Algorithms and Combinatorics of Geometric Graphs (`Geomgraphs`) 2025-2026

TD1 (exercises)
Luca Castelli Aleardi

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Exercise 1 – Euler formula and Pick theorem

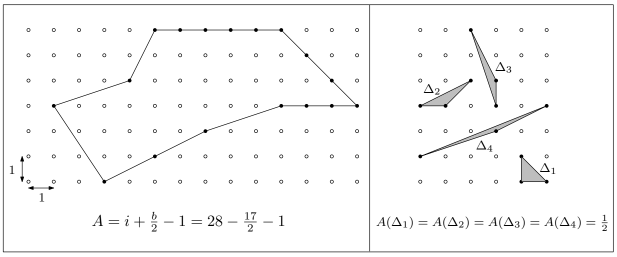

Figure 1: Illustration of the Pick theorem.

Let us consider a simple polygon (without holes or self-intersections) whose vertices are located at integer coordinates on a regular grid. We want to solve the problem of computing its area.

Pick theorem states that it suffices to compute the number of inner vertices and the number of boundary vertices (as illustrated in Fig. 1).

Question 1.1. *Let us consider a simple polygon P having vertices located at integer coordinates on a regular grid, and denote by i the number of grid points that are lying in the interior of P, and by b the number of points that are on the boundary of P. Show that the are of P is given by:*

$$A = i + \frac{b}{2} - 1$$

Hint. *You can make use of the following property: given a triangle whose three vertices are grid points and not containing any other grid point (in its interior or on its boundary) then its area is $\frac{1}{2}$ (for such a triangle we have $i = 0$ et $b = 3$)*

Exercise 2 – 3-connectedness of plane triangulations

In a given planar embedding of a graph G, we define a *separating triangle* as a cycle of length 3 which is not the boundary of a (triangle) face in the embedding of G.

Question 2.1. *Let T be a plane triangulation with $n \geq 4$ vertices. Let E_s be the set of the edges of T which are not belonging to a separating triangle of T. Show that E_s contains at least n edges.*

Question 2.2. *Show that a planar triangulation with at least four vertices is 3-connected.*

Exercise 3 – Euler formula: alternative proof

Question 3.1. *Provide a proof of Euler formula for planar graphs involving an induction on the faces.*

Hint: *first show that in a planar triangulation one can remove the faces one by one in such a way the resulting graphs are planar quasi-triangulations (e.g. the outer boundary is simple - without self-intersections - and all inner faces are triangles).*

Exercise 4 – Twelve Pentagon theorem

Figure 2: Two examples of genus 0 polyhedra satisfying the twelve pentagon theorem: the fullerene $C60$ and a common soccer ball

In this section we consider 3D shapes whose polyhedral structure consists only of pentagonal and hexagonal faces (a few examples are illustrated in Fig. 2).

Question 4.1. *Let us consider a genus* 0 *polyhedron (e.g. homeomorphic to a sphere), where every face is a pentagon or a hexagon, and such that every vertex has degree* 3*. Show that there must be exactly twelve pentagonal faces.*

Exercise 5 – Euler formula and spherical geometry

A *spherical triangle* T with vertices $\{A, B, C\}$ is defined by the portion of a sphere bounded by three geodesic arcs (A, B), (B, C) et (C, A). Similarly, one can define a *spherical polygon* whose sides are geodesic arcs. An important property of spherical geometry is that the sum of the interior angles of a spherical polygon depends on its area, as illustrated by Fig. 3 and expressed by Girard theorem (recall that in the Euclidean plane, the sum of interior angles only depends on the number of sides: in a triangle the sum of all its three angles is π).

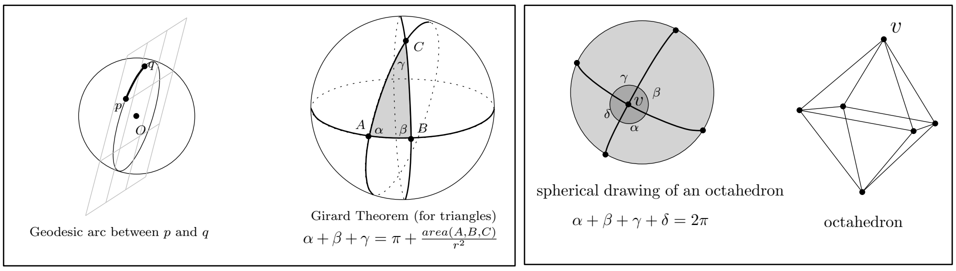

Figure 3: (left) Sum of interior angles for spherical polygons. (right) spherical drawing of a convex polyhedral surface.

Proposition 1 (Girard)**.** *Given a sphere having radius* r *and a spherical polygon* P *with* k *interior angles* $\{\alpha_j\}_{j\leq k}$*, the sums of angles is given by:*

$$\sum_{j=1}^{k} \alpha_j = (k-2)\pi + \frac{area(P)}{r^2}$$

Question 5.1. *We want to provide a geometric proof of Euler formula: let us consider a convex polyhedron* M *(which corresponds to a genus* 0 *closed mesh, whose vertices are in convex position), with* n *vertices,* e *edges and* f *faces. By using a* central projection *(from the center of the sphere) we can obtain a* spherical drawing *of* M *on the sphere such that all the faces are mapped to non-overlapping spherical polygons (edges are drawn as non-crossing geodesic arcs);*

1. *using Girard's theorem, compute the overall sum of the interior angles of all spherical polygons;*
2. *give a proof of Euler formula: recall that given a vertex* v *of a graph drawn on the sphere, the sum of all angles around* v *is* 2π*, and that the total area of a sphere of radius* r *is* $4\pi r^2$*.*