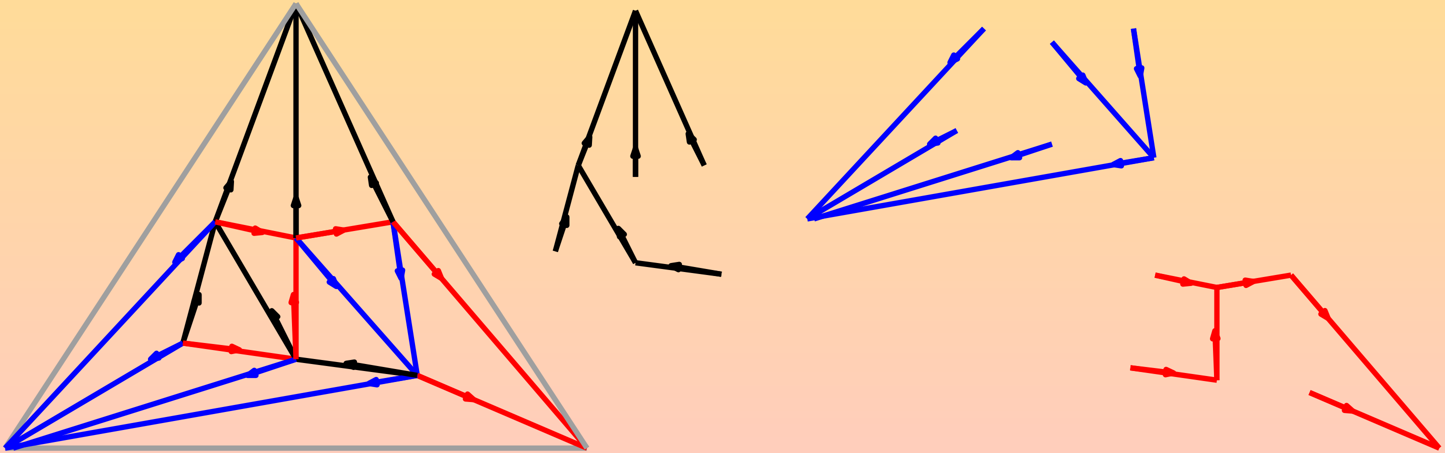
Schnyder woods: important facts

Theorem

Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

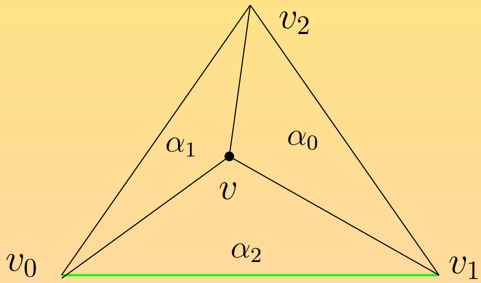
Theorem

The three set T_0, T_1, T_2 are spanning trees of (the inner nodes of) T:



Schnyder woods and (planar) **barycentric coordinates**

Geometric interpretation of barycentric coordinates

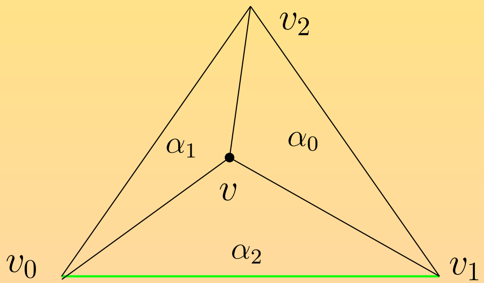


$$v = \alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2$$

$$v = \frac{area(v,v_1,v_2)v_0 + area(v_0,v,v_2)v_1 + area(v_0,v_1,v)v_2}{area(v_0,v_1,v_2)}$$

Schnyder woods and (planar)
barycentric coordinates

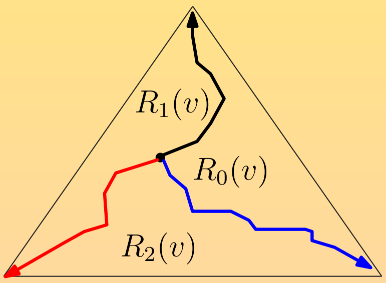
Geometric interpretation of barycentric coordinates



$$v = \alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2$$

$$v = \frac{area(v,v_1,v_2)v_0 + area(v_0,v,v_2)v_1 + area(v_0,v_1,v)v_2}{area(v_0,v_1,v_2)}$$

Combinatorial interpretation



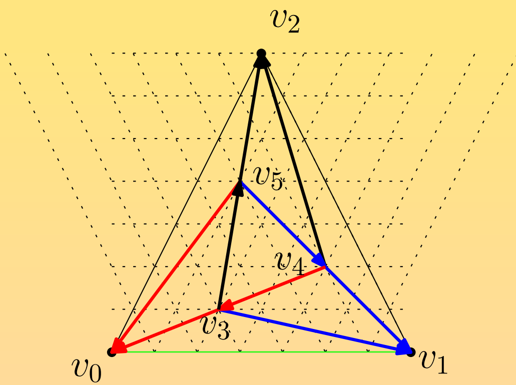
$$v = \frac{R_0}{2n-5} v_0 + \frac{R_1}{2n-5} v_1 + \frac{R_2}{2n-5} v_2$$

$R_i(v)$:=number of triangles in region i

Theorem (Schnyder, Soda '90)

For a triangulation $\mathcal{T}$ having n vertices, we can draw it on a grid of size $(2n-5) \times (2n-5)$, by setting $v_0 = (2n-5, 0)$, $v_1 = (0,0)$ and $v_2 = (0, 2n-5)$.

Schnyder drawing: a small example



v_3 $(1, 2, 4)$

v_4 $(2, 4, 1)$

v_5 $(4, 1, 2)$

three total orders

$\alpha_0(v_3) < \alpha_0(v_4) < \alpha_0(v_5)$ L_0

$\alpha_1(v_5) < \alpha_1(v_3) < \alpha_1(v_4)$ L_1

$\alpha_2(v_4) < \alpha_2(v_5) < \alpha_2(v_3)$ L_2

Theorem (Schnyder, Soda '90)

For a triangulation $\mathcal{T}$ having n vertices, we can draw it on a grid of size $(2n-5) \times (2n-5)$, by setting $v_0 = (2n-5, 0)$, $v_1 = (0, 0)$ and $v_2 = (0, 2n-5)$.