Schnyder woods for higher genus triangulated surfaces

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Motivation and applications
Schnyder woods (or Schnyder trees or realizers)

edge orientation and coloration for triangulations

Combinatorics of maps
- enumeration problems

Graph drawing
- draw a planar graph, with vertices having integer coordinates (using as few as possible coordinates)

Compression and succinct encoding
- Reduce the amount of (memory) space used by the connectivity of a graph.
- Supporting efficient navigation, using small space
  Example: adjacency queries between vertices
A nice characterization of planar graphs: Schnyder woods

Edge orientations, tree decompositions and dimension of a graph

**Theorem** (Schnyder, Felsner, Trotter)
A graph $G$ is planar if and only if its dimension is at most 3

**Theorem** (Schnyder ’90)
A graph $G$ is planar if and only if the dimension of its incidence poset is at most 3
The definition in the planar (triangulated) case
A nice characterization of planar graphs

(Schnyder ’90)

Let $T$ be a triangulation having outer face $\{x_0, x_1, x_{n-1}\}$.

with $n$ nodes
A nice characterization of planar graphs

A Schnyder wood of a triangulation is a partition $T_0, T_1, T_2$ of the internal edges of $T$ s.t.:

i) edge are colored and oriented in such a way that each inner node has exactly one outgoing edge of each color.
A nice characterization of planar graphs

(Schnyder '90)

A partition $T_0, T_1, T_2$ of the internal edges of $T$ s.t.:

i) edge are colored and oriented in such a way that each inner nodes has exactly one outgoing edge of each color

ii) colors and orientations around each inner node must respect the local Schnyder condition

$n$ nodes

$x_0 \ldots x_n$
Important facts about Schnyder woods
A first fundamental fact: 3 tree decomposition

$T_0$, $T_1$, $T_2$ are spanning trees of (the inner nodes of) $T$:
Second fact: dimension of a graph

$L_0, L_1, L_2$ are three orders on the vertices of $T$:

$L_0 : v_1 < v_2 < v_3 < v_4 < v_5 < v_6$
$L_1 : v_2 < v_3 < v_6 < v_4 < v_5 < v_1$
$L_2 : v_2 < v_3 < v_6 < v_1 < v_3 < v_2$
The first motivation: barycentric drawing
Combinatorial interpretation

How to use Schnyder woods:
• Let $P_i(v)$ be the path from $v$ to $x_i$ in $T_i$.
• Let $R_i(v)$ be the region defined by $P_{i+1}(v)$, $P_{i+2}(v)$ and $(x_{i+1}, x_{i+2})$.

The combinatorial equivalent of the area is given by the number of triangles enclosed in each region $R_i$:

$$v_i = \frac{|R_i(v)|}{|T|}$$

**Theorem**
For a triangulation $T$ having $n$ vertices, we can draw it on a grid of size $(2n-5) \times (2n-5)$, by setting $x_1 = (2n-5, 0)$, $x_2 = (0, 0)$ and $x_3 = (0, 2n-5)$. 
Application: graph compression and succinct encoding

Optimal compression (Poulalhon-Schaeffer, Icalp ‘03)

Succinct encoding (Chuang-Garg-He-Kao-Lu Icalp ’98)
(Barbay-Castelli Aleardi-He-Munro Isaac’07)
Tree decompositions in higher genus
Related works: tree decompositions of toroidal graphs the "tambourine" solution
(Bonichon, Gavoille, Labourel, ICGT '05)

Main idea of this approach:
Compute a pair of adjacent non contractible cycles

Result: \( T_0, T_1, T_2 \) vertex spanning trees

Inconvenients:

- valid only for toroidal triangulation (genus 1)
- potentially large number of vertices not satisfying the local condition
Our main contribution
a generalized higher genus definition
Given a rooted triangulated surface of genus $g$
Schnyder Woods: the higher genus case

i) a small set $\mathcal{E}^s$ of *special* edges, doubly oriented and colored

$$|\mathcal{E}^s| = 2g$$

$\mathcal{E}^s = \{e_1, e_2\}$
Schnyder Woods: the higher genus case

i) a small set $\mathcal{E}^s$ of *special* edges, doubly oriented and colored

\[ \mathcal{E}^s = \{ e_1, e_2 \} \]

(\(u, v, w\)) at most \(2 \cdot 2g\) multiple vertices (incident to special edges)

ii) a new local condition for edges in a sector incident to a multiple vertex
Computing Schnyder Woods (in the plane)
Incremental vertex conquest (Brehm’s approach)

The traversal starts from the root face

In our example, the root face coincides with the exterior (infinite) face
Incremental vertex conquest

conquer\((v_{n-1}) +\)

colororient\((v_{n-1})\)
Incremental vertex conquest

\[ \text{conquer}(v_{n-2}) + \text{colororient}(v_{n-2}) \]

\( v_{n-2} \) is a free vertex

no incident chordal edges

\( w \) is not free
Incremental vertex conquest

\[ conquer(v_{n-3}) + colororient(v_{n-3}) \]
Incremental vertex conquest
Incremental vertex conquest
Incremental vertex conquest
Incremental vertex conquest
Incremental vertex conquest
Computing $g$-Schnyder Woods
New handle operators: split and merge

conquer \( (w) \)  

chordal edge \( (u, w) \)

\( (u, w) \) chordal edge

\( (u, w) \) defines a non-trivial cycle

split \( (u, w) \)  

merge \( (u, w) \)
New handle operators: split and merge

$(u, w)$ chordal edge defining a non-trivial cycle

$(u, w)$ split edge

split a boundary into 2 boundary components

$(u, w)$ merge edge

merge two different boundary components
Example of execution of our algorithm (toroidal case)

Starts the traversal from the root face (green edge)
Example of execution of our algorithm (toroidal case)

\[ \text{conquer}(w) + \text{colororient}(w) \]

\[ S \]
Example of execution of our algorithm (toroidal case)

$conquer(w) + colororient(w)$
Example of execution of our algorithm (toroidal case)

\[ \text{conquer}(w) + \text{colororient}(w) \]
Example of execution of our algorithm (toroidal case).
After a maximal sequence of vertex conquest operations . . . no more free vertices... the (planar) traversal gets stuck.

$S^{in}$ is a topological disk.

$S^{out}$ is a face connected map with a boundary component.
Example of execution of our algorithm (toroidal case)

Let us perform a \textit{split}(u, w) operation

$S^{\text{out}} \setminus (u, w)$ is a face connected map of genus 1 with two boundary components
Example of execution of our algorithm (toroidal case)

We can now perform a conquer($u$) operation

$S^{out} \setminus (u, w)$ is a face connected map of with two boundary components
Example of execution of our algorithm (toroidal case)

Let us perform a \texttt{merge}(w, v) operation

\( \mathcal{S}^{\text{out}} \setminus (w, v) \) is a face connected topological disk
Example of execution of our algorithm (toroidal case)

Let us see in a better way...

$S_{\text{out}} \setminus (w, v)$ is a face connected topological disk

Execution ends performing a sequence of conquer operations
Example of execution of our algorithm (toroidal case)

Let us see in a better way...

$S^{out} \setminus (w, v)$ is a face connected topological disk

Execution ends performing a sequence of conquer operations
Example of execution of our algorithm (toroidal case)

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Example of execution of our algorithm (toroidal case)
Correctness and termination

Existence of split and merge edges

$T$ vertex spanning tree of the primal graph, containing the boundary (blue) edges

$T^*$ is a 1 face map containing $2g$ non-trivial cycles

$T^*$ contains $g$ split and merge edges
A fundamental lemma about chordal edges (the planar case)

**Lemma**
There always exists at least one boundary vertex, not incident to chordal edges

**Proof:** by induction on the size of the boundary

**Theorem** Each planar triangulation admits a *canonical ordering* on the vertices
Correctness and termination (higher genus)

Properties of chordal edges in genus $g$

- **Case 1**
  - $S_{in}$
  - $w'$
  - $z'$

- **Case 2**
  - $S_{in}$
  - $w'$
  - $z'$
  - $z''$

- **Case 3**
  - $S_{in}$
  - $w'$

**Impossible case**
- no loops, no multiple edges allowed
Correctness and termination
Coloring invariants

\[
\text{split}(u, w) + \text{conquer}(w) + \text{colororient}(w) + \\
\text{conquer}(u) + \text{colororient}(u)
\]
Our main result

**Theorem**  (Castelli Aleardi, Fusy and Lewiner, 2008)
Given a (simple) rooted triangulation $\mathcal{T}$ of genus $g$ and size $n$, we can compute in $O(n)$ time a $g$-Schnyder wood of $\mathcal{T}$.

The local Schnyder condition is true almost everywhere in the graph at the exception of multiple vertices.
From plane trees to genus $g$ maps
A new characterization in term of g-maps

**Theorem** (Castelli Aleardi, Fusy and Lewiner, 2008)
The three sets of edges $T_0$ and $T_1$ (red and blue edges), as well as the set $T_2 \cup \mathcal{E}$ (black edges and special edges) are maps of genus $g$ satisfying:

- $T_0, T_1$ are maps with at most $1 + 2g$ faces;
- $T_2 \cup \mathcal{E}$ is a 1 face map
A new encoding application
Corollary

A triangulation of genu $g$ having $n$ vertices can be encoded with $4n + O(g \log n)$ bits.
Futur works and open questions

optimal encoding in higher genus

lattice structure for the set of Schnyder woods

extension to the 3-connected case (polygonal meshes)
planar graphs

3-connected graph

bijections between planar maps and trees

Schnyder wood

Futur works

higher genus

bijection between planar maps and trees

graph drawing
Optimal coding and sampling (planar case)

**Theorem.** (Tutte 62) The number of planar triangulation with $n+2$ vertices is

$$\frac{2(4n-3)!}{(3n-1)!n!} \asymp \left( \frac{256}{27} \right)^n.$$

**Théorème.** (Poulalhon–Schaeffer Icalp 03)
Il existe une bijection entre la classe des arbres de taille $n$ ayant deux bourgeons par noeud, et la classe des triangulations planaires enracinées à $n+2$ sommets.


a new nice interpretation of Tutte's formula:

$$|\mathcal{T}_n| = \frac{2}{2n} \cdot |\mathcal{A}_n^{(2)}|.$$
Optimal coding (genus $g$)

$S$

triangulated graph of genus $g$

$\mathcal{T}^g$

one face map of genus $g$
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