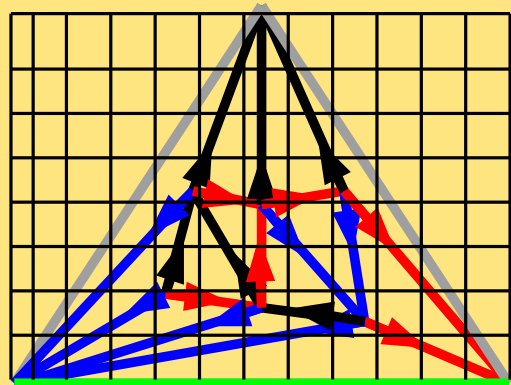


Schnyder woods for higher genus triangulated surfaces

11 june 2008, SoCG 2008, Univ. of Maryland, College Park



Luca Castelli Aleardi
CS Departement, ULB, Bruxelles



Eric Fusy

Simon Fraizer University, Vancouver

Thomas Lewiner

PUC University, Rio de Janeiro

Motivation and applications

Schnyder woods (or Schnyder trees or realizers)

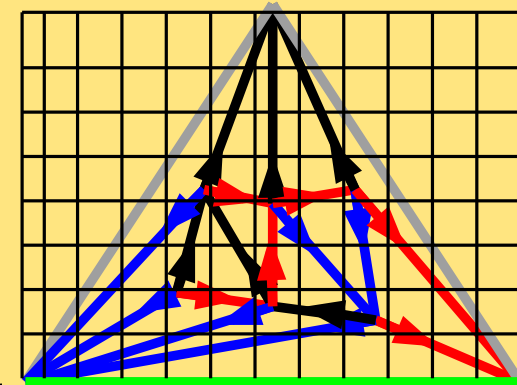
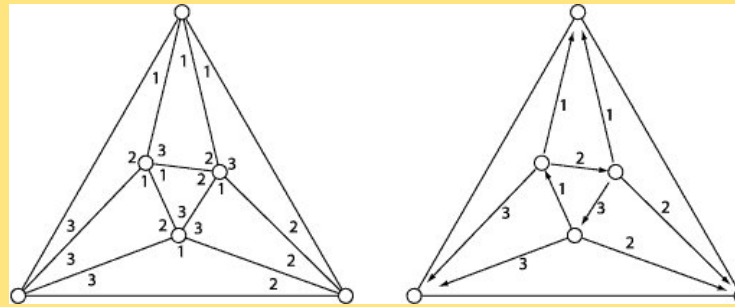
edge orientation and coloration for triangulations

Combinatorics of maps

- enumeration problems

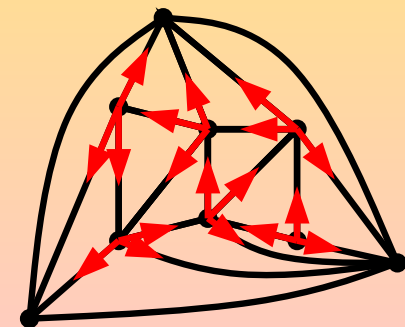
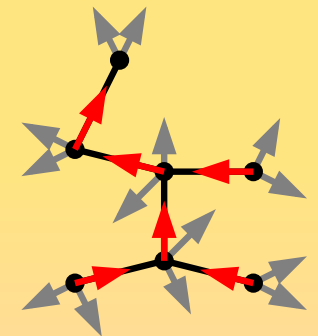
Graph drawing

- draw a planar graph, with vertices having integer coordinates (using as few as possible coordinates)



Compression and succinct encoding

- Reduce the amount of (memory) space used by the connectivity of a graph.
- Supporting efficient navigation, using small space
Example: adjacency queries between vertices



A nice characterization of planar graphs: Schnyder woods

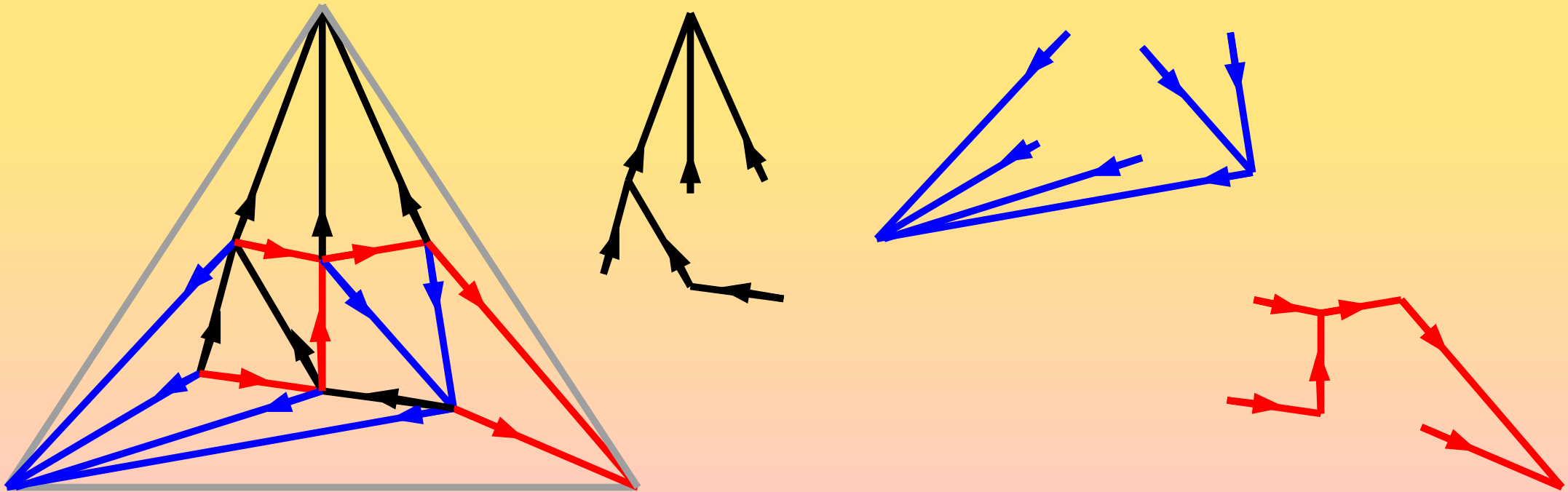
Edge orientations, tree decompositions and dimension of a graph

Theorem (Schnyder, Felsner, Trotter)

A graph G is planar if and only if its dimension is at most 3

Theorem (Schnyder '90)

A graph G is planar if and only if the dimension of its incidence poset is at most 3

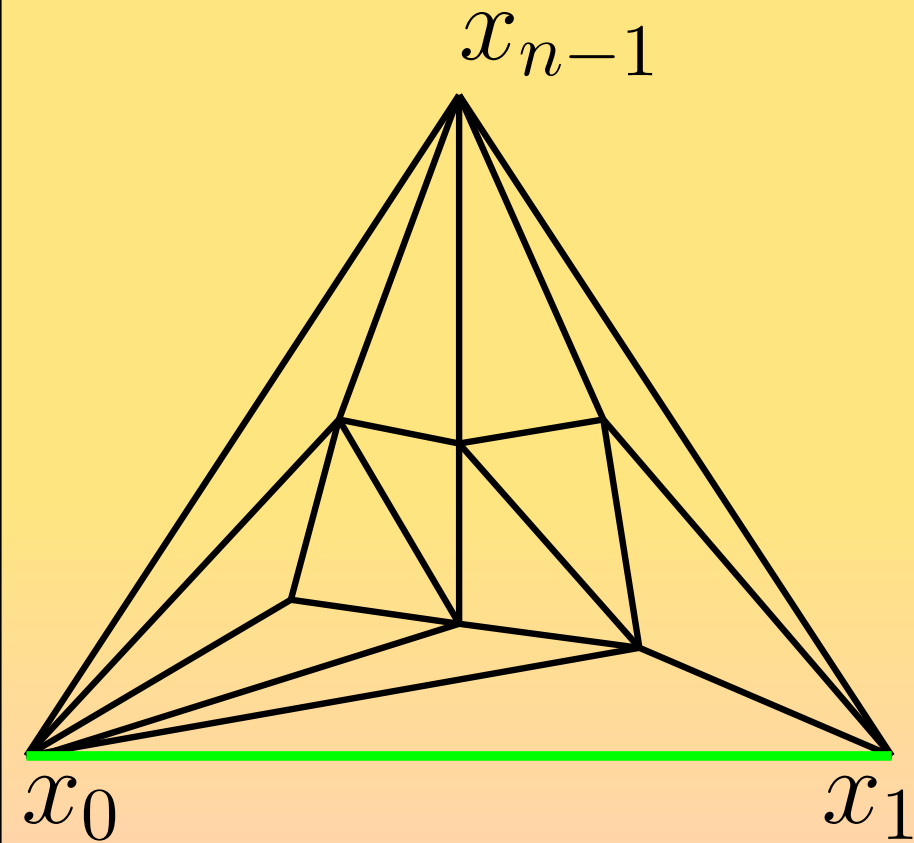


The definition in the planar (triangulated) case

A nice characterization of planar graphs

(Schnyder '90)

Let T be a triangulation having outer face $\{x_0, x_1, x_{n-1}\}$.
with n nodes



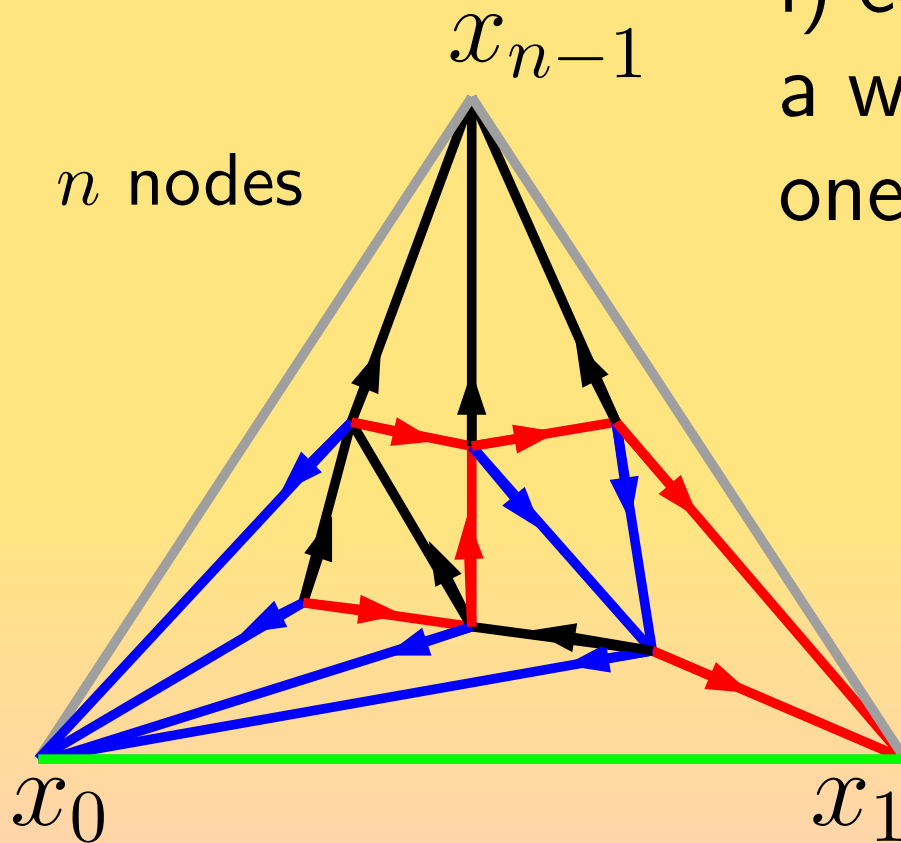
A nice characterization of planar graphs

(Schnyder '90)

A Schnyder wood of a triangulation is

a partition T_0, T_1, T_2 of the internal edges of T s.t. :

i) edge are colored and oriented in such a way that each inner node has exactly one outgoing edge of each color



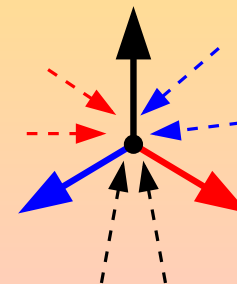
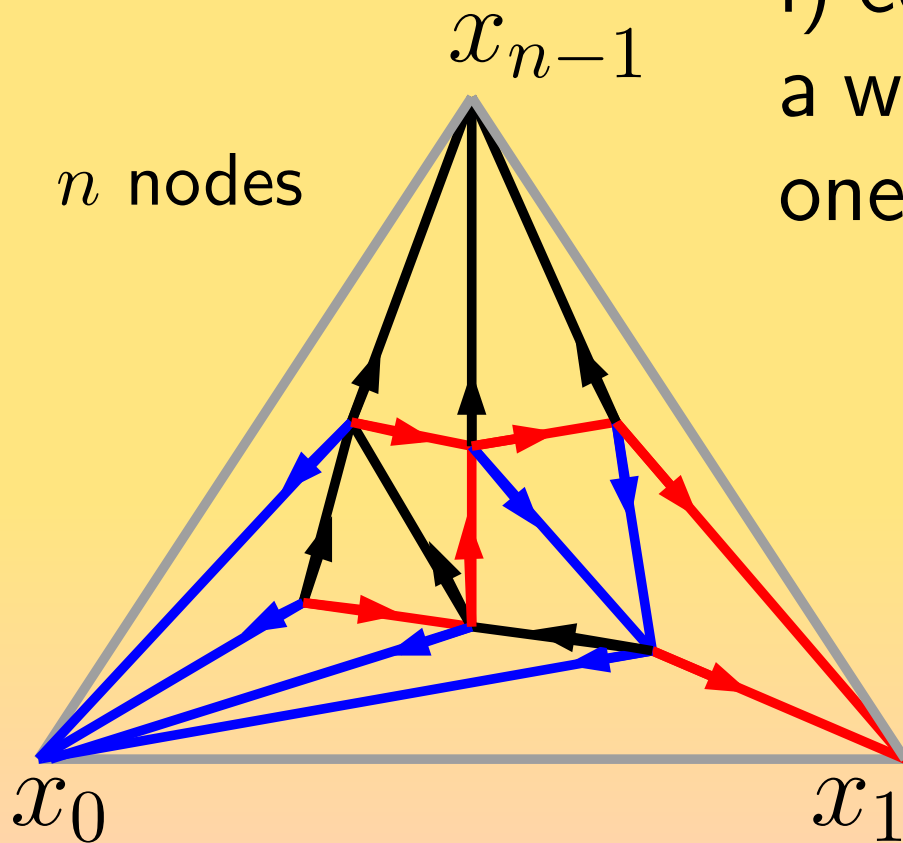
A nice characterization of planar graphs

(Schnyder '90)

A partition T_0, T_1, T_2 of the internal edges of T s.t. :

i) edge are colored and oriented in such a way that each inner nodes has exactly one outgoing edge of each color

ii) colors and orientations around each inner node must respect the local Schnyder condition

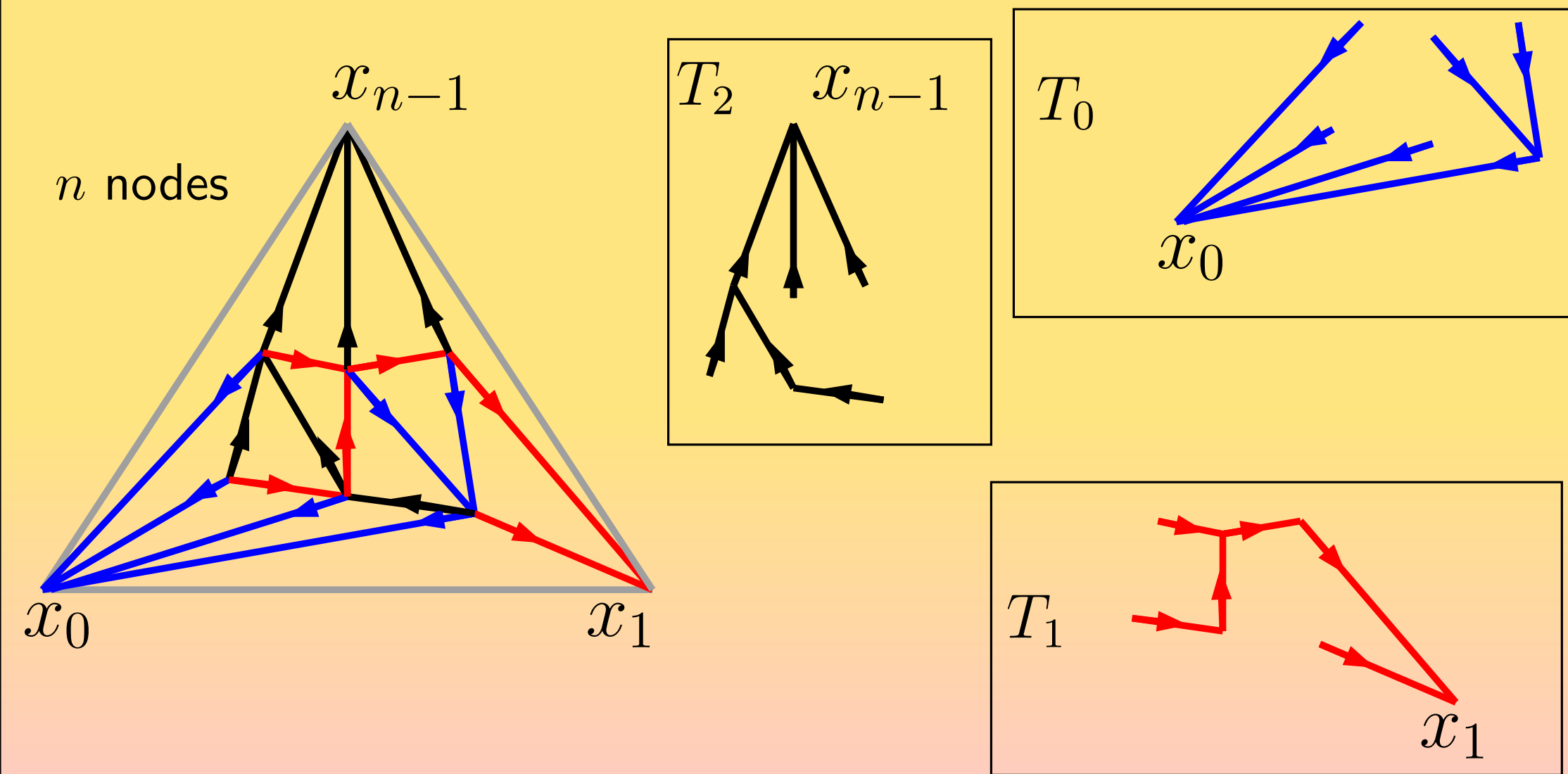


Important facts about Schnyder woods

A first fundamental fact: 3 tree decomposition

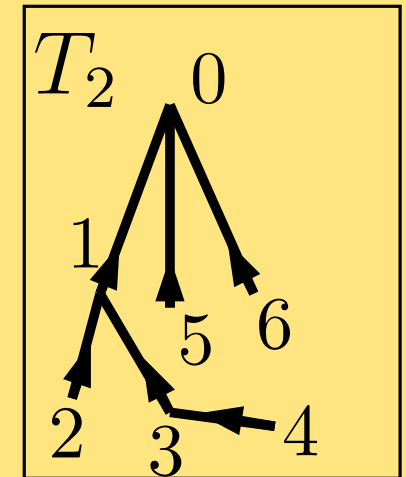
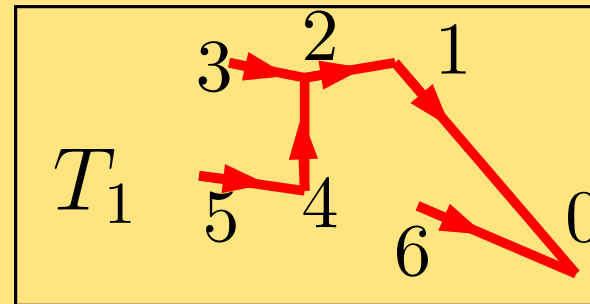
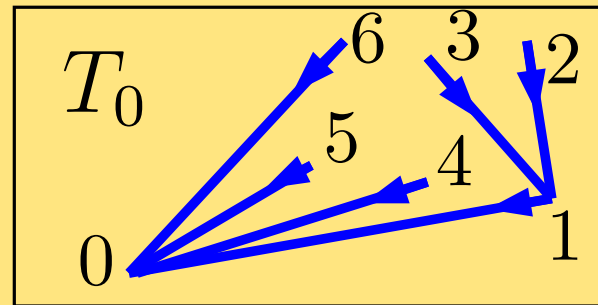
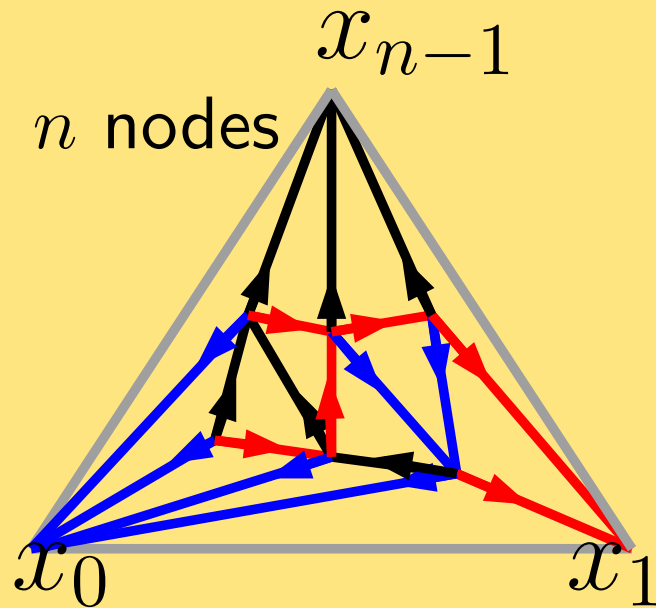
(Schnyder '90)

T_0 , T_1 , T_2 are spanning trees of (the inner nodes of) T :



Second fact: dimension of a graph

L_0, L_1, L_2 are three orders on the vertices of T :



$$L_0 : v_1 < v_2 < v_3 < v_4 < v_5 < v_6$$

$$L_1 : v_2 < v_3 < v_6 < v_4 < v_5 < v_1$$

$$L_2 : v_2 < v_3 < v_6 < v_1 < v_3 < v_2$$

The first motivation: barycentric drawing

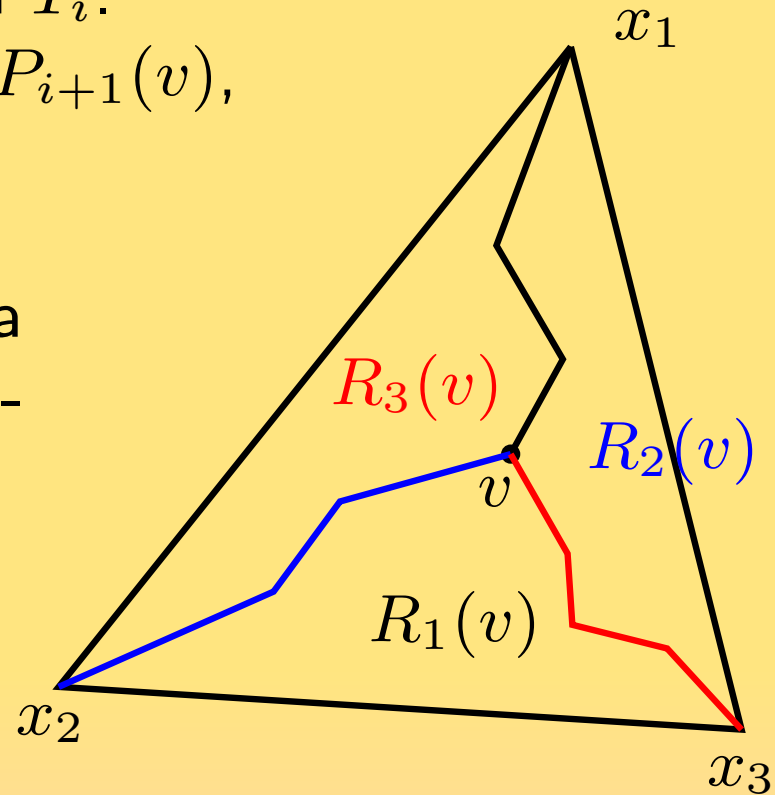
Combinatorial interpretation

How to use Schnyder woods:

- Let $P_i(v)$ be the path from v to x_i in T_i .
- Let $R_i(v)$ be the region defined by $P_{i+1}(v)$, $P_{i+2}(v)$ and (x_{i+1}, x_{i+2}) .

The combinatorial equivalent of the area is given by the number of triangles enclosed in each region R_i :

$$v_i = \frac{|R_i(v)|}{|T|}$$



Theorem

For a triangulation \mathcal{T} having n vertices, we can draw it on a grid of size $(2n - 5) \times (2n - 5)$, by setting $x_1 = (2n - 5, 0)$, $x_2 = (0, 0)$ and $x_3 = (0, 2n - 5)$.

Tree decompositions in higher genus

Related works: tree decompositions of toroidal graphs

the "tambourine" solution

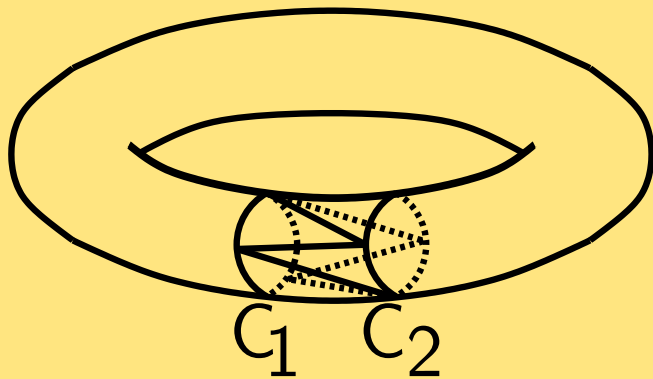
(Bonichon, Gavaille, Labourel, ICGT '05)



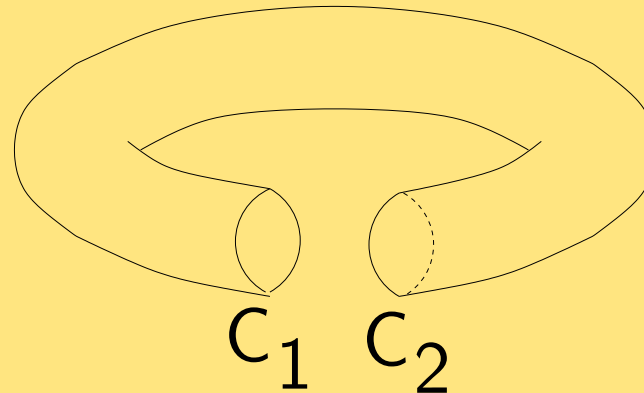
Main idea of this approach:

Compute a pair of adjacent non contractible cycles

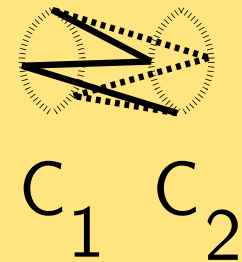
Graph G



Graph H



Tambourine T



Result: T_0 , T_1 , T_2 vertex spanning trees

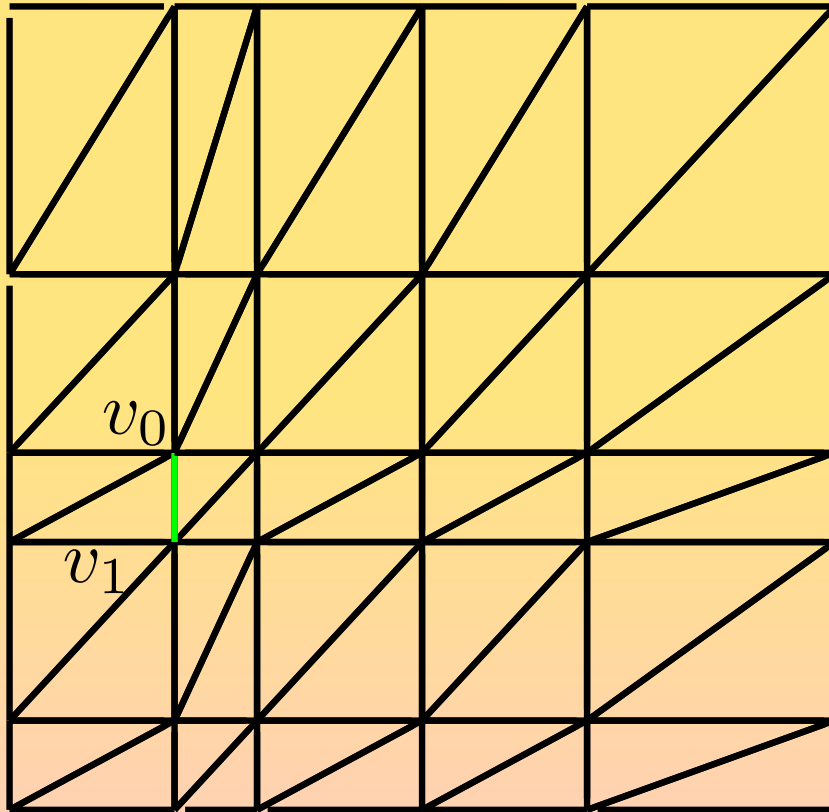
Inconvenients:

- valid only for toroidal triangulation (genus 1)
- potentially large number of vertices not satisfying the local condition

Our main contribution
a generalized higher genus definition

Schnyder Woods: the higher genus case

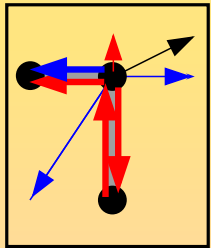
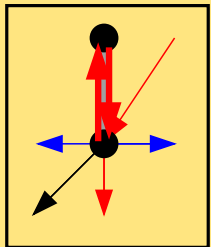
Given a rooted triangulated surface of genus g



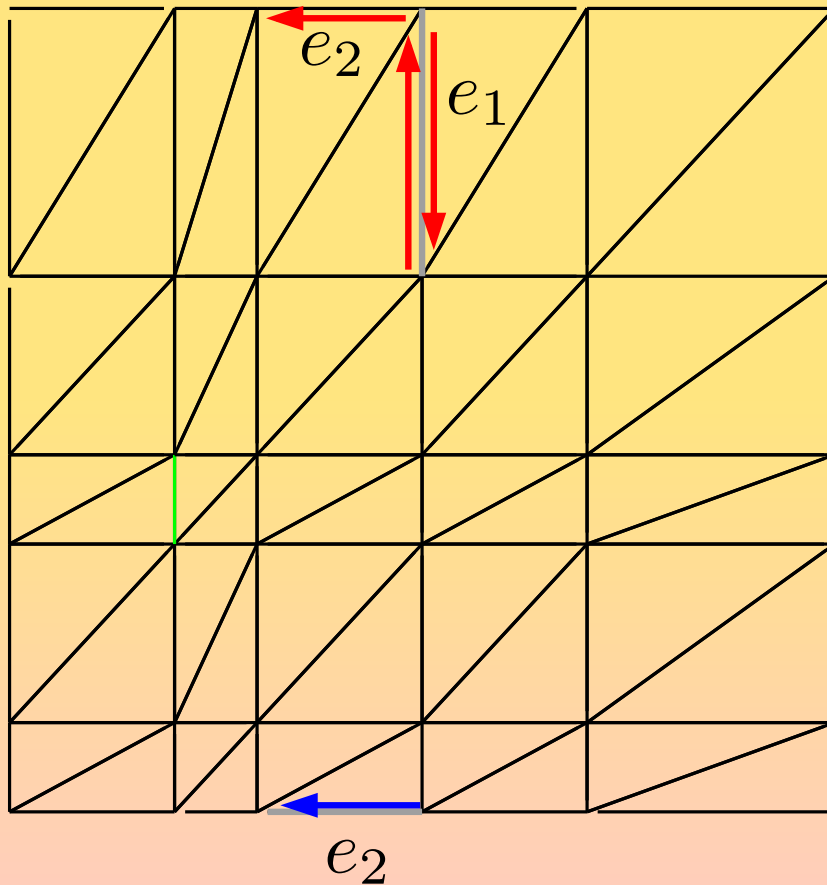
Schnyder Woods: the higher genus case

i) a small set \mathcal{E}^s of *special* edges, doubly oriented and colored

$$|\mathcal{E}^s| = 2g$$



$$\mathcal{E}^s = \{e_1, e_2\}$$

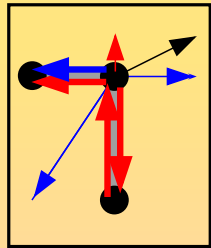
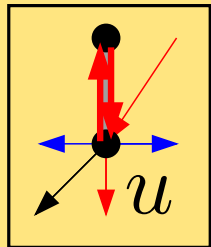


Schnyder Woods: the higher genus case

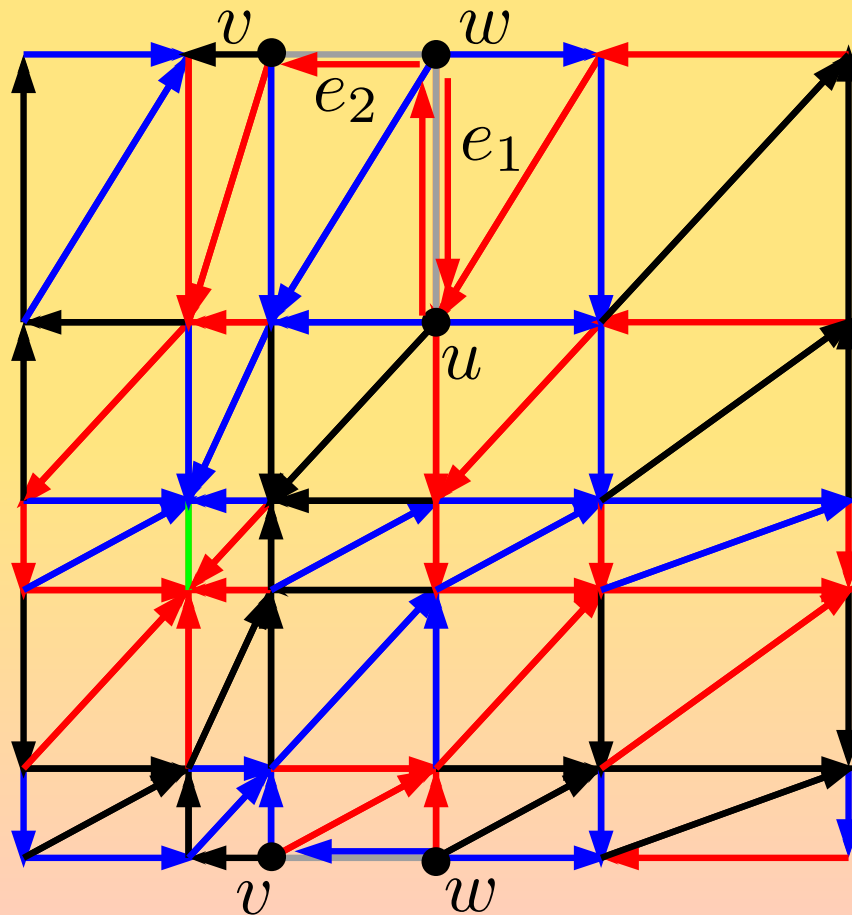
i) a small set \mathcal{E}^s of *special* edges,
doubly oriented and colored

(u, v, w)

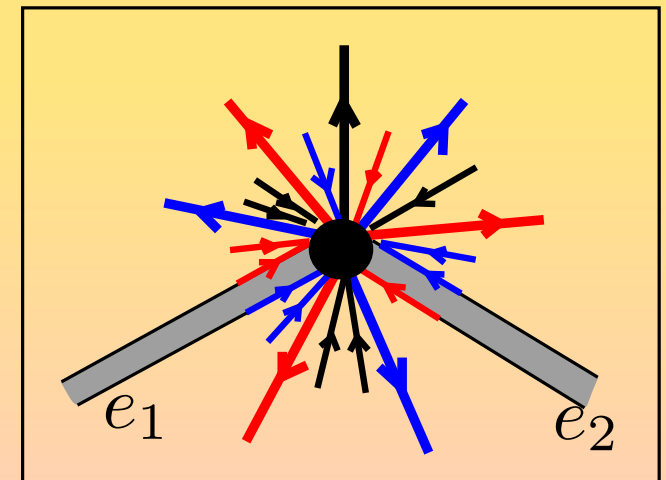
at most $2 \cdot 2g$ multiple vertices
(incident to special edges)



$$\mathcal{E}^s = \{e_1, e_2\}$$



ii) a new local condition for
edges in a sector incident to
a multiple vertex

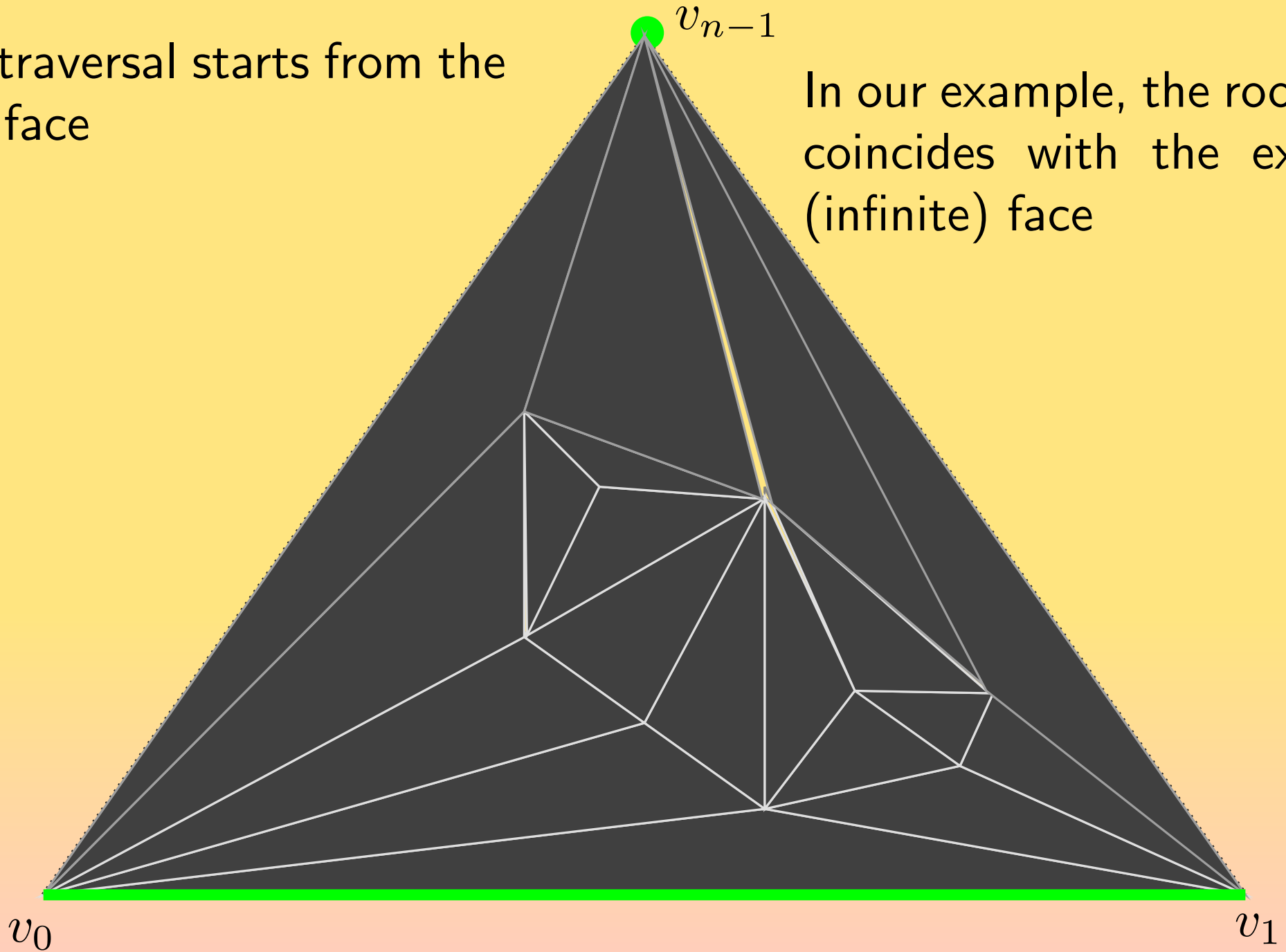


Computing Schnyder Woods (in the plane)

Incremental vertex conquest (Brehm's approach)

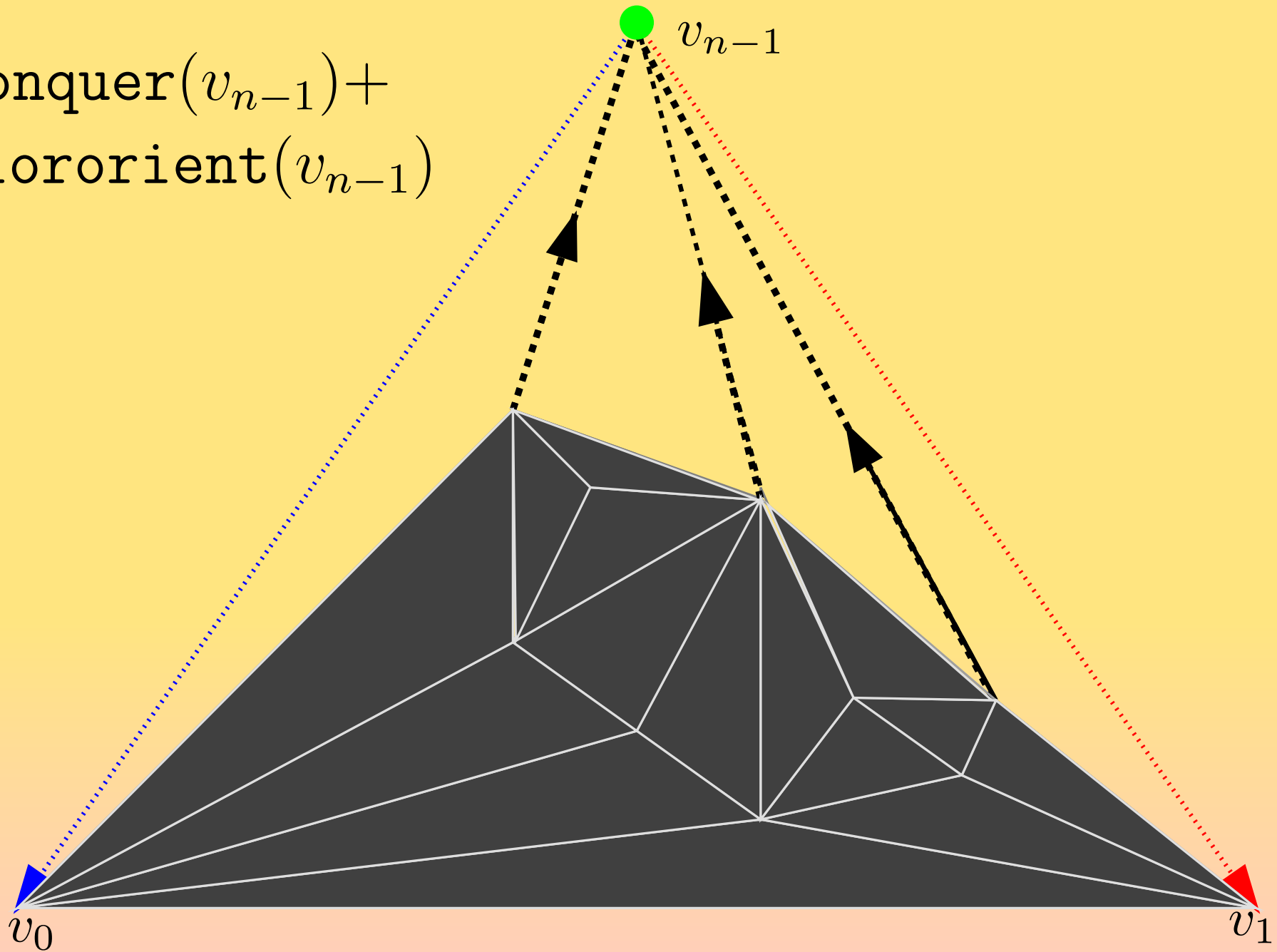
The traversal starts from the root face

In our example, the root face coincides with the exterior (infinite) face



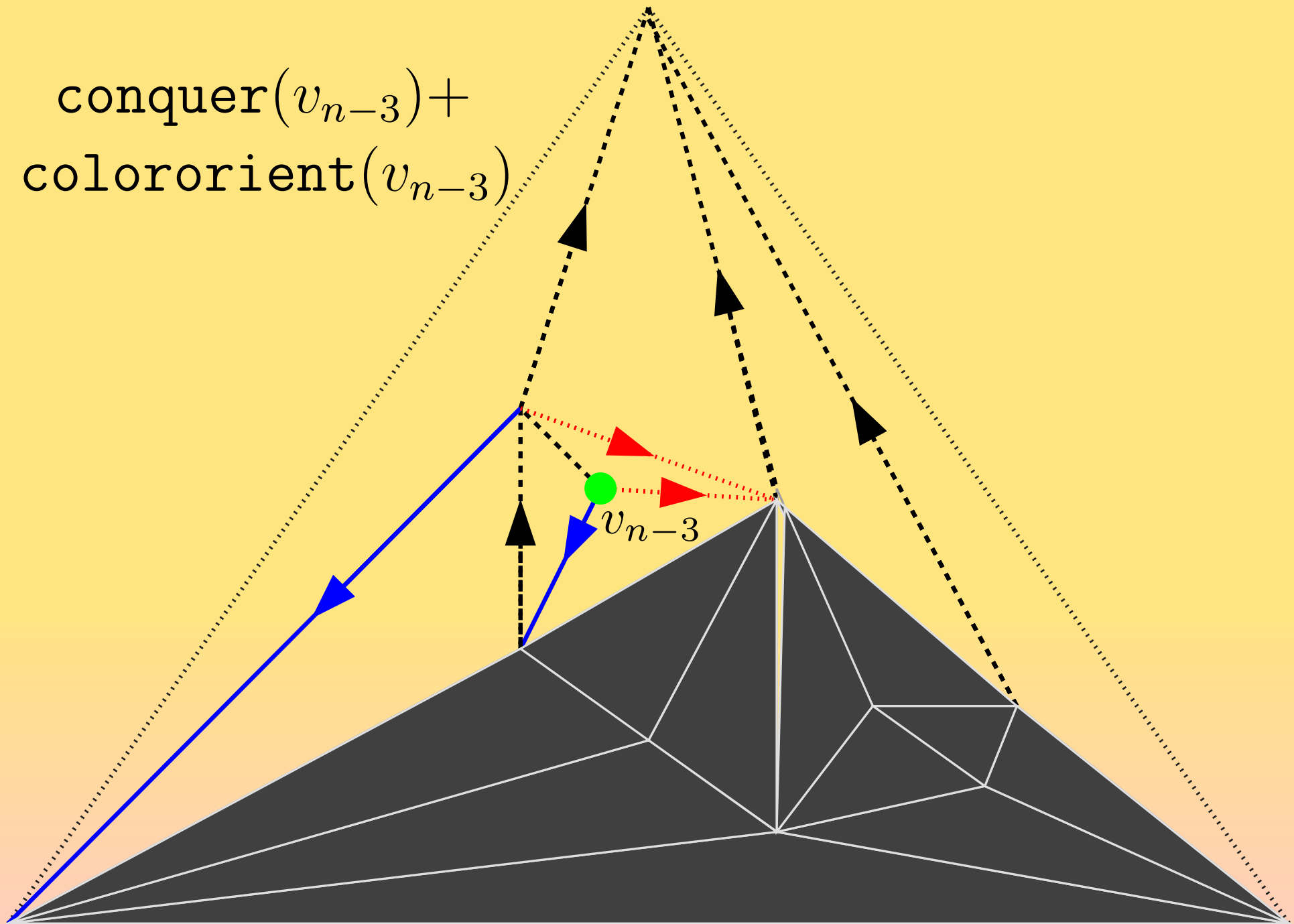
Incremental vertex conquest

`conquer(v_{n-1}) +`
`colororient(v_{n-1})`

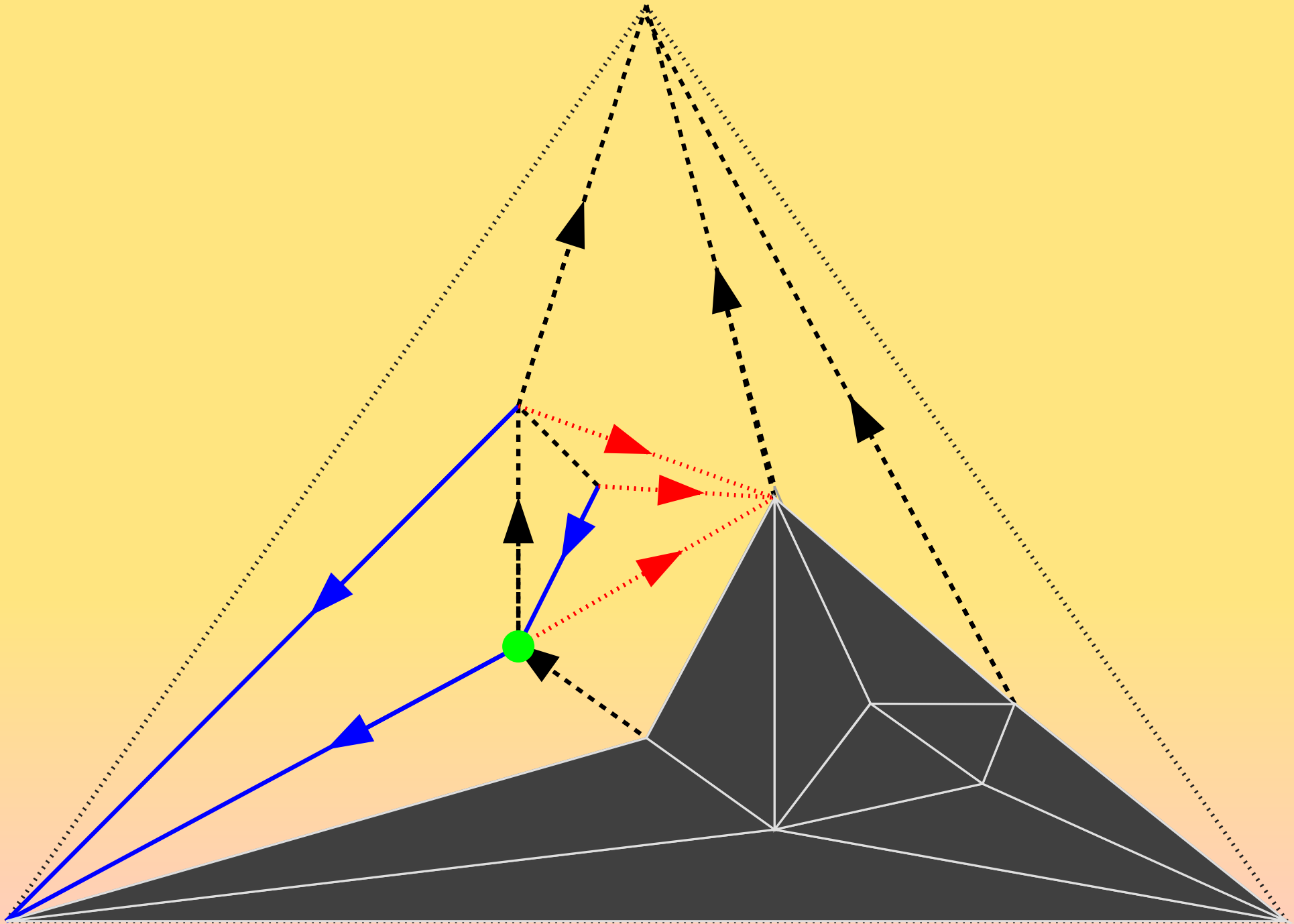


Incremental vertex conquest

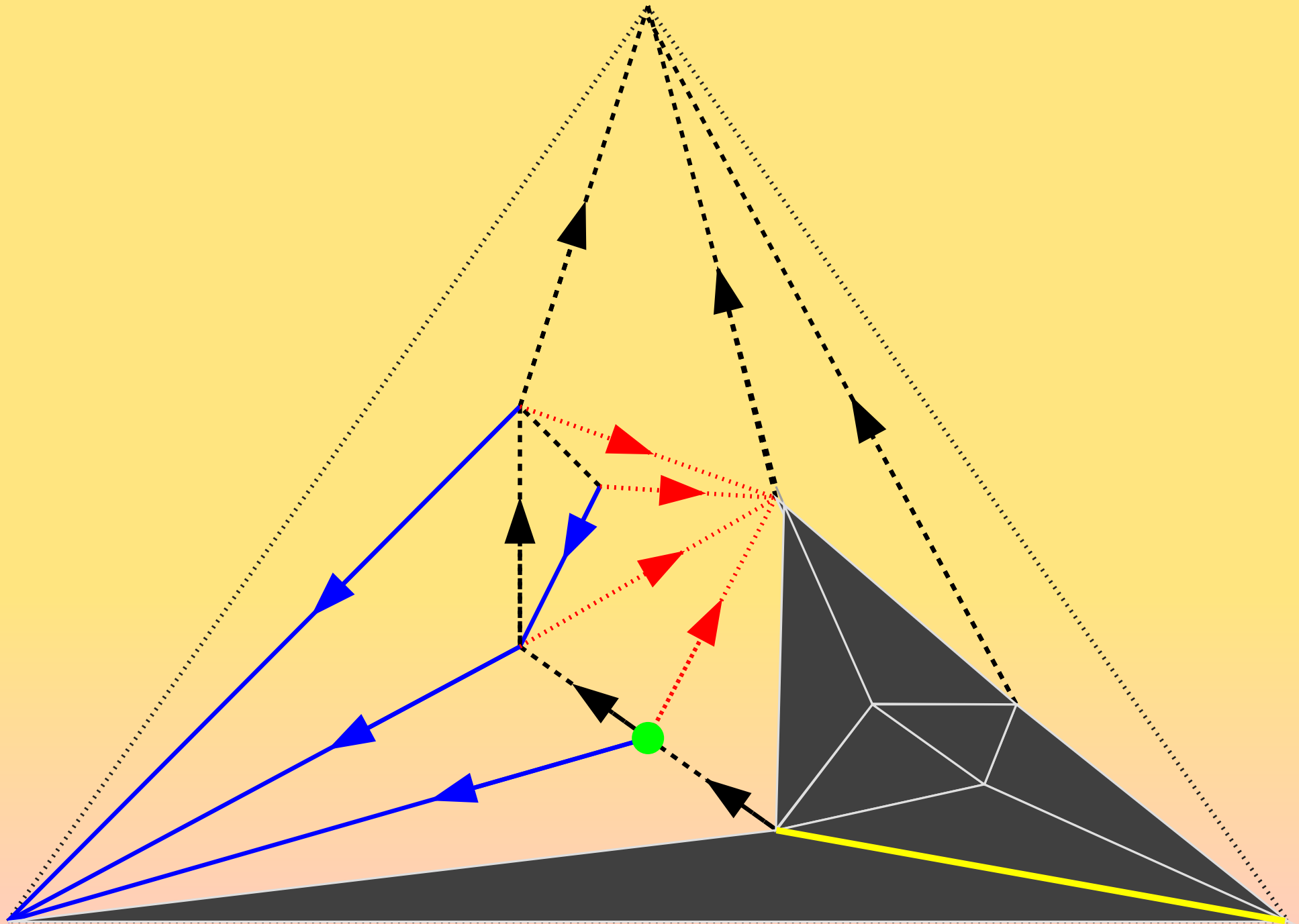
$\text{conquer}(v_{n-3}) +$
 $\text{colororient}(v_{n-3})$



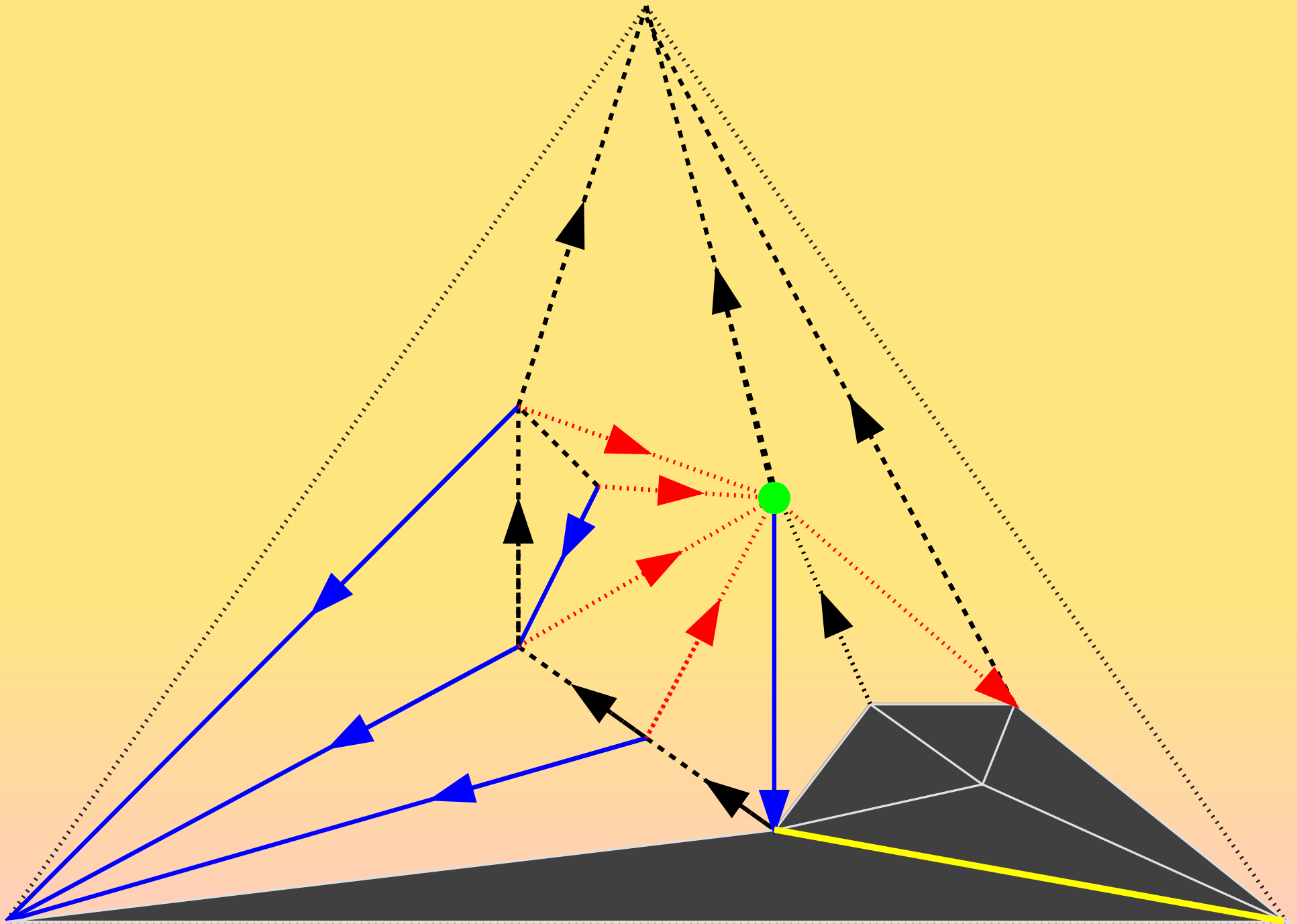
Incremental vertex conquest



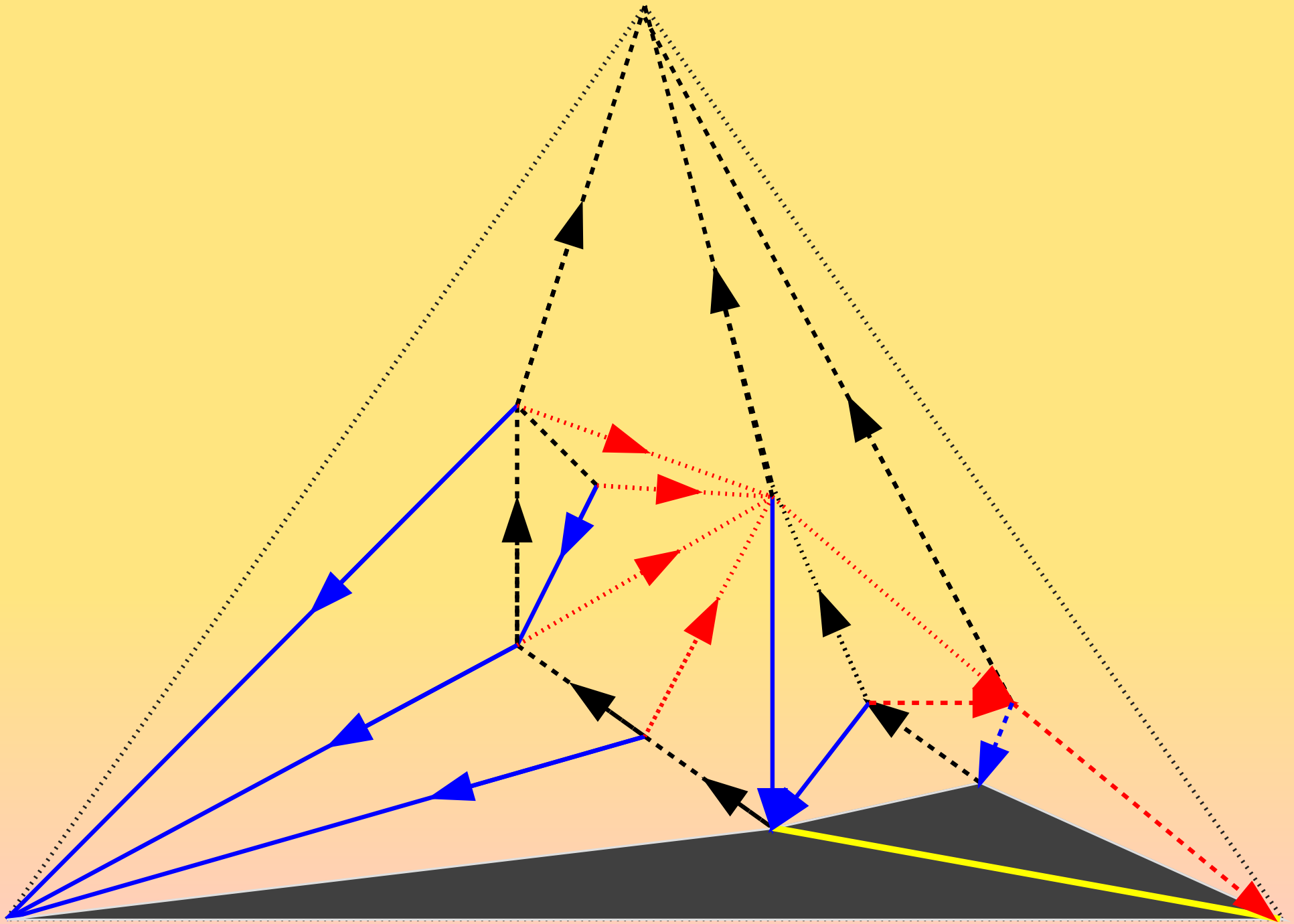
Incremental vertex conquest



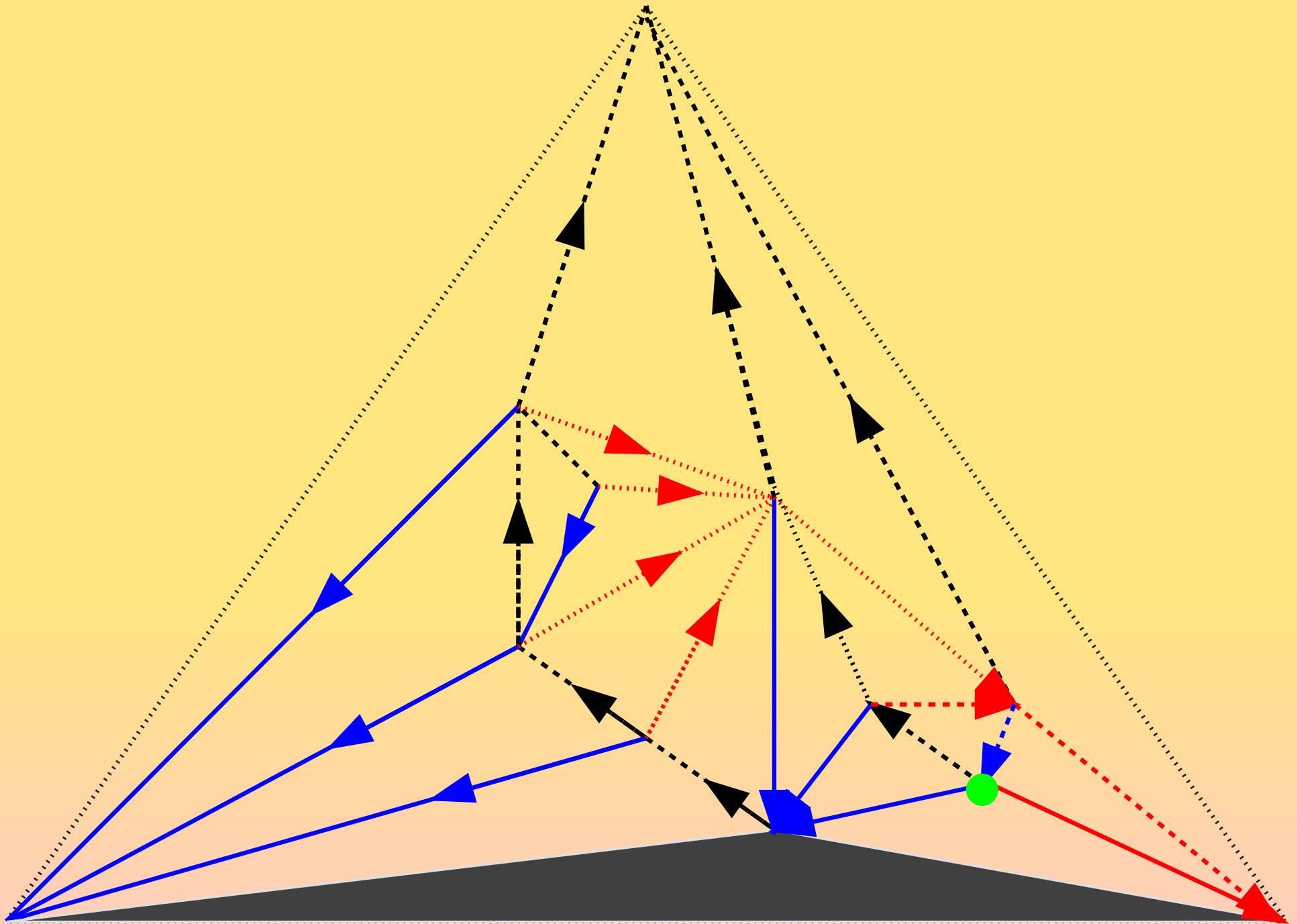
Incremental vertex conquest



Incremental vertex conquest



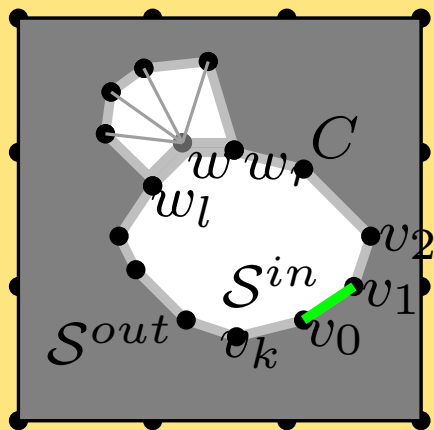
Incremental vertex conquest



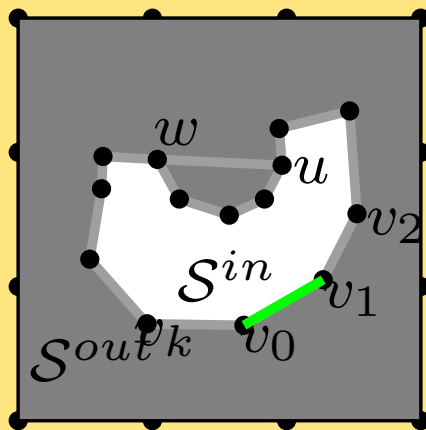
Computing g -Schnyder Woods

New handle operators: split and merge

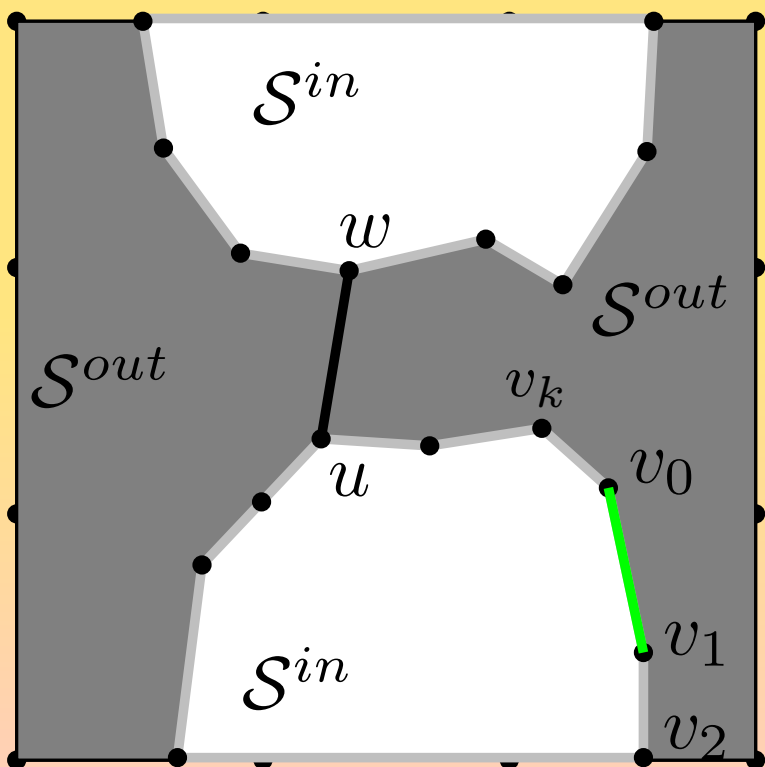
conquer(w)



chordal edge (u, w)

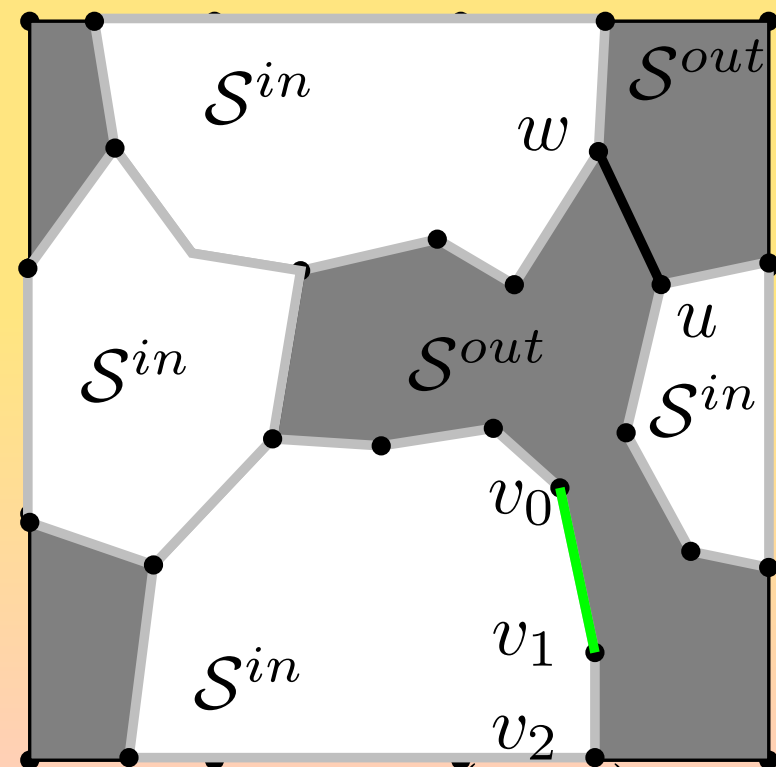


(u, w) chordal edge



split(u, w)

(u, w) defines a non-trivial cycle



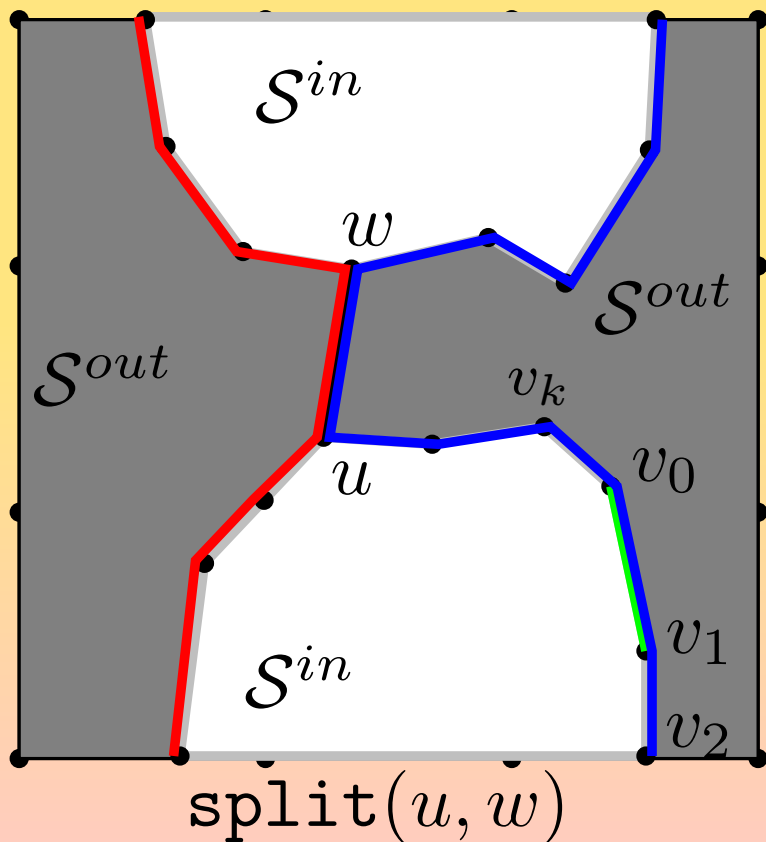
merge(u, w)

New handle operators: split and merge

(u, w) chordal edge defining a non-trivial cycle

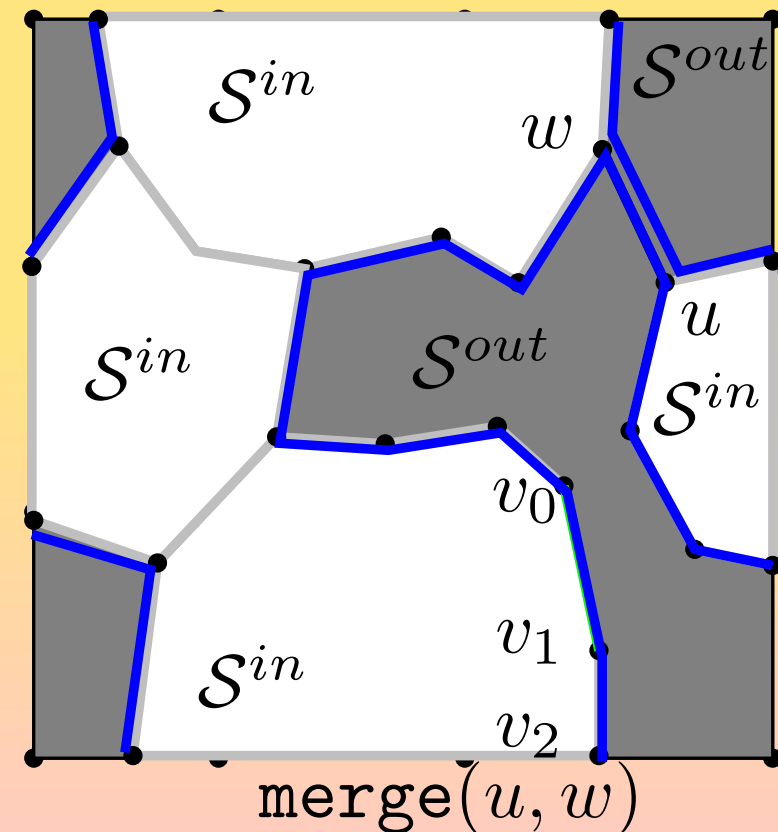
(u, w) split edge

split a boundary into 2 boundary components



(u, w) merge edge

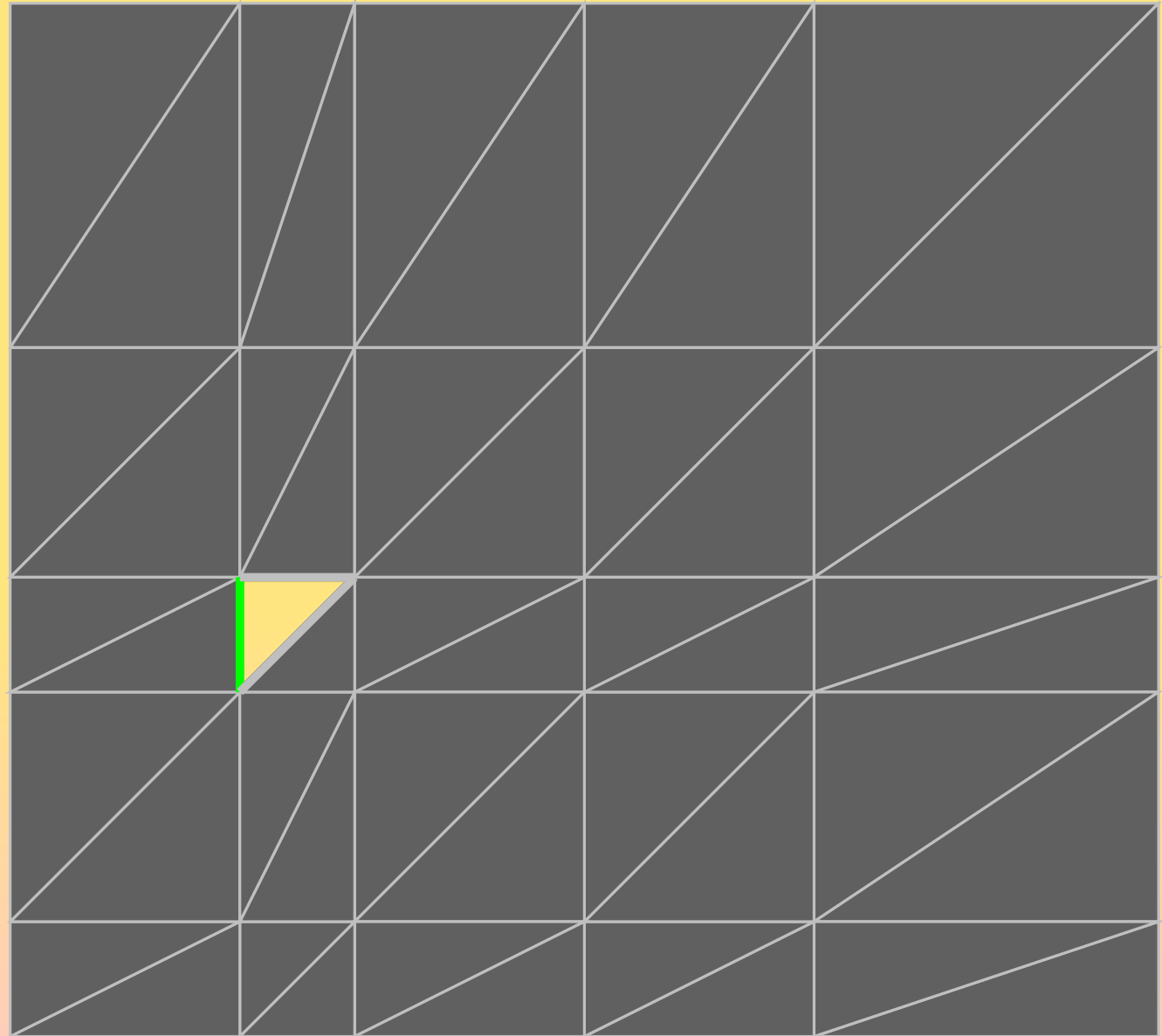
merge two different boundary components



Example of execution of our algorithm (toroidal case)

Starts the traversal from the root face (green edge)

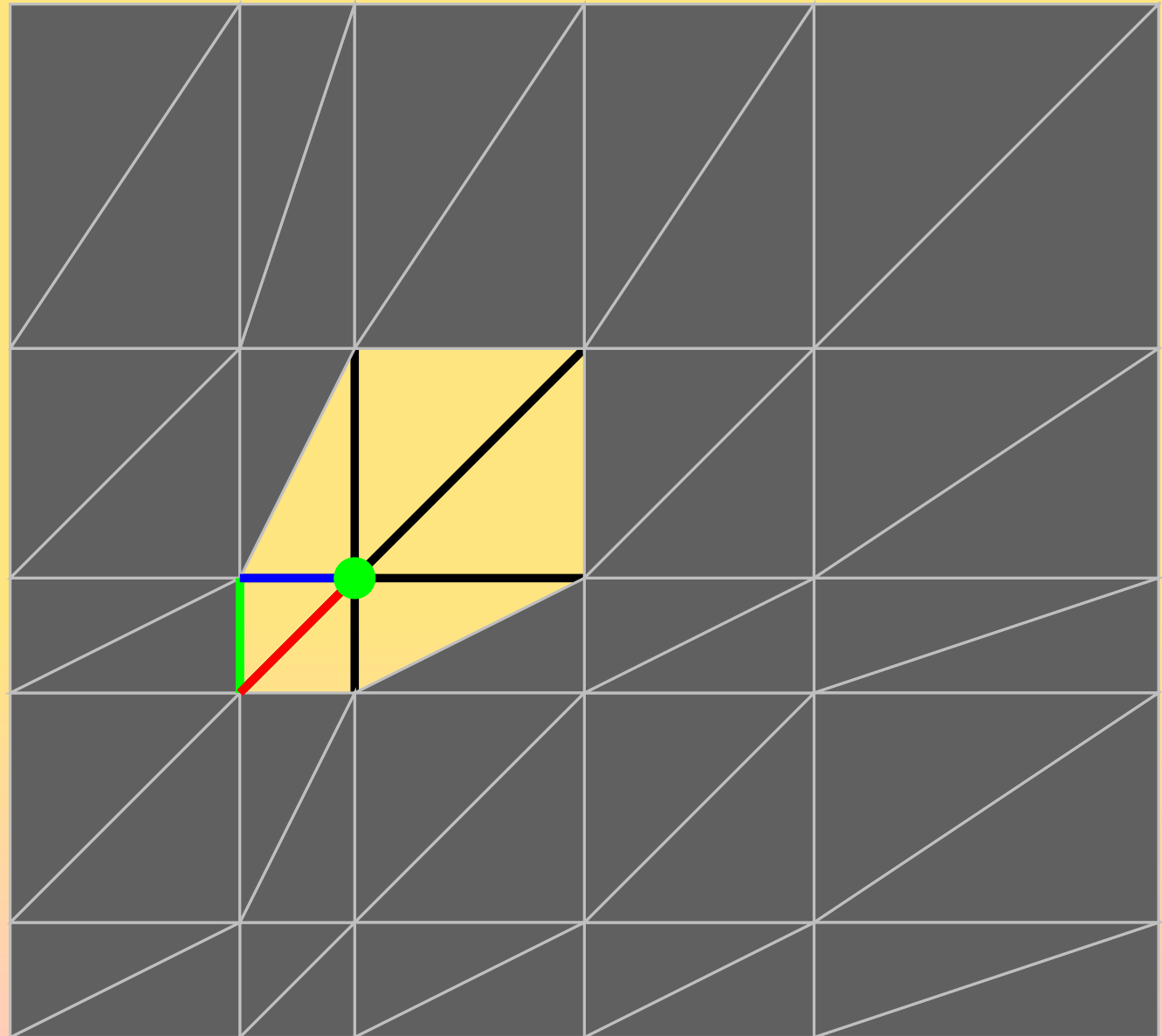
\mathcal{S}



Example of execution of our algorithm (toroidal case)

$\text{conquer}(w) +$
 $\text{colororient}(w)$

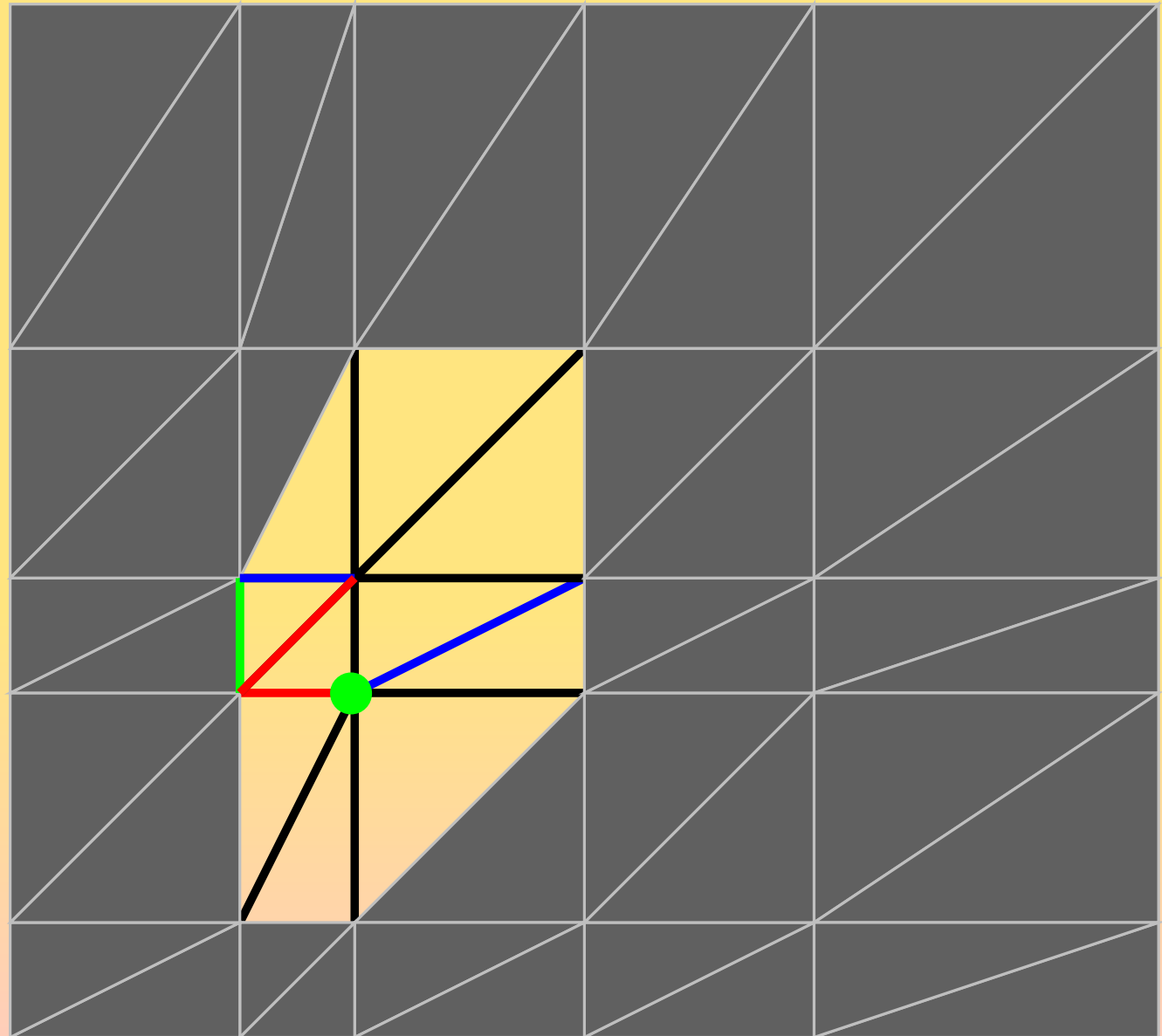
\mathcal{S}



Example of execution of our algorithm (toroidal case)

$\text{conquer}(w) +$
 $\text{colororient}(w)$

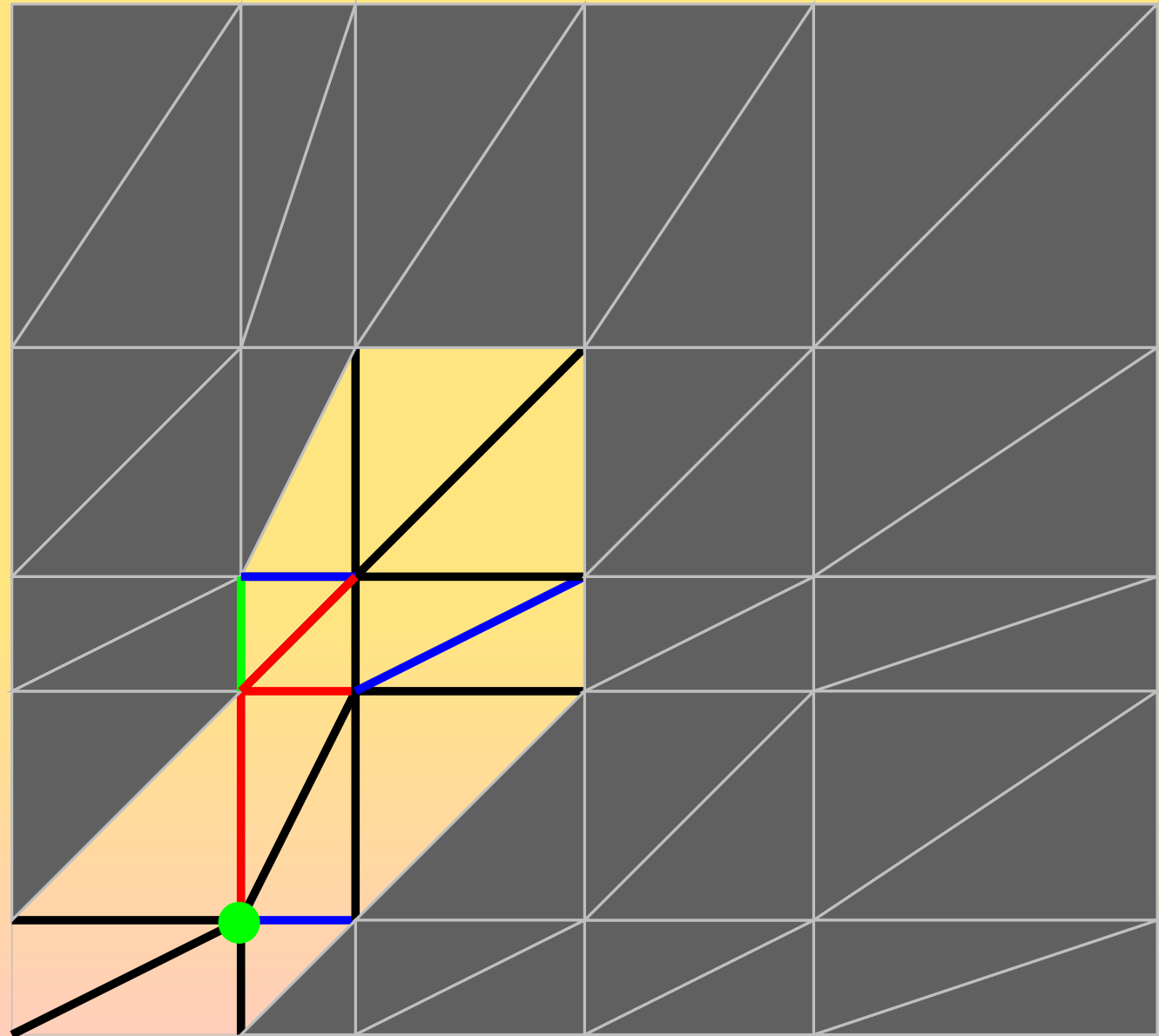
\mathcal{S}



Example of execution of our algorithm (toroidal case)

$\text{conquer}(w) +$
 $\text{colororient}(w)$

\mathcal{S}

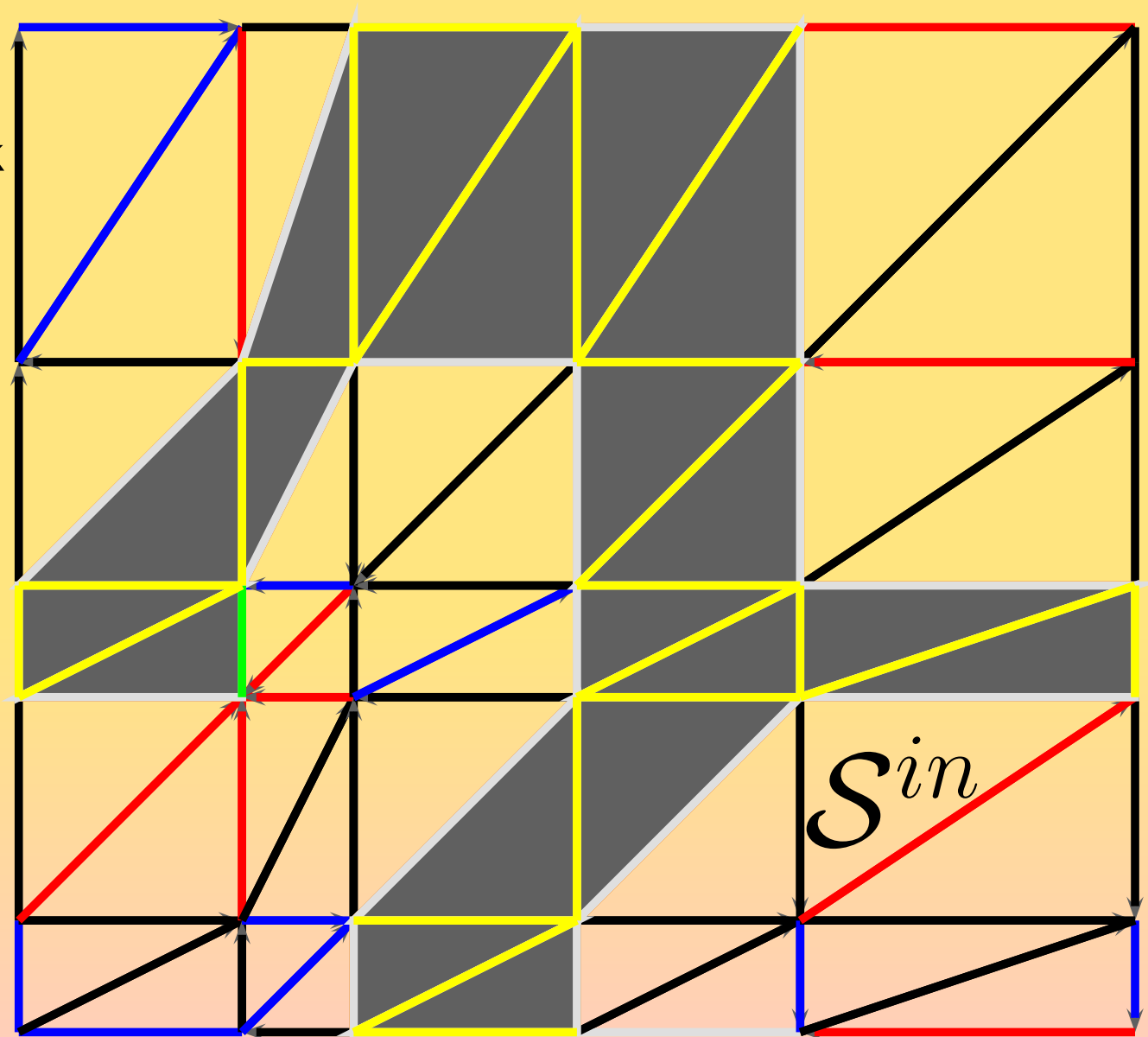


Example of execution of our algorithm (toroidal case)

After a maximal sequence of vertex conquest operations ...
no more free vertices... the (planar) traversal gets stuck

\mathcal{S}^{in} is a topological disk

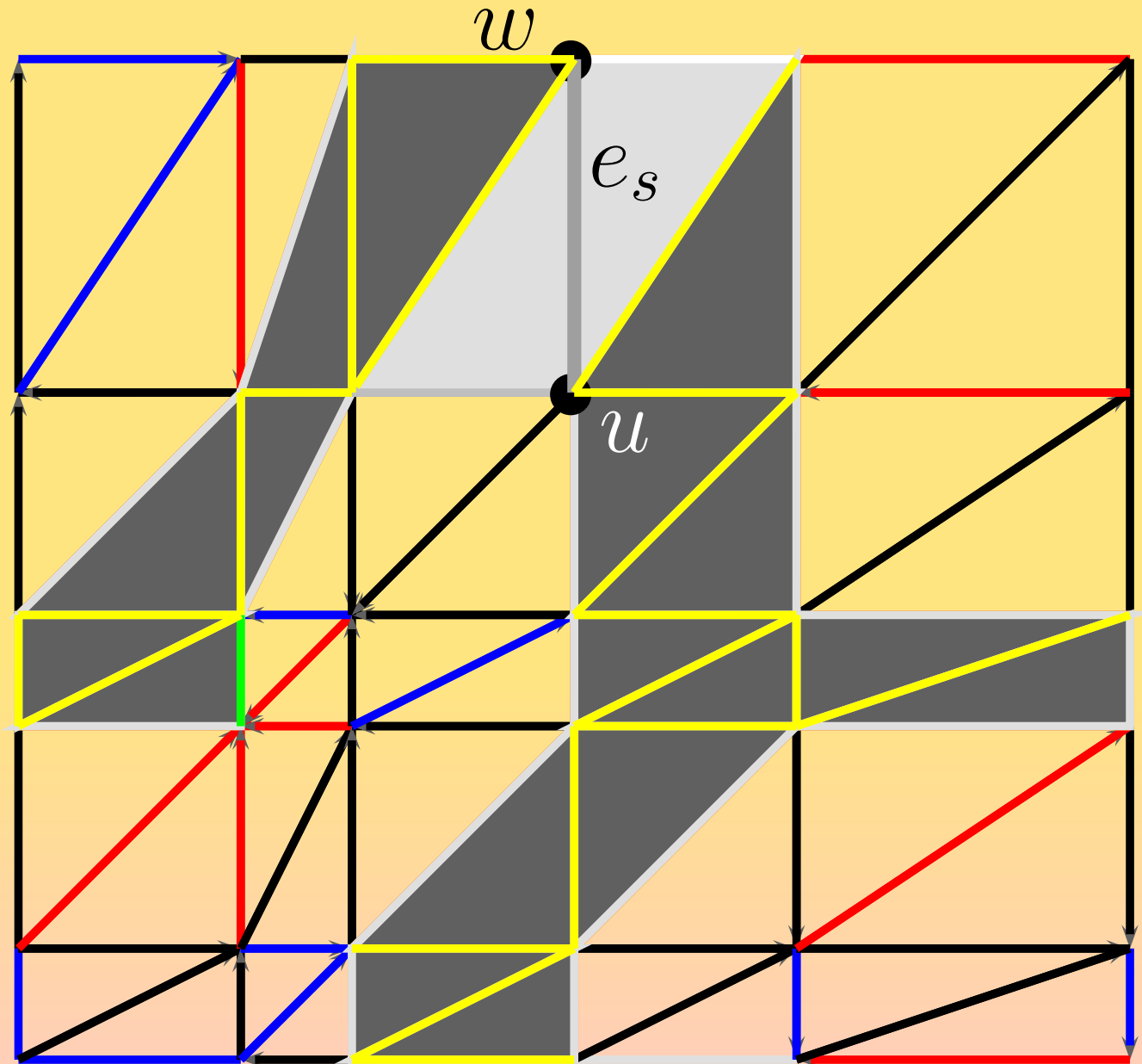
\mathcal{S}^{out} is a face connected map with a boundary component



Example of execution of our algorithm (toroidal case)

Let us perform a $\text{split}(u, w)$ operation

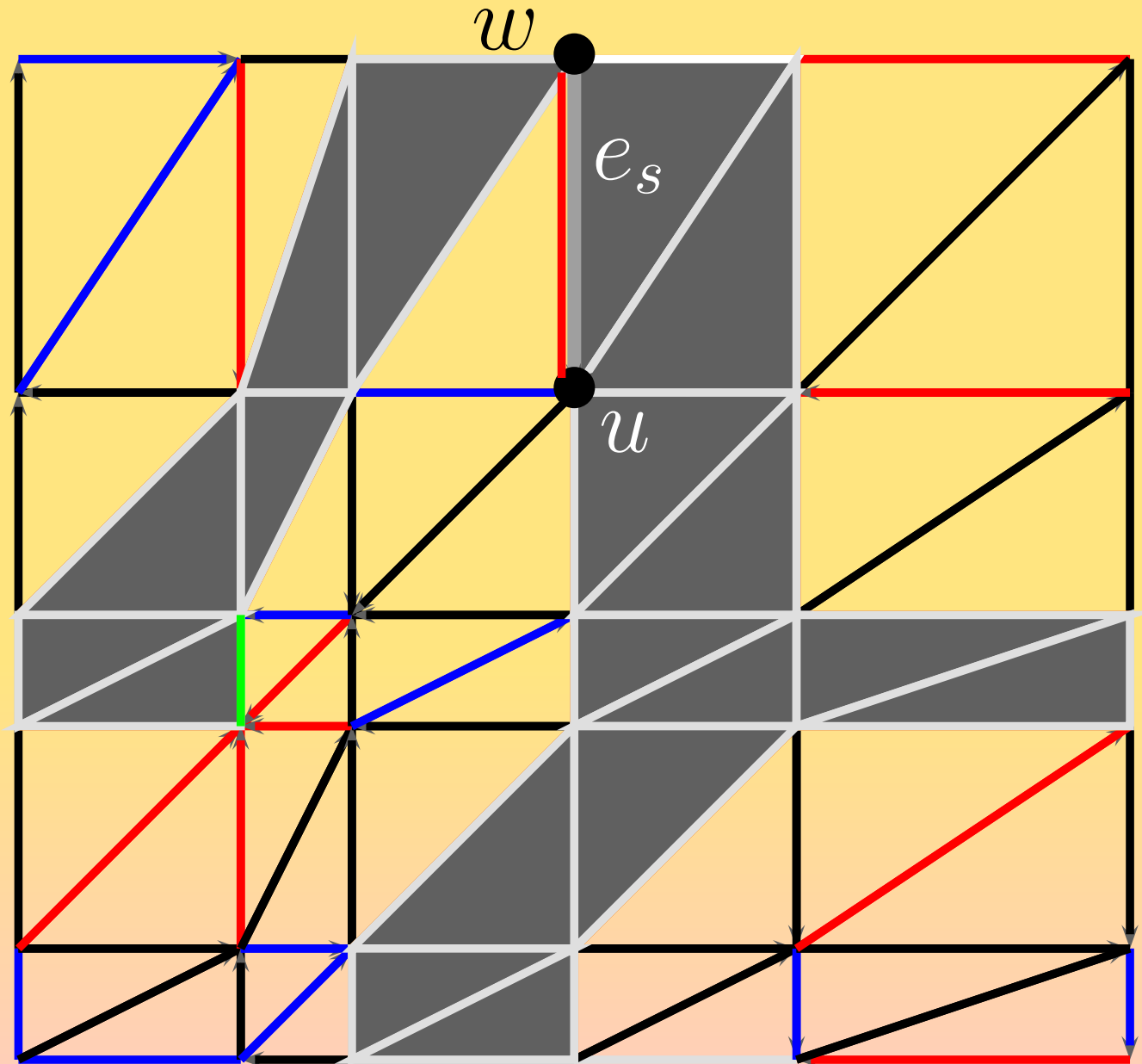
$\mathcal{S}^{out} \setminus (u, w)$ is a face connected map of genus 1 with two boundary components



Example of execution of our algorithm (toroidal case)

We can now perform a $\text{conquer}(u)$ operation

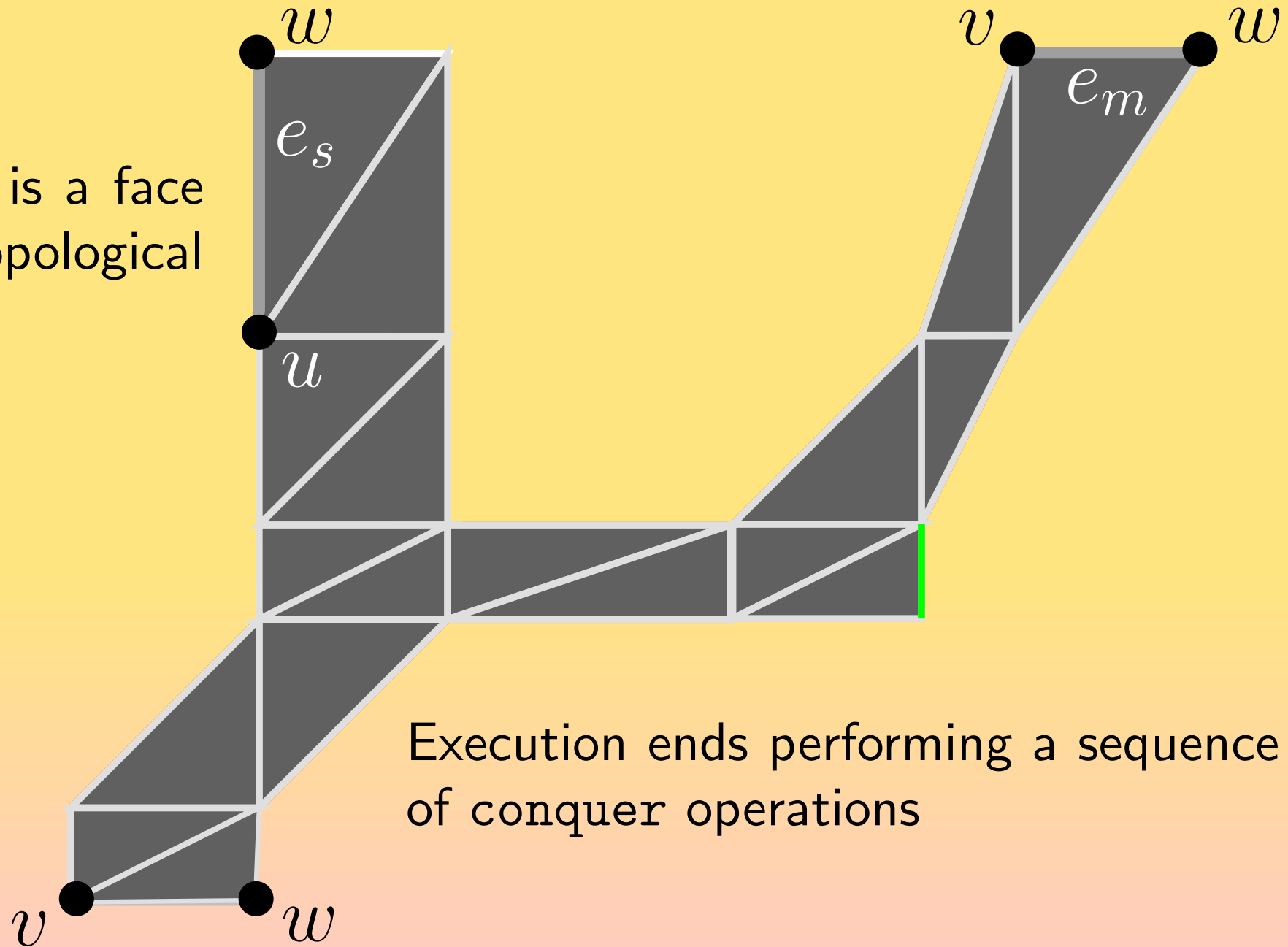
$\mathcal{S}^{out} \setminus (u, w)$ is a face connected map of with two boundary components



Example of execution of our algorithm (toroidal case)

Let us see in a better way...

$\mathcal{S}^{out} \setminus (w, v)$ is a face connected topological disk

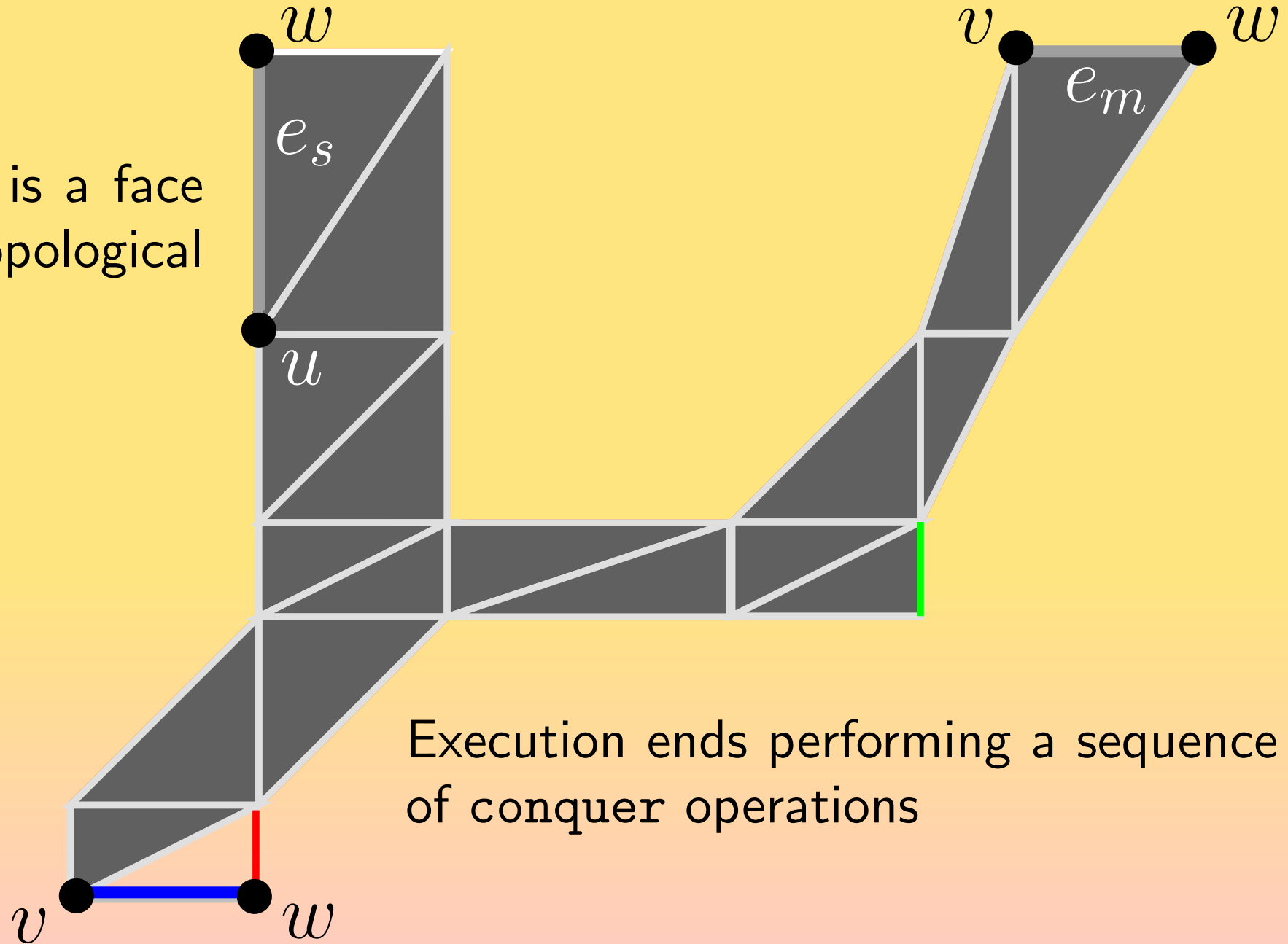


Execution ends performing a sequence of conquer operations

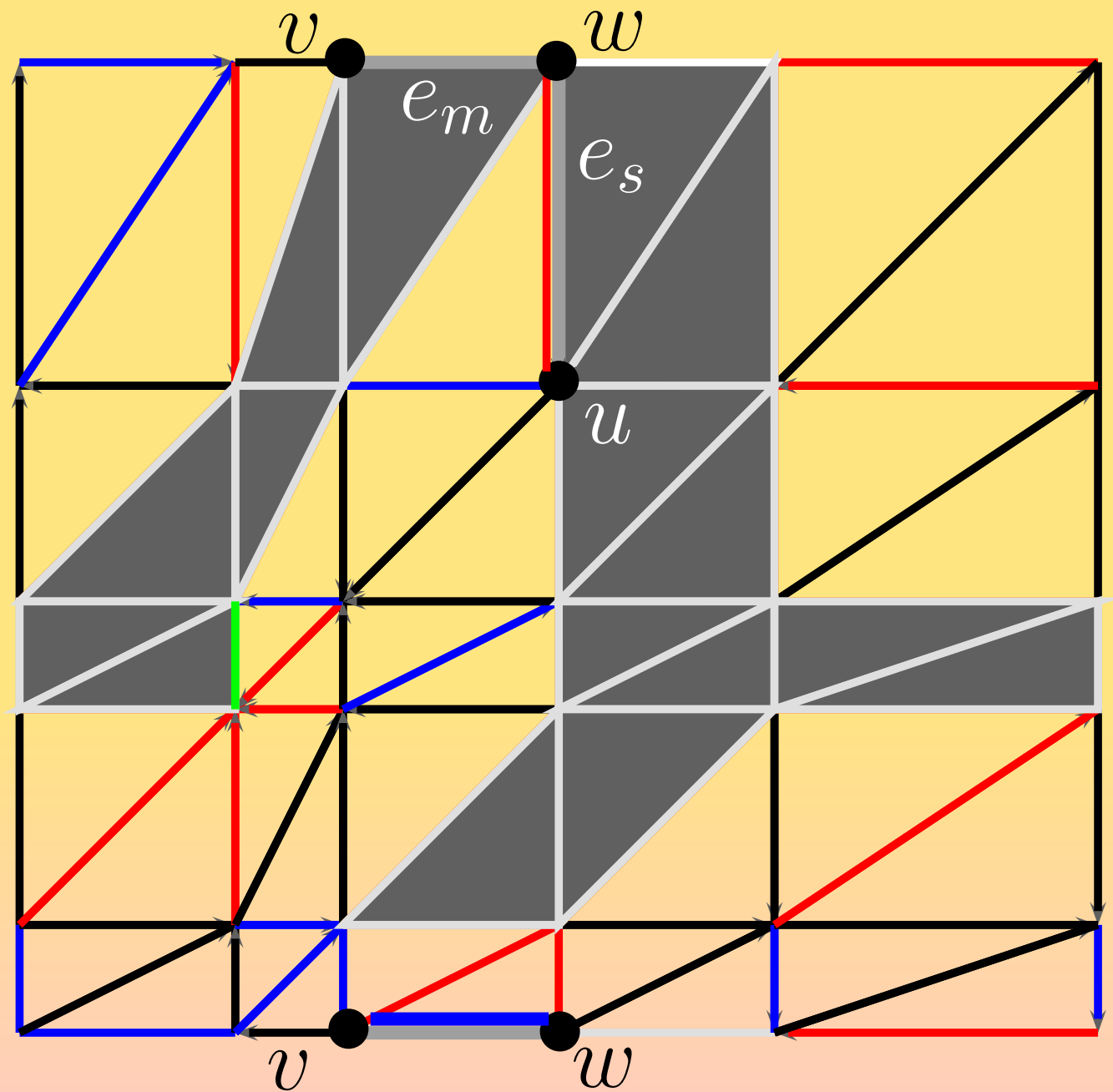
Example of execution of our algorithm (toroidal case)

Let us see in a better way...

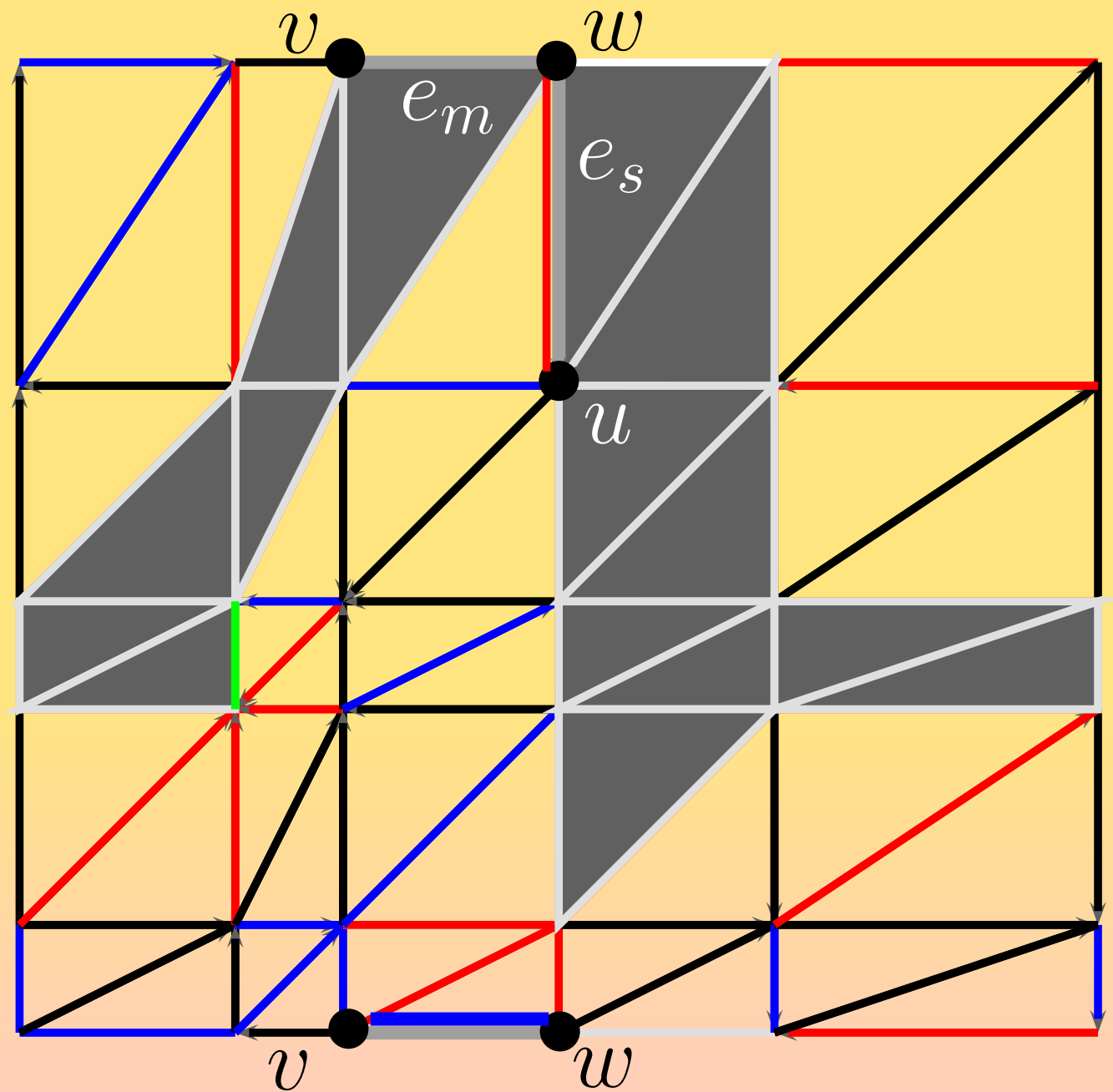
$\mathcal{S}^{out} \setminus (w, v)$ is a face connected topological disk



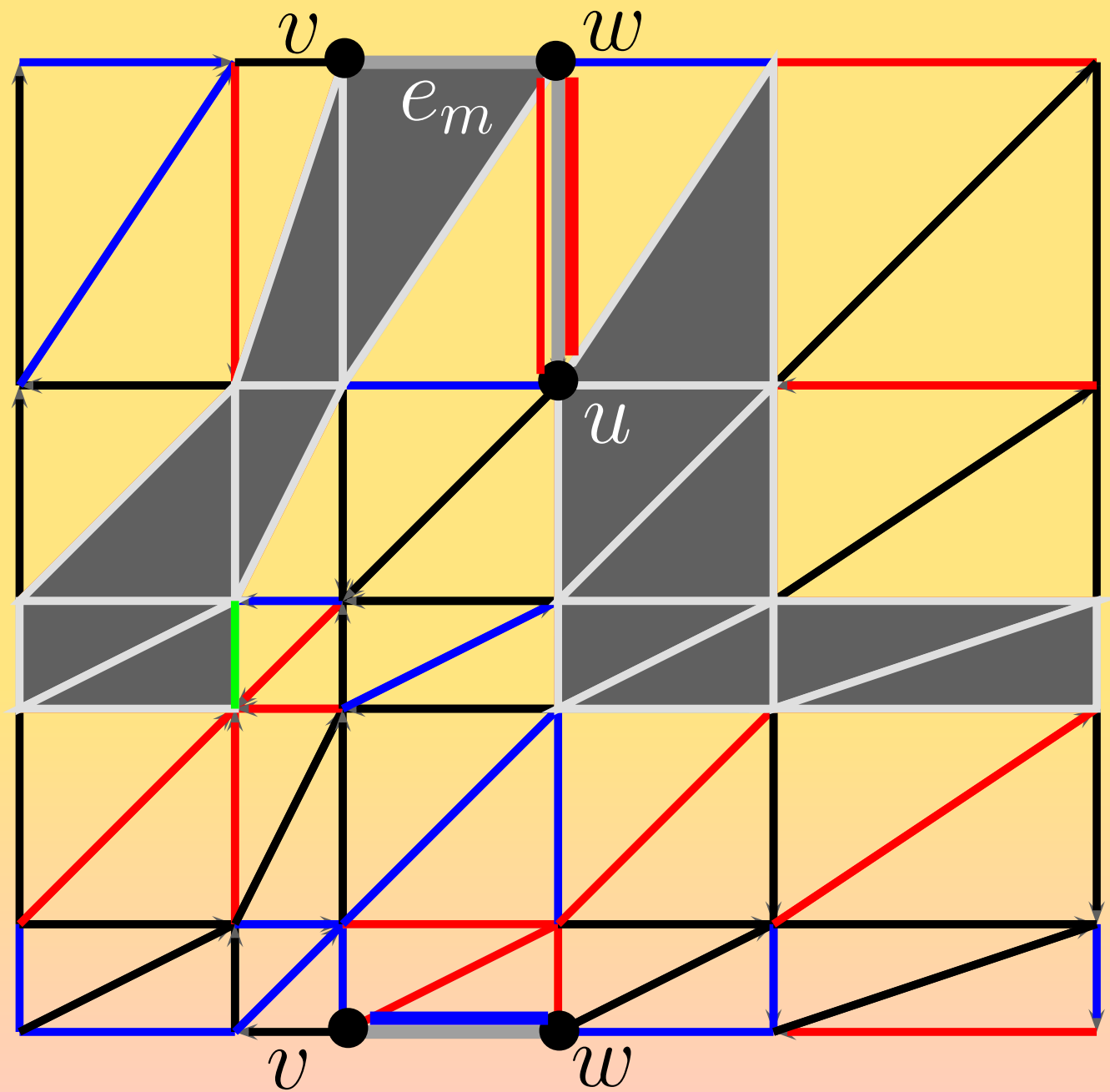
Example of execution of our algorithm (toroidal case)



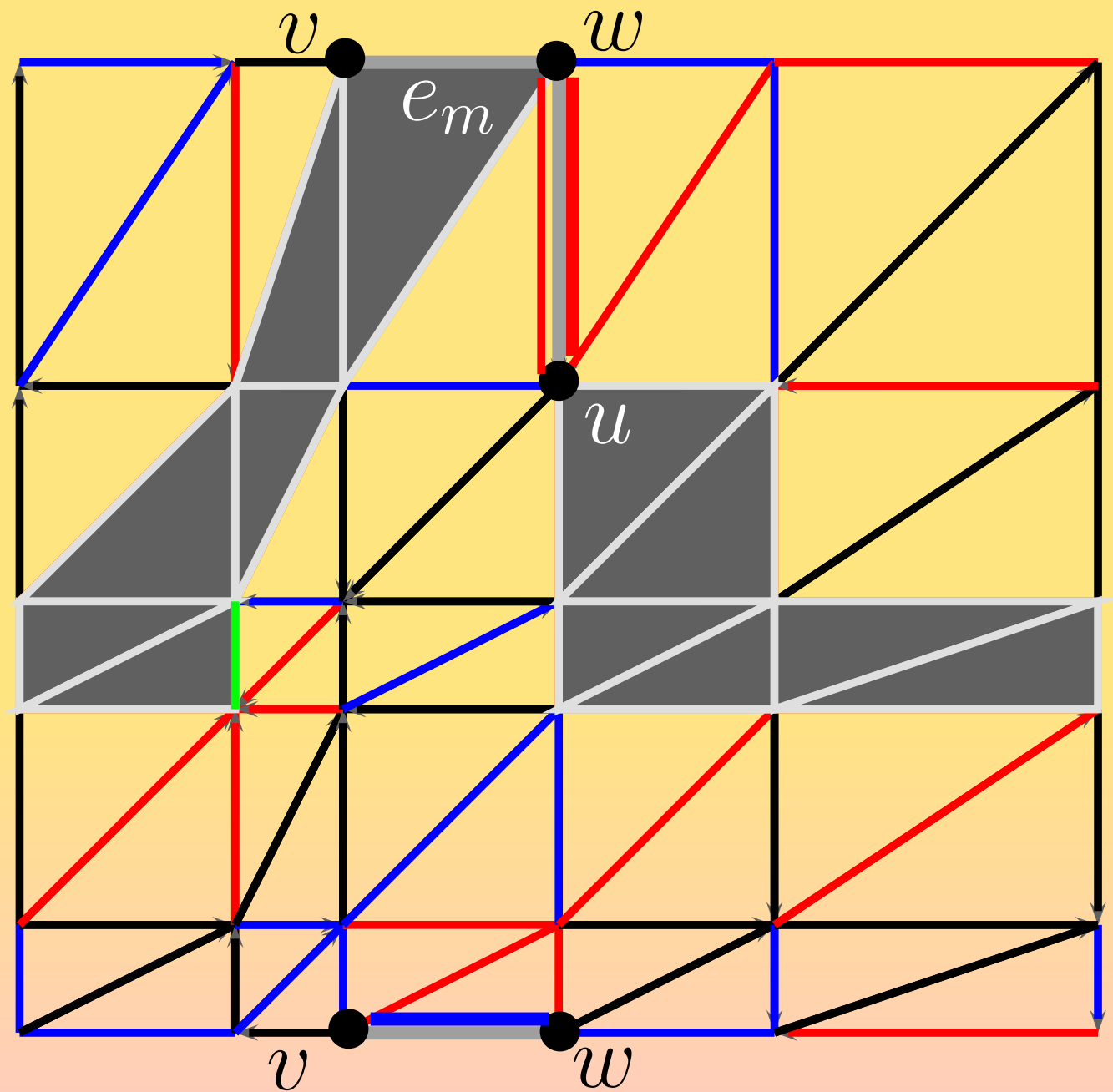
Example of execution of our algorithm (toroidal case)



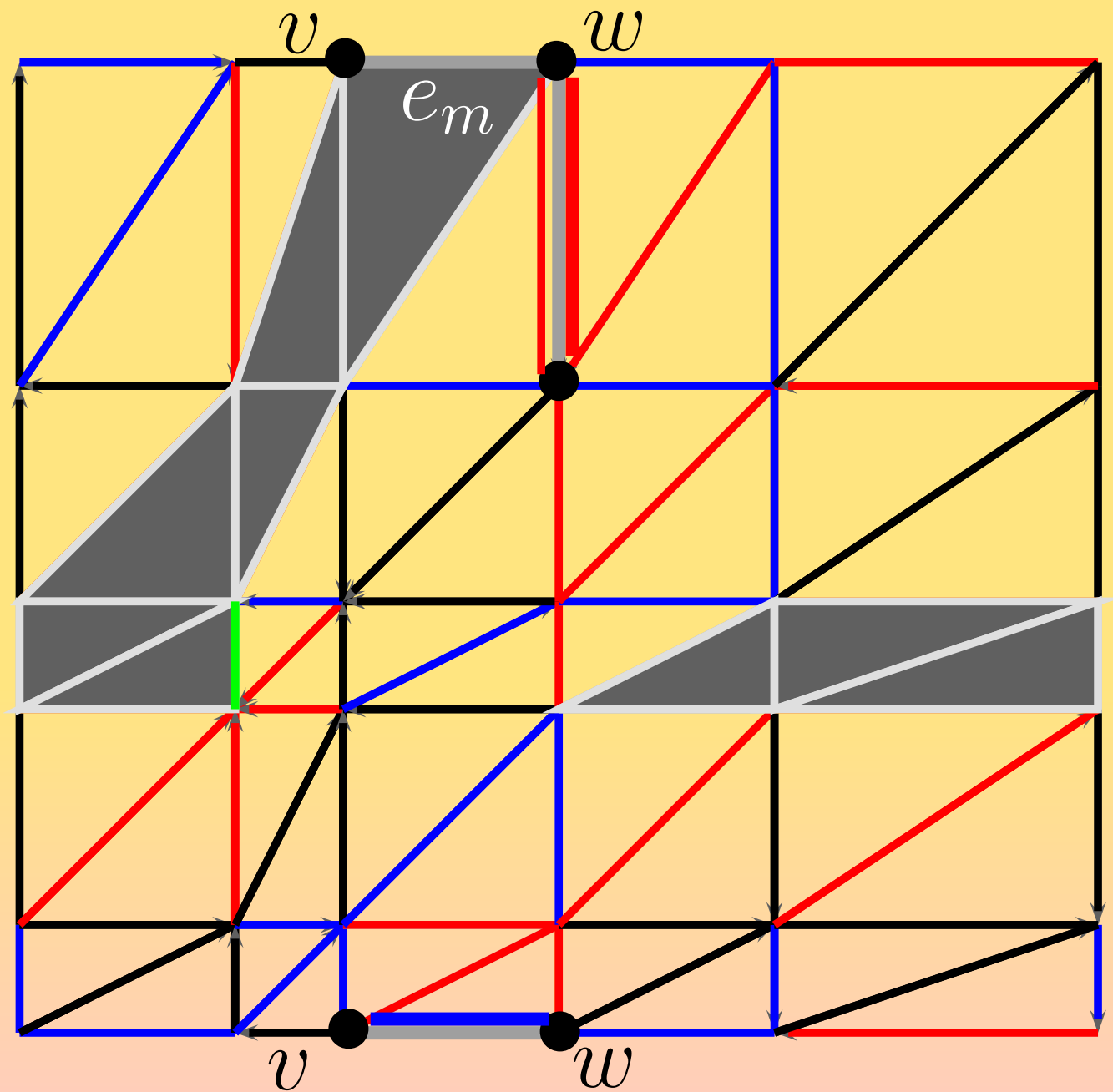
Example of execution of our algorithm (toroidal case)



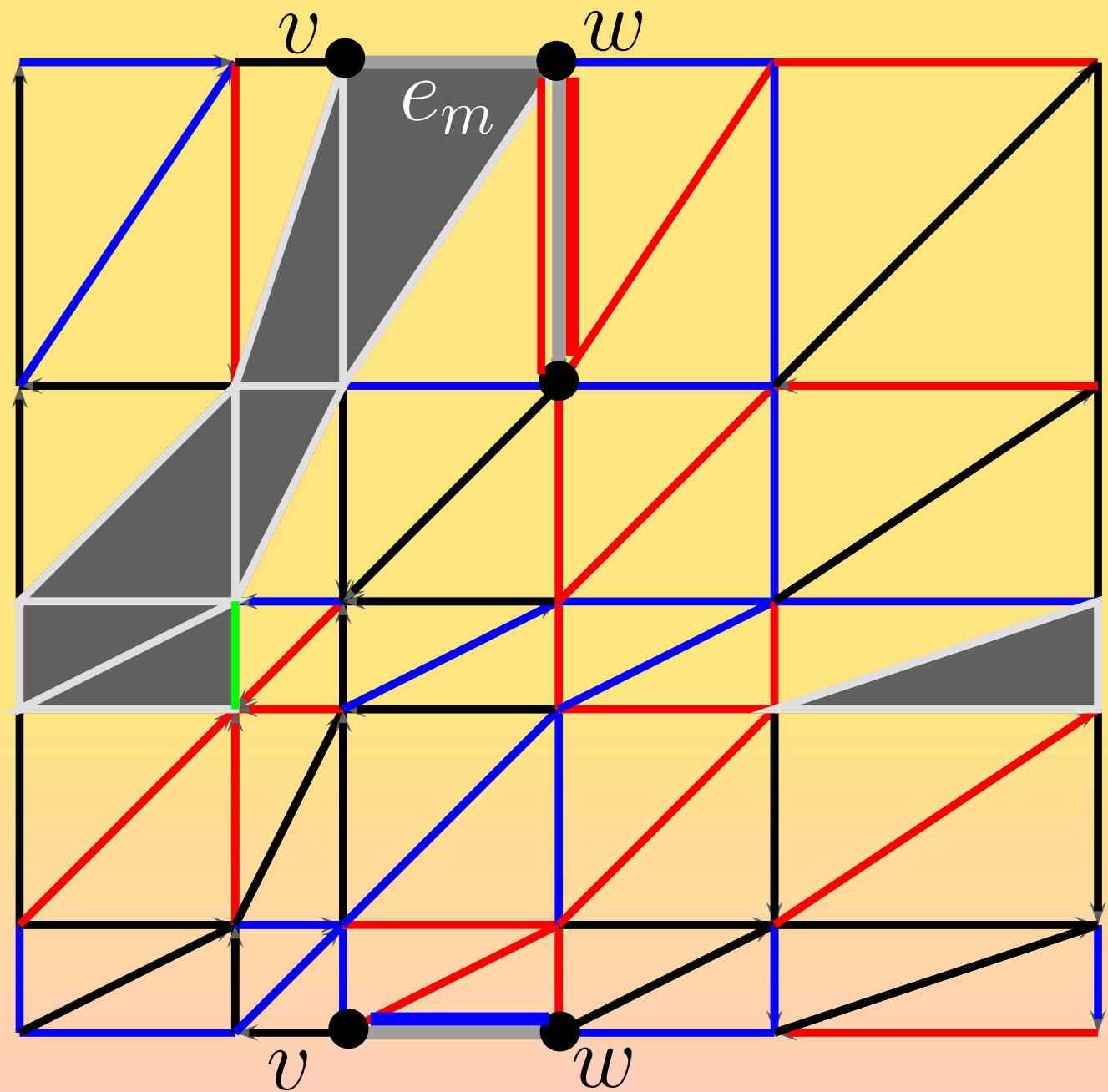
Example of execution of our algorithm (toroidal case)



Example of execution of our algorithm (toroidal case)

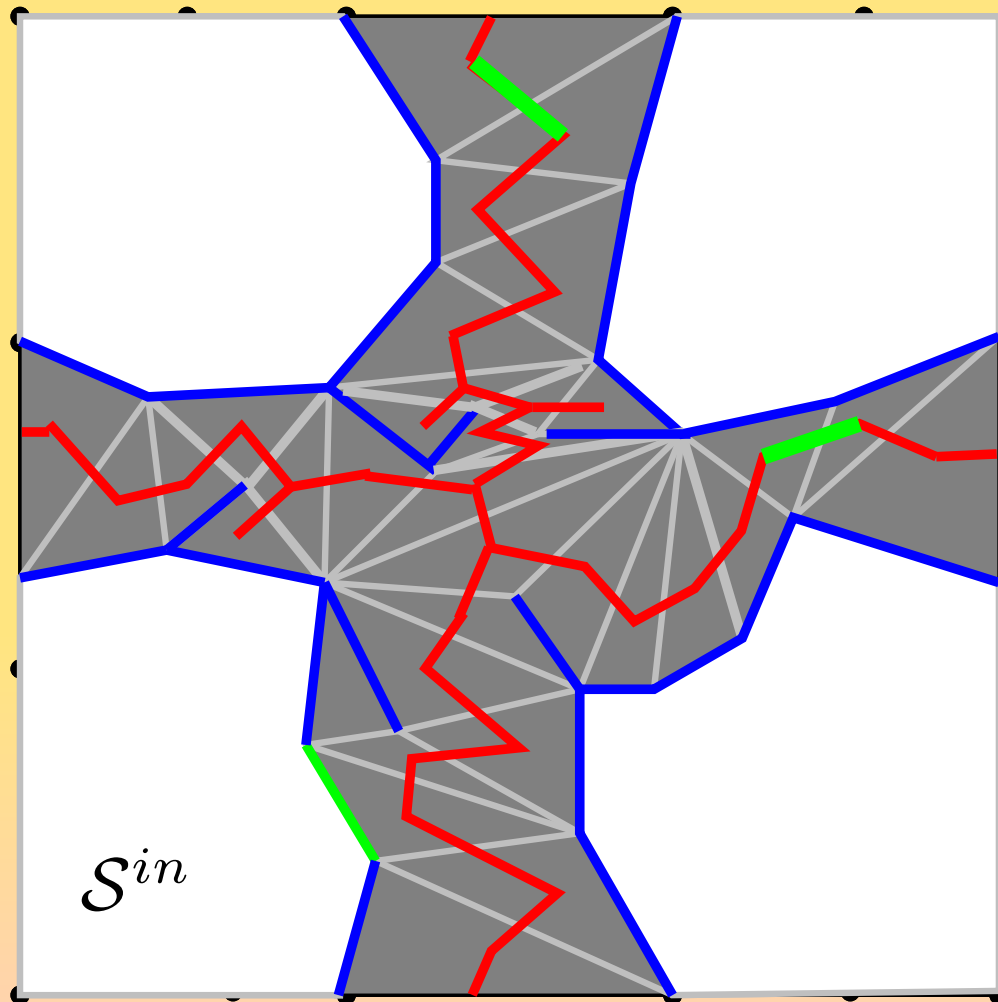


Example of execution of our algorithm (toroidal case)



Correctness and termination

Existence of split and merge edges



T vertex spanning tree of the primal graph, containing the boundary (blue) edges

T^* is a 1 face map containing $2g$ non-trivial cycles

T^* contains g split and merge edges

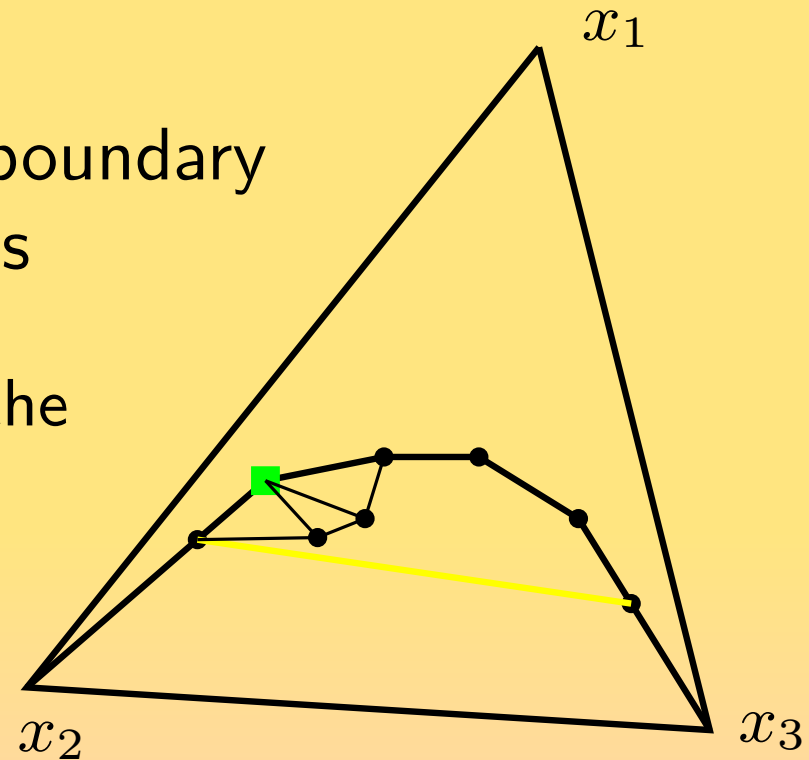
Correctness and termination

A fundamental lemma about chordal edges
(the planar case)

Lemma

There always exists at least one boundary vertex, not incident to chordal edges

Proof: by induction on the size of the boundary

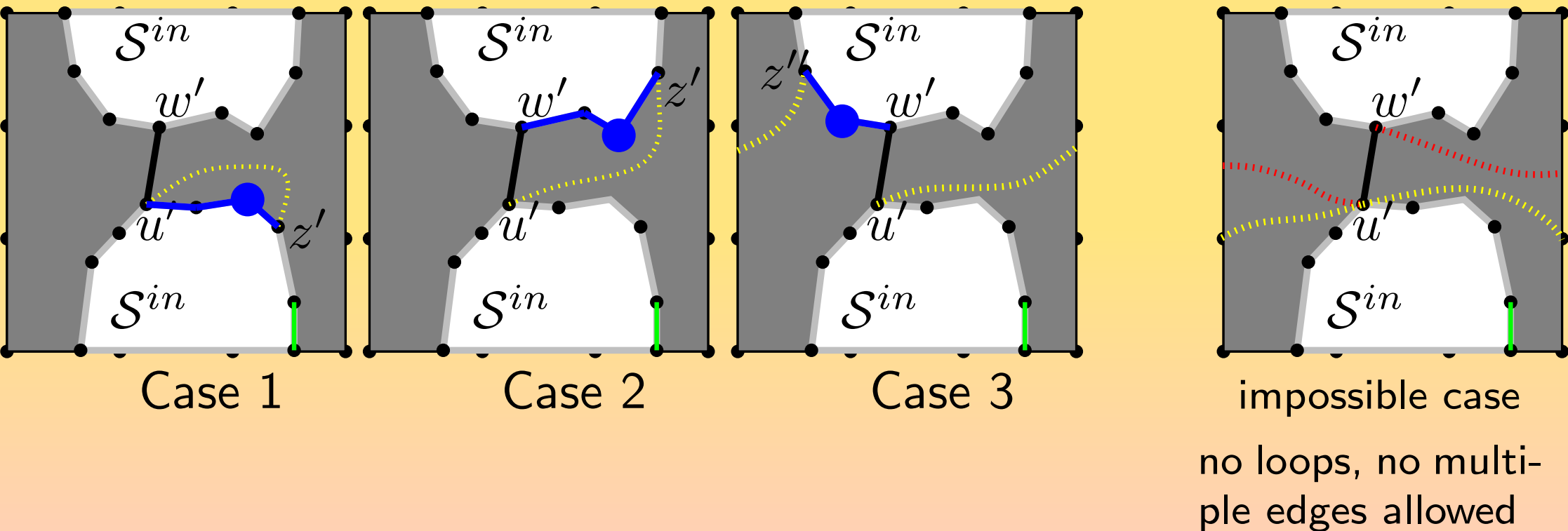


Theorem Each planar triangulation admits
a *canonical ordering* on the vertices

Correctness and termination

(higher genus)

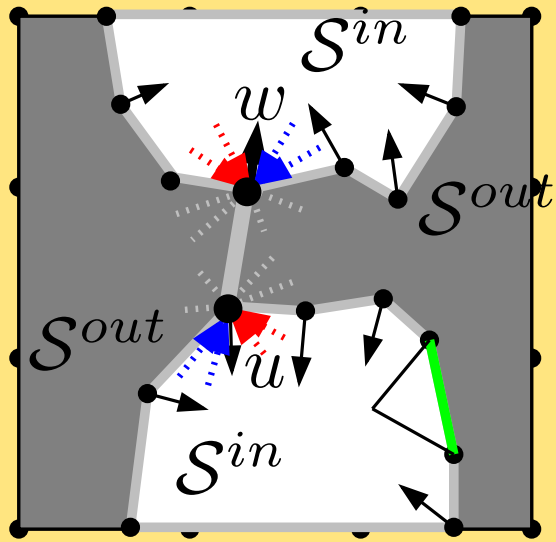
Properties of chordal edges in genus g



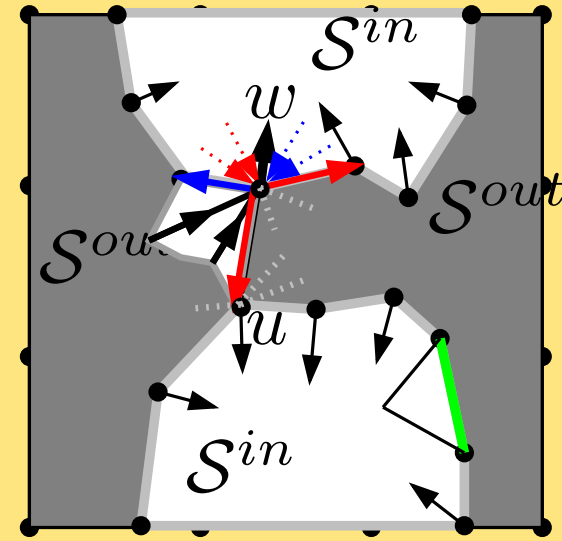
Correctness and termination

(higher genus)

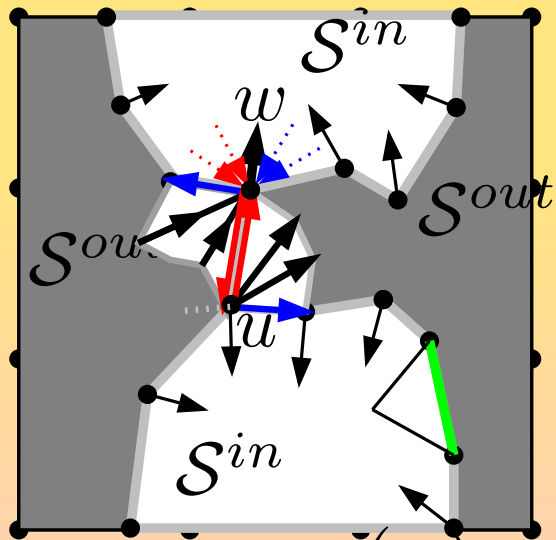
Coloring invariants



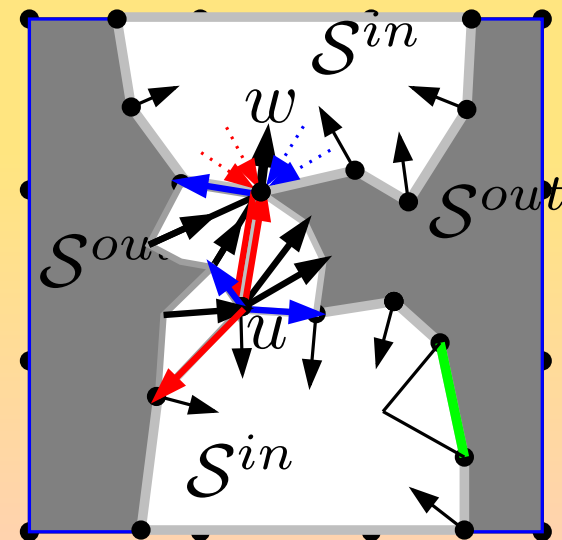
$\text{split}(u, w)$



$\text{conquer}(w) + \text{colororient}(w)$



$\text{conquer}(u) + \text{colororient}(u)$



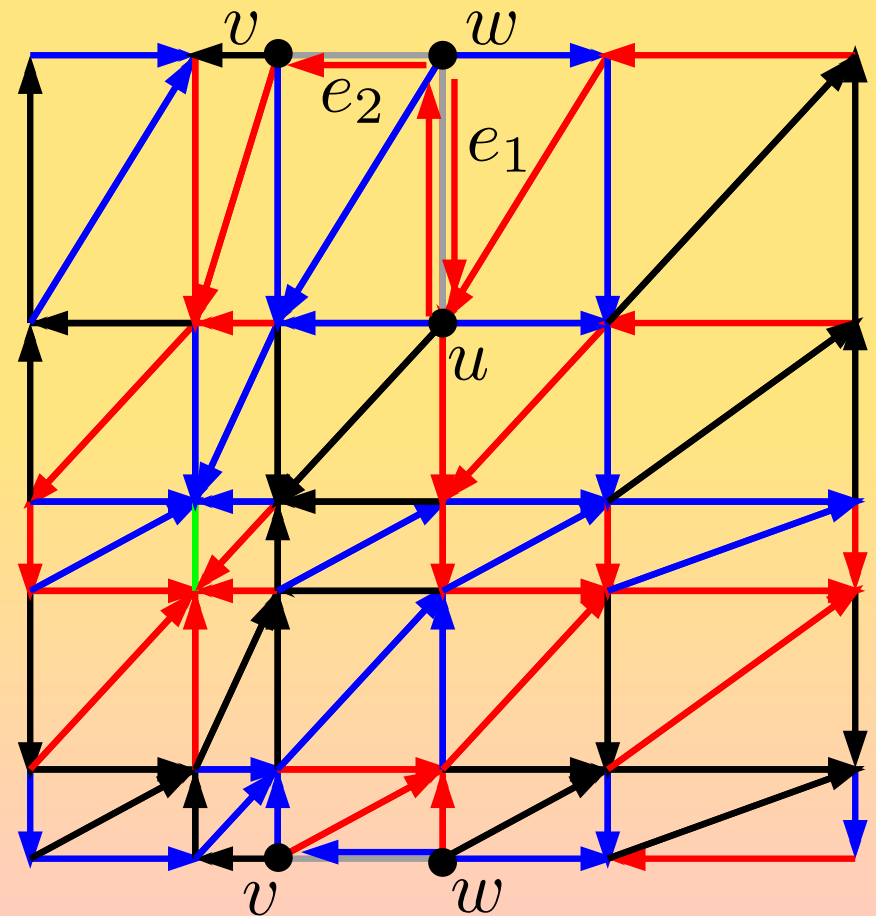
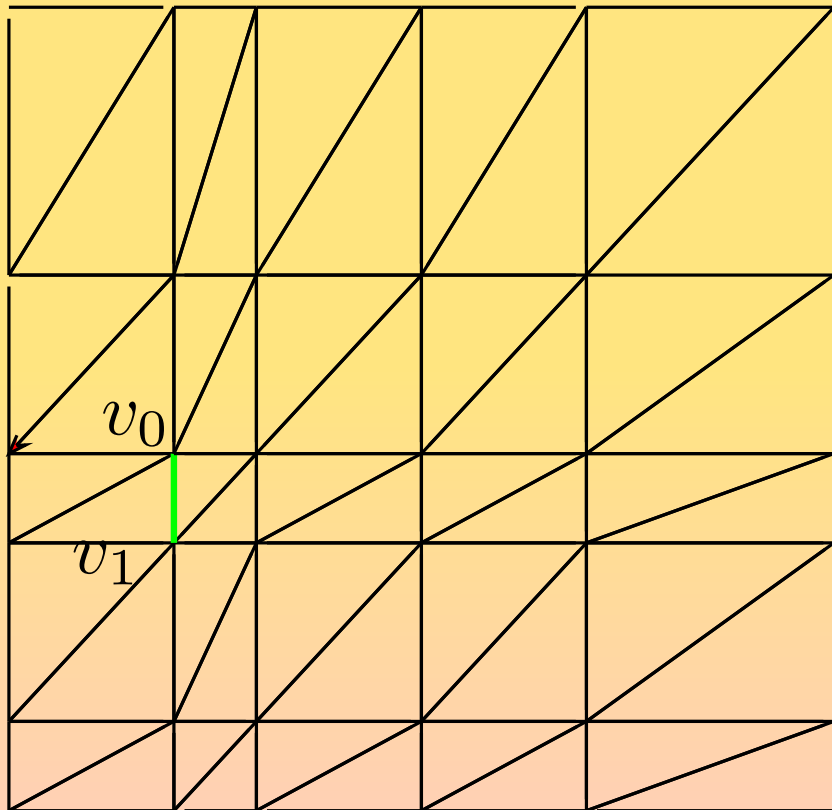
$\text{conquer}(u) + \text{colororient}(u)$

Our main result

Theorem (Castelli Aleardi, Fusy and Lewiner, 2008)

Given a (simple) rooted triangulation \mathcal{T} of genus g and size n , we can compute in $O(n)$ time a g -Schnyder wood of \mathcal{T} .

The local Schnyder condition is true almost everywhere in the graph at the exception of multiple vertices



From plane trees to genus g maps

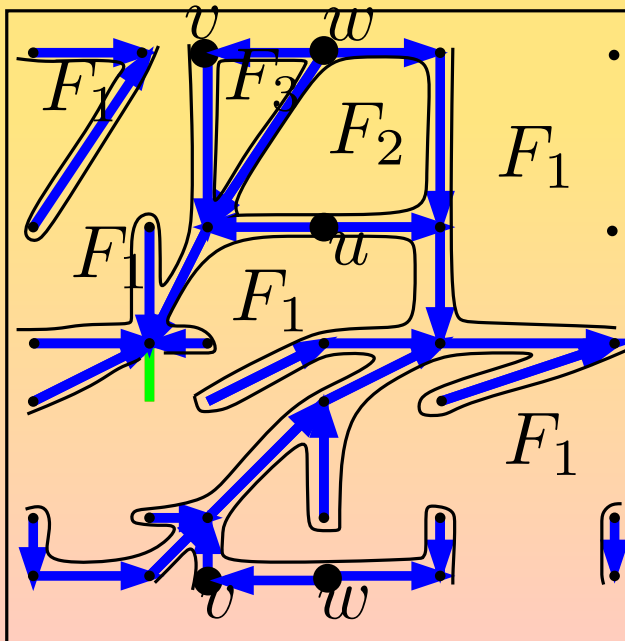
A new characterization in term of g-maps

Theorem (Castelli Aleardi, Fusy and Lewiner, 2008)

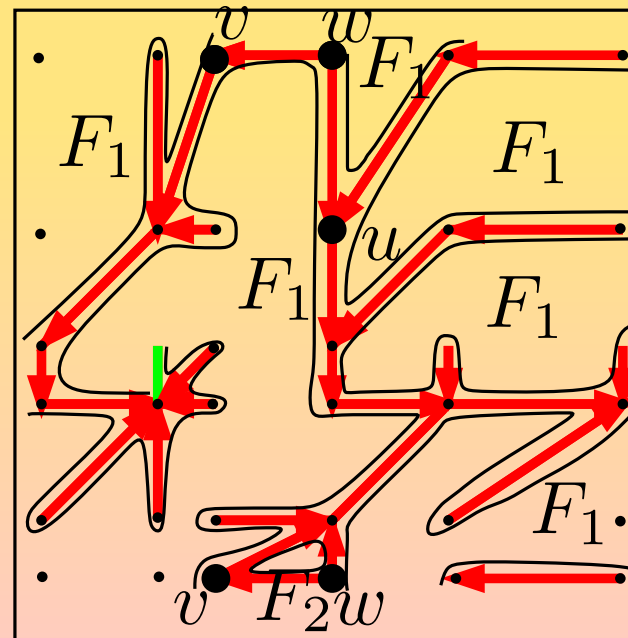
The three sets of edges T_0 and T_1 (red and blue edges), as well as the set $T_2 \cup \mathcal{E}$ (black edges and special edges) are maps of genus g satisfying:

- T_0, T_1 are maps with at most $1 + 2g$ faces;
- $T_2 \cup \mathcal{E}$ is a 1 face map

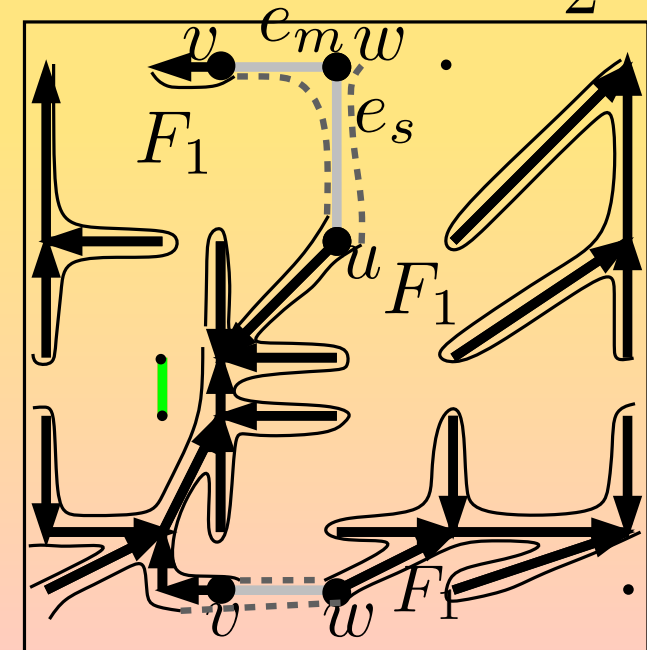
T_0



T_1



T_2



A new encoding application

Futur works and open questions

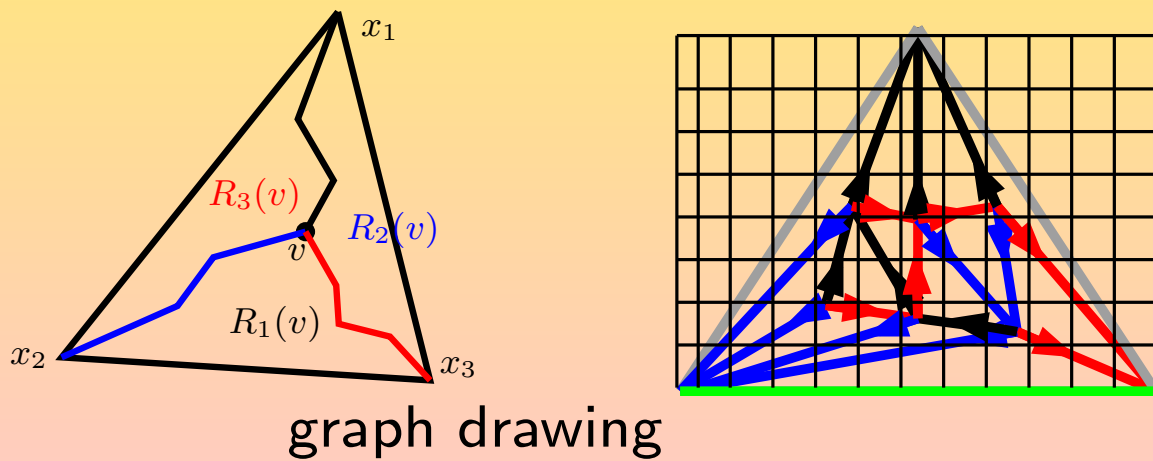
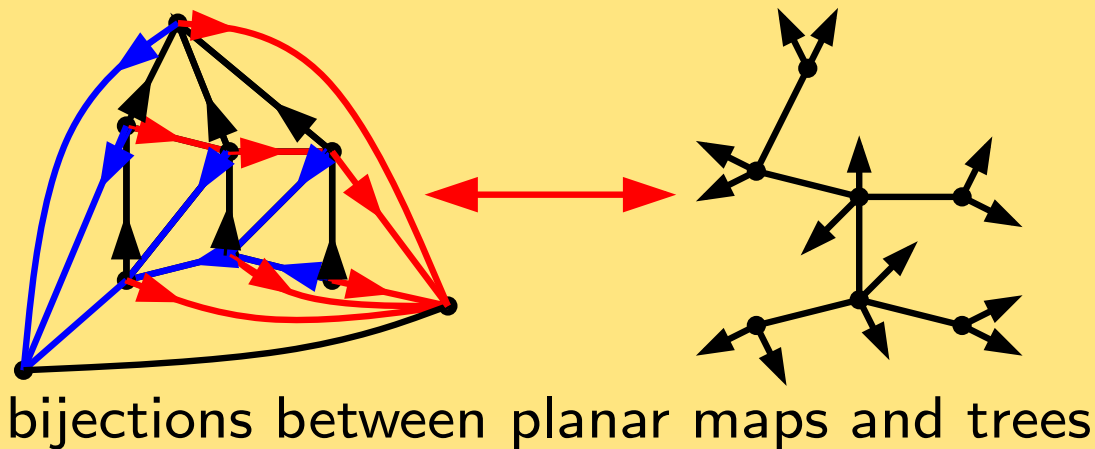
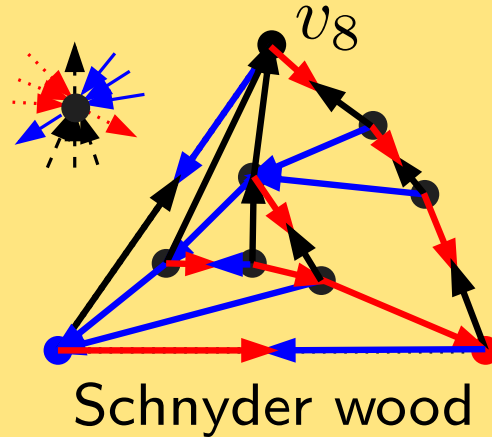
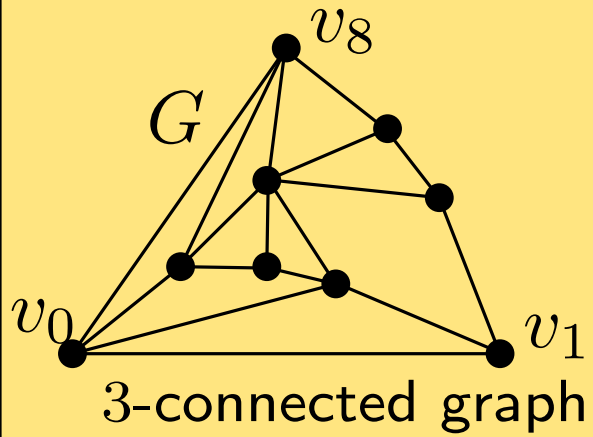
optimal encoding in higher genus

lattice structure for the set of Schnyder woods

extension to the 3-connected case (polygonal meshes)

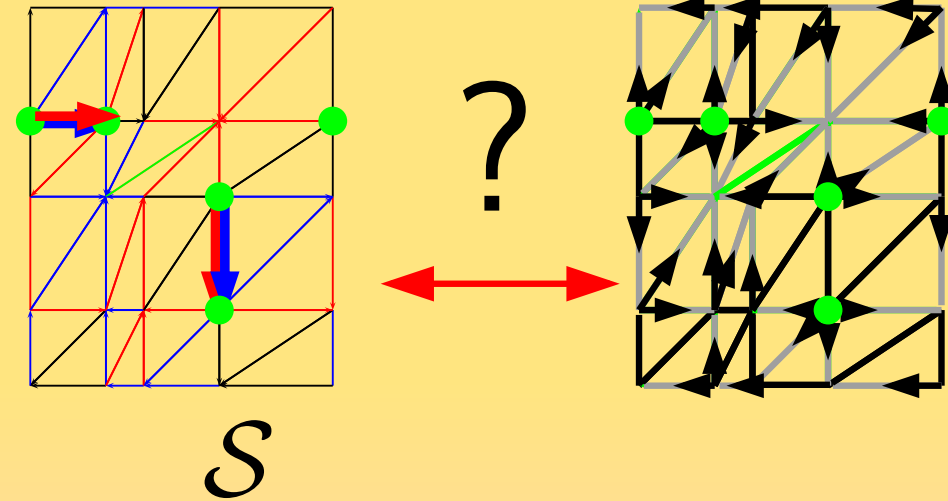
Futur works

planar graphs



higher genus

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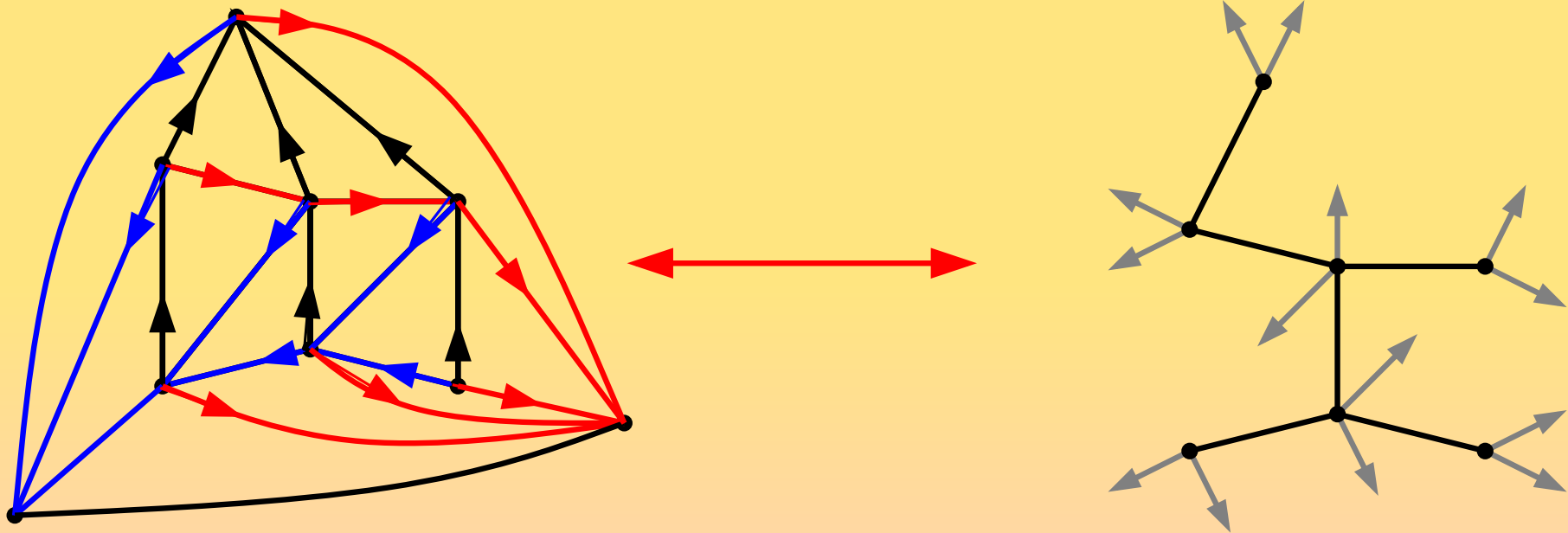
Optimal coding and sampling (planar case)

Theorem. (Tutte 62) The number of planar triangulation with $n + 2$ vertices is

$$\frac{2(4n-3)!}{(3n-1)!n!} \asymp \left(\frac{256}{27}\right)^n .$$

Théorème. (Poulalhon–Schaeffer Icalp 03)

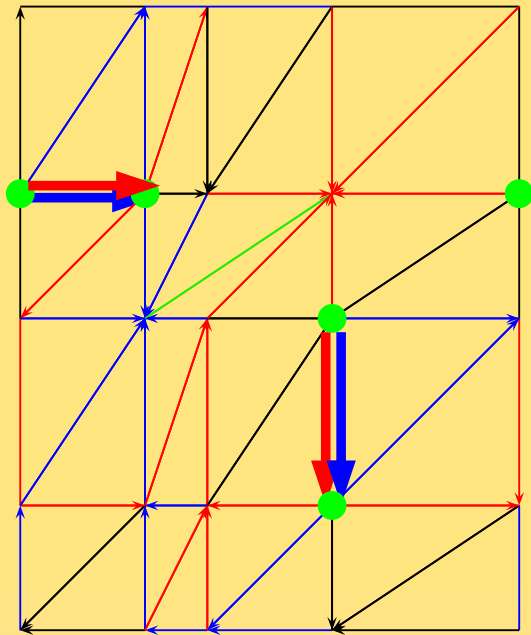
Il existe une bijection entre la classe des arbres de taille n ayant deux bourgeons par noeud, et la classe des triangulations planaires enracinées à $n + 2$ sommets.



a new nice interpretation of Tutte's formula:

$$|\mathcal{T}_n| = \frac{2}{2n} \cdot |\mathcal{A}_n^{(2)}| .$$

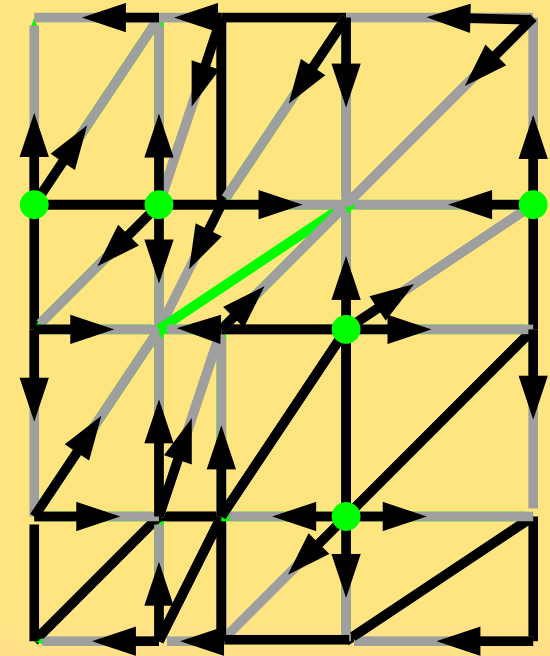
Optimal coding (genus g)



\mathcal{S}

triangulated graph of genus g

?



\mathcal{T}^g

one face map of genus g