Algorithms and Combinatorics of Geometric Graphs (`Geomgraphs`) 2025-2026

TD5 (exercises)

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1 Schnyder woods of 4-connected planar triangulations (LCA):

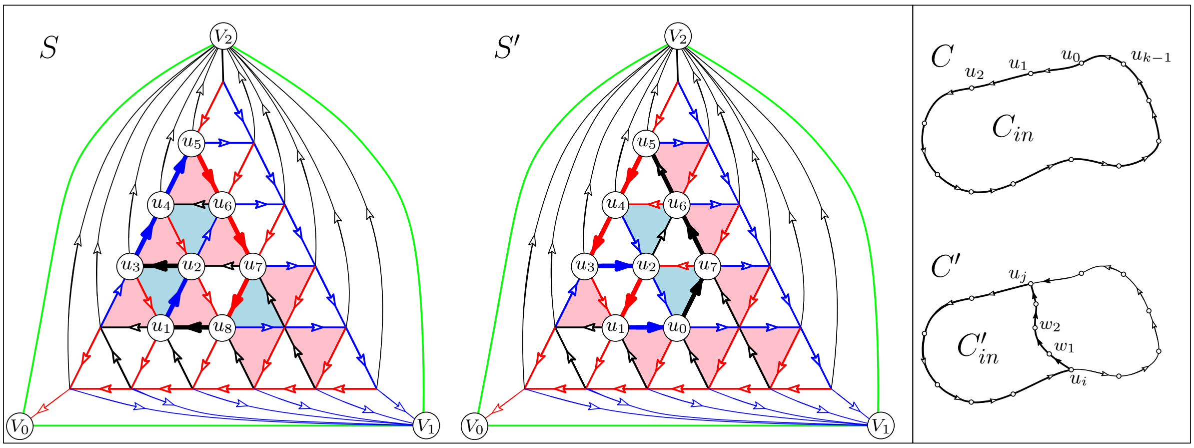

Figure 1: (Left) A (4-connected) planar triangulation $\mathcal{T}$ with root face (V_0, V_1, V_2), endowed with two distinct Schnyder woods S and S'. The 3-colored faces are shaded: *cw*-oriented triangles are red and *ccw*-oriented triangles are blue. Bold lines represent the edges of a cycle of $C = \{(u_0, u_1), \ldots, (u_6, u_7), (u_7, u_0)\}$ of length 8, whose inner region C_{in} contains 6 triangles. The cycle C is *cw*-oriented in S and *ccw*-oriented in S'. (Right) Illustration of Question 4.

We will assume that the input plane triangulation $\mathcal{T}$ is endowed with a Schnyder wood (Fig. **??**(left) shows two Schnyder woods of the same triangulation).

A *cycle* is defined by an ordered list of edges $\{(u_0, u_1), (u_1, u_2), \ldots (u_{k-1}, u_0)\}$ such that each vertex is shared by exactly two consecutive edges (no self-intersections). A cycle partitions the set of faces of $\mathcal{T}$ into two regions: the *outer region* (containing the root outer face) and the *inner region* C_{in} containing all remaining faces (inside C). The cycles of length 3 are called *triangles*. A triangle does not necessarily defines a face in the embedding of $\mathcal{T}$ (such a triangle is called *separating*). Consider a cycle $C = \{(u_0, u_1), \ldots (u_{k-1}, u_0)\}$ and for an edge $e = (u, v) \in C$, oriented from u to v, and denote by $t_{in}(e)$ the face containing e that is lying in the inner region C_{in}. We say that C is *cw*-oriented (i.e. oriented in clockwise direction) if for each edge $e \in C$ the corresponding face $t_{in}(e)$ is on the right of e. Similarly, the cycle C is *ccw*-oriented (i.e. oriented in counter-clockwise direction) if for each edge $e \in C$ the face $t_{in}(e)$ is on the left of e). A triangle is 3-*colored* if the three edges $\{(u_0, u_1), (u_1, u_2), (u_2, u_0)\}$ have different colors (see Fig. **??**(left) for an illustration).

1. Let $C = \{(u_0, u_1), \ldots (u_{k-1}, u_0)\}$ be a cycle in $\mathcal{T}$ and assume that no edge in the inner region of C is oriented outgoing from a vertex u_i of C. Show that C must have length 3.

[*Hint: use a double counting of the edges in the inner region of* C]

2. Show that the boundary of a 3-colored triangle must define an oriented cycle (either *cw*-oriented or *ccw*-oriented).

3. Show that if $\mathcal{T}$ contains an oriented cycle C then the inner region of C also contains a 3-colored triangle.

Flips of faces in 4-connected planar triangulations

From now on we will assume that the input plane triangulation $\mathcal{T}$ is 4-*connected* (and endowed with a Schnyder wood), which means that $\mathcal{T}$ **does not contain separating triangles**.

4. Let C be an oriented cycle of $\mathcal{T}$. Show that there exists a path of $p \geq 1$ directed edges $P = \{(u_i, w_1), (w_1, w_2), \ldots (w_{p-1}, u_j)\}$ (where $u_i, u_j \in C$ and $i \neq j$) whose edges are all lying in the inner region of C and define a new oriented cycle C' such its inner region C'_{in} is strictly contained in the inner region of C (see Fig. **??**(right) for an illustration).

Given a Schnyder wood S of $\mathcal{T}$ and a 3-colored face $t = \{(u_0, u_1), (u_1, u_2), (u_2, u_0)\}$ the operation of *flip* of t consists in changing the orientation of its boundary edges. If the boundary of t is *cw*-oriented then we have a so-called *cw-flip*, and the edge of color i gets color $i + 2$ (indices are modulo 3). Otherwise, if t is *cww*-oriented, we have a *ccw-flip* (the edge of color i gets color $i + 1$). Observe that after flipping a 3-colored face in S we obtain another valid Schnyder wood S' (distinct from S). So, starting from an initial Schnyder wood S_0 we can construct a sequence Schnyder woods $\mathcal{S} = \{S_0, S_1, S_2, \ldots\}$ by flipping 3-colored faces (where S_{i+1} is obtained from S_i by flipping exactly one face). Observe that, in principle, the sequence $\mathcal{S}$ may contain the same Schnyder wood more than once.

5. Consider the two Schnyder woods S and S' of Fig. **??**. Give a list of face *cw*-flips that allows to obtain the Schnyder wood S' from S (the expected result is an ordered list of triangles, whose vertices are labeled as in Fig. **??**).

[*Remark: observe that the edge orientations of S and S' only differ on the oriented cycle C*]

Let us denote by $|C_{in}|$ the number of faces in the inner region bounded by C.

6. Let us consider two Schnyder woods S and S' of $\mathcal{T}$ and a *cw*-oriented cycle C in S. Assume that C is *ccw*-oriented in S' and that all remaining edges in $\mathcal{T} \setminus C$ have the same orientation in S and S'. Show that there is a sequence of face *cw*-flips [1] of length $|C_{in}|$ in which every inner face of C appears exactly once, that allows to reverse the orientation of C and to obtain S' starting from S.

The graph of Schnyder woods (of a 4-connected plane triangulation)

Our goal is to show that flipping 3-colored faces we can obtain all distinct Schnyder woods of $\mathcal{T}$ and, moreover, we want to get a bound on the length of flipping sequences. Observe that performing both *cw*-flips and *ccw*-flips of 3-colored faces one can get flip sequences of infinite length: this does not occur if we restrict ourselves to only *cw*-flips (or to only *ccw*-flips).

7. Let t be a face of $\mathcal{T}$ and consider a sequence of Schnyder woods $\{S_0, S_1, S_2, \ldots\}$ obtained performing *cw*-flips of 3-colored faces. Assume that the face t has been flipped more than once, which means that S_{i+1} is obtained from S_i flipping t and also S_{j+1} is obtained from S_j after flipping t (for some $j > i \geq 0$). Show that after *cw*-flipping t the first time and before *cw*-flipping t the second time there must occur a *cw*-flip of each neighboring face of t.

8. Show that the number of times a given face of $\mathcal{T}$ can be *cw*-flipped is bounded.

[*Hint: use previous question and observe that the faces incident to the outer vertices V_0, V_1 and V_2 cannot be flipped.*]

Given a fixed 4-connected planar triangulation $\mathcal{T}$, let $G_{\mathcal{T}}$ be the graph whose nodes are all Schnyder woods of $\mathcal{T}$ and such that two Schnyder woods S_1 and S_2 are neighbors if and only if one can be obtained from another by a single face flip (either a *cw*-flip or a *ccw*-flip).

9. Show that $\mathcal{T}$ admits a Schnyder wood without *cw*-oriented cycles (and, similarly, a Schnyder wood with no *ccw*-oriented cycles).

10. Show that the graph $G_{\mathcal{T}}$ of the Schnyder woods is connected (given two Schnyder woods S and S', one can go from S to S' with a sequence of face flips).

[*Hint: you may use the fact (**not** to be proved) that the Schnyder wood without cw-oriented (resp. ccw-oriented) triangles is unique.*]

[1] A sequence of *cw*-flips is just a defined by a sequence of 3-colored and *cw*-oriented faces.