MPRI 2-38-1: Algorithms and combinatorics for geometric graphs

Lecture 8

Drawing graphs embedded on surfaces

november 13, 2024

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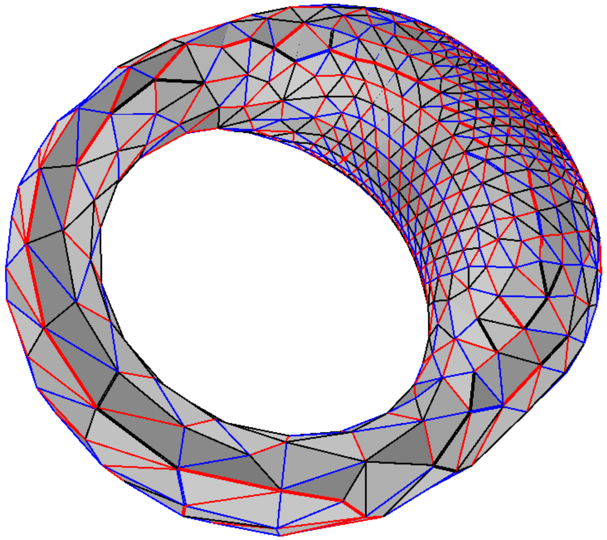
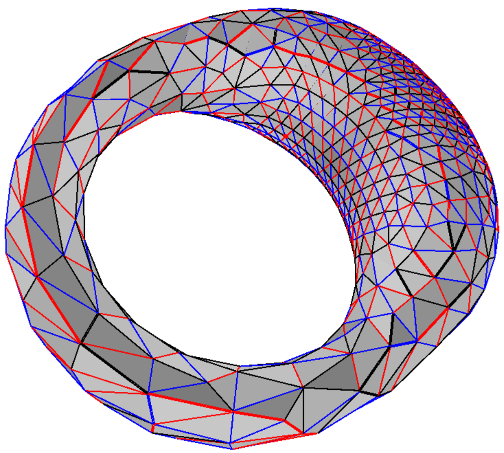
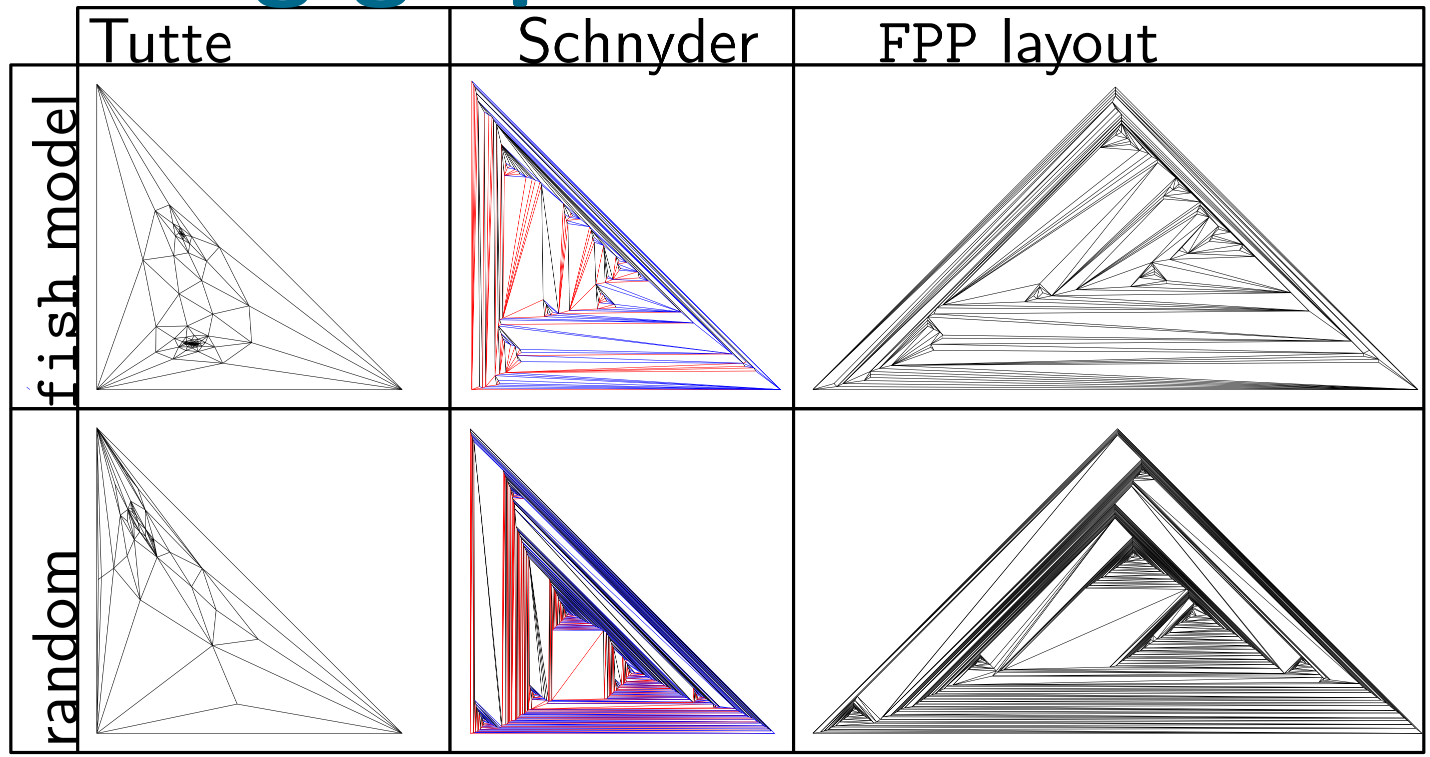


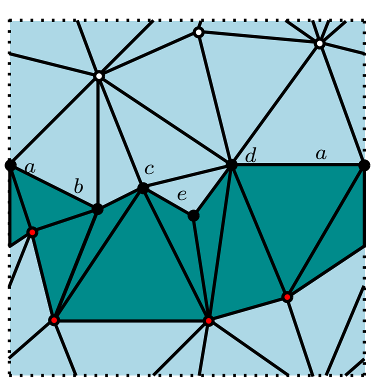
Goal: drawing graphs on surfaces

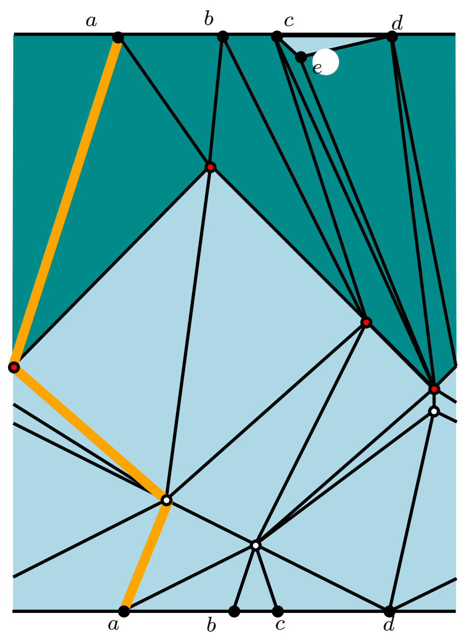
Schnyder woods and canonical orderings for higher genus surfaces



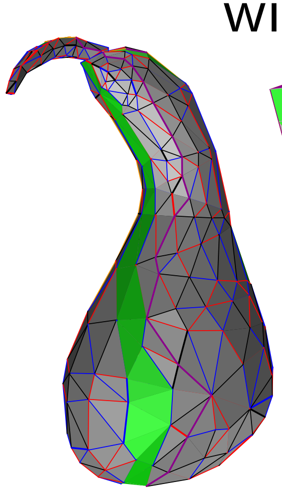


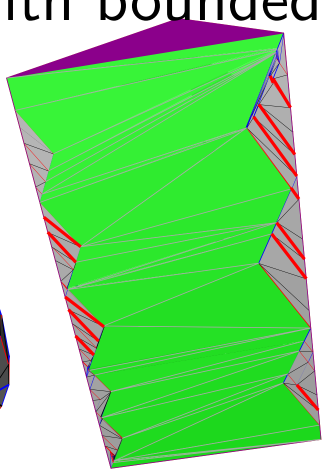
periodic toroidal drawings

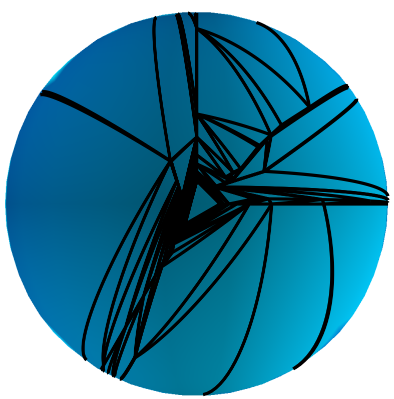




Spherical drawings with bounded resolution



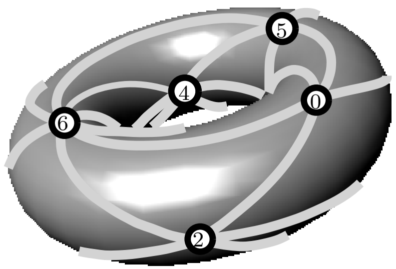




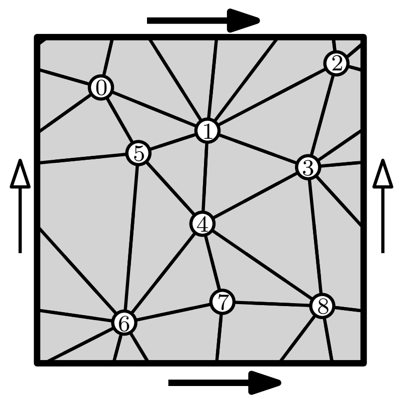
Graphs on surfaces

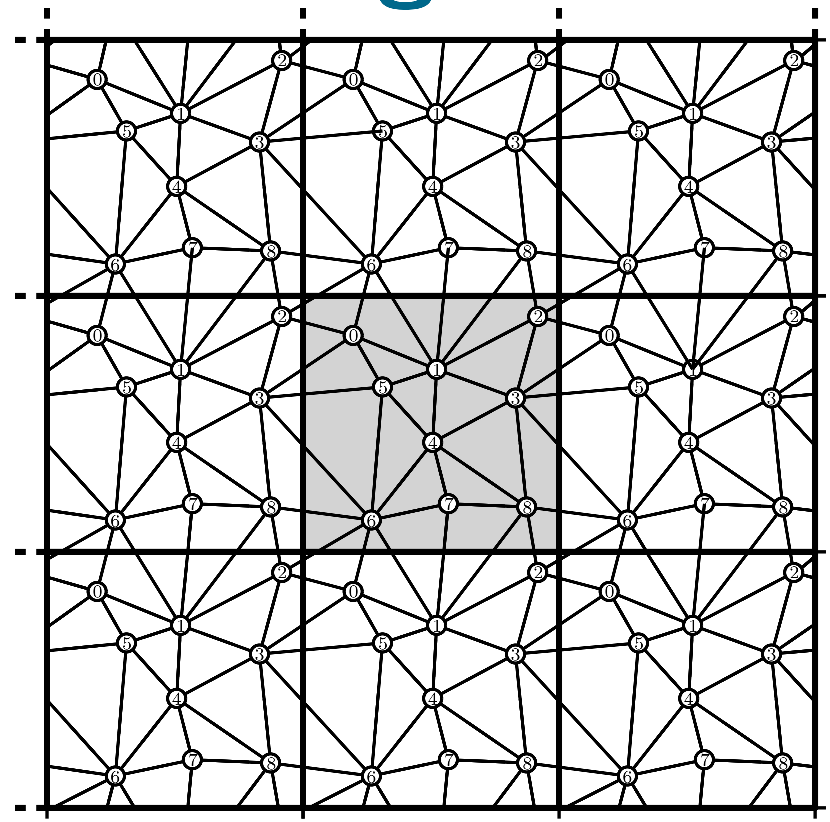
(some definitions and notations)

Periodic (planar) drawings



flat torus

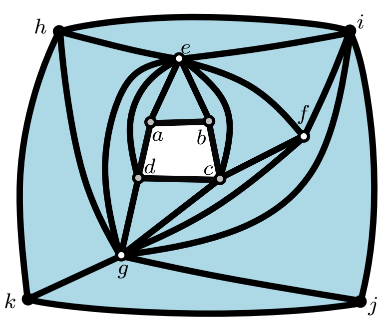


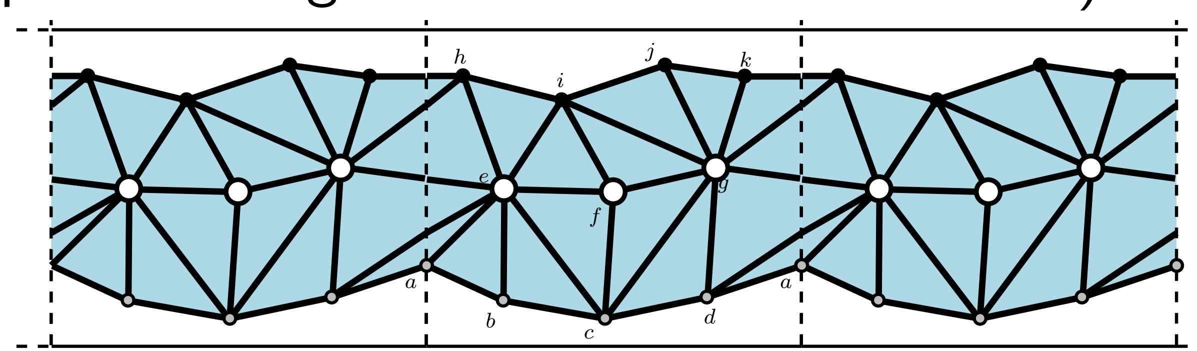


G: (simple) toroidal triangulation

G^∞ (infinite graph) universal cover

G: cylindric triangulation (planar triangulation with two boundaries)

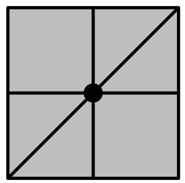


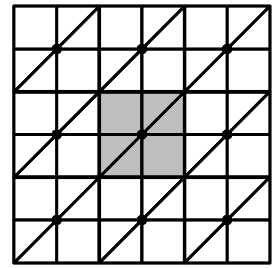


annular representation of G

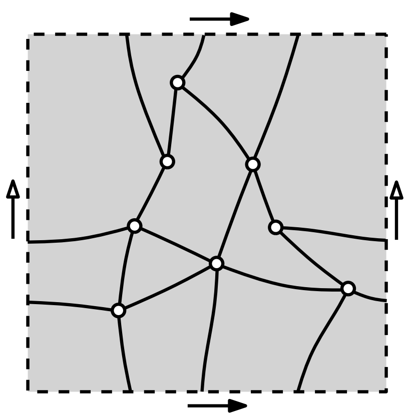
x-periodic drawing of G

Simple and 3-connected graphs on surfaces

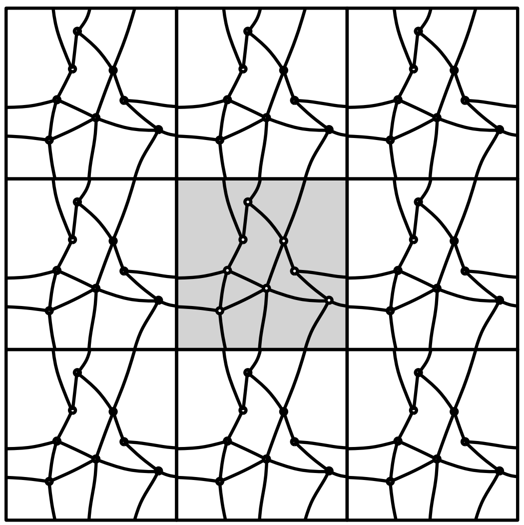




G: essentially simple graph

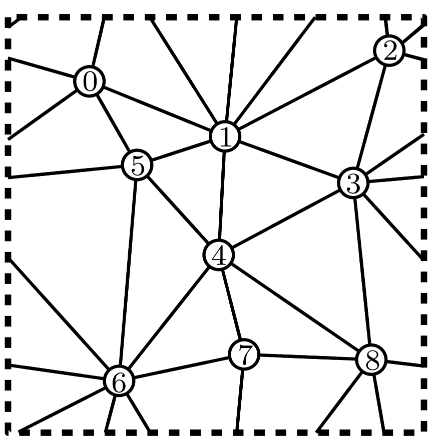


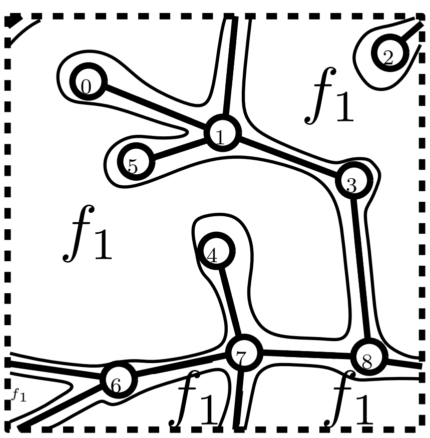
G: essentially 3-connected graph



G^∞ is 3-connected (in the universal cover)

G





G' (its endowed embedding) has a unique face f_1

$\mathcal{S} \setminus G'$ is homeomorphic to a topological disk

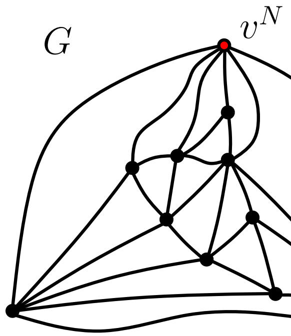
$G' :=$ cut-graph of G (in this example S is a sub-graph spanning all vertices)

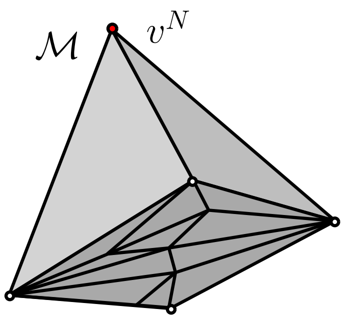
Drawing graphs on surfaces

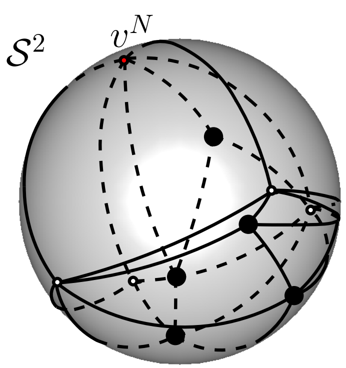
(periodic straight line drawings)

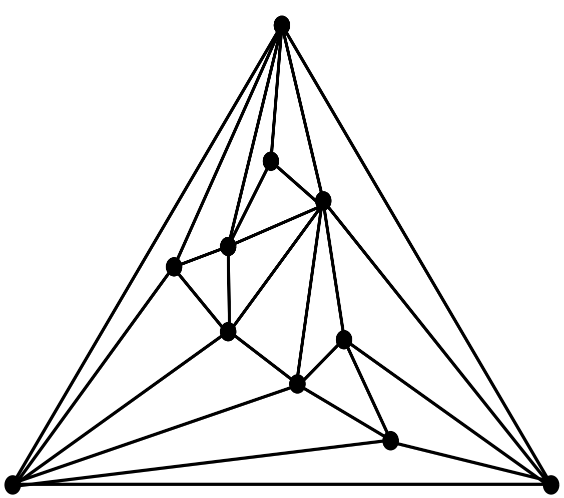
Drawing higher genus graphs

$g = 0$



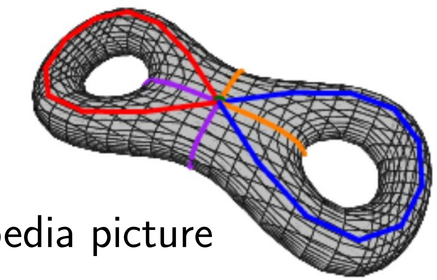




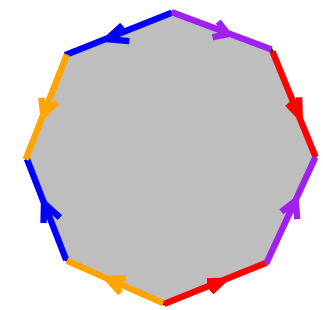


Let us try planarize the graph

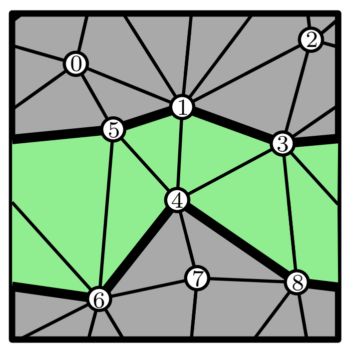
1- compute (canonical) polygonal schemes

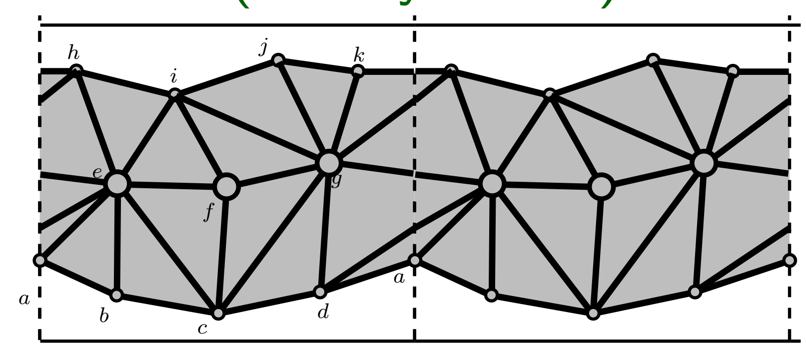


Wikipedia picture

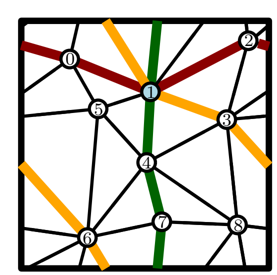


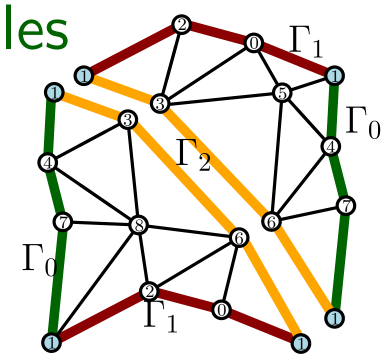
2- compute a tambourine (two cylinders)





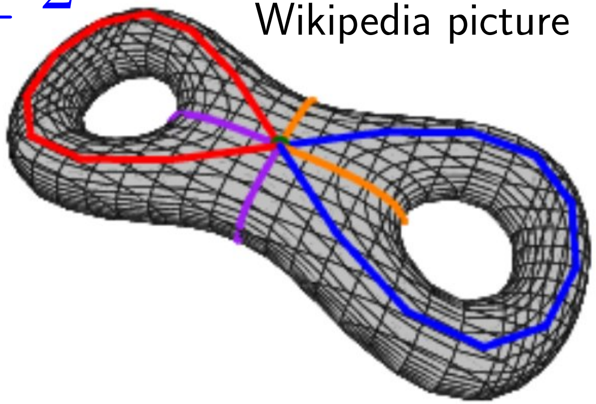
3- compute 3 non homotopic cycles





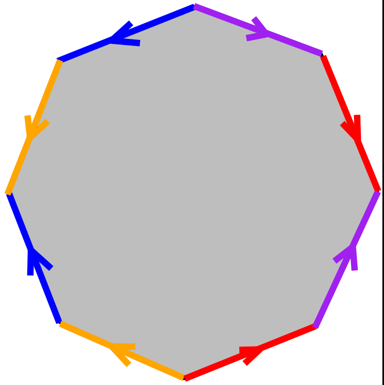
Drawing higher genus graphs

$g \geq 2$



Wikipedia picture

Polygonal scheme



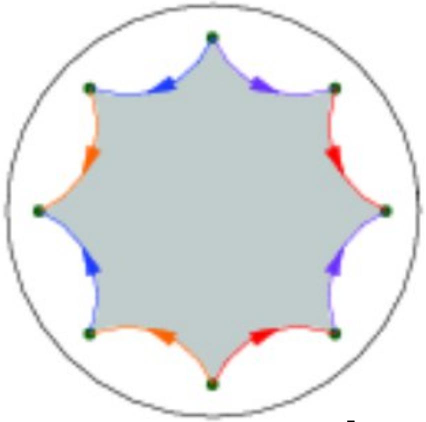
drawing in polynomial area

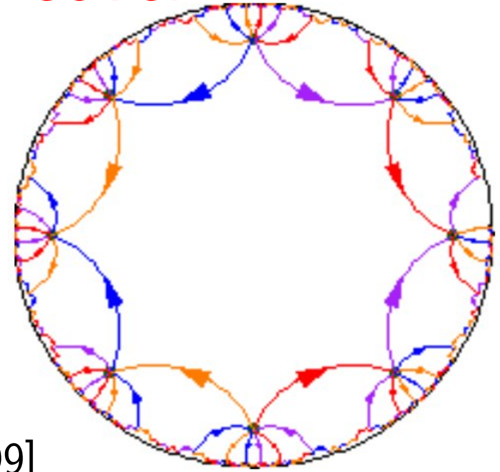
[Duncan, Goodrich, Kobourov, GD'09]

[Chambers, Eppstein, Goodrich, Löffler, GD'10]

(Palais de la Découverte,Fête de la Science, October 2013)

Universal cover





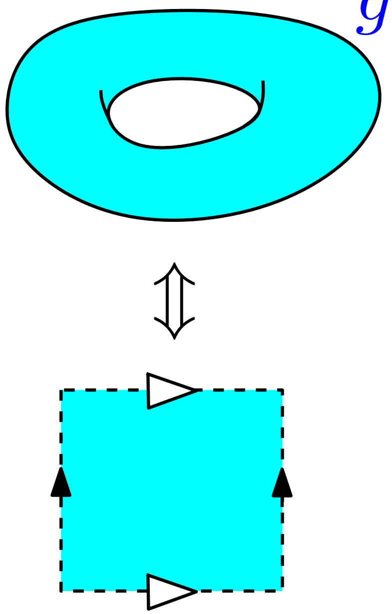
[Mohar'99]

periodic drawing

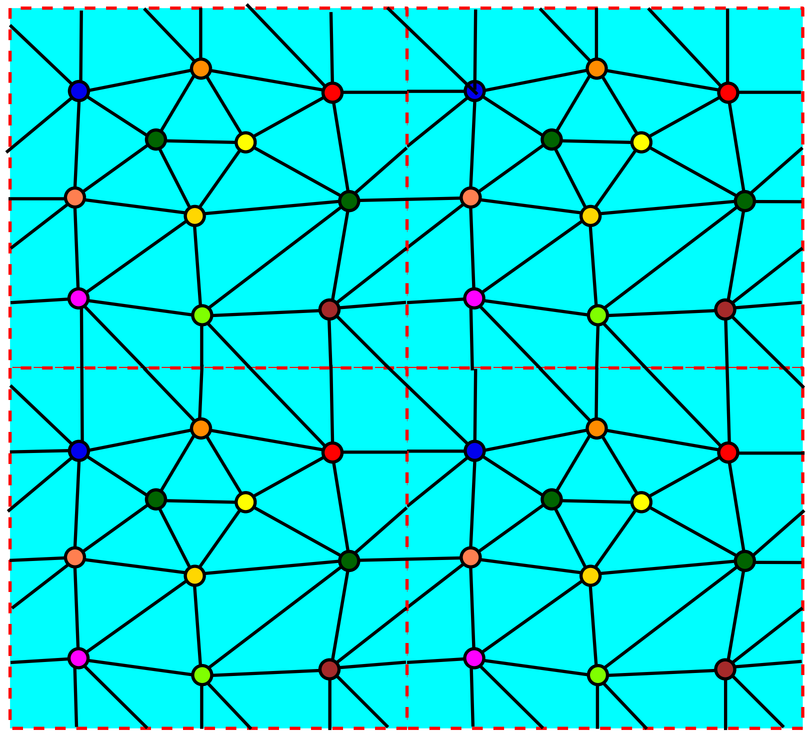
out of circle packing

Drawing toroidal graphs

On the torus



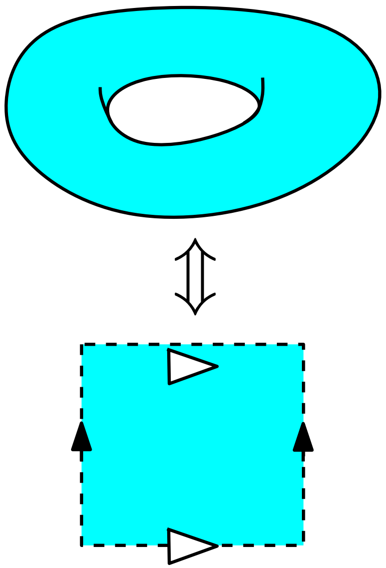
$g = 1$

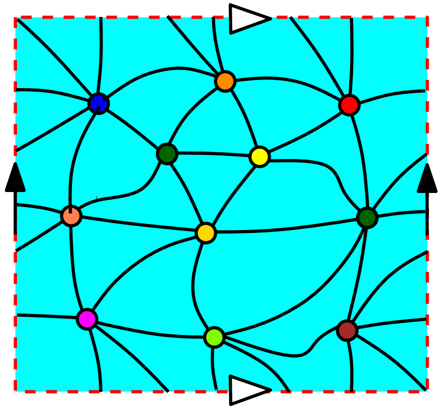


(Palais de la Découverte, Fête de la Science, October 2013)

Periodic straight-line drawings

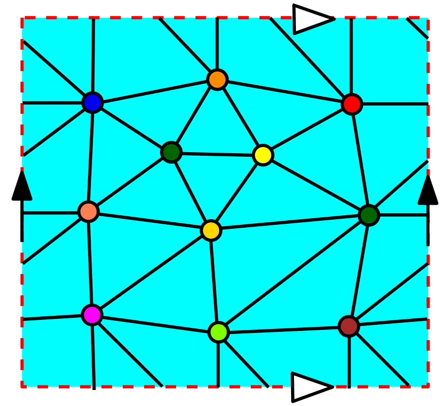
On the torus





drawing on the flat torus

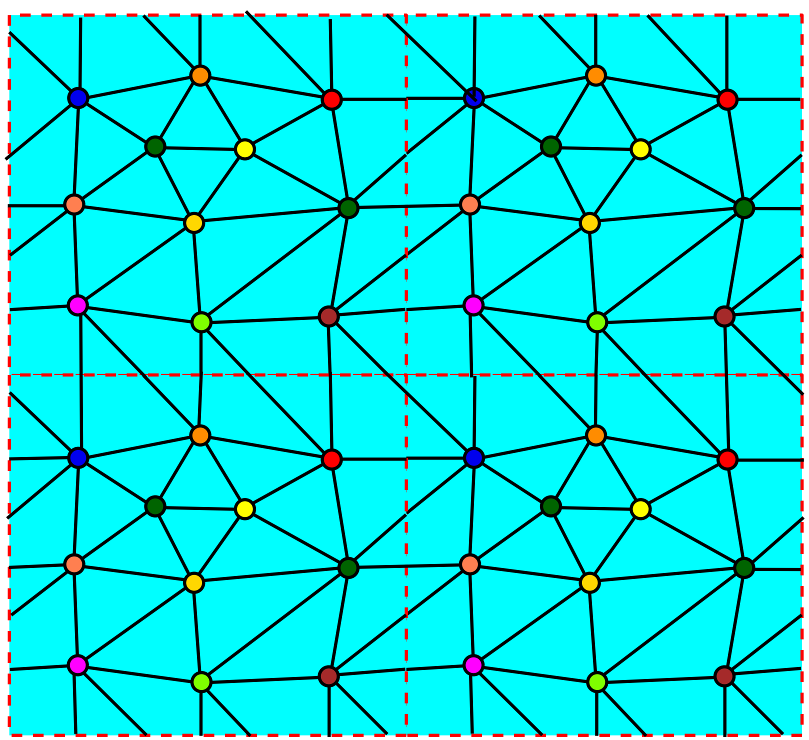
$\Rightarrow$

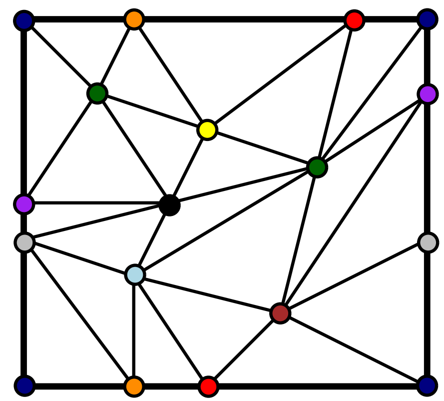


straight-line drawing
x-periodic and
y-periodic drawing

[Castelli Devillers Fusy, GD'12]
$O(n \times n^{\frac{3}{2}})$ **grid**

[Goncalves Lévêque, DCG]
$O(n^2 \times n^2)$ **grid**



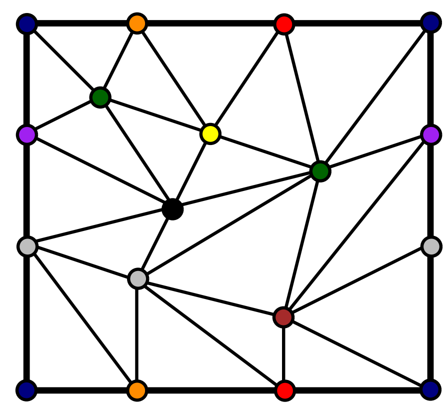


straight-line frame
not x-periodic
not y-periodic

[Chambers et al., GD'10]

[Duncan et al., GD'09]

$O(n \times n^2)$ **grid**



straight-line frame
x-periodic and
y-periodic drawing

[Castelli Fusy Kostrygin, Latin'14]

Toroidal drawing I: the shift algorithm on the torus

A shift-algorithm for the torus

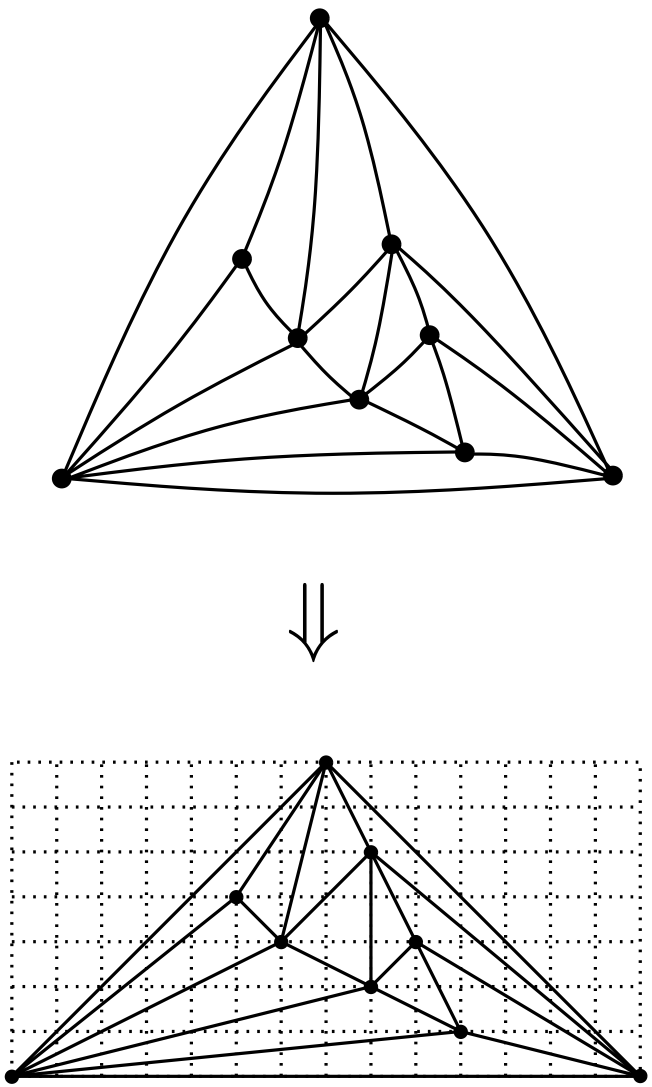
1. Recall algorithm of

[De Fraysseix et al'89]

2. Extend to the cylinder

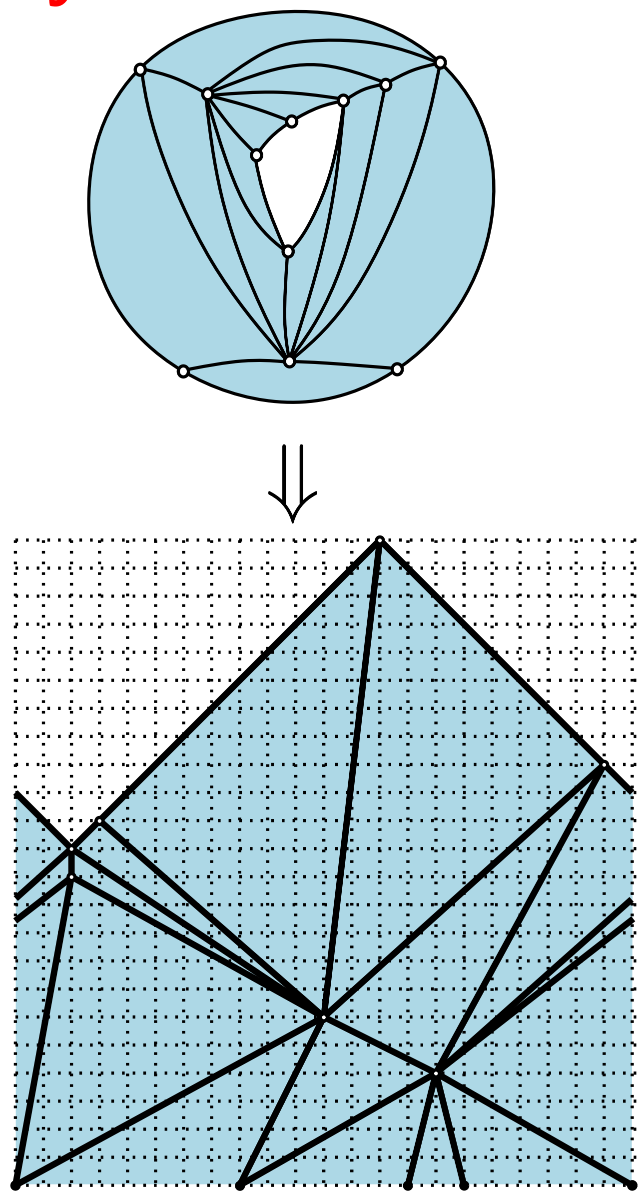
3. Get toroidal drawings

Plane



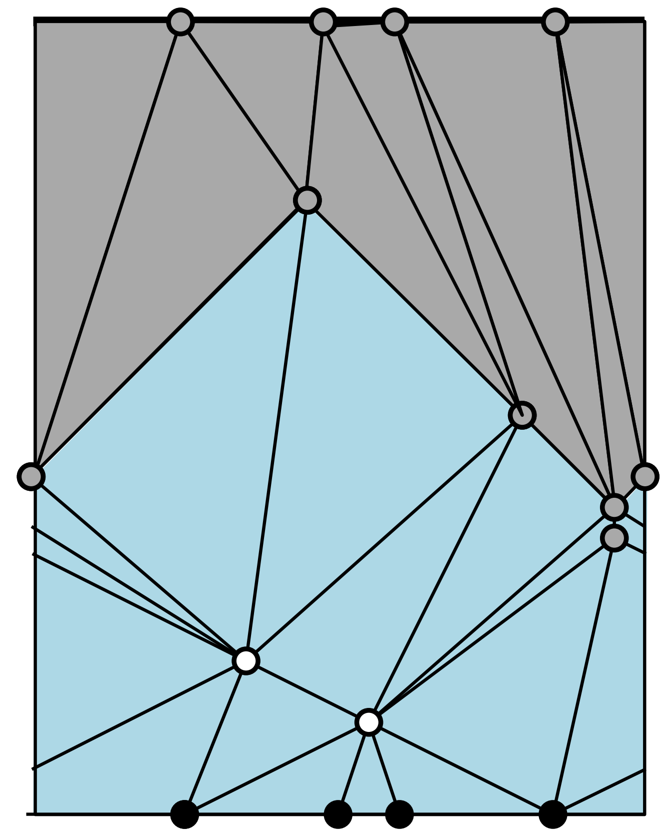
Grid $2n-4 \times n-2$

Cylinder



Grid $\leq 2n \times n(2d+1)$

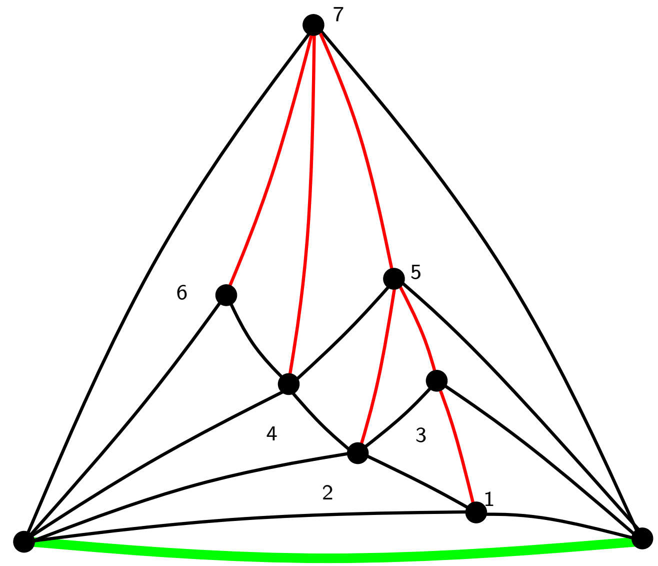
Torus

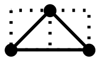


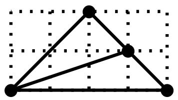
Grid$\leq 2n \times (1+n(2c+1))$

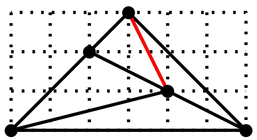
Incremental drawing algorithm

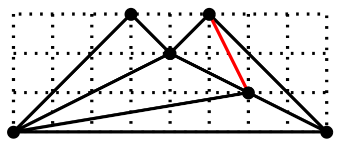
[de Fraysseix, Pollack, Pach'89]

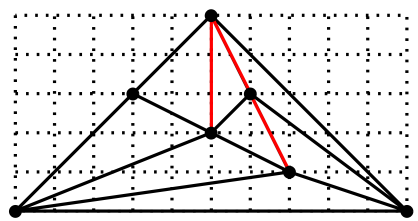


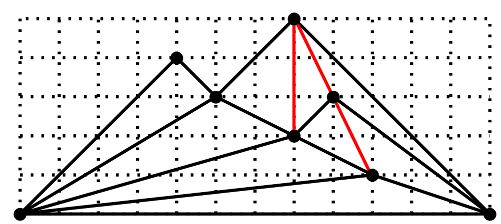
1. 

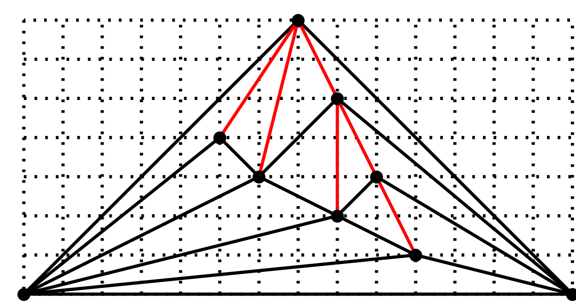
2. 

3. 

4. 

5. 

6. 

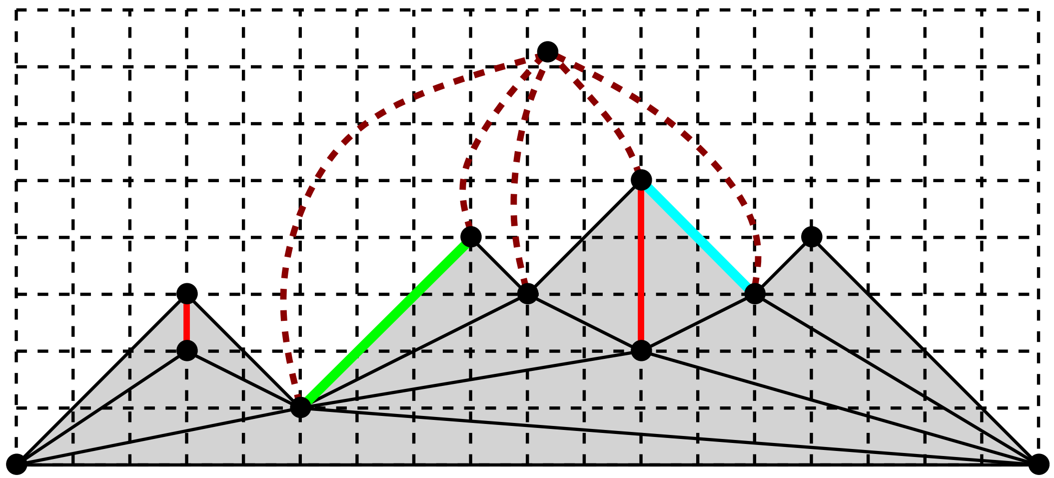
7. 

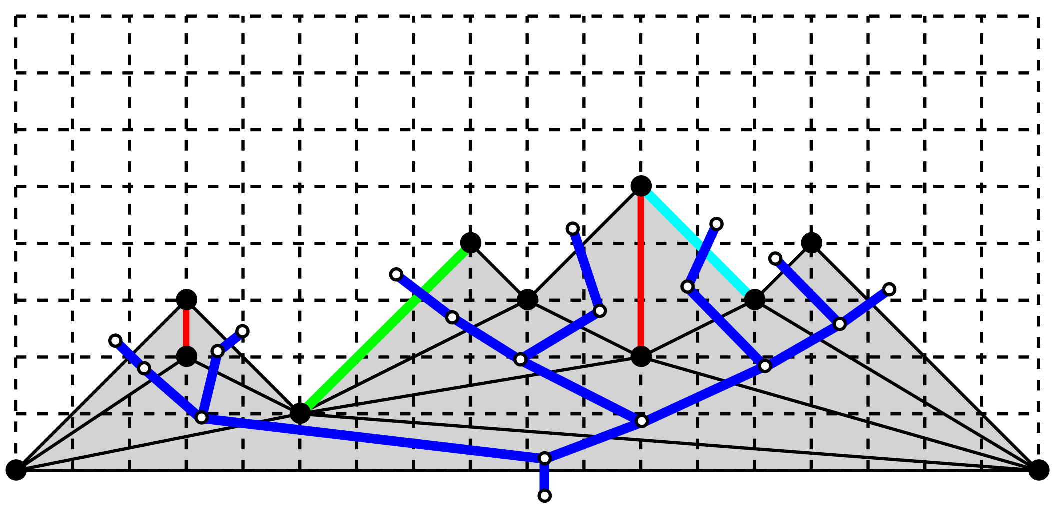
Grid size of G_k: $2k \times k$

Reformulation of the shift-step

At each step: insert two vertical strips of width 1 using the dual tree

G_{k-1}





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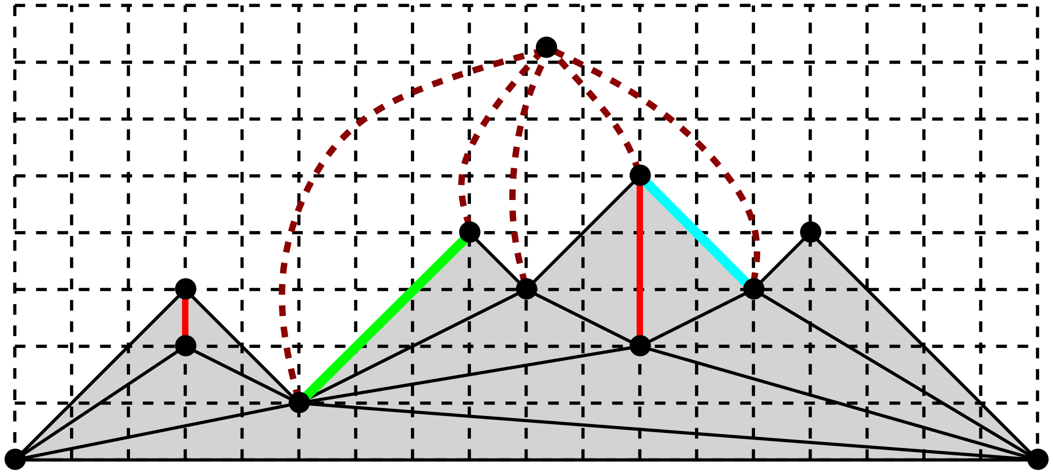

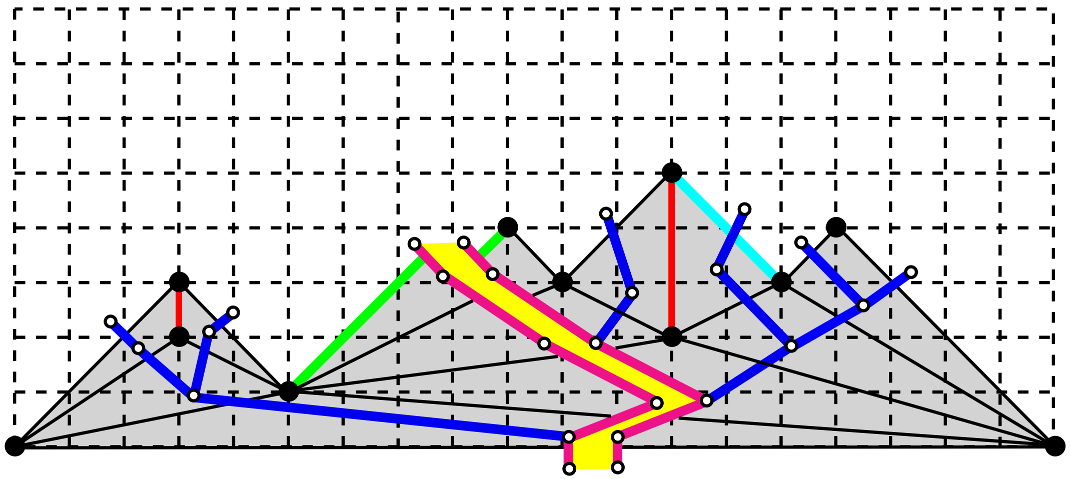

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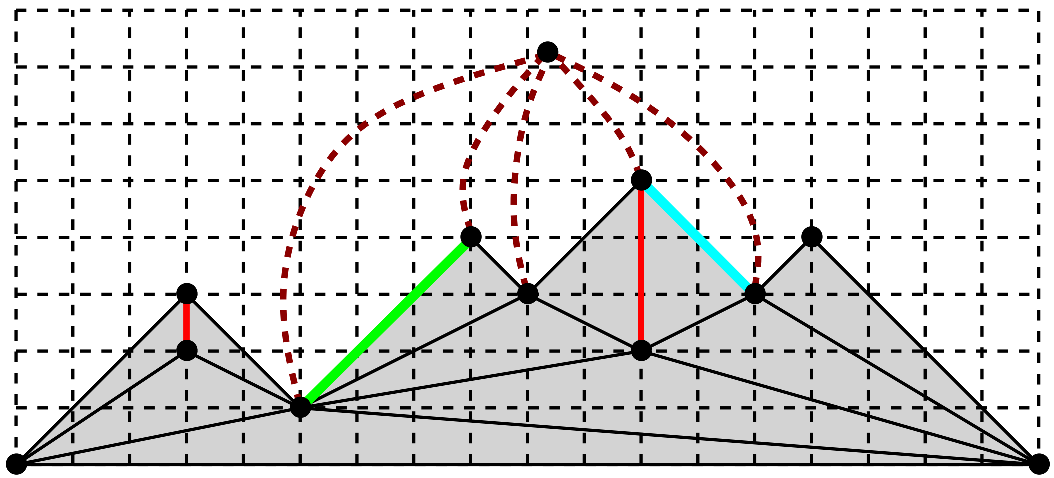

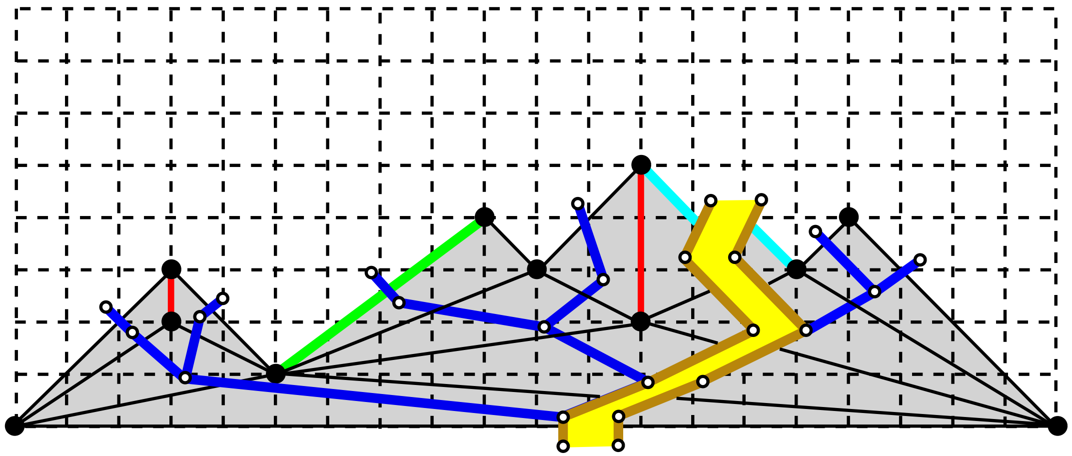

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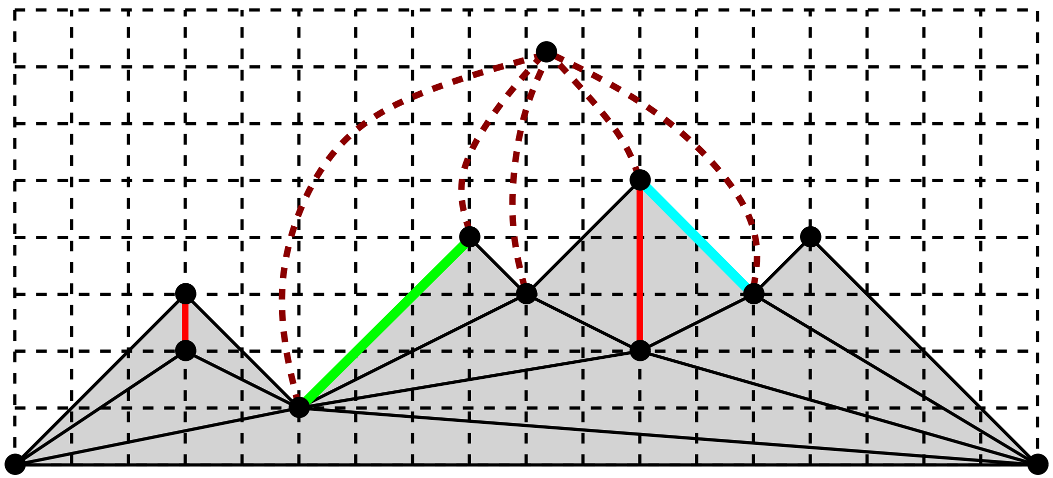
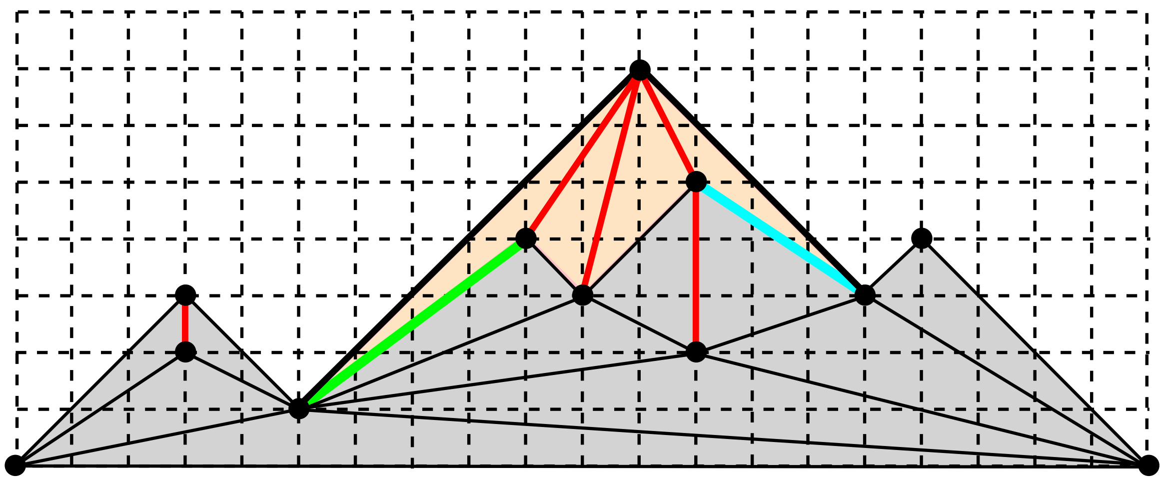
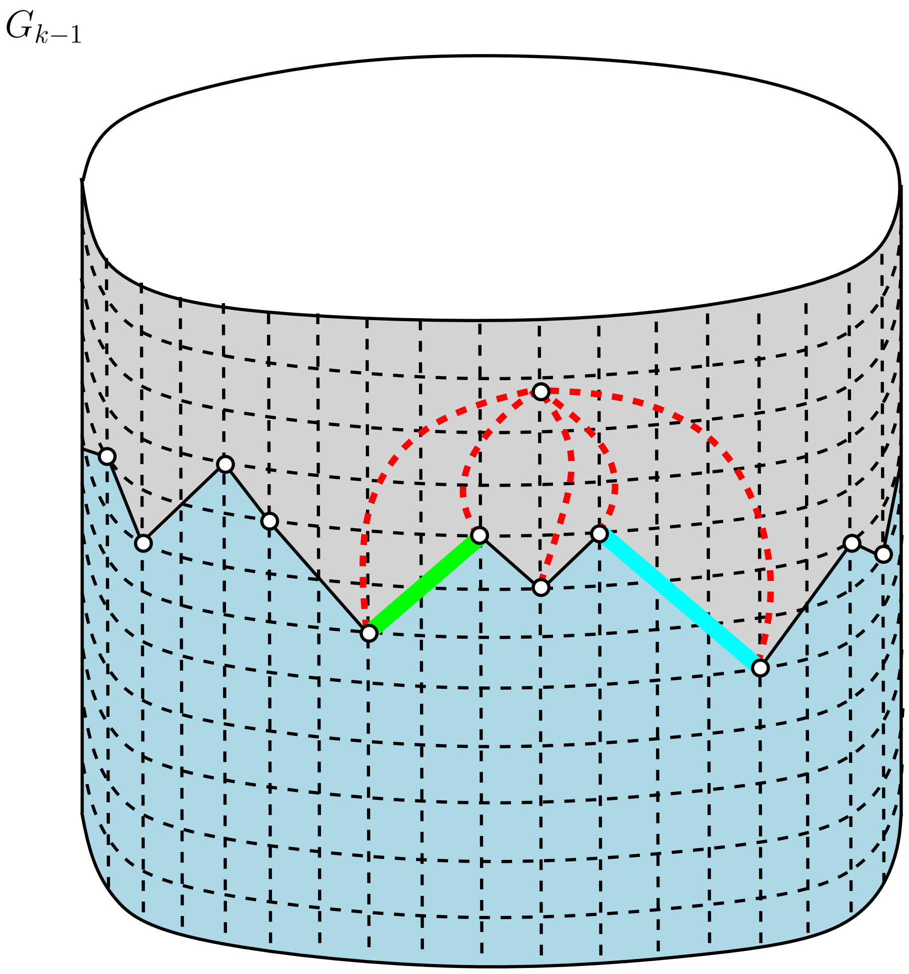


G_k

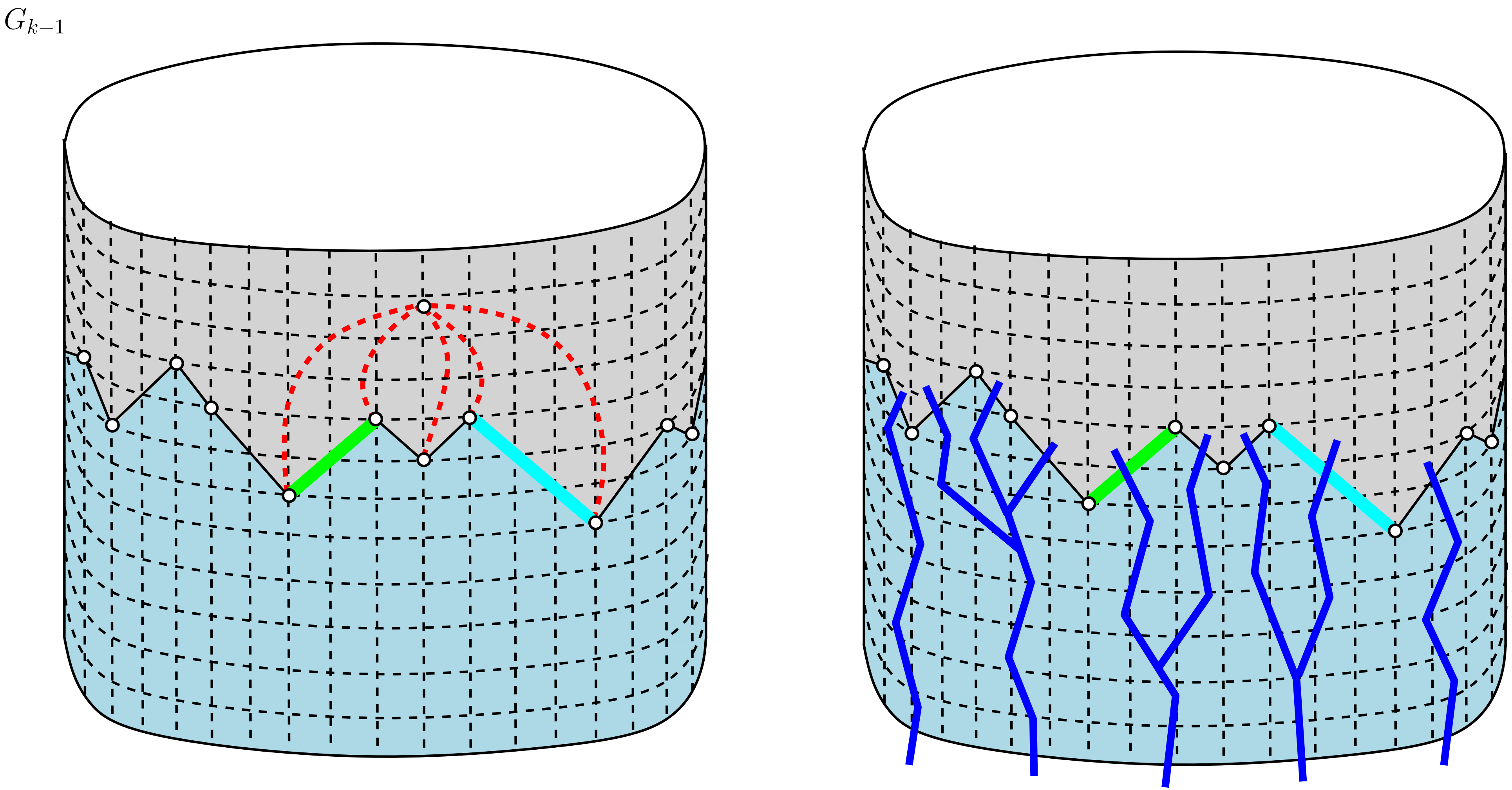


Extension to the cylinder: drawing algorithm



At each step: - insert two vertical strips of width 1
- insert the next vertex as in the planar case

Extension to the cylinder: drawing algorithm



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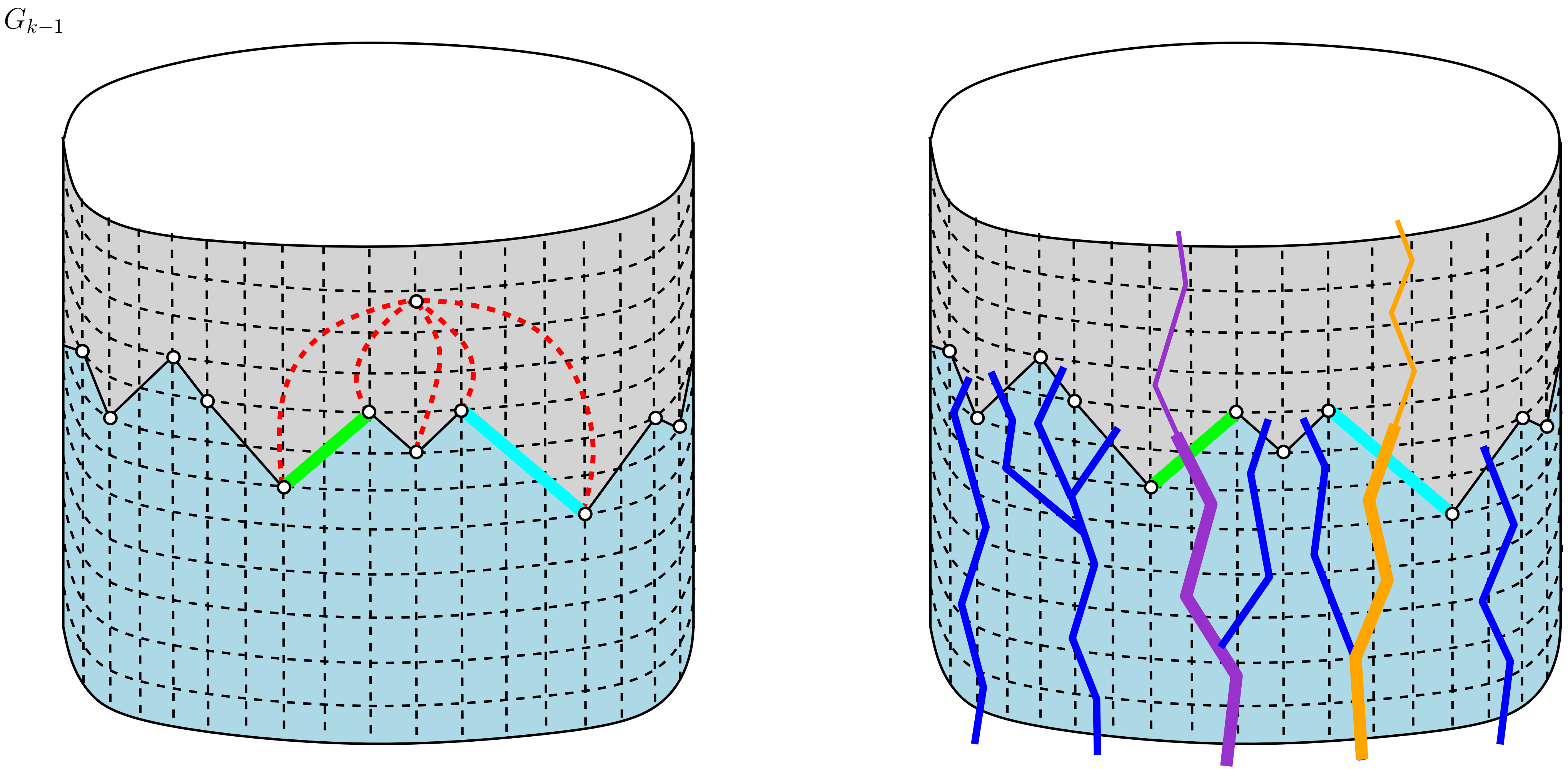
Extension to the cylinder: drawing algorithm



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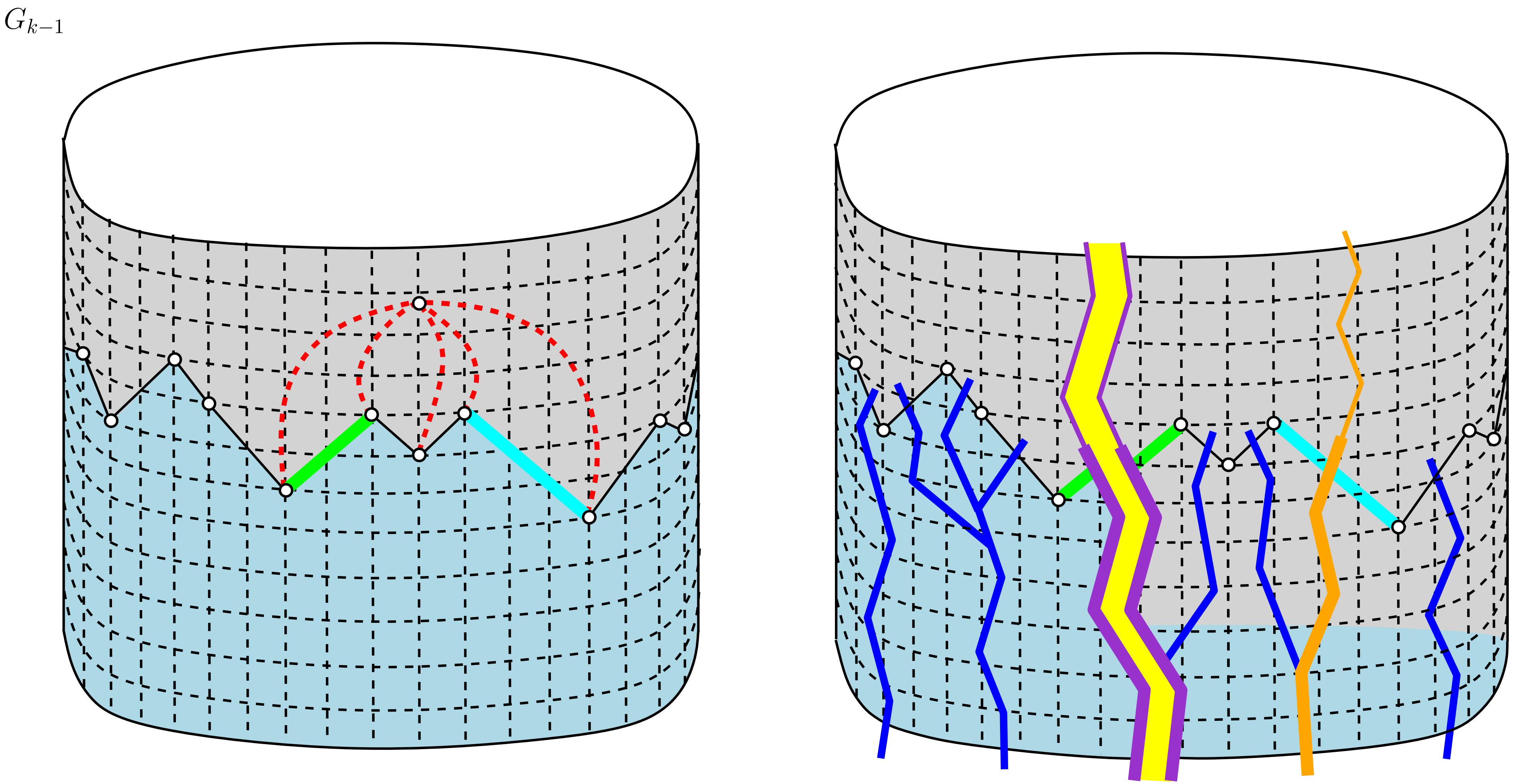
Extension to the cylinder: drawing algorithm



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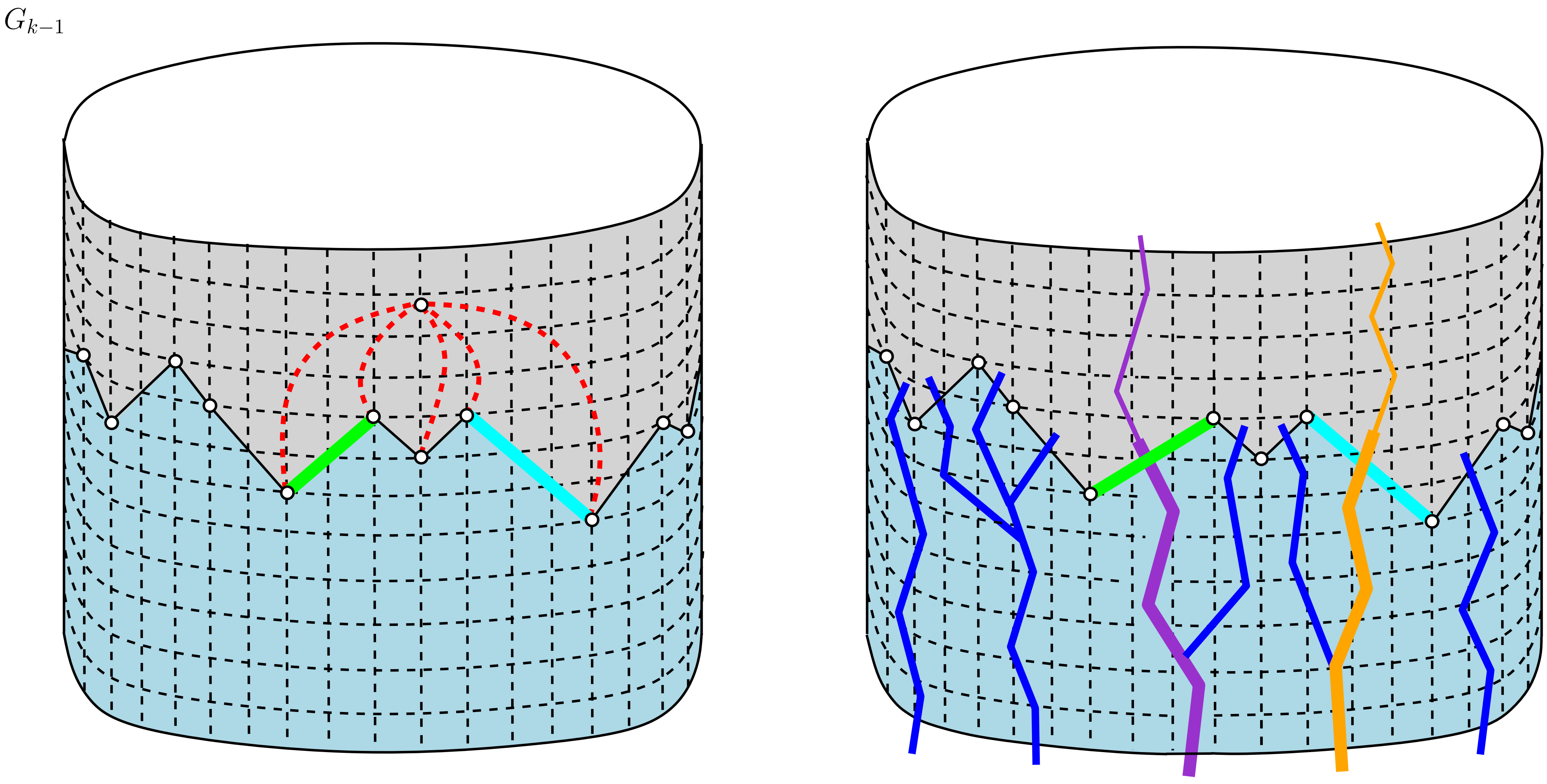
Extension to the cylinder: drawing algorithm



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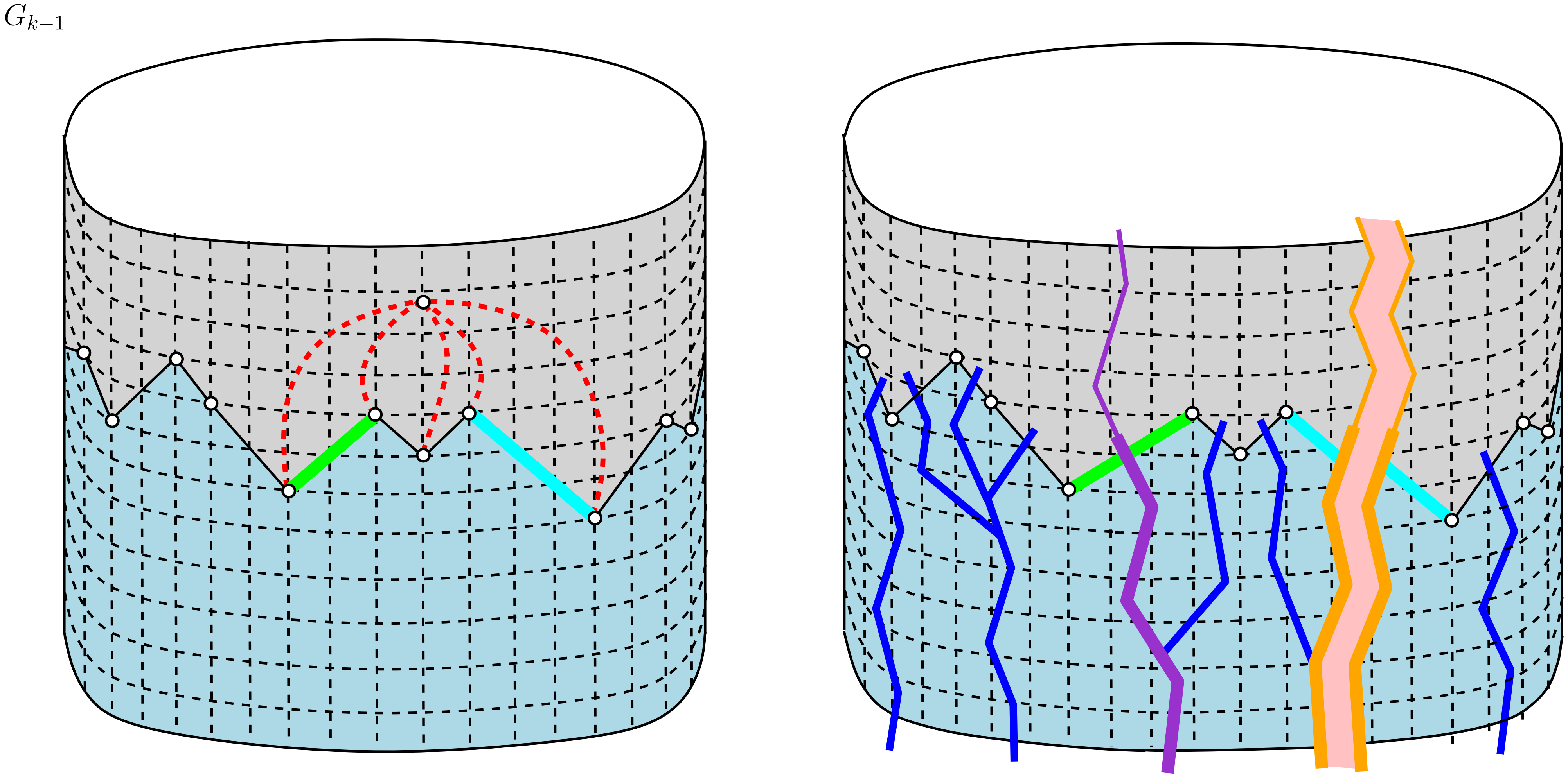

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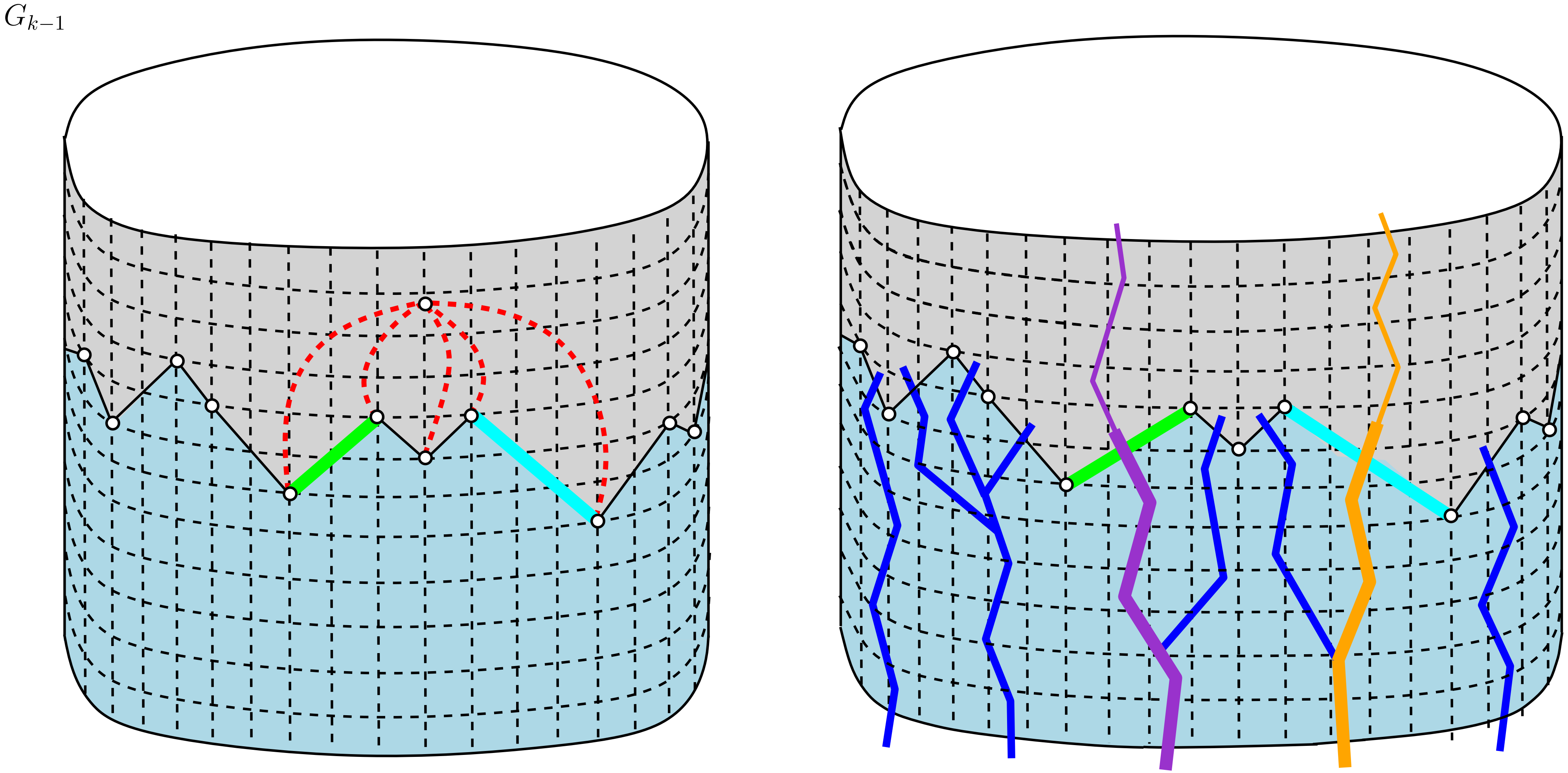
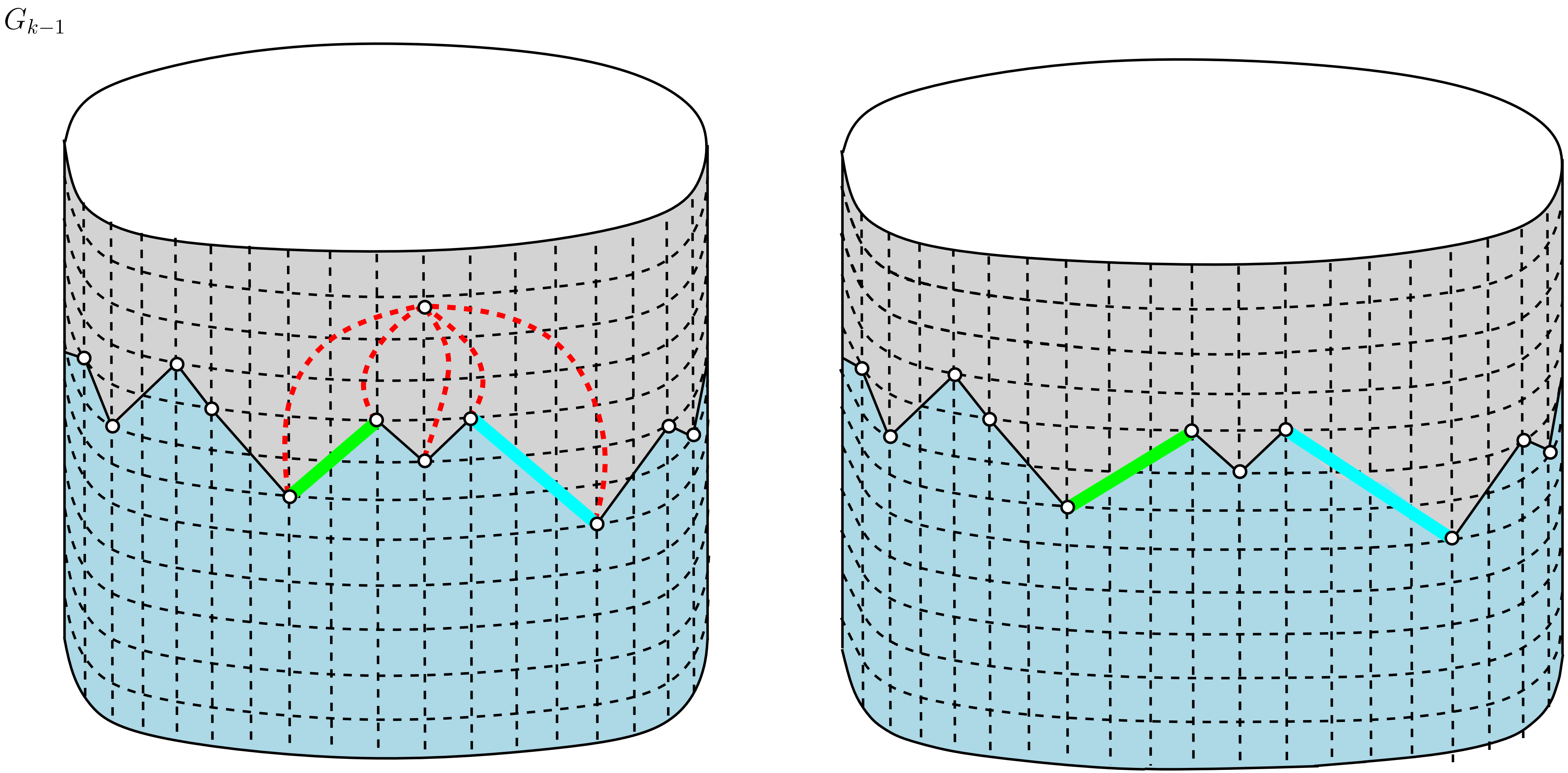
- insert the next vertex as in the planar case

Extension to the cylinder: drawing algorithm



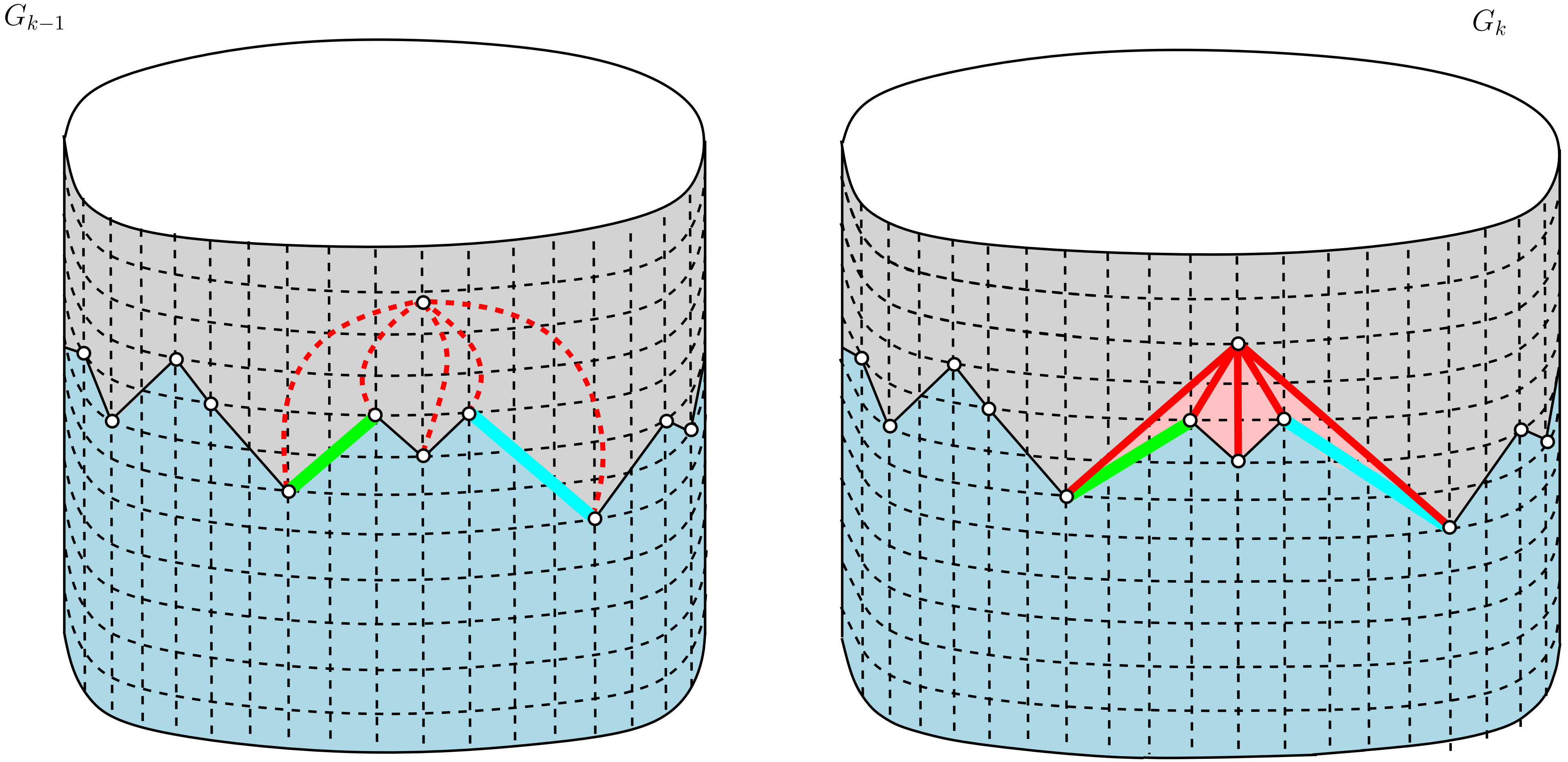
At each step: - insert two vertical strips of width 1
- insert the next vertex as in the planar case

Extension to the cylinder: drawing algorithm



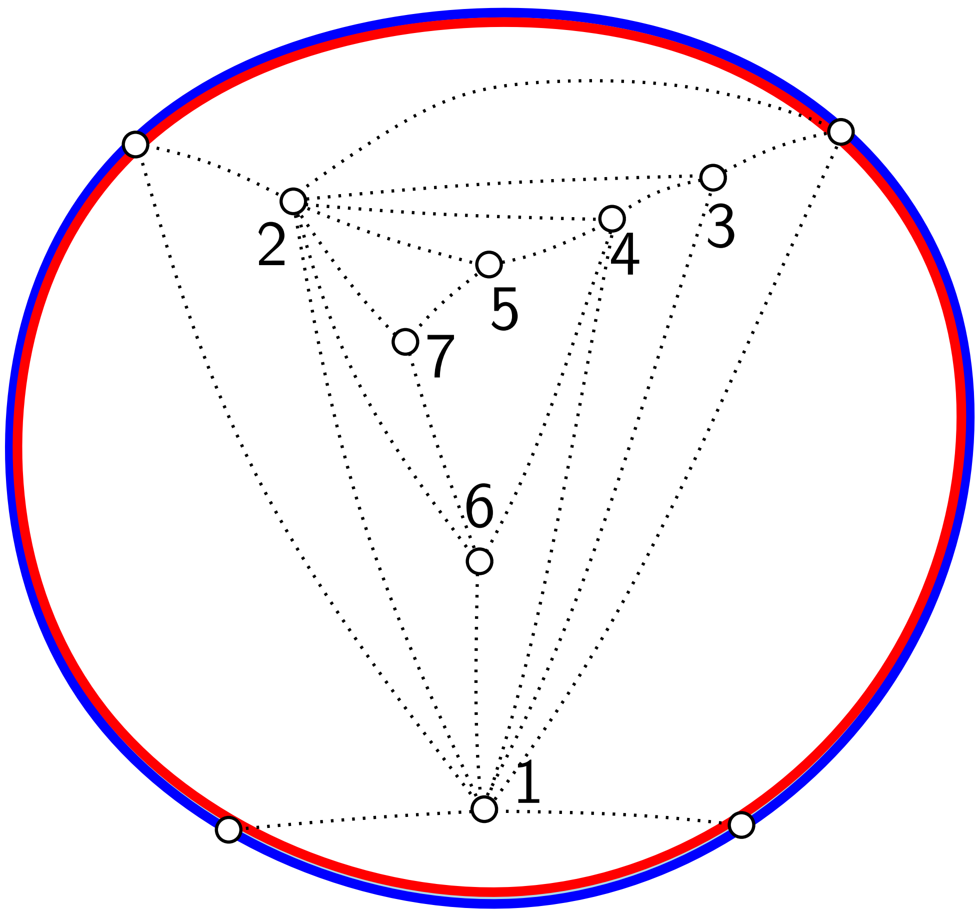
At each step: - insert two vertical strips of width 1
- insert the next vertex as in the planar case

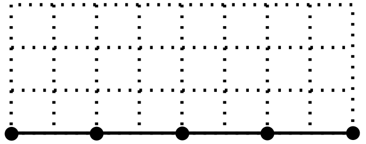
Extension to the cylinder: drawing algorithm



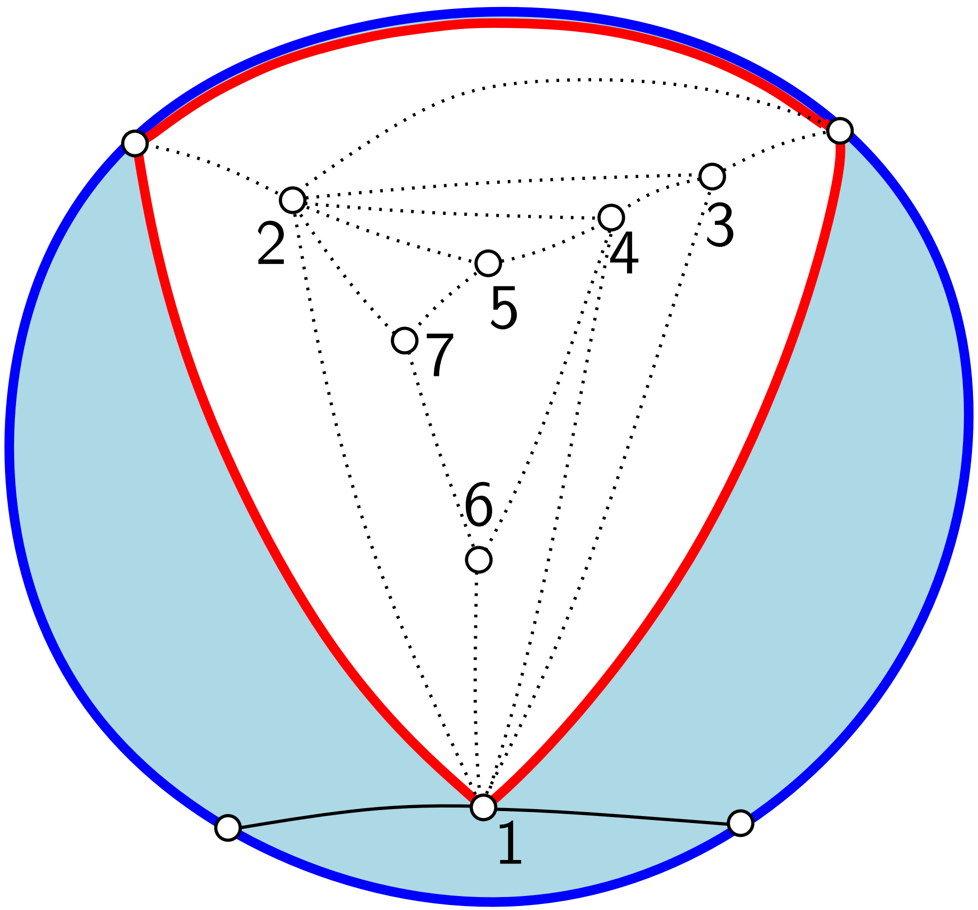
At each step: - insert two vertical strips of width 1
- insert the next vertex as in the planar case

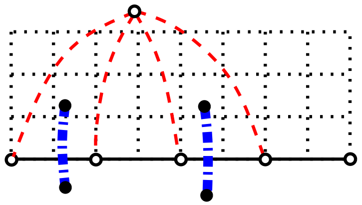
Extension to the cylinder: drawing algorithm

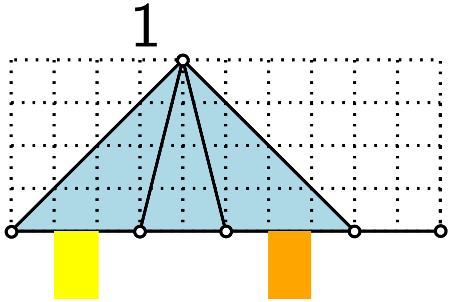




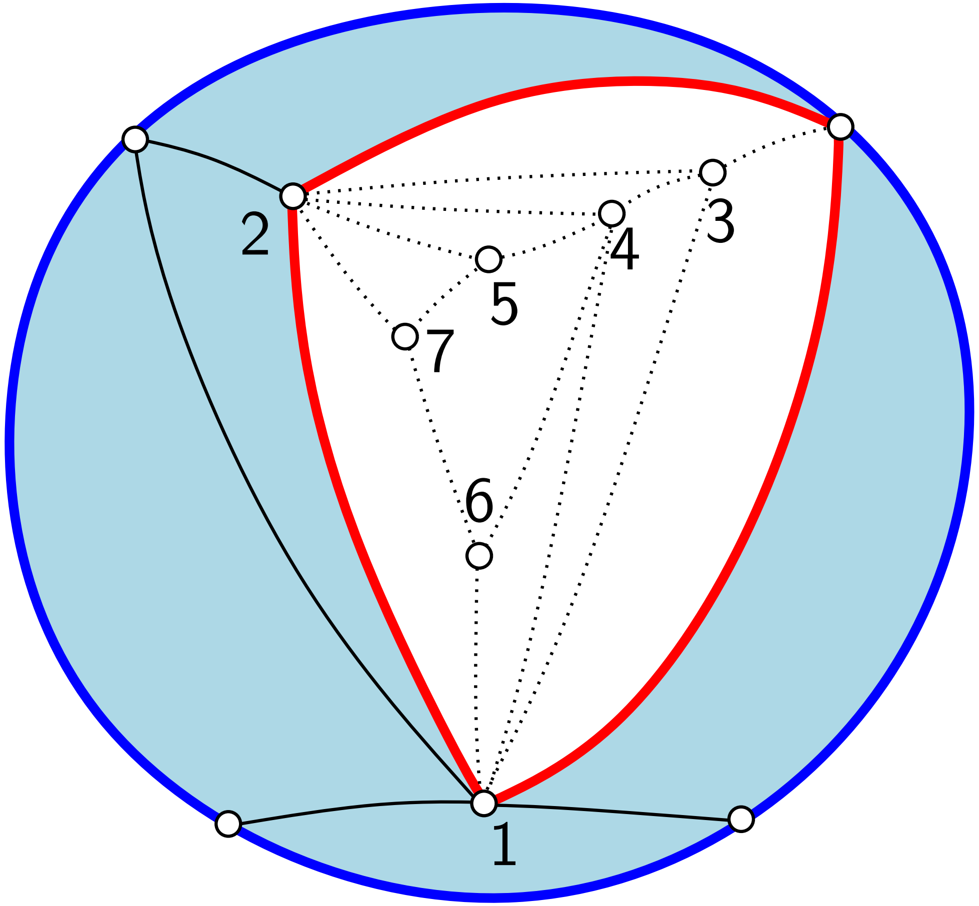
Extension to the cylinder: drawing algorithm

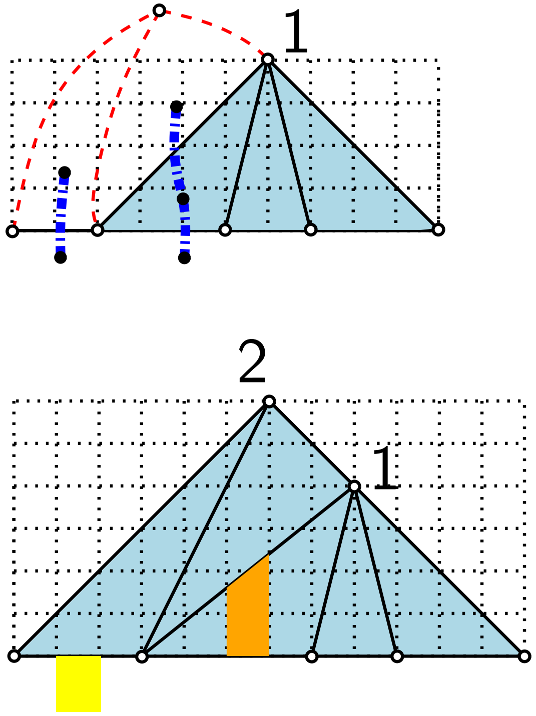




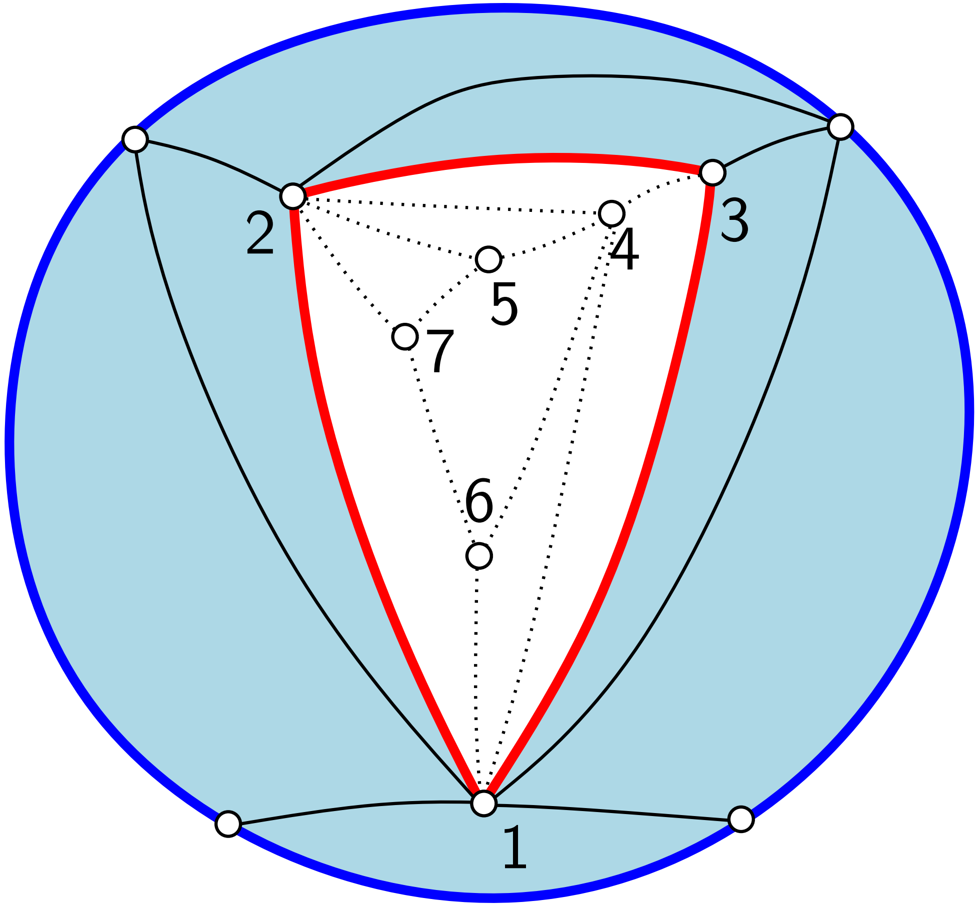


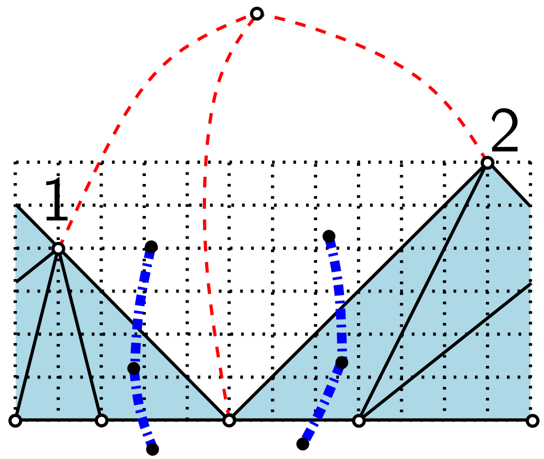
Extension to the cylinder: drawing algorithm

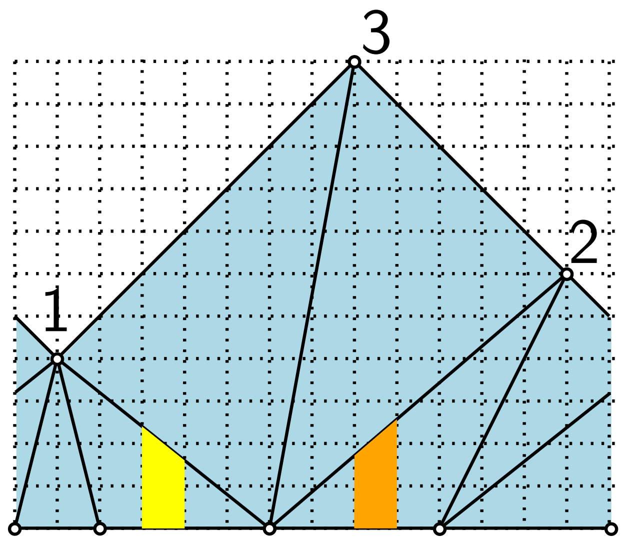




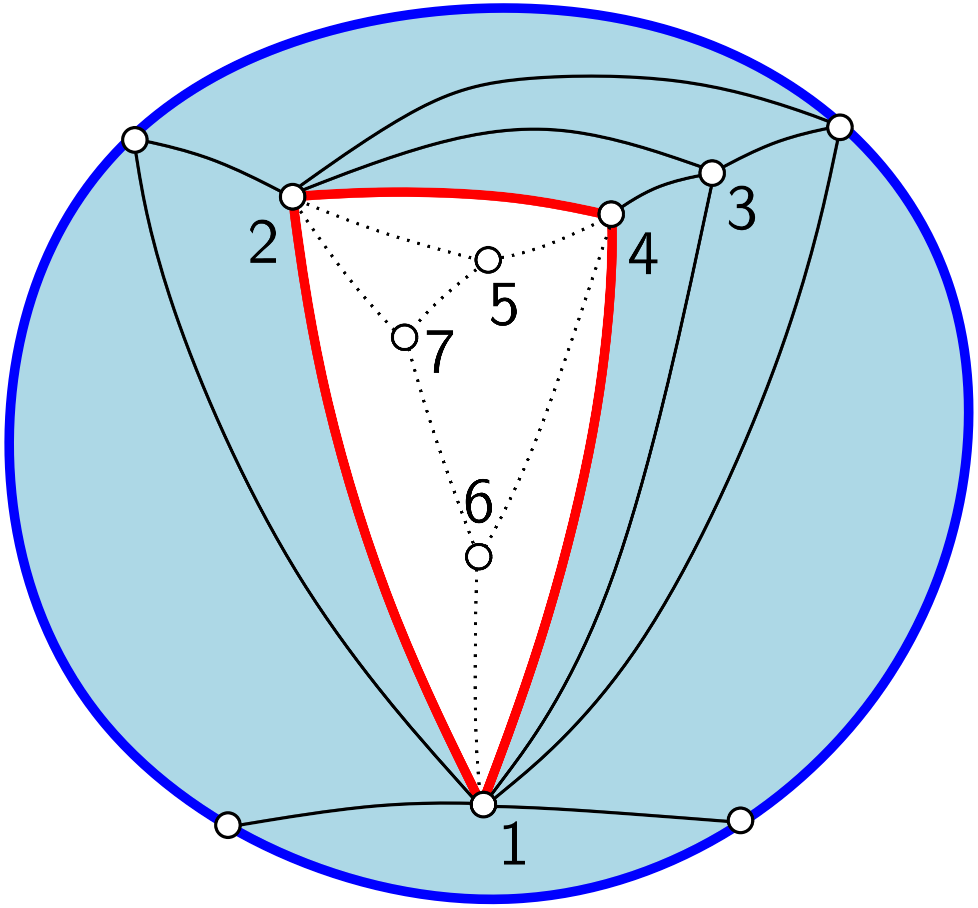
Extension to the cylinder: drawing algorithm

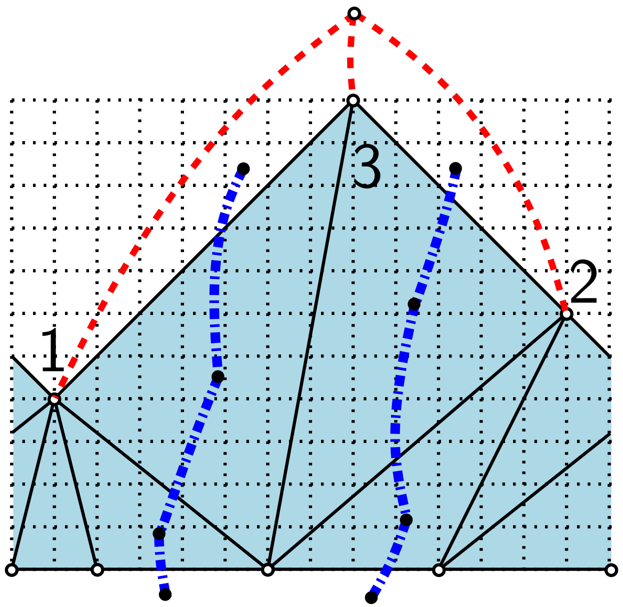


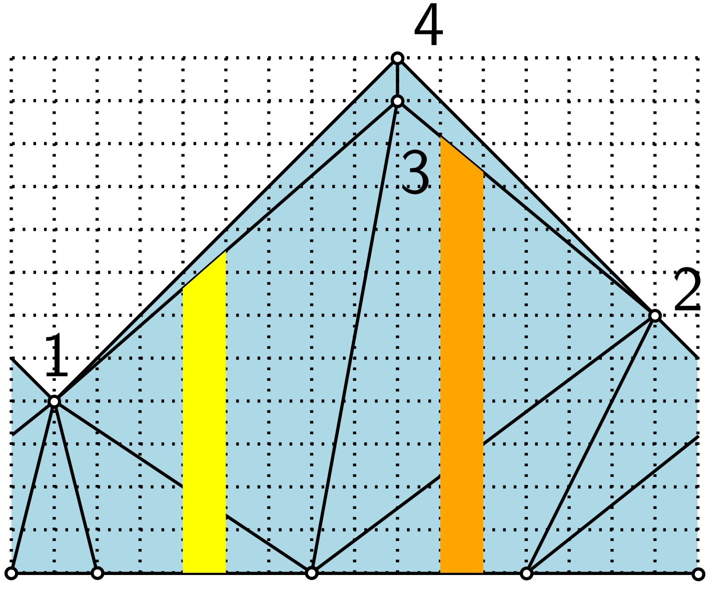




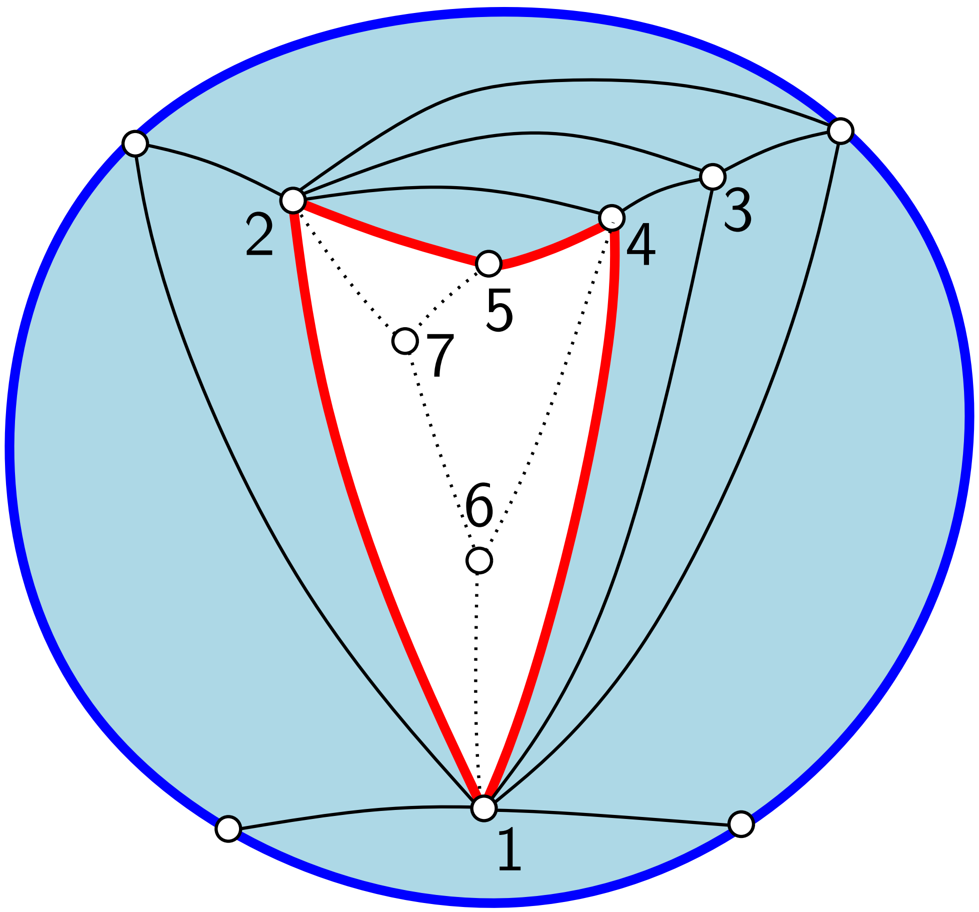
Extension to the cylinder: drawing algorithm

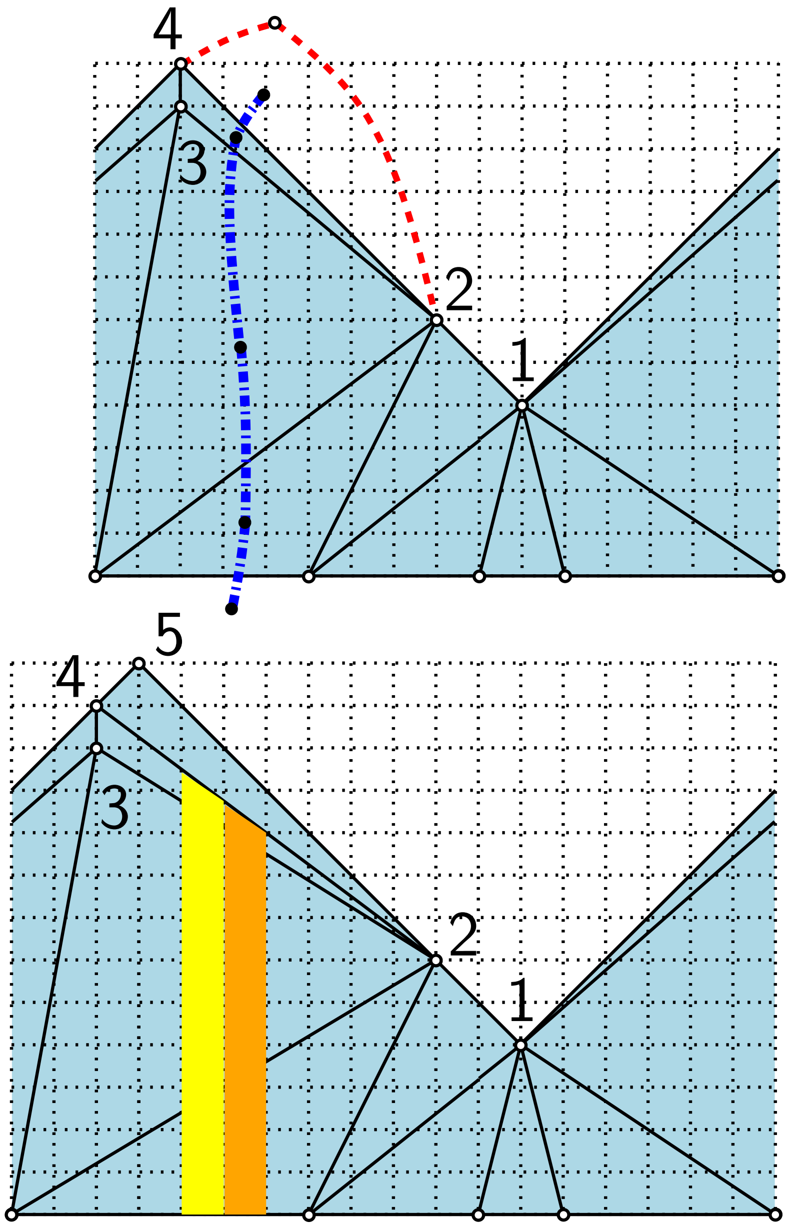




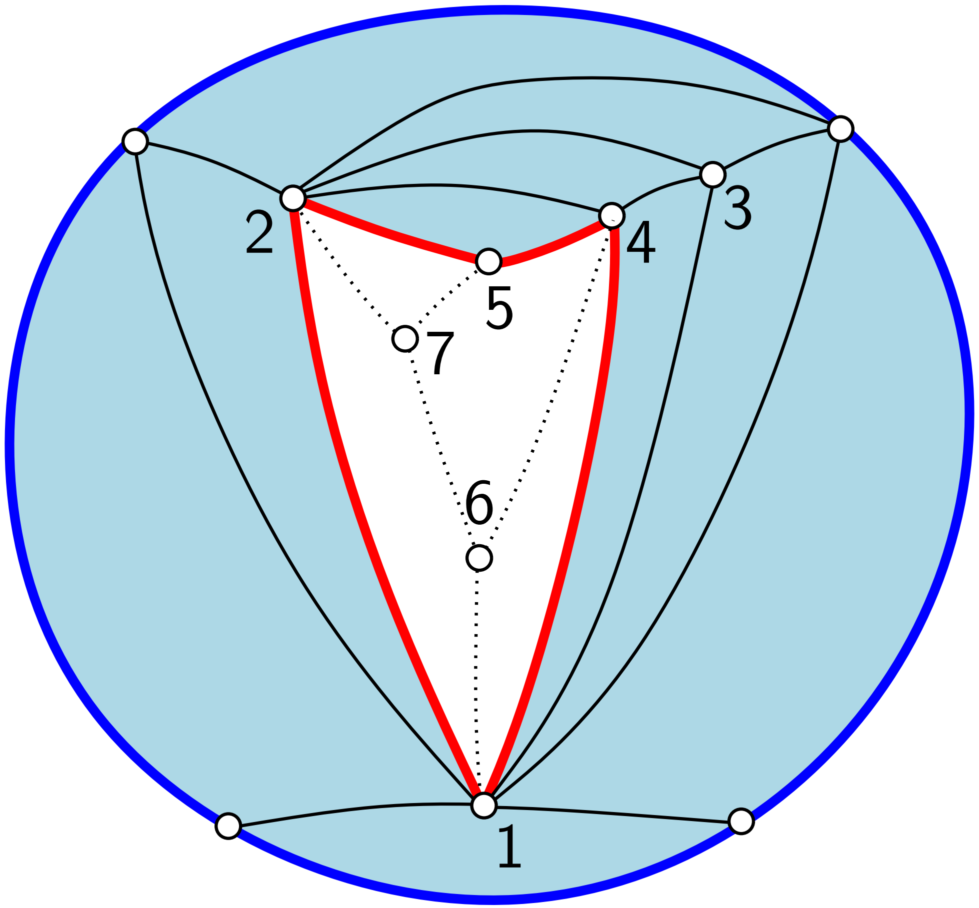


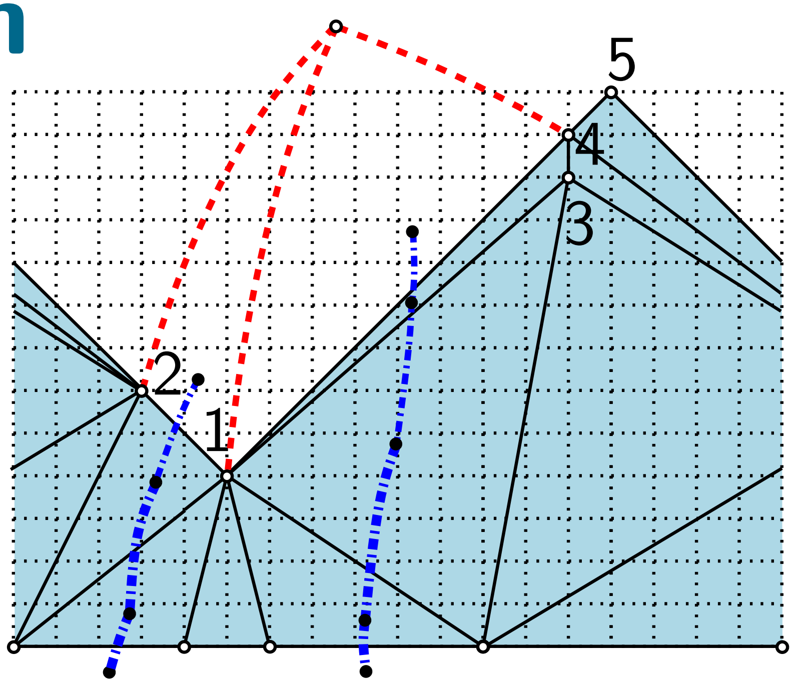
Extension to the cylinder: drawing algorithm



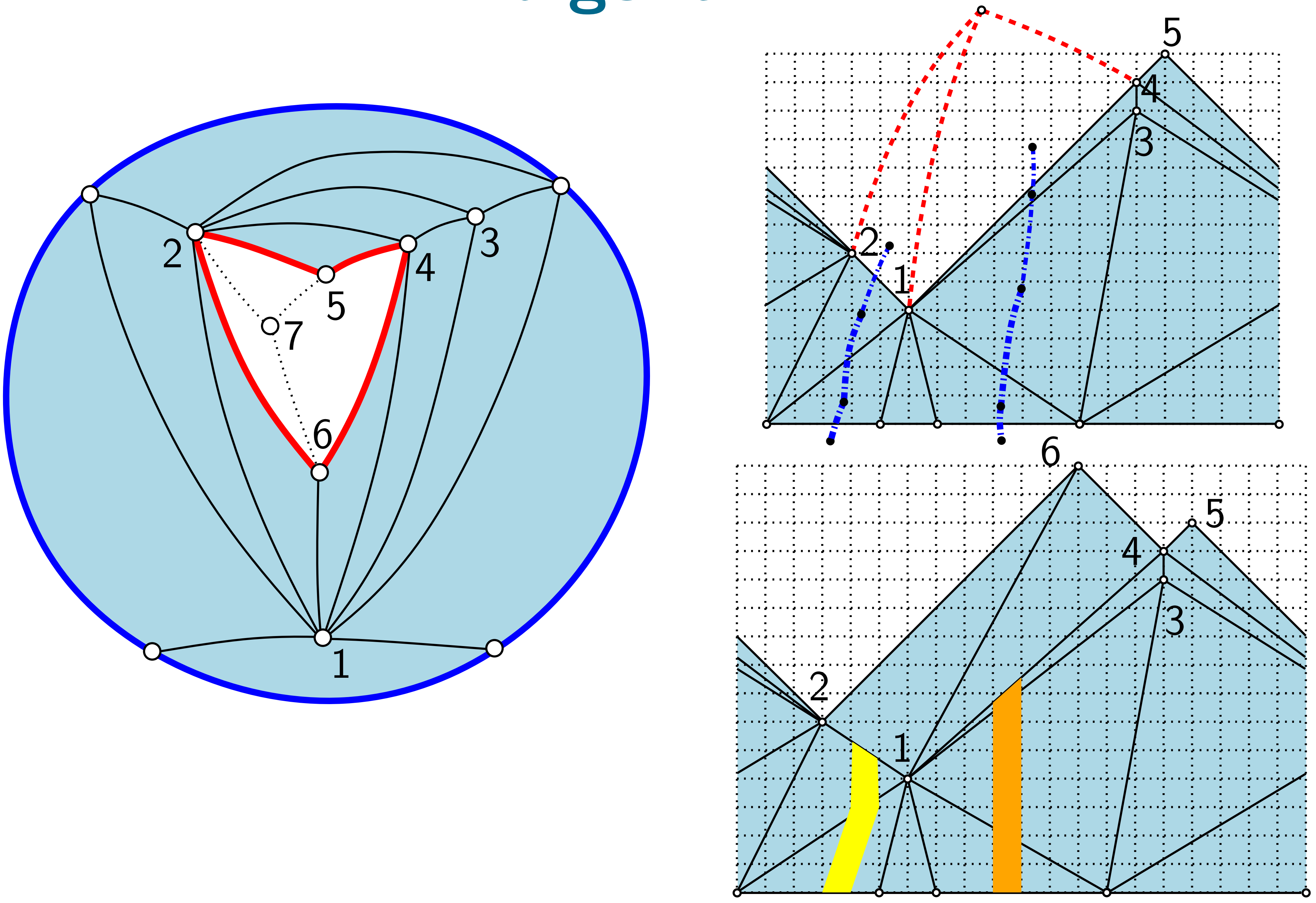


Extension to the cylinder: drawing algorithm

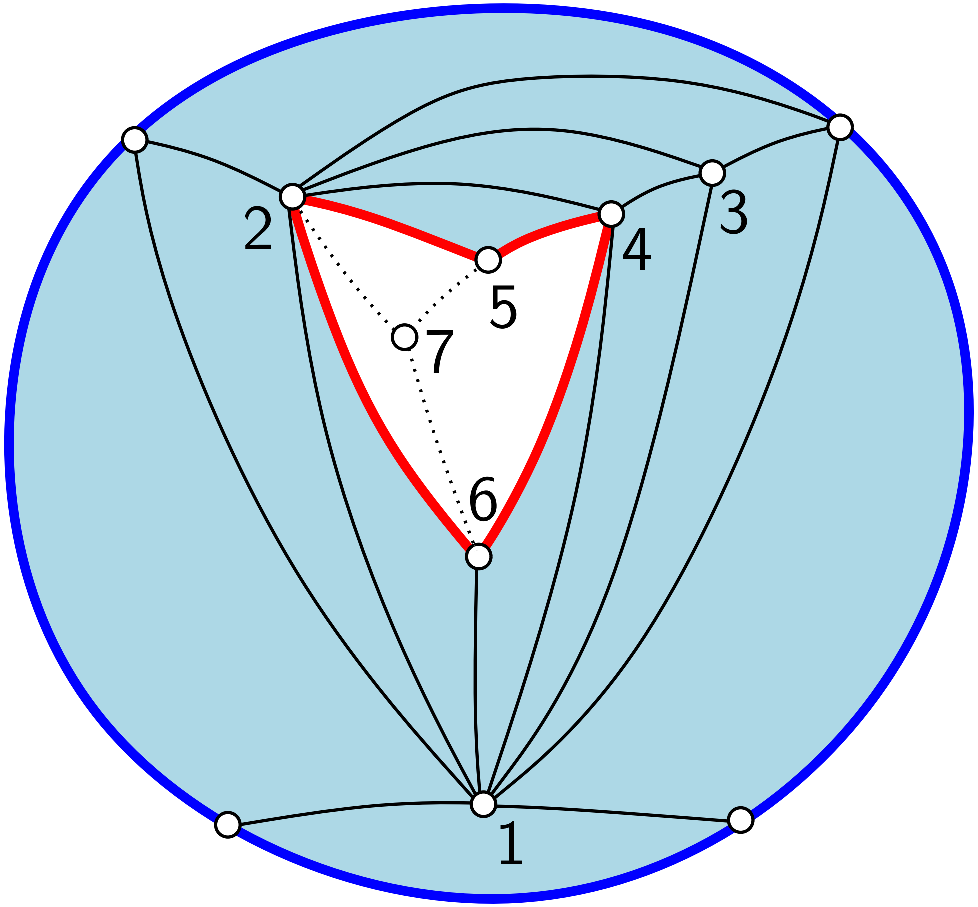


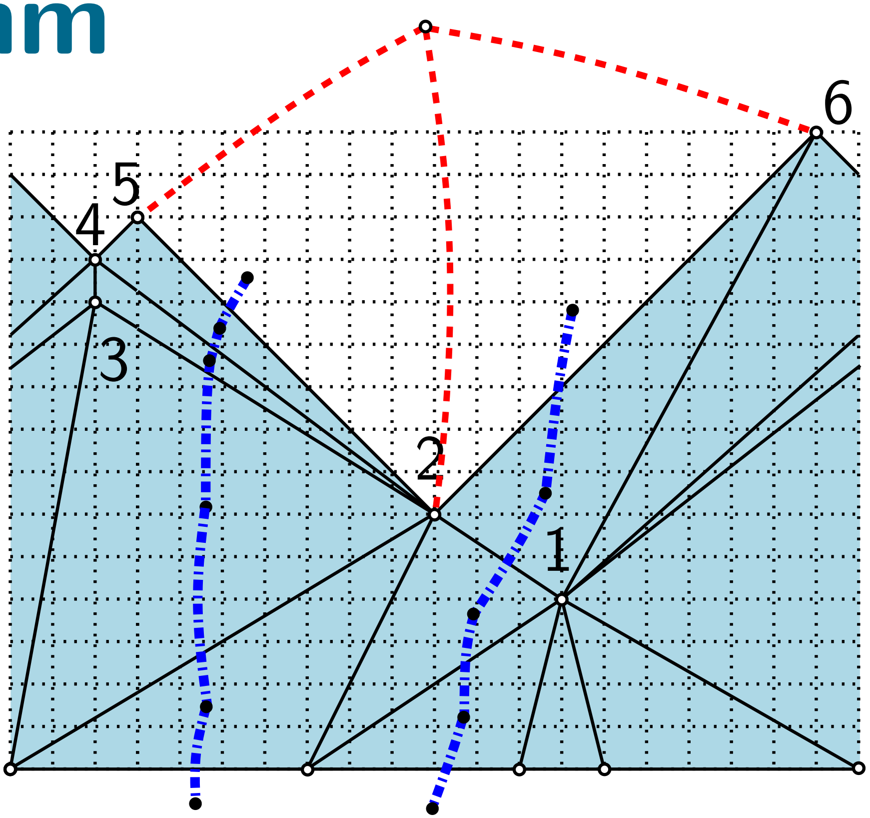


Extension to the cylinder: drawing algorithm

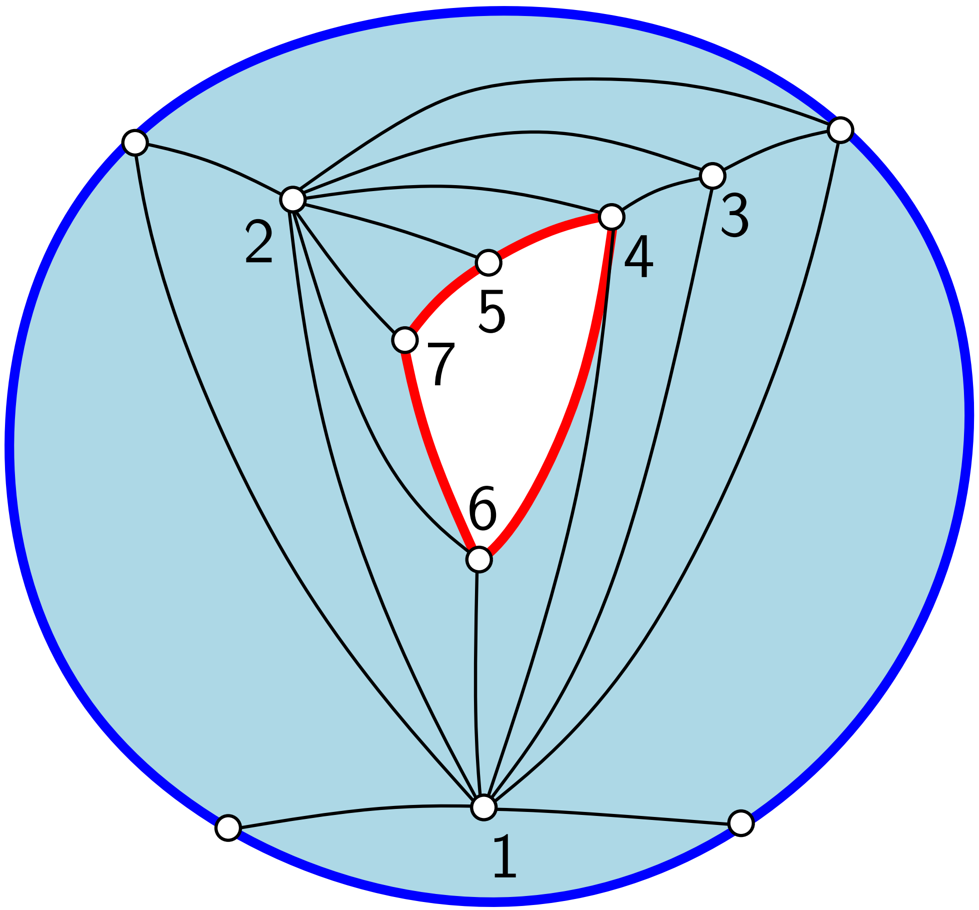


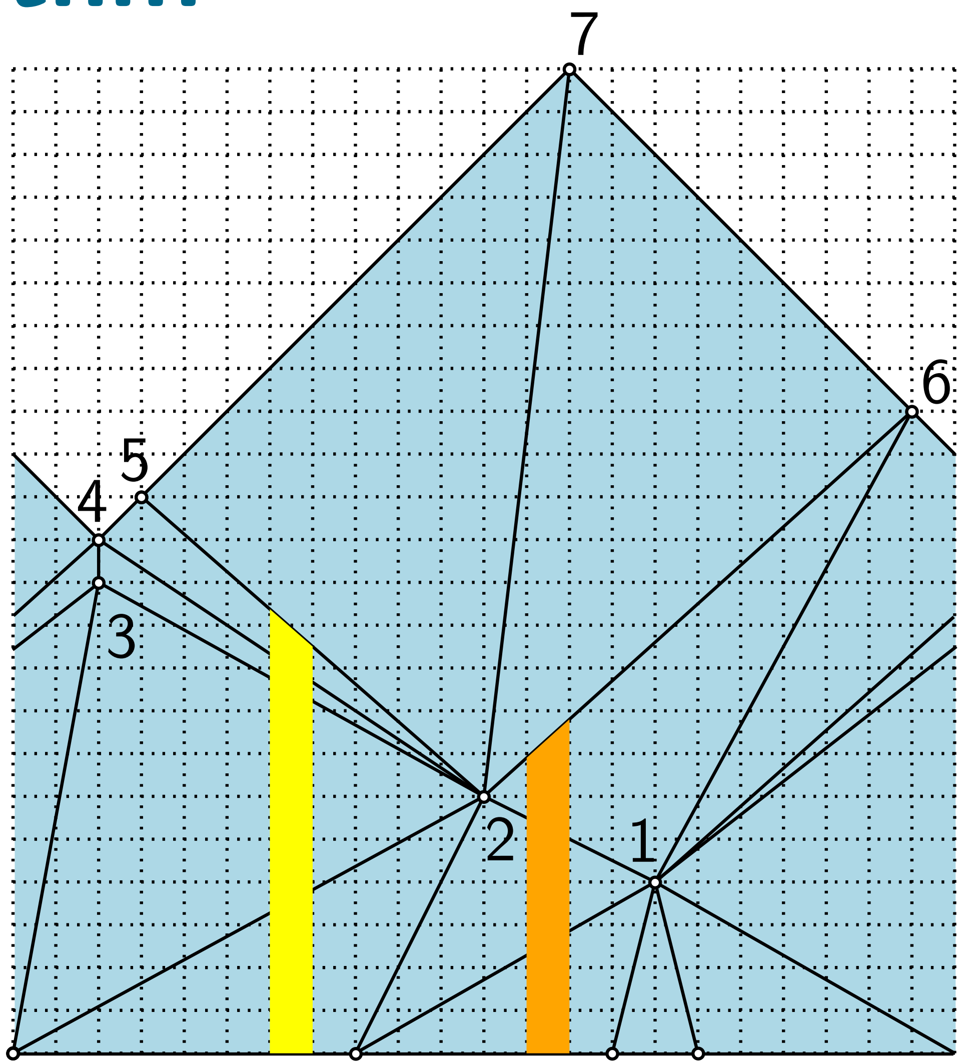
Extension to the cylinder: drawing algorithm





Extension to the cylinder: drawing algorithm

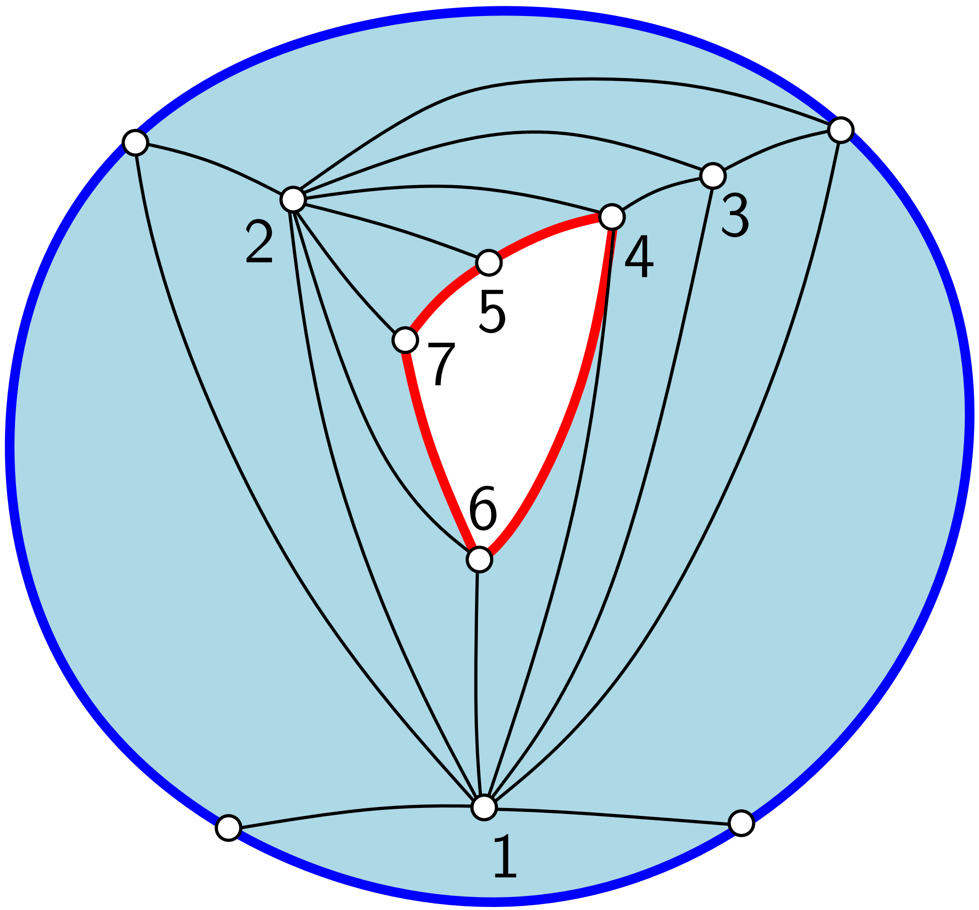


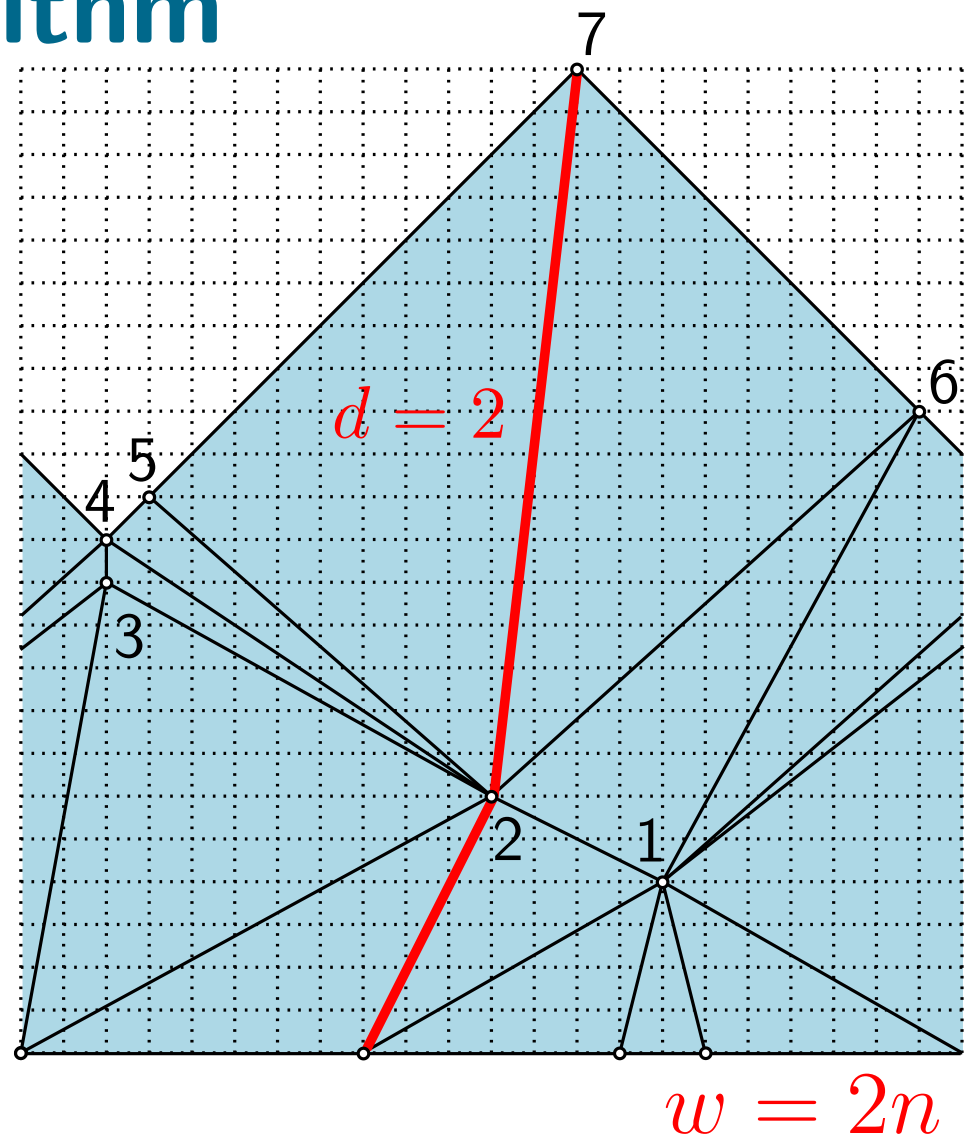


Width $= 2n$ Height $\leq n(n-3)/2$

Can also deal with chordal edges incident to outermost cycle

Extension to the cylinder: drawing algorithm





Each edge has vertical extension at most w

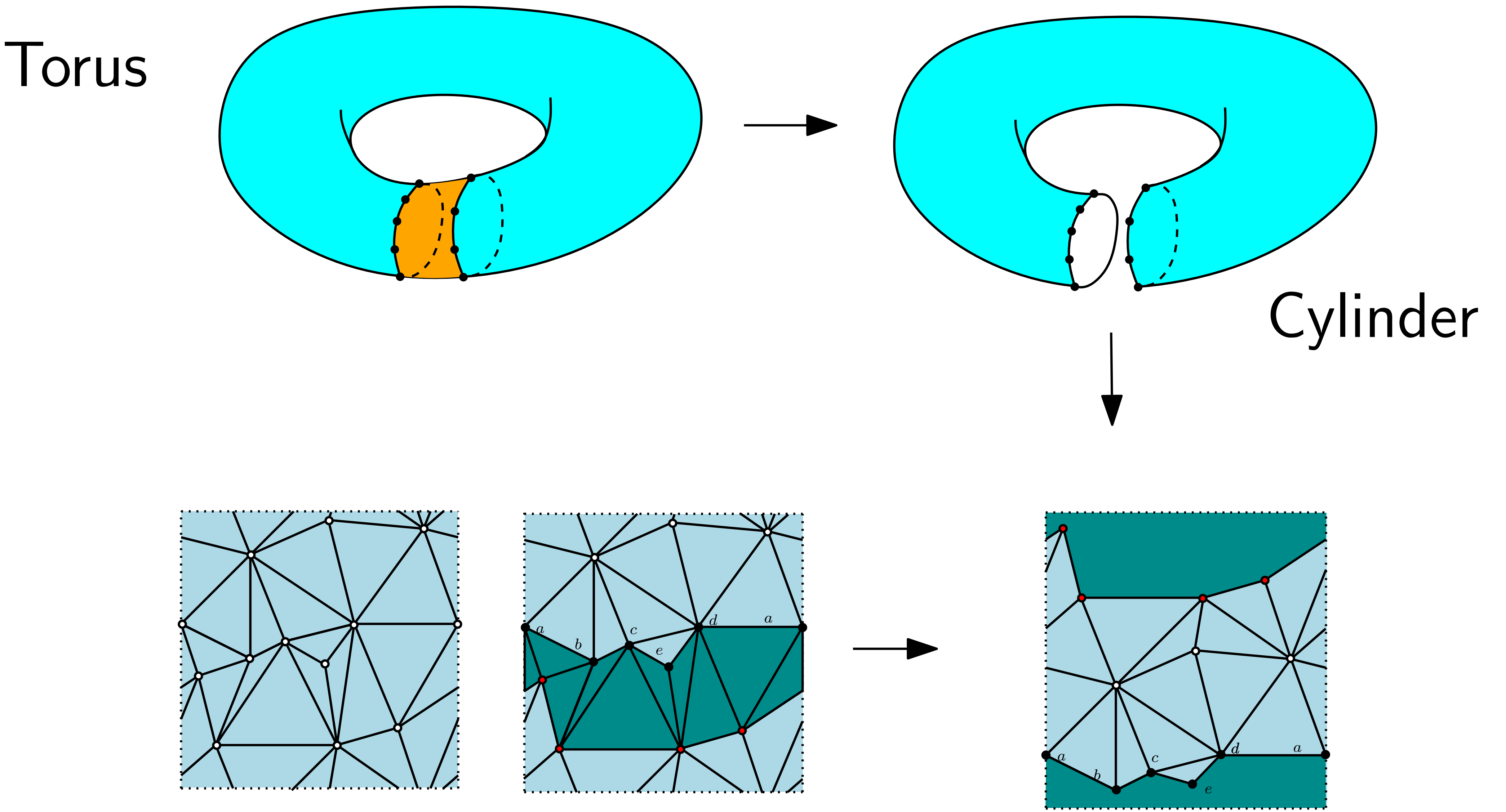
$$\Rightarrow h \le n(2d + 1)$$

with d the graph-distance between the two boundaries

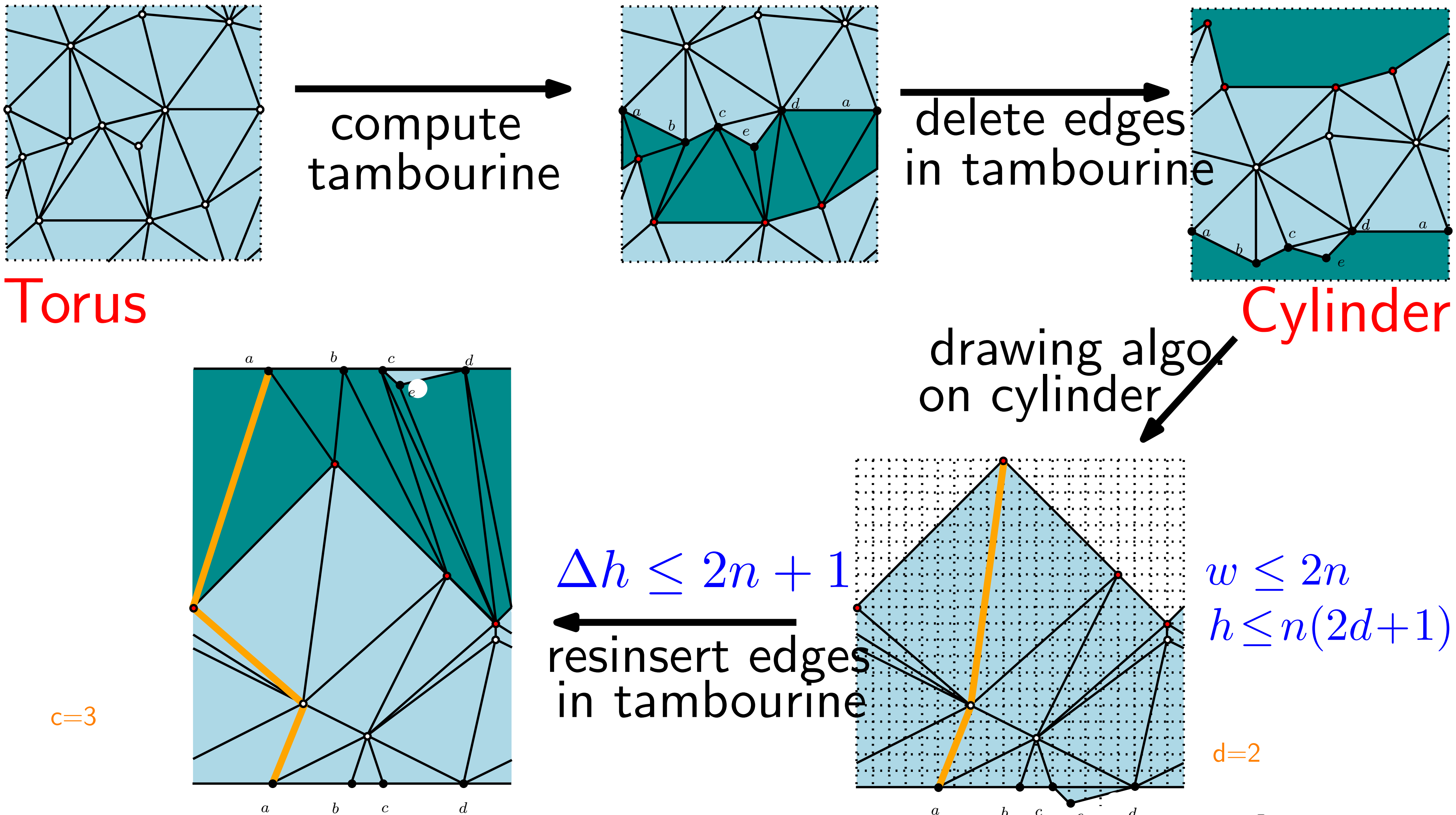
Getting toroidal drawings

Every toroidal triangulation admits a "tambourine"

$[\textbf{Bonichon, Gavoille, Labourel'06}]$



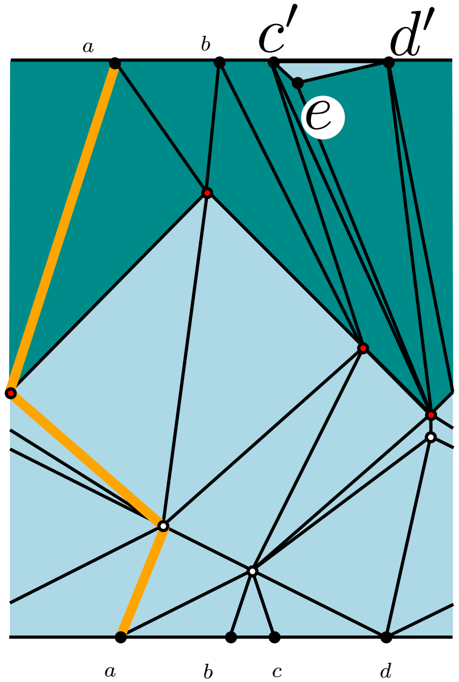
Getting toroidal drawings

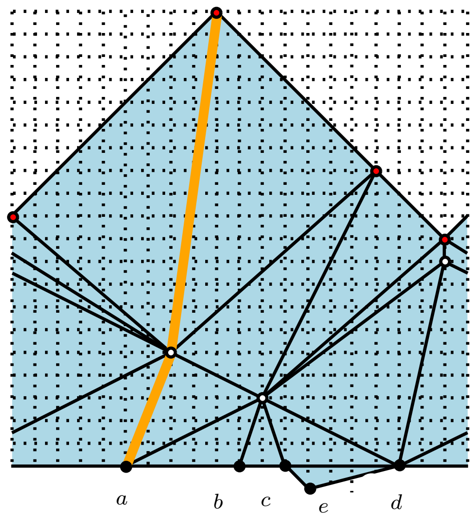


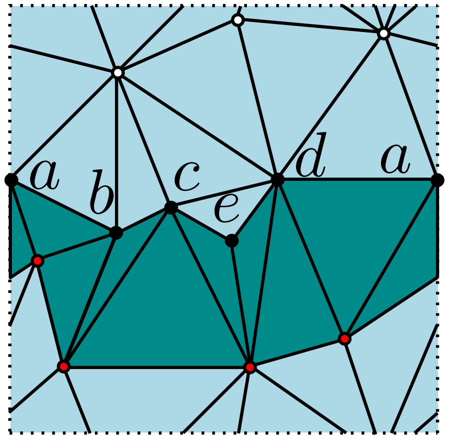
Let c =length shortest non-contractible cycle, $c \le \sqrt{2n}$ **[Hutchinson, Albert'78]**

Can choose tambourine so that $d < \boxed{c \Rightarrow h = O(n^{3/2})}$

Ensuring x-periodicity







Possible issue:

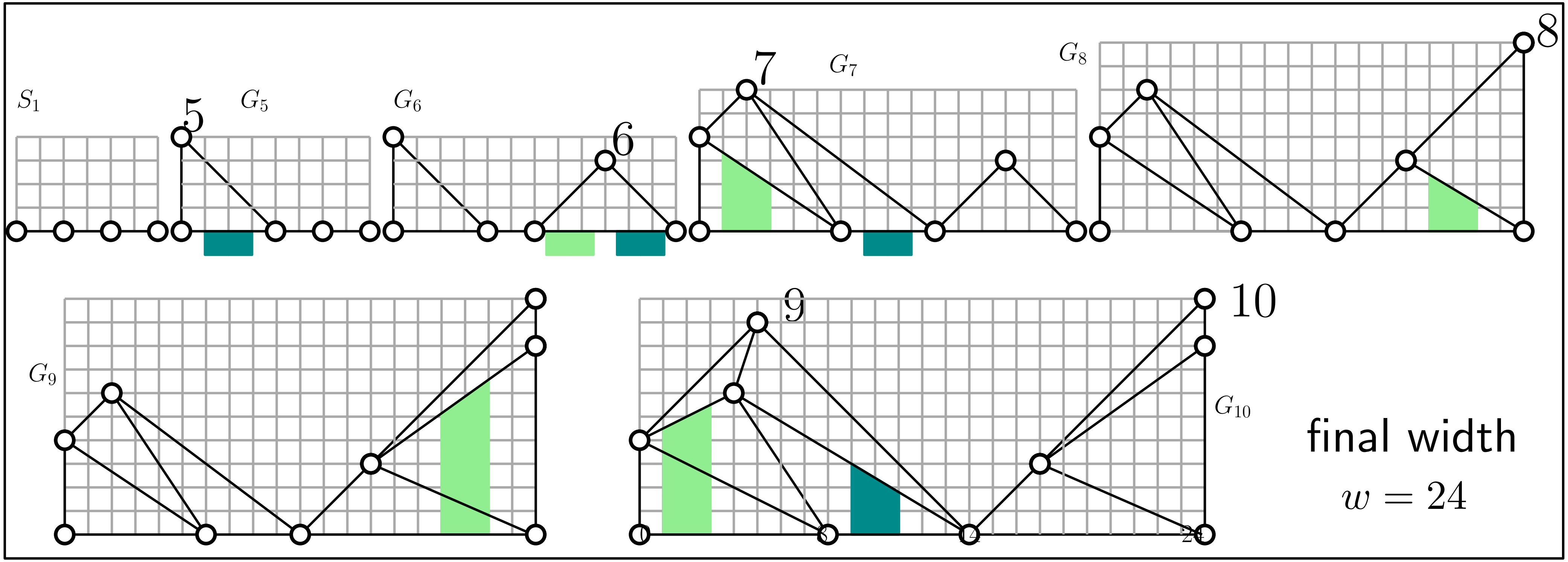
after gluing the upper and bottom boundary, the edge lengths must coincide. The width of (c', d') depends on the number of vertices in the triangle (c, e, d)

Ensuring x-periodicity: 2-pass shift

Goal: compute a grid drawing (via the shift-algorithm, with prescribed edge lengths (for horizontal edges)

Prescribed edge lengths: $\mathbf{F} = (48, 72, 96)$
(in the final drawing)

First pass: with initial vector $\mathbf{I}_1 = (2, 2, 2)$

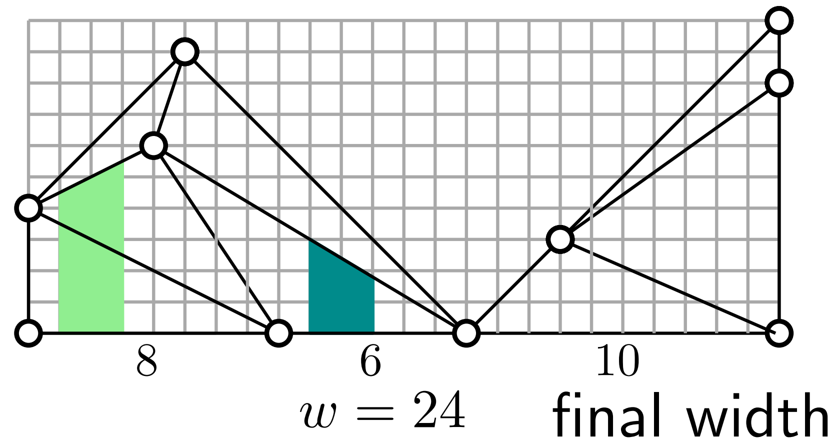


$\mathbf{L} = (8, 6, 10)$

Ensuring x-periodicity: 2-pass shift

Goal: compute a grid drawing (via the shift-algorithm, with prescribed edge lengths (for horizontal edges)

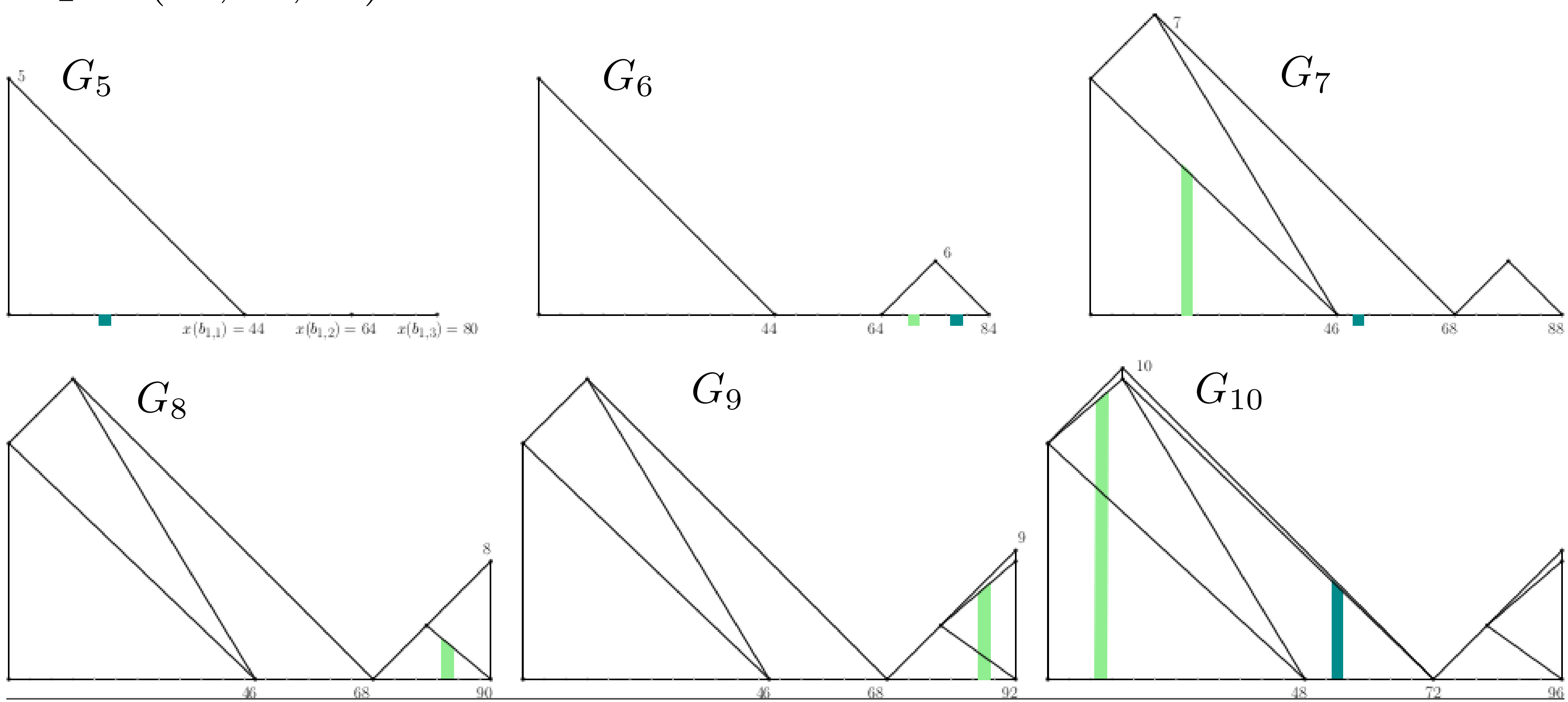
Prescribed edge lengths: $\mathbf{F} = (48, 72, 96)$
(in the final drawing)



Second pass:
use a new initial vector
$\mathbf{I}_2 = (44, 64, 80)$

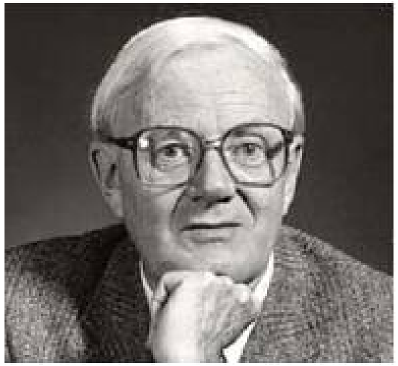
$\mathbf{S} = (6, 4, 8)$

$\mathbf{I}_1 = (2, 2, 2)$



Tutte drawings on surfaces

Tutte drawings (in the plane)



Thm (Tutte barycentric method, 1963)

Every 3-connected planar graph G admits a convex representation ρ in R^2.

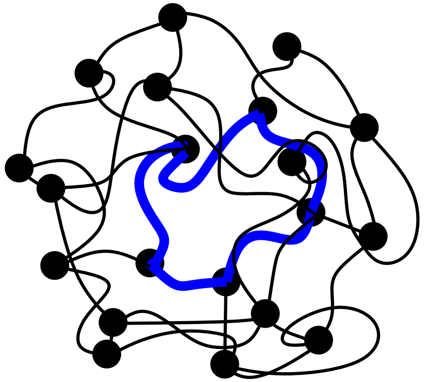
$$\rho : (V_G) \longrightarrow R^2$$

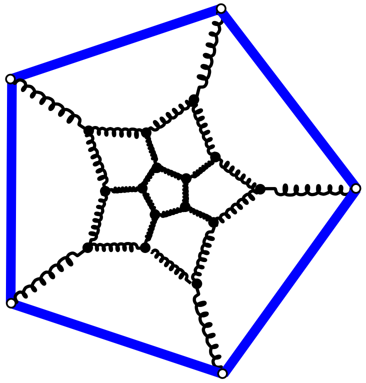
the images of interior vertices are barycenters of their neighbors

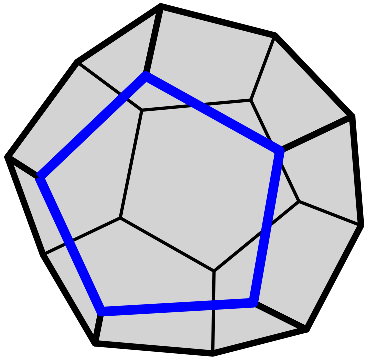
$$\rho(v_i) = \sum_{j \in N(i)} w_{ij} \rho(v_j)$$

where w_{ij} satisfy $\sum_j w_{ij} = 1$, and $w_{ij} > 0$

according to Tutte: $w_{ij} = \frac{1}{deg(v_i)}$



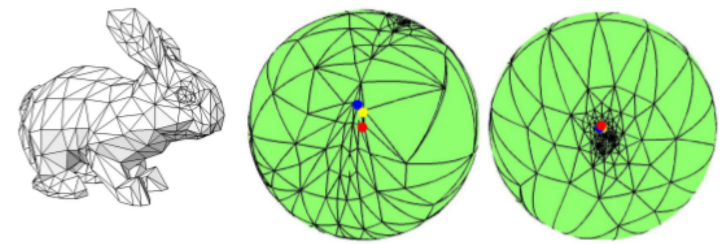




Spherical parameterization (Tutte on the sphere)

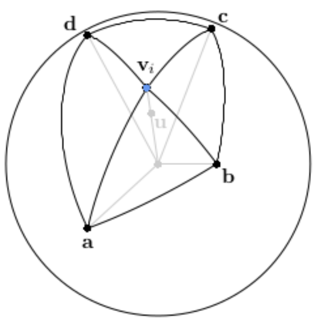
$$\mathbf{v}_i = \frac{\mathbf{u}_i}{||\mathbf{u}_i||}, \quad \text{with} \quad \mathbf{u}_i = \sum_{j=1}^{n} w_{ij}\mathbf{v}_j, \quad i = 1, 2, \ldots, N.$$

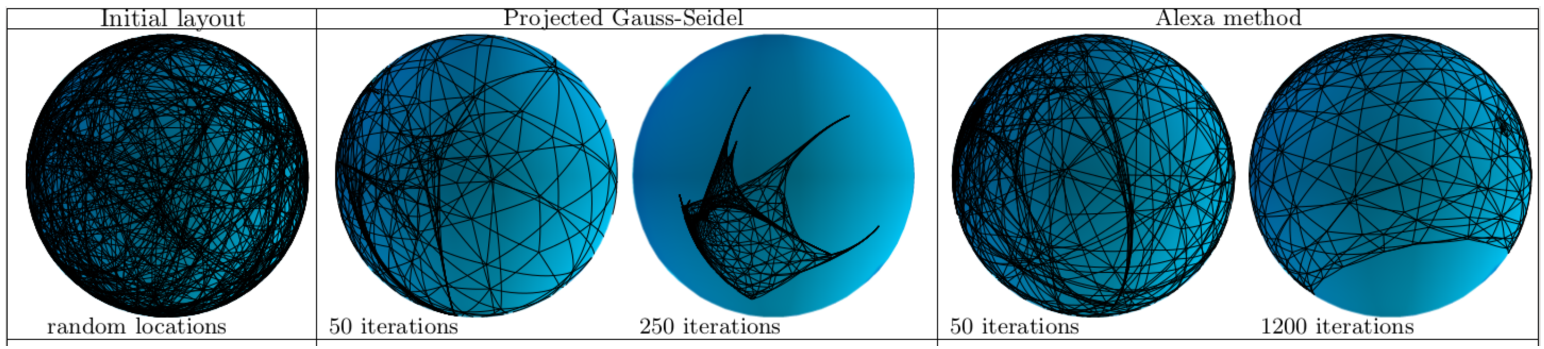
(system of quadratic equations)



(Gotsman Gu Sheffer, 2003)

$$\mathbf{v}_i = \frac{\mathbf{u}}{||\mathbf{u}||}.$$





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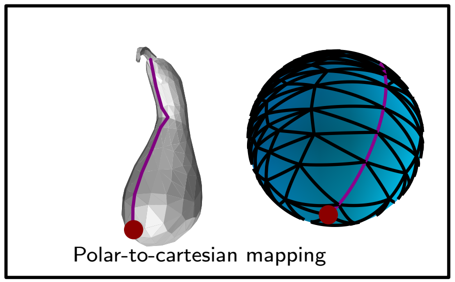
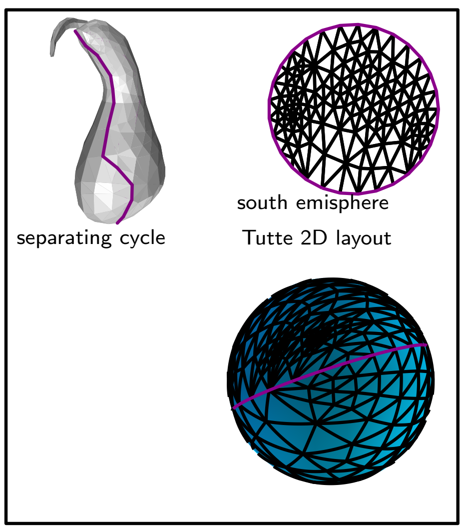
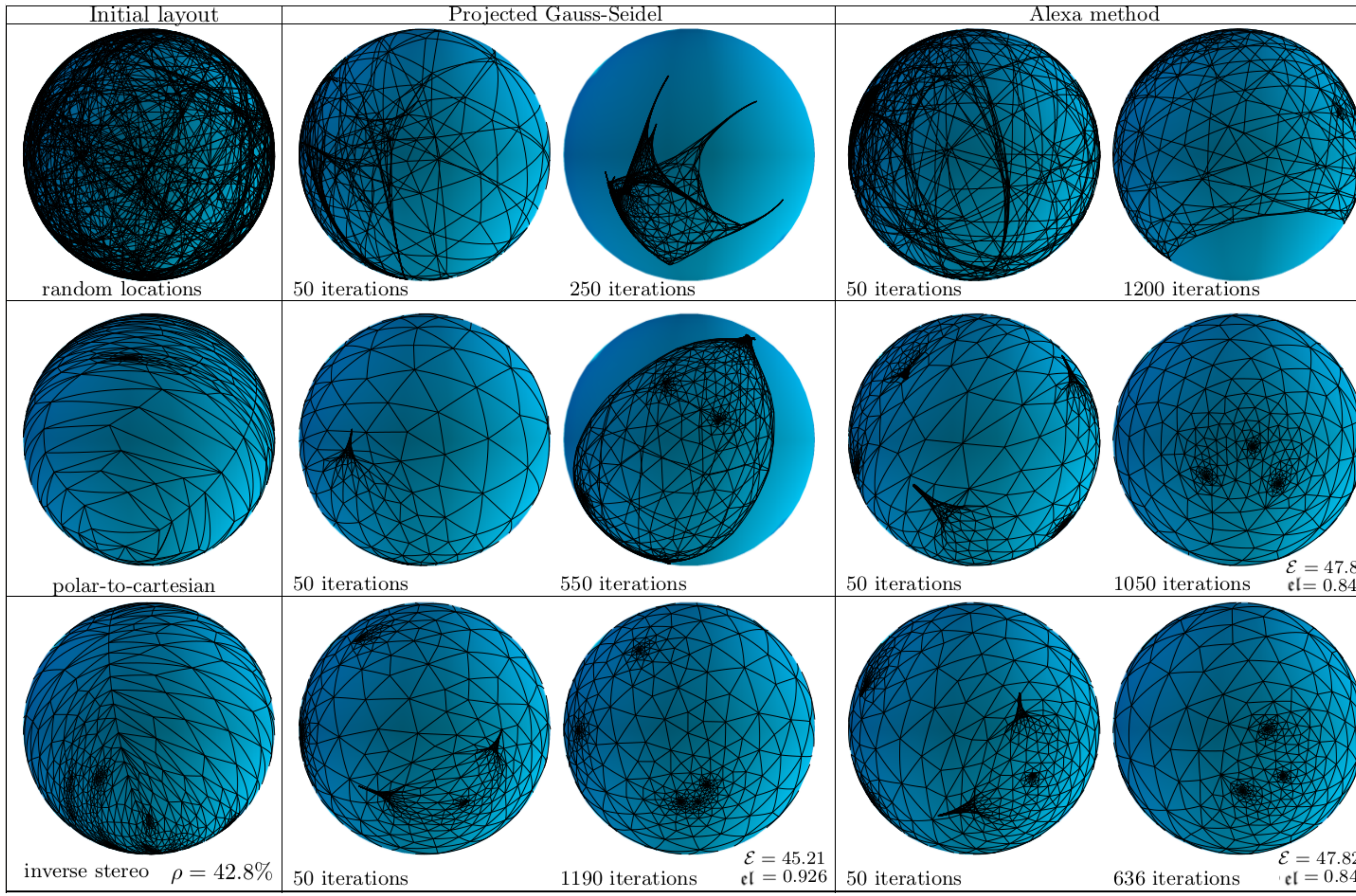
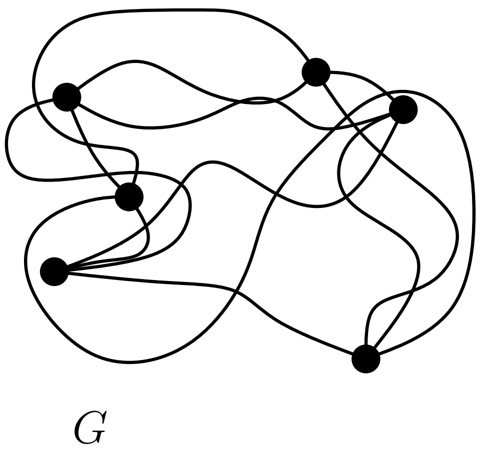


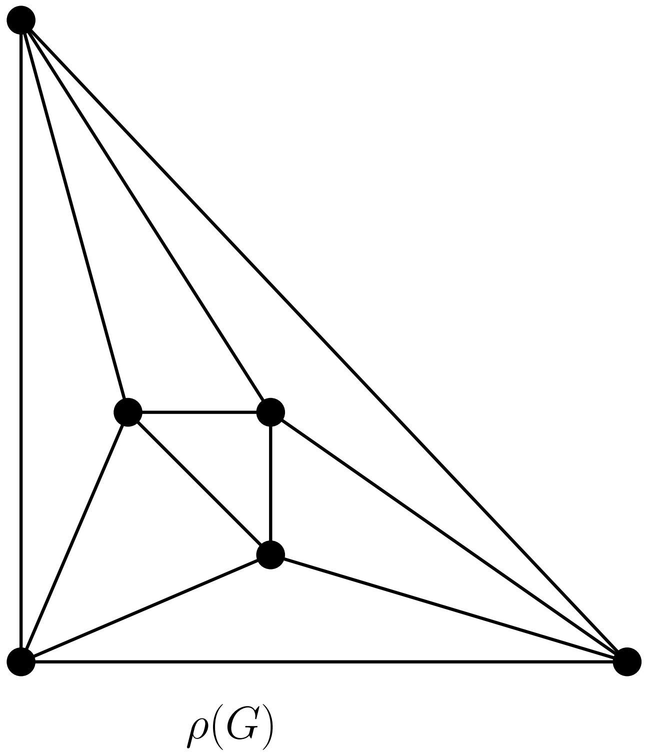




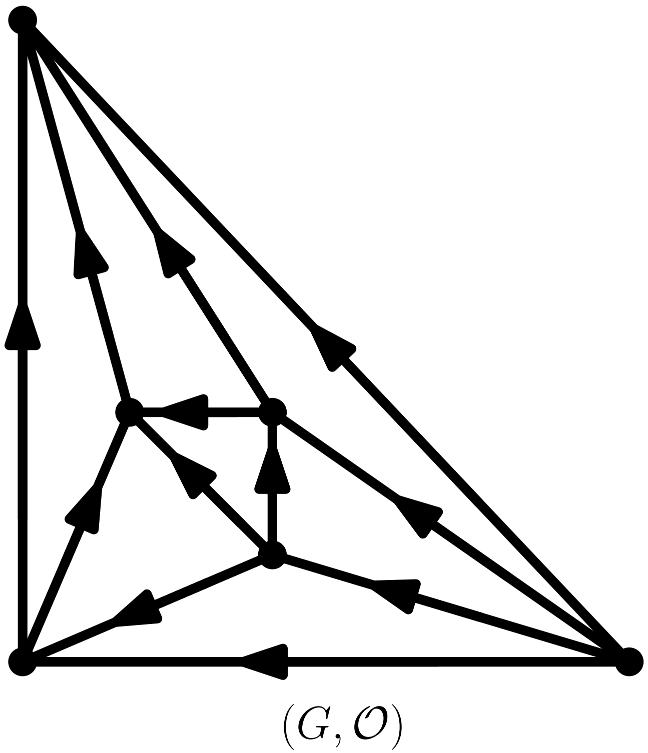
Tutte drawings, from another point of view

Planar drawings and edge orientations





planar drawing



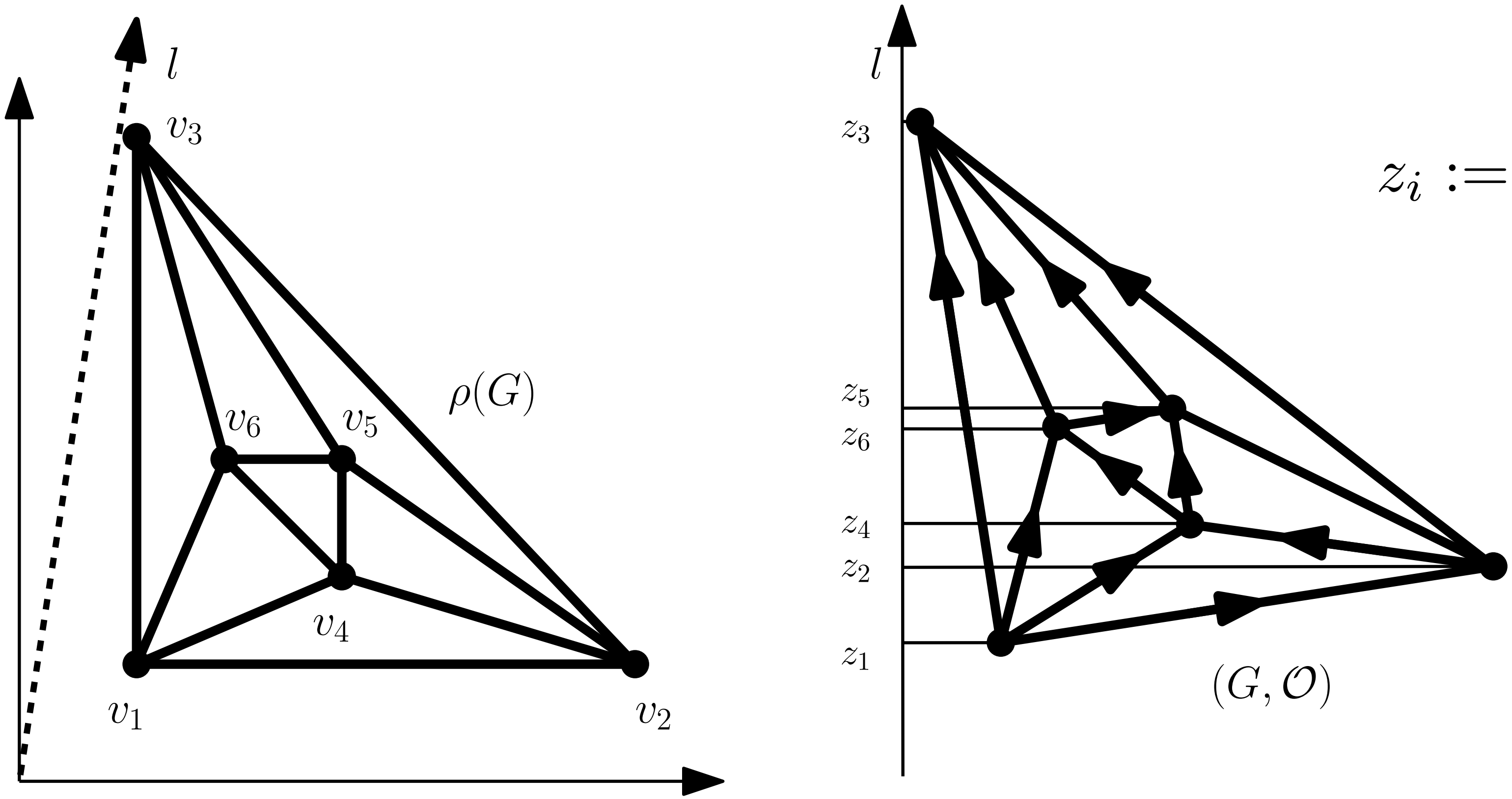
edge orientation

Planar drawings and edge orientations

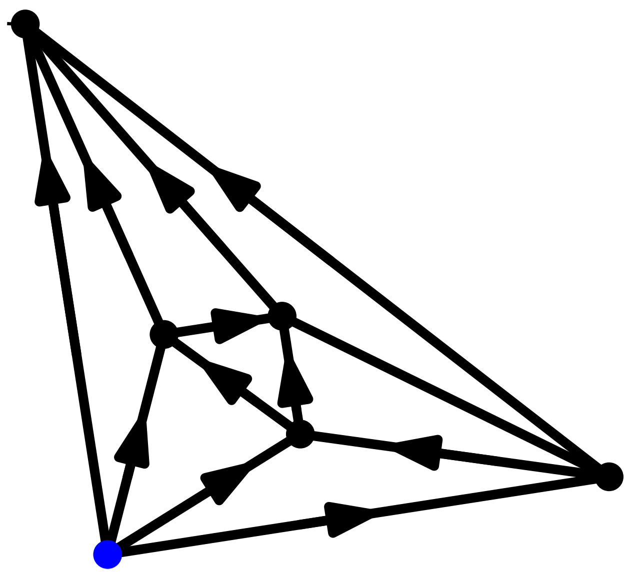
Choose a generic line l and project all vertices on l
(such that the images z_i of vertices are distinct)

$$R_{\alpha\beta} = \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix}$$

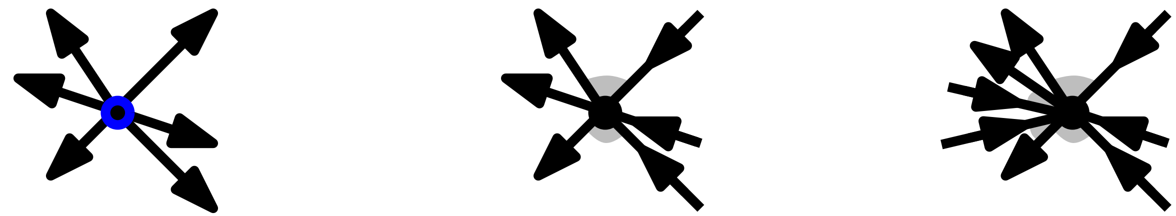
$$z_i := \alpha x_i + \beta y_i$$



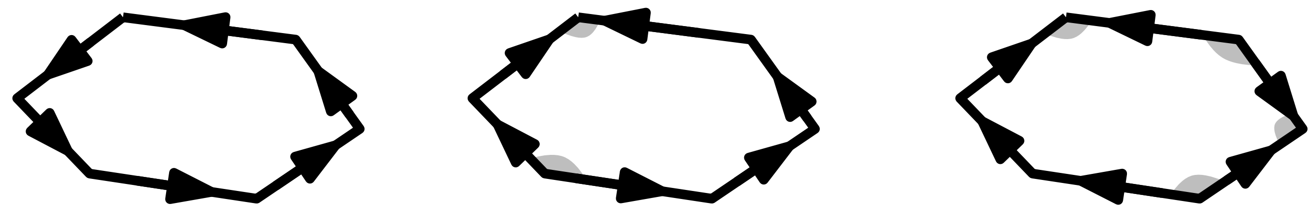
Vertex and face classification



$(G, \mathcal{O})$



$ind(v) = 1$, $sc(v) = 0$ source	$ind(v) = 0$, $sc(v) = 2$ non singular	$ind(v) < 0$, $sc(v) = 4$ saddle



$ind(v) = 1$, $sc(f) = 0$ vortex	$ind(v) = 0$, $sc(f) = 2$ non singular	$ind(v) < 0$, $sc(f) = 4$ saddle

Index of a vertex $ind(v) := \frac{(2 - sc(v))}{2}$

$sc(v) :=$ number of sign changes around vertex v

Index of a face $ind(f) := \frac{(2 - sc(f))}{2}$

$sc(f) :=$ number of sign changes around face f

Discrete Index Theorem (Poincaré-Hops)

Thm Given a closed manifold mesh of genus g:

$$\sum_{v\in V} ind(v) + \sum_{f\in F} ind(f) = 2 - 2g$$

Proof

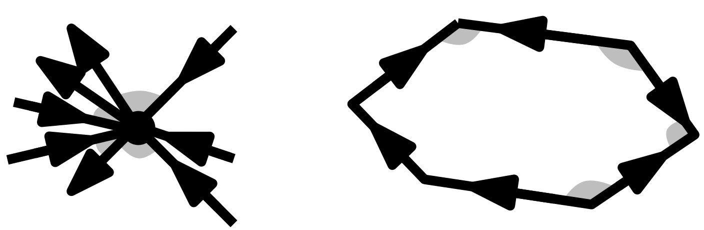
$$\sum_{v\in V} ind(v) + \sum_{f\in F} ind(f) = \frac{1}{2}\sum_{v\in V}(2 - sc(v)) + \frac{1}{2}\sum_{f\in F}(2 - sc(f))$$

$$= V + F - \frac{1}{2}\left[\sum_{v\in V} sc(v) + \sum_{f\in F} sc(f)\right]$$

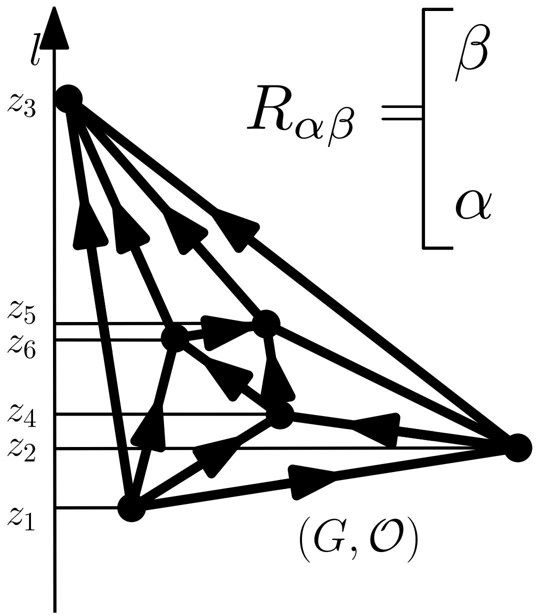
$$= V + F - \frac{1}{2}[2 \cdot E]$$

$$= 2 - 2g$$

Claim: the total number of changes of edge direction (around each vertex and around each face) is equal to the number of half-edges



Discrete one-forms and Tutte equations



$$R_{\alpha\beta} = \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix}$$

$$z_i := \alpha x_i + \beta y_i$$

$$\Delta z_{ij} := z_j - z_i$$

(value of the one-form Δ for edge (i, j))

Remark: $\Delta_{ij} = \Delta_{ji}$ and $\Delta_{ij} \neq 0$

Fact: Given a Tutte barycentric drawing of a planar graph G, we have:

$\sum_{v_j \in N(i)} \frac{1}{d_i} \Delta z_{ij} = 0$, for each inner vertex v_i

$\sum_{(u,v) \in \partial f} \Delta z_{uv} = 0$ for each face f of G

Discrete one-forms and Tutte equations

$\sum_{v_j \in N(i)} \frac{1}{d_i} \Delta z_{ij} = 0$, for each inner vertex v_i $\qquad z_i := \alpha x_i + \beta y_i$

$\sum_{(u,v) \in \partial f} \Delta z_{uv} = 0$ for each face f of G $\qquad \Delta z_{ij} := z_j - z_i$

Proof: inner vertices are placed at the barycenter of their neighbors

$$v_i = \sum_{j \in N(i)} \frac{1}{d_i} v_j, \quad \text{for any inner vertex } v_i$$

which is equivalent to: $x_i = \sum_{j \in N(i)} \frac{1}{d_i} x_j$ et $y_i = \sum_{j \in N(i)} \frac{1}{d_i} y_j$. By definition we have:

$$z_i := \alpha x_i + \beta y_i = \alpha \sum_j \frac{1}{d_i} x_j + \beta \sum_j \frac{1}{d_i} y_j = \sum_j \frac{1}{d_i} (\alpha x_j + \beta y_j) = \sum_j \frac{1}{d_i} z_j$$

implying:

$$\sum_{v_j \in N(i)} \frac{1}{d_i} \Delta z_{ij} = \sum_{v_j \in N(i)} \frac{1}{d_i} (z_j - z_i) = \Big(\sum_{v_j \in N(i)} \frac{1}{d_i} z_j \Big) - d_i z_i = 0$$

Discrete one-forms and Tutte equations

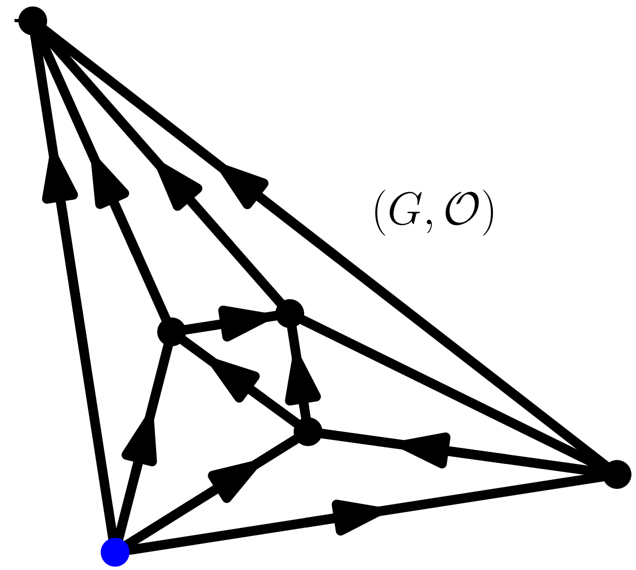
$\sum_{v_j \in N(i)} \frac{1}{d_i} \Delta z_{ij} = 0$, for each inner vertex v_i

$z_i := \alpha x_i + \beta y_i$

$\sum_{(u,v) \in \partial f} \Delta z_{uv} = 0$ for each face f of G

$\Delta z_{ij} := z_j - z_i$

Fact (exercise): In an orientation induced by a Tutte drawing there are no saddle vertices and no saddle faces



Corollary (exercise): all faces in a Tutte drawing are convex

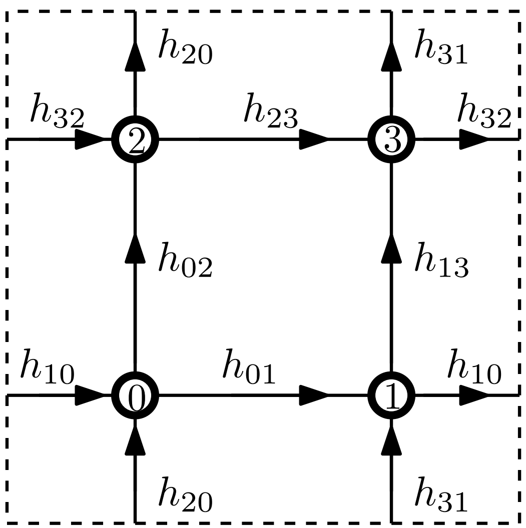
Tutte equations on the torus

$\Delta z_h :=$ one-forme associated to half-edge h

$$\begin{cases} \sum_{h\in\partial f} \Delta z_h = 0 \text{ for each face } f \text{ of } G & E \text{ unknowns} \\ \sum_{h\in N(i)} \frac{1}{d_i}\Delta z_h = 0, \text{ for } \textbf{each} \text{ vertex } v_i & \text{F+V equations} \end{cases}$$

The rank of the two systems is $(F-1)$ and $(V-1)$

The solution space has dimension
$E - [(V-1) + (F-1)] = 2g = 2$



$$\begin{cases} h_{01} + h_{01} - h_{10} - h_{20} = 0 & \text{for vertex } v_0 \\ h_{10} + h_{13} - h_{01} - h_{31} = 0 & \text{for vertex } v_1 \\ h_{23} + h_{20} - h_{32} - h_{02} = 0 & \text{for vertex } v_2 \\ h_{32} + h_{31} - h_{23} - h_{13} = 0 & \text{for vertex } v_3 \end{cases}$$

$$Eq_3 = -(Eq_0 + Eq_1 + Eq_2)$$

Computing coordinates on the torus

$\Delta z_h :=$ one-forme associated to half-edge h

$$\begin{cases} \sum_{h \in \partial f} \Delta z_h = 0 \text{ for each face } f \text{ of } G \\ \sum_{h \in N(i)} \frac{1}{d_i} \Delta z_h = 0, \text{ for } \textbf{each} \text{ vertex } v_i \end{cases}$$

E unknowns

F+V equations

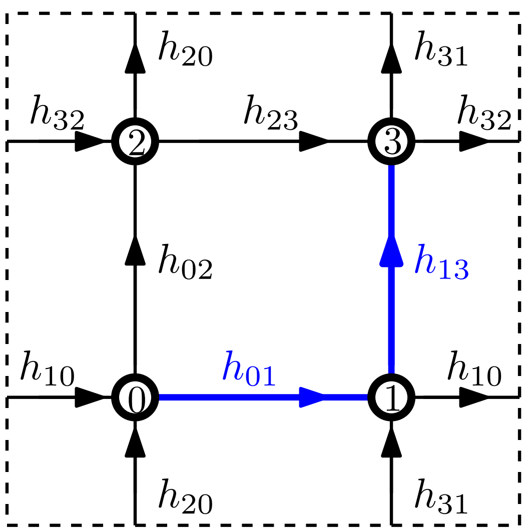
Compute the null-space of the matrix

Choose any pair of linearly indenpendent one-forms Δx and Δy

Choose a vertex v_0 as origin and compute coordinates relatives to v

$$(x_0, y_0) = (0, 0)$$

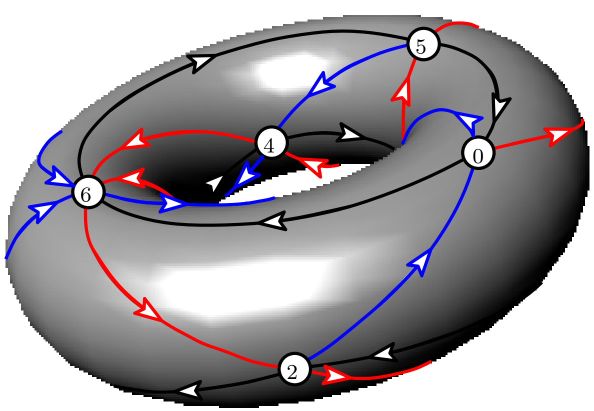
$$(x_i, y_i) = \left(\sum_{h \in P(v_0, v_i)} \Delta x_h, \sum_{h \in P(v_0, v_i)} \Delta y_h\right)$$

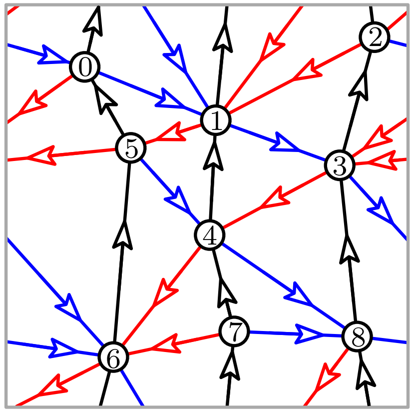


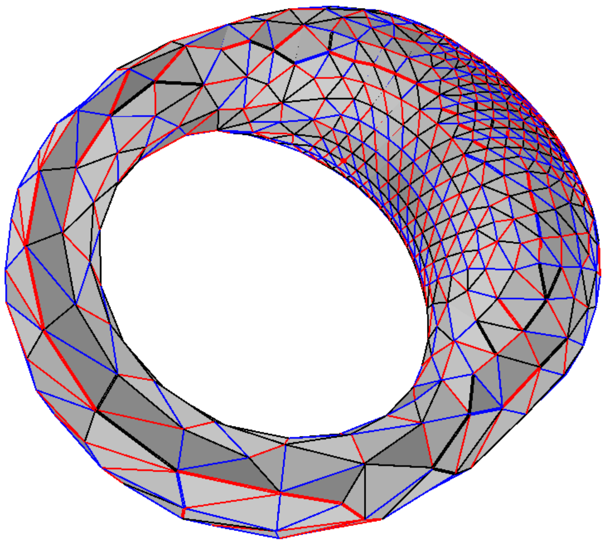
$$\Delta x = (1, 1, 1, 1, 0, 0, 0, 0)$$
$$\Delta y = (0, 0, 0, 0, 1, 1, 1, 1)$$

$$\begin{cases} h_{01} + h_{01} - h_{10} - h_{20} = 0 \\ h_{10} + h_{13} - h_{01} - h_{31} = 0 \\ h_{23} + h_{20} - h_{32} - h_{02} = 0 \\ h_{32} + h_{31} - h_{23} - h_{13} = 0 \end{cases}$$

Toroidal drawings III: toroidal Schnyder woods







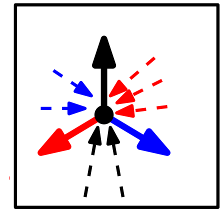
Toroidal Schnyder woods: definition

Toroidal Schnyder woods [Goncalves Lévêque, DCG'14]

$g = 1 \qquad e = 3n$

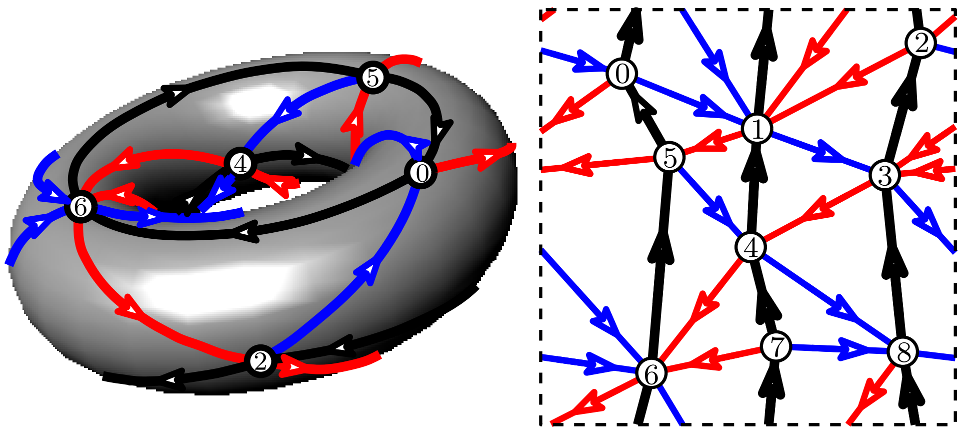
$n - e + f = 2 - 2g$

- 3-orientation + Schnyder local rule valid at each vertex

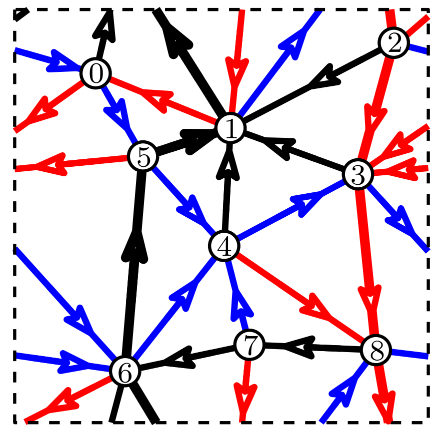


Toroidal Schnyder woods are **crossing** if

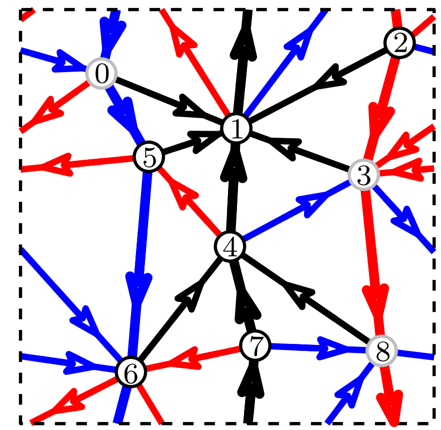
- every monochromatic cycle intersects at least one monochromatic cycle of each color



crossing Schnyder wood



half-crossing Schnyder wood

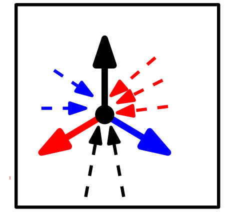


the Schnyder wood is **not half-crossing**

Toroidal Schnyder woods vs. 3-orientations

Toroidal Schnyder woods [Goncalves Lévêque, DCG'14]

- 3-orientation + Schnyder local rule valid at each vertex



$g = 1 \qquad e = 3n$

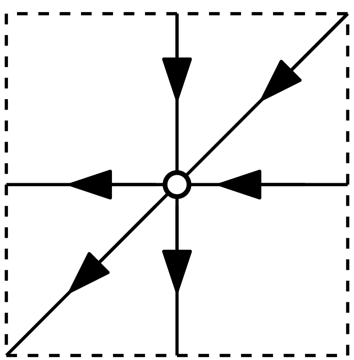
$n - e + f = 2 - 2g$

Toroidal Schnyder woods are **crossing** if

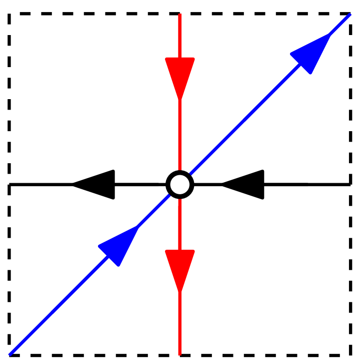
- every monochromatic cycle intersects at least one monochromatic cycle of each color

Remark: unlike the planar case, some 3-orientations do not lead to valid Schnyder woods

toroidal triangulation (one vertex, 3 loops)

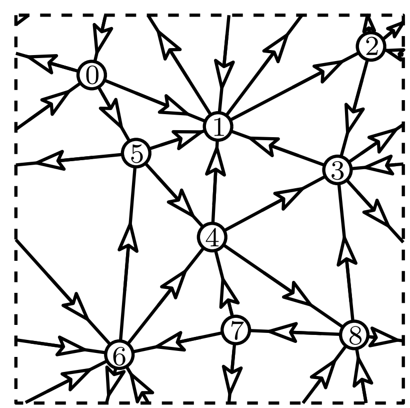


3-orientation
not admitting a valid
Schnyder wood

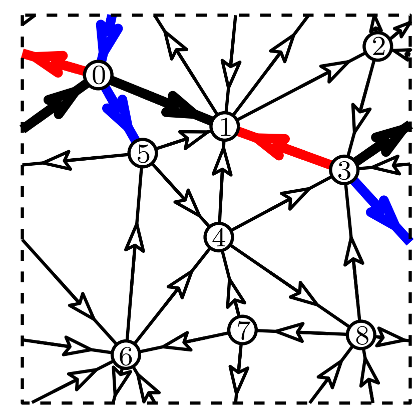


3-orientation
admitting a valid
Schnyder wood

simple toroidal triangulation



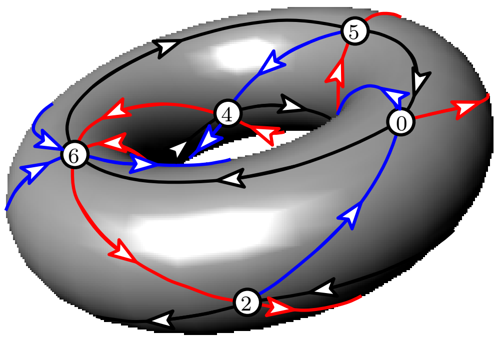
valid 3-orientation

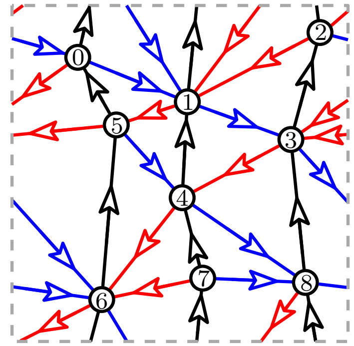


not valid Schnyder wood

(local Schnyder rule cannot be propagated everywhere)

Toroidal Schnyder woods: cycles

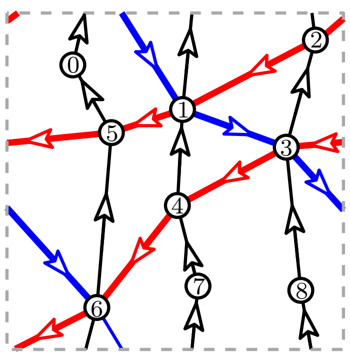


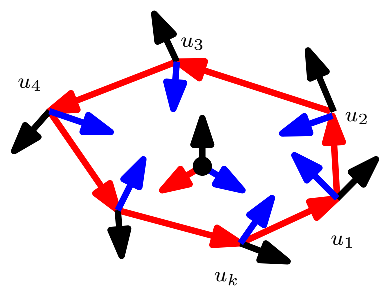


toroidal Schnyder wood

$g = 1$ $\quad e = 3n$
$n - e + f = 2 - 2g$

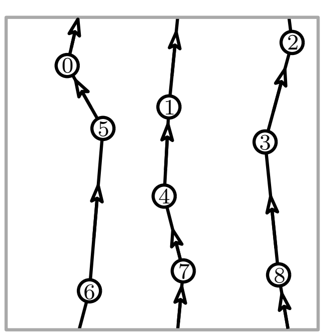
- toroidal Schnyder woods must contain a (mono-chromatic) cycle in each color: $e = 3n$ (n edges in each color)
- mono-chromatic cycles are non-contractibles





Remark: the inner region of a contractible mono-chromatic cycle is a topological disk

- some colors may define disconnected components



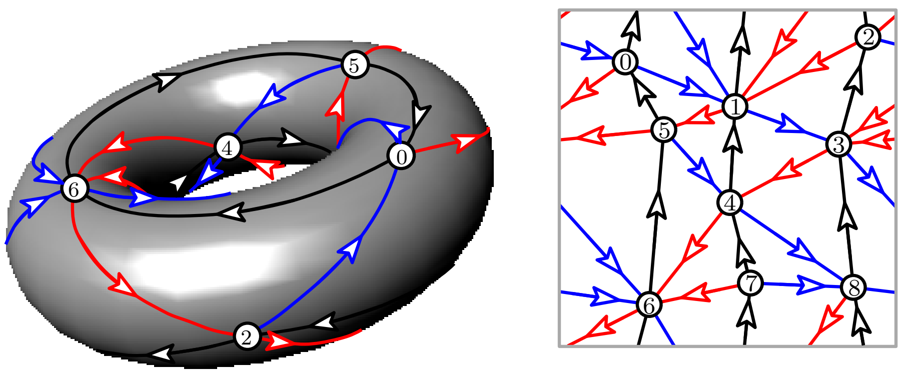
(there are 3 disjoint mono-chromatic cycles of color 2)

Open problem: is it possible to find (at least) one toroidal Schnyder wood with connected mono-chromatic components

n	# irreducible triangulations	#triangulations (g = 1)
7	1	1
8	4	7
9	15	112
10	1	2109
11	–	37867

(true for all triangulations of size at most $n = 11$)

Toroidal Schnyder woods: cycles

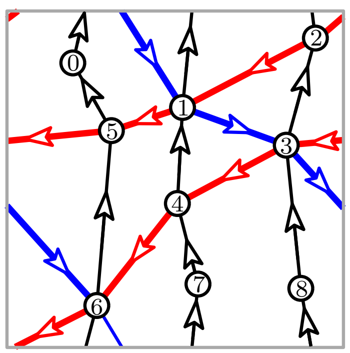


toroidal Schnyder wood

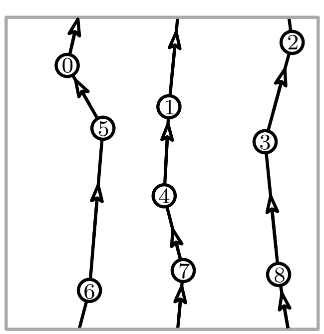
$g = 1$	$e = 3n$
$n - e + f = 2 - 2g$	

- toroidal Schnyder woods must contain a (mono-chromatic) cycle in each color: $e = 3n$

- mono-chromatic cycles are non-contractibles

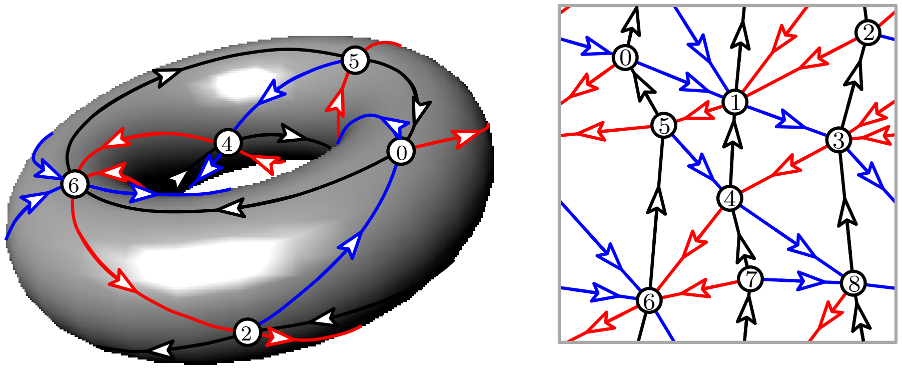


- all mono-chromatic cycles of the same color are homotopic (parallel) and oriented in one direction



Toroidal Schnyder woods: cycles

$g = 1 \qquad e = 3n$
$n - e + f = 2 - 2g$



toroidal Schnyder wood

- toroidal Schnyder woods must contain a (mono-chromatic) cycle in each color: $\qquad e = 3n$

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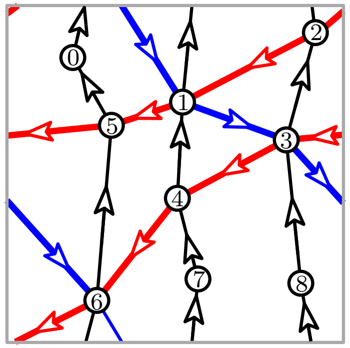
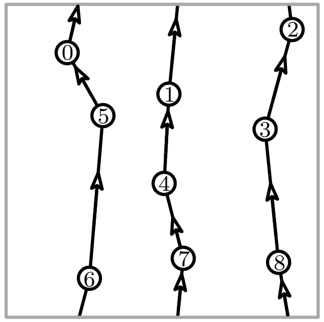
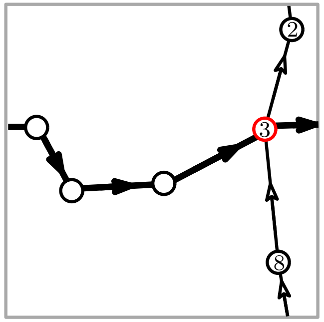
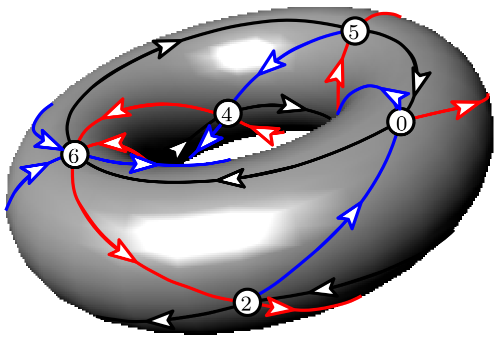


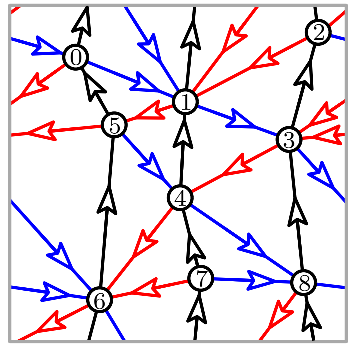
- all mono-chromatic cycles of the same color are:
  homotopic and disjoint (parallel) and oriented in one direction





Toroidal Schnyder woods: cycles



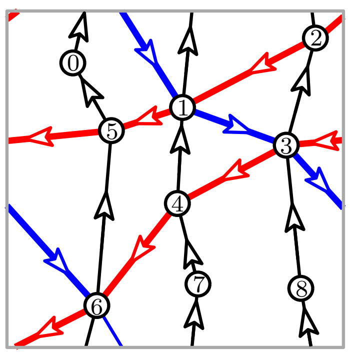


toroidal Schnyder wood

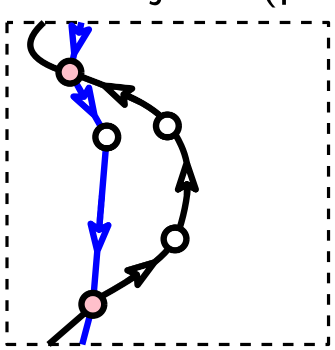
$g = 1 \quad e = 3n$
$n - e + f = 2 - 2g$

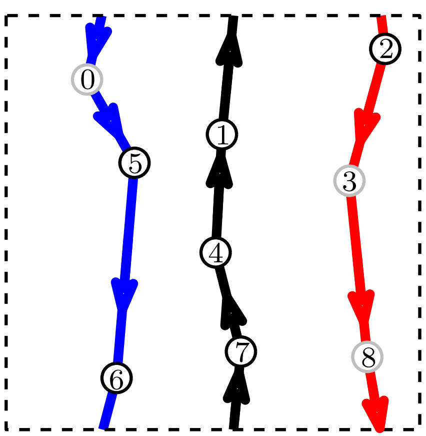
- toroidal Schnyder woods must contain a (mono-chromatic) cycle in each color: $e = 3n$

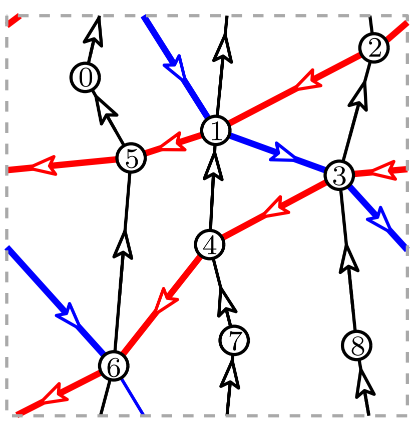
- mono-chromatic cycles are non-contractibles



- all mono-chromatic cycles of different colors are:
  either homotopic and disjoint (parallel) or crossing





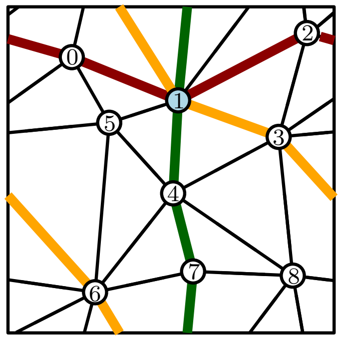


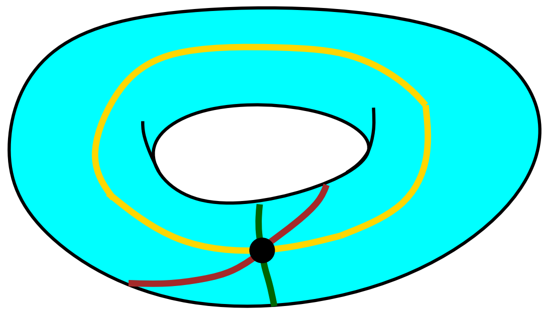
Toroidal Schnyder woods: existence I

Thm[Fijavz, unpublished]

(for simple toroidal triangulations)

A simple toroidal triangulation contains three non-contractible and non-homotopic cycles that all intersect on one vertex and that are pairwise disjoint otherwise.

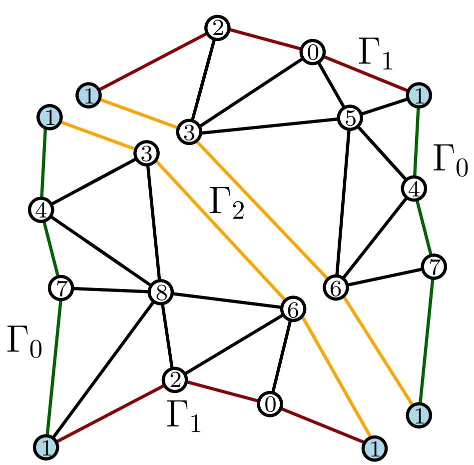




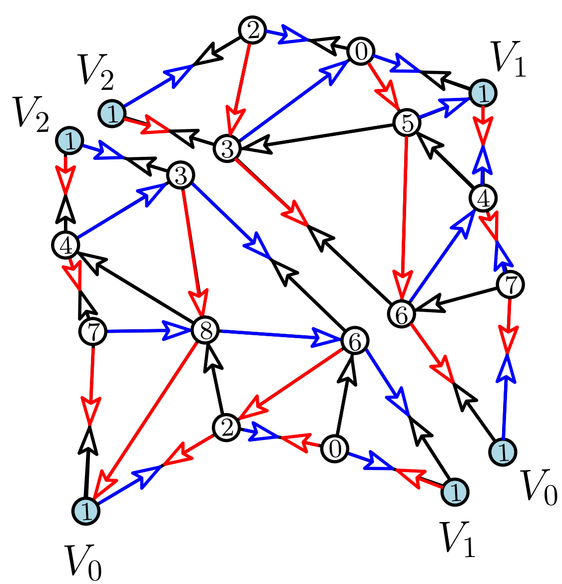
Corollary[Goncalves Lévêque, DCG'14]

Any simple toroidal triangulation admits a toroidal **crossing Schnyder wood**

split along Γ_0, Γ_1, Γ_2

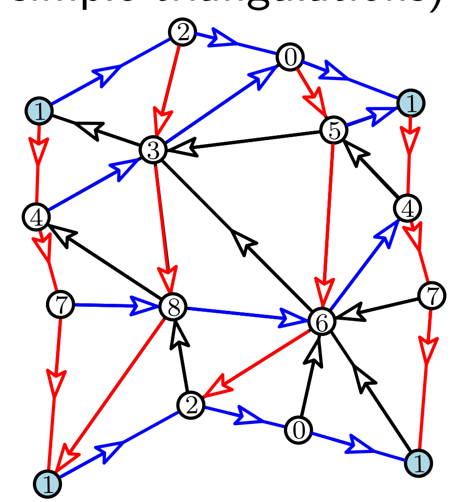


(two planar quasi-triangulations)



crossing toroidal Schnyder wood

(for simple triangulations)



Toroidal Schnyder woods: existence I

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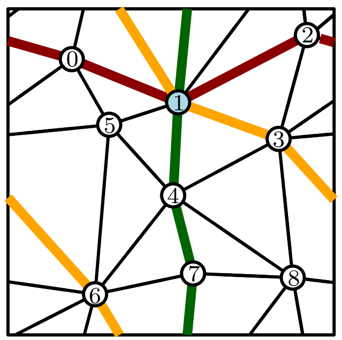
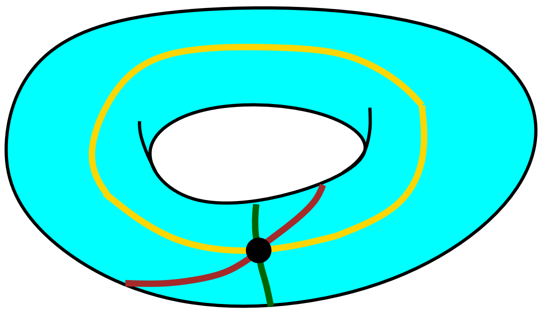
A simple toroidal triangulation contains three non-contractible and non-homotopic cycles that all intersect on one vertex and that are pairwise disjoint otherwise.



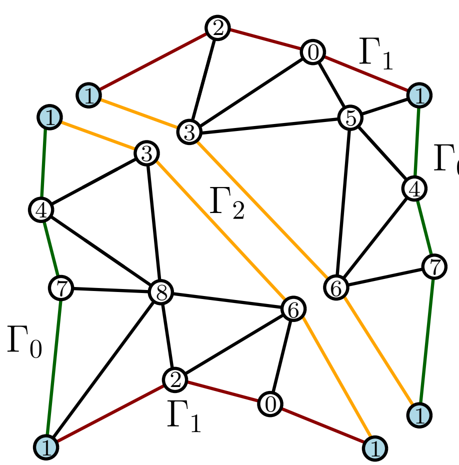


Conjecture: is it possible to find (at least) one toroidal Schnyder wood with connected mono-chromatic components and such the intersection of the three cycles is a single vertex?

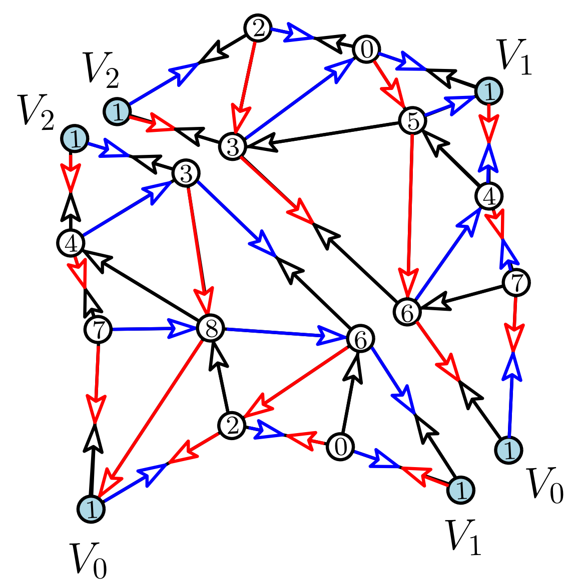
Corollary[Goncalves Lévêque, DCG'14]

Any simple toroidal triangulation admits a toroidal **crossing Schnyder wood**

split along Γ_0, Γ_1, Γ_2

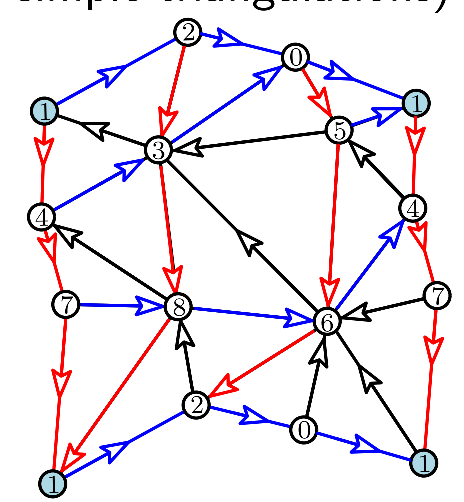


(two planar quasi-triangulations)



crossing toroidal Schnyder wood

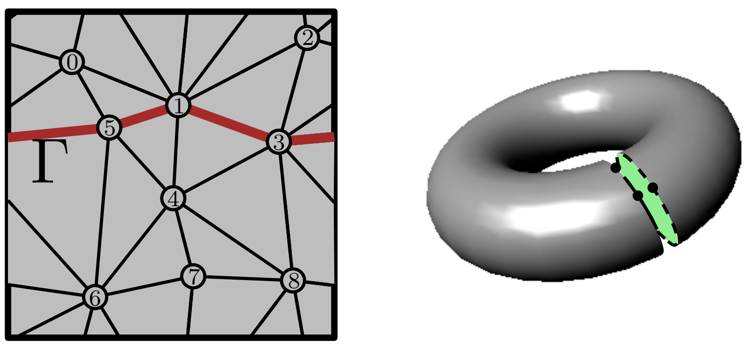
(for simple triangulations)



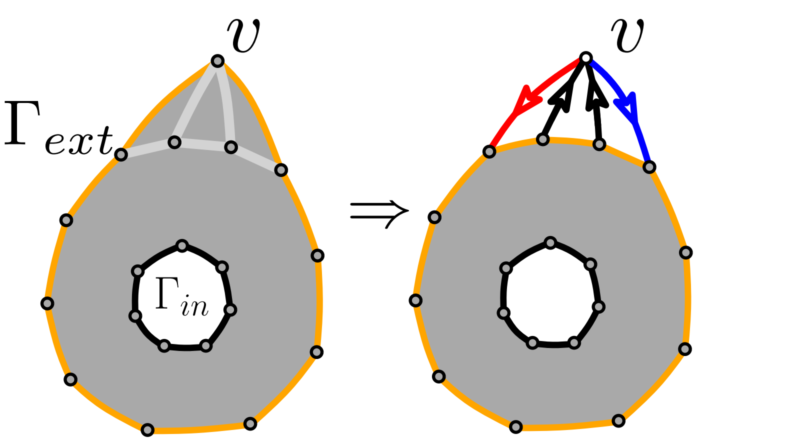
Toroidal Schnyder woods: existence II

(not necessarily crossing Schnyder woods)

First step: cut G along a non contractible cycle Γ
(getting a cylindric triangulation)

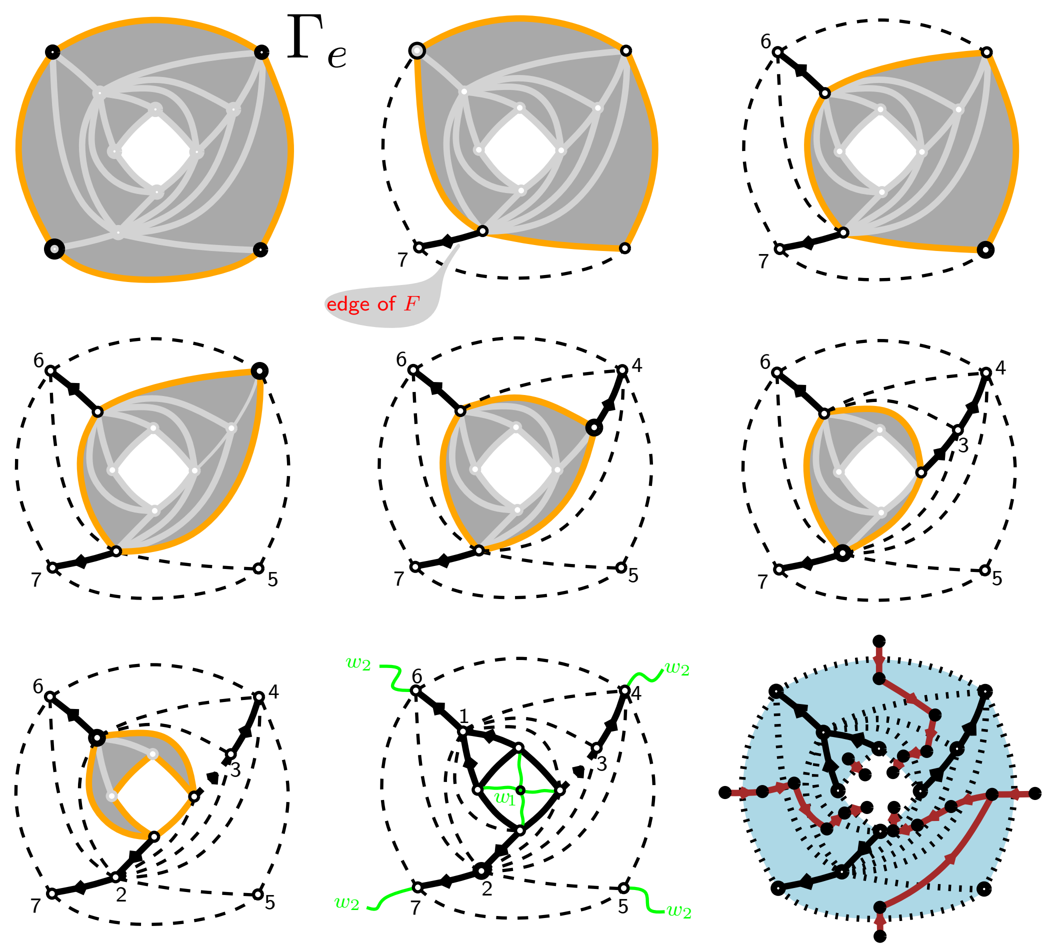


Γ is split into two copies: Γ_{ext} and Γ_{in}



At each step colorient and oriented edges

Second step: compute a cylindric canonical ordering
Performing vertex shelling, starting from Γ_{ext}

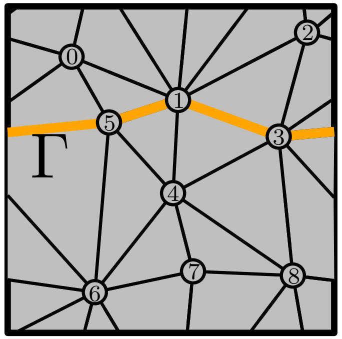


Corollary

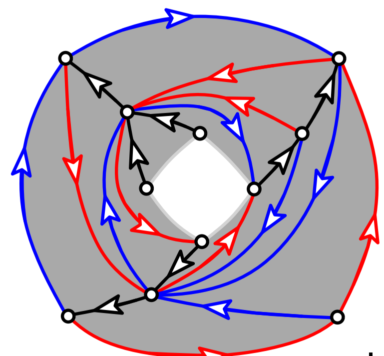
Any simple toroidal triangulation admits a toroidal **(not necessarily crossing) Schnyder wood**

Toroidal Schnyder woods: existence II

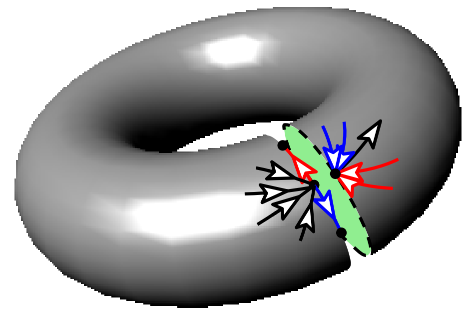
(not necessarily crossing Schnyder woods)



⇒

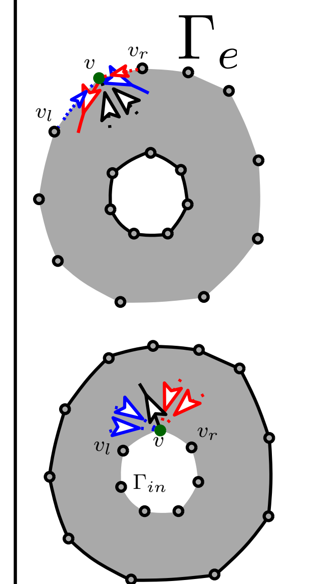


⇒



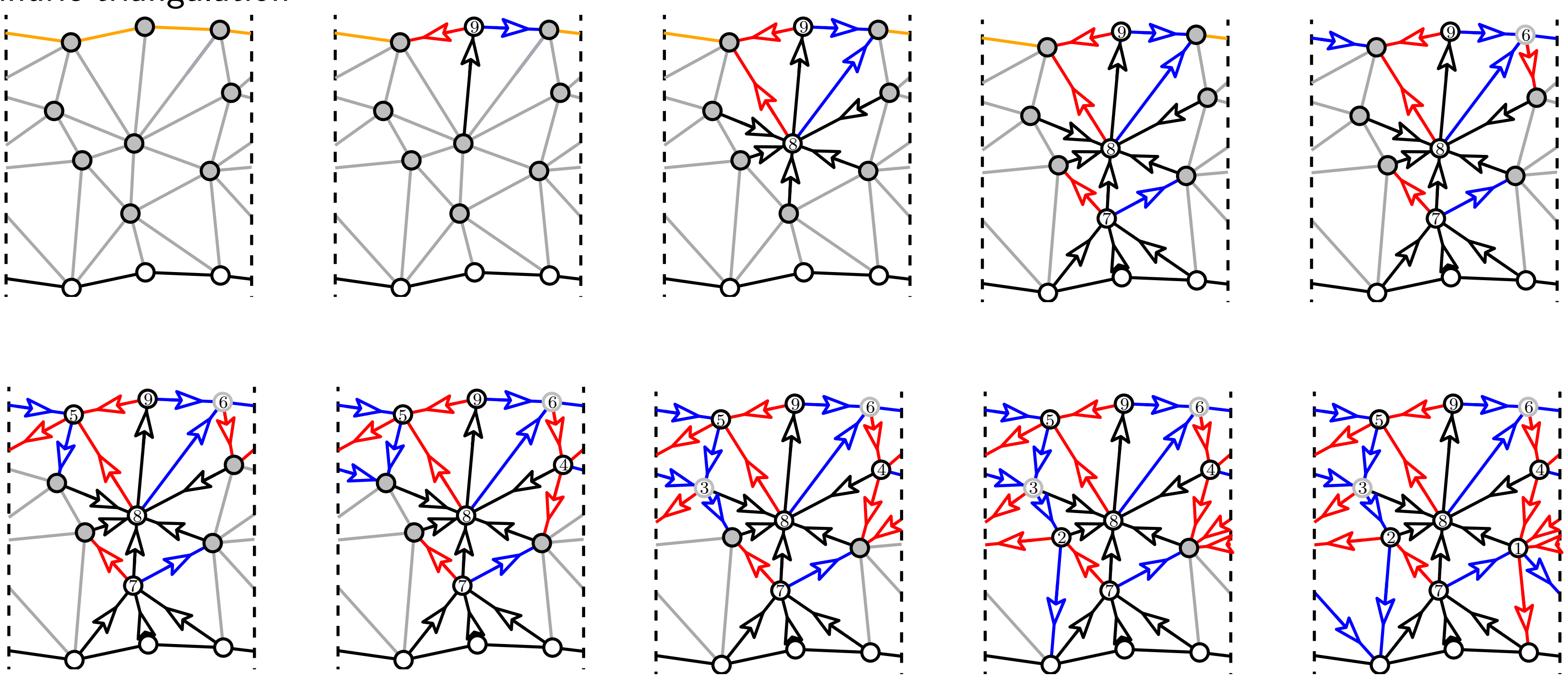
glue together the two boundaries

(the local Schnyder woods remains satisfied on Γ)

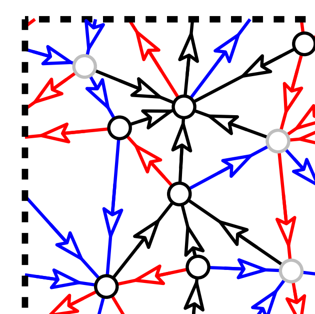


local invariant

Cylindric triangulation

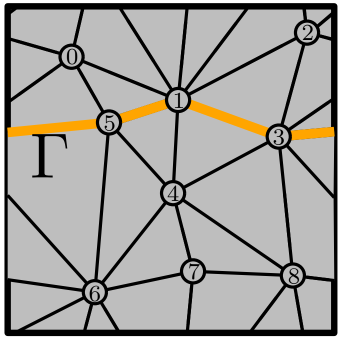


Toroidal Schnyder wood

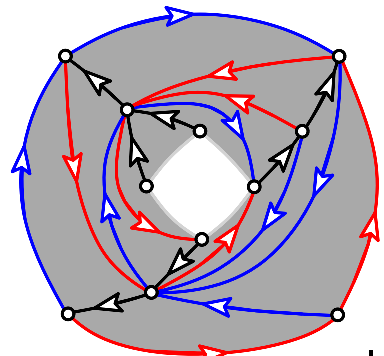


Toroidal Schnyder woods: existence II

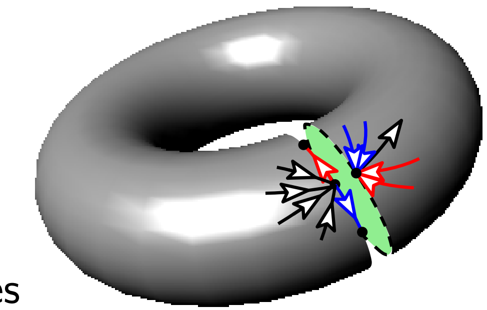
(**not necessarily crossing** Schnyder woods)



⇒



⇒



glue together the two boundaries

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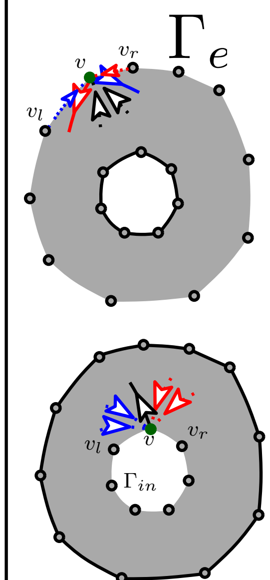
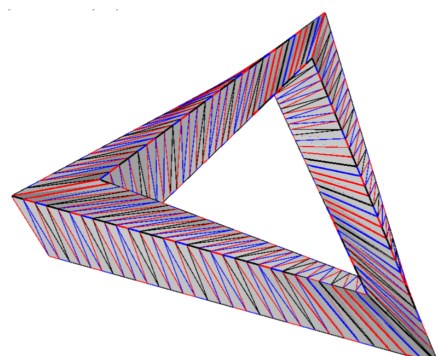


local invariant

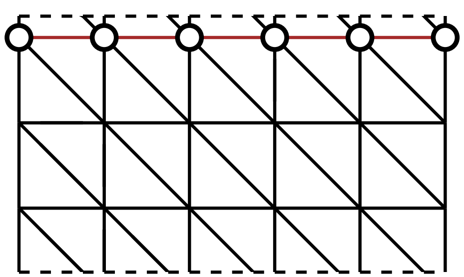
Thm

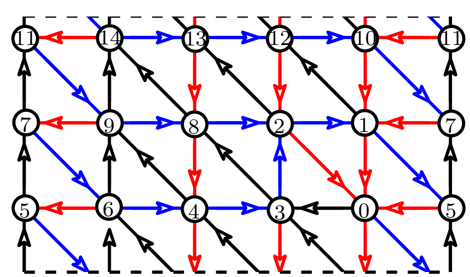
Any simple toroidal triangulation admits a toroidal **(not necessarily crossing) Schnyder wood**
(which can be computed in linear time)



Penrose triangle

$n = 360$





red and black cycles are homotopic (not crossing)

Open problem

Is it possible to modify the vertex shelling order to preserve the crossing condition?

Toroidal Schnyder woods: existence III

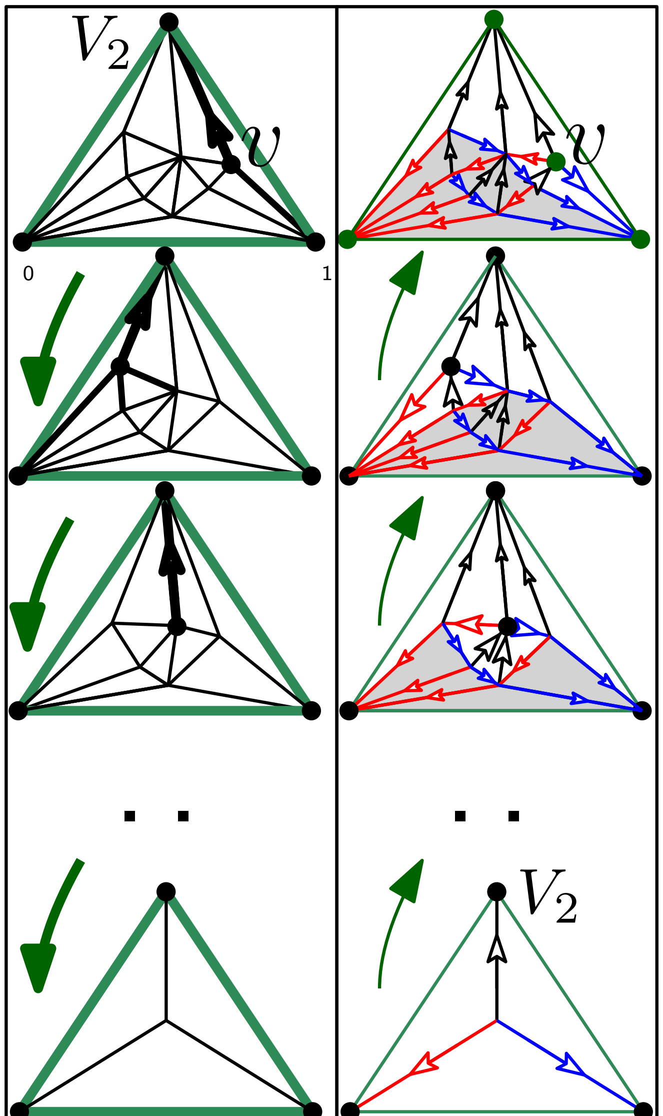
Thm[Goncalves Lévêque, DCG'14]

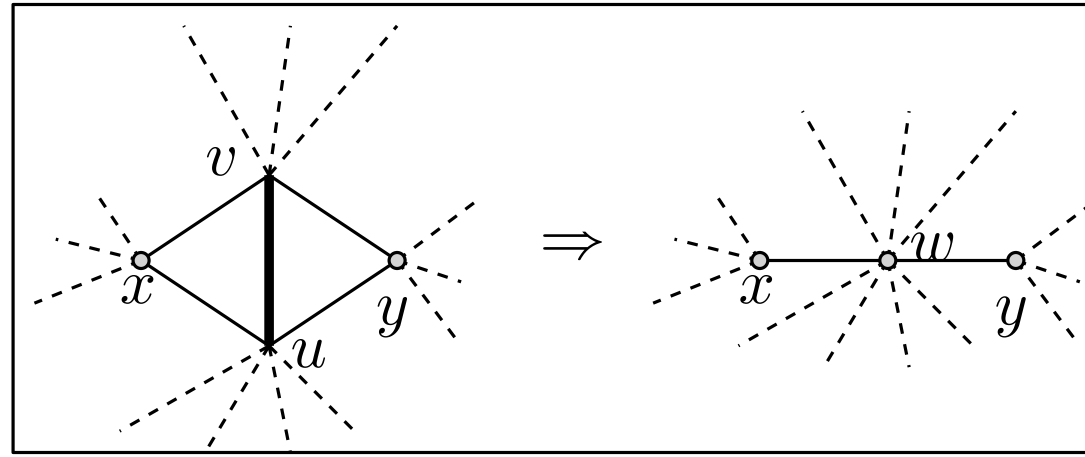
(for general toroidal triangulations)

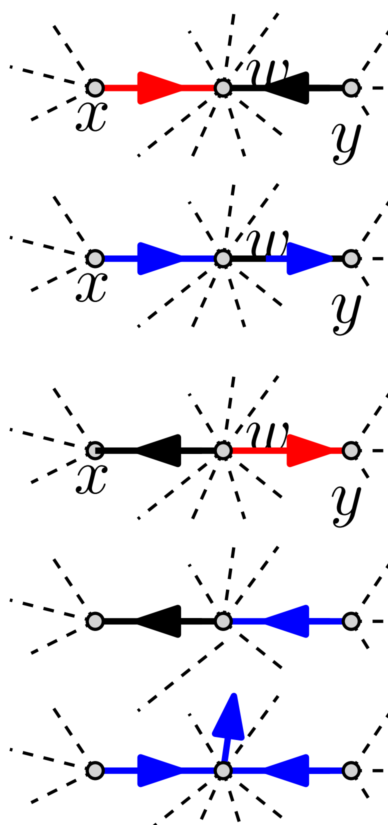
Any toroidal triangulation admits a toroidal **crossing Schnyder wood**

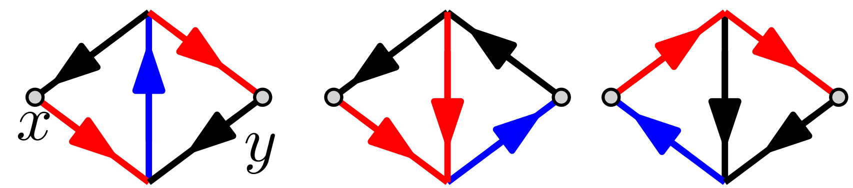
computation of (planar) Schnyder woods

first phase: perform edge contractions
second phase: perform edge expansion+edge coloring









(3 possible choices of coloring)

(5 cases to distinguish)

Toroidal Schnyder woods: existence III

Thm[Goncalves Lévêque, DCG'14] (for general toroidal triangulations)

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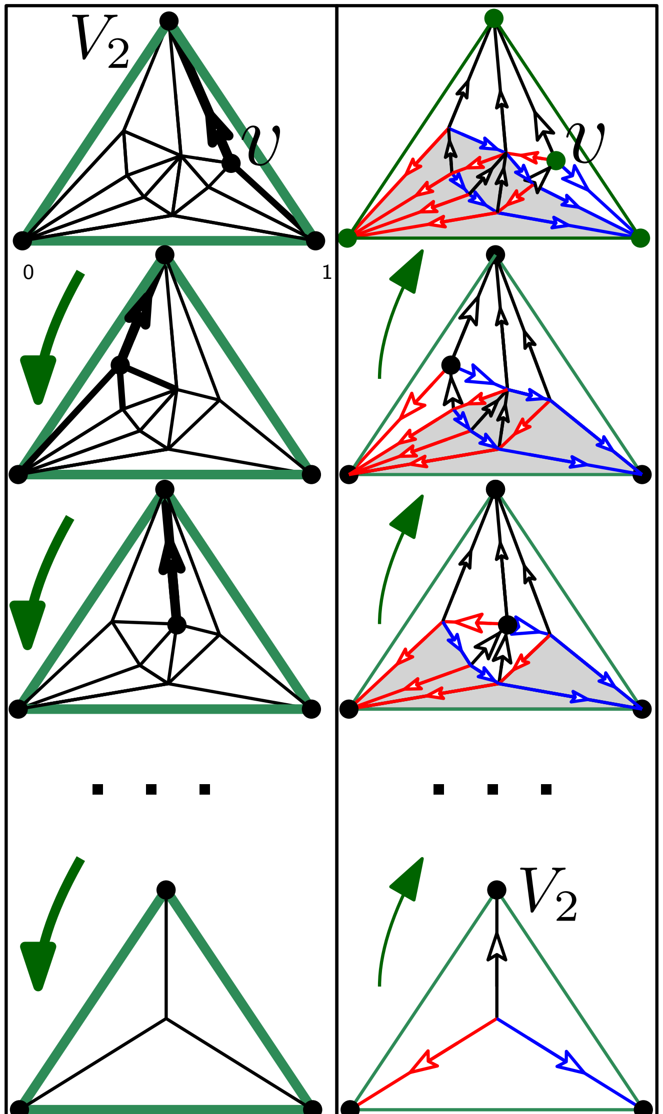
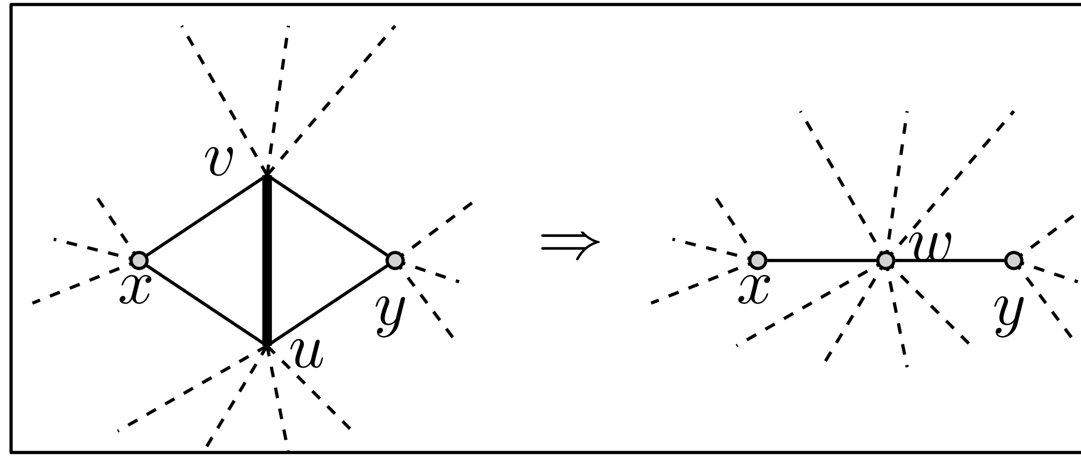
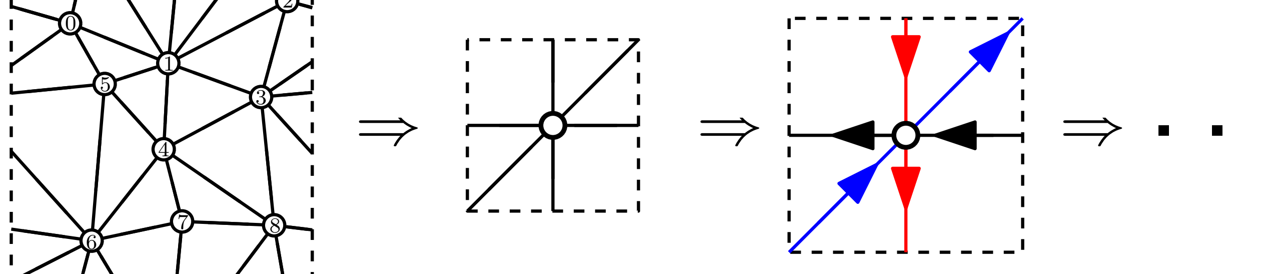




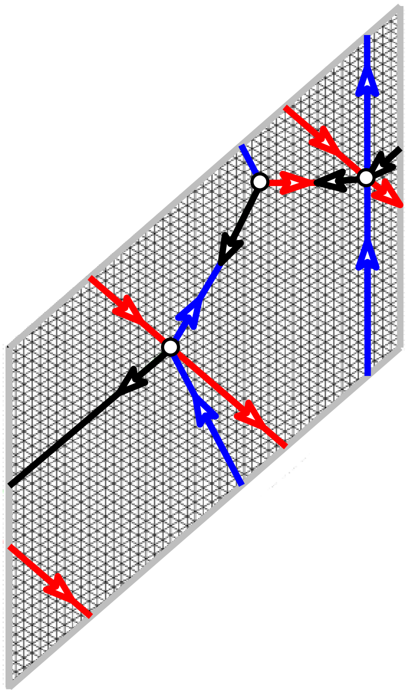
color and orient



perform a sequence of $n-1$ edge contractions

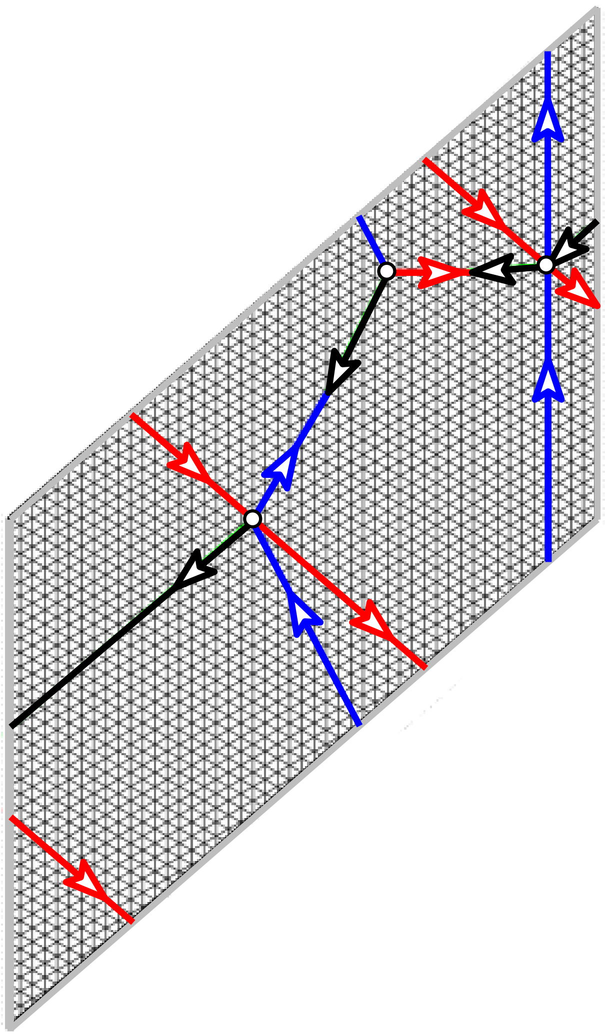
decontract (and color/orient)

Periodic (planar) Schnyder drawings of toroidal graphs



Toroidal Schnyder drawings

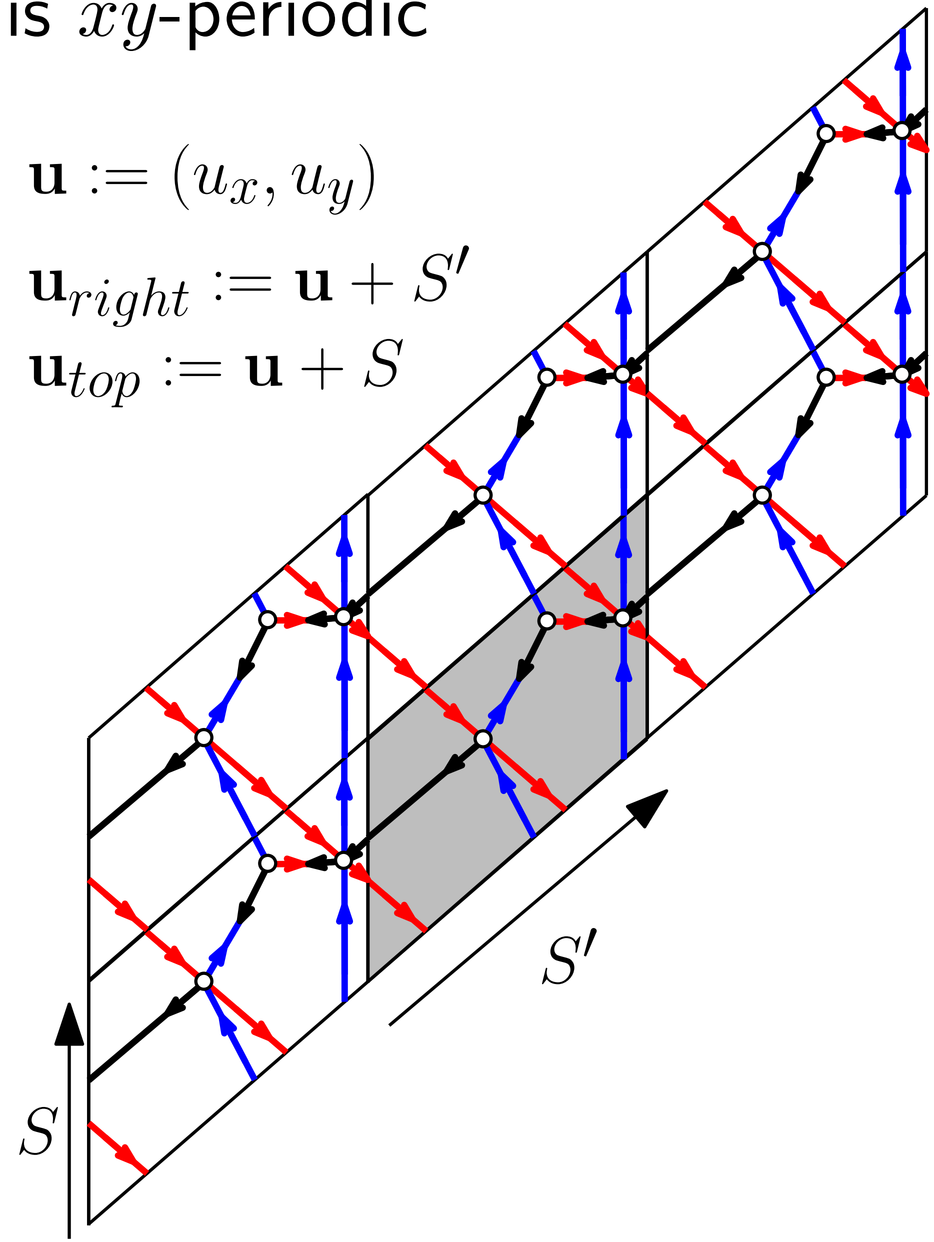
Goal: try to generalize the region counting method to obtain a straight-line grid drawing which is xy-periodic



$$\mathbf{u} := (u_x, u_y)$$

$$\mathbf{u}_{right} := \mathbf{u} + S'$$

$$\mathbf{u}_{top} := \mathbf{u} + S$$



Region counting on the torus

How regions are defined on the torus?

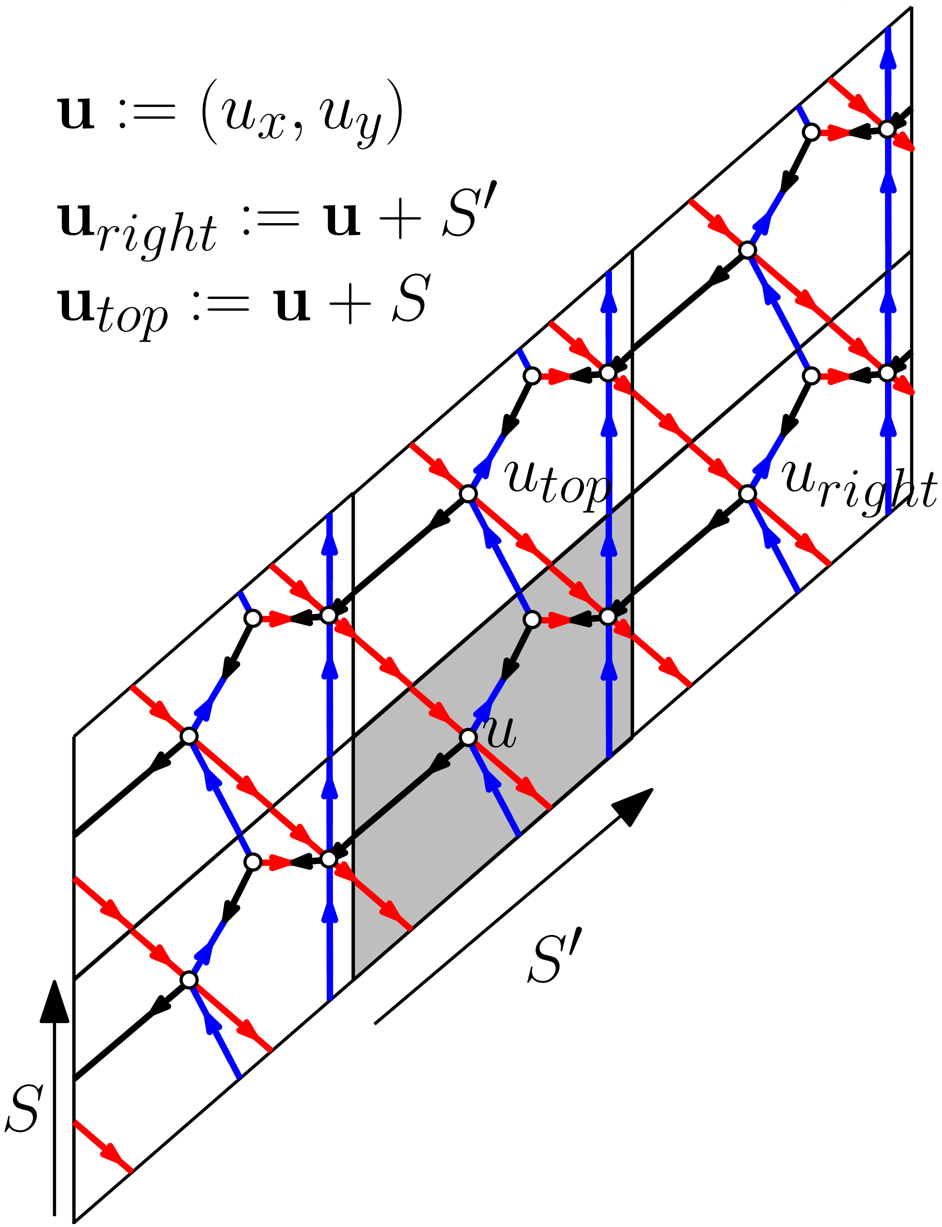
How to assign coordinates to vertices to ensure periodicity?

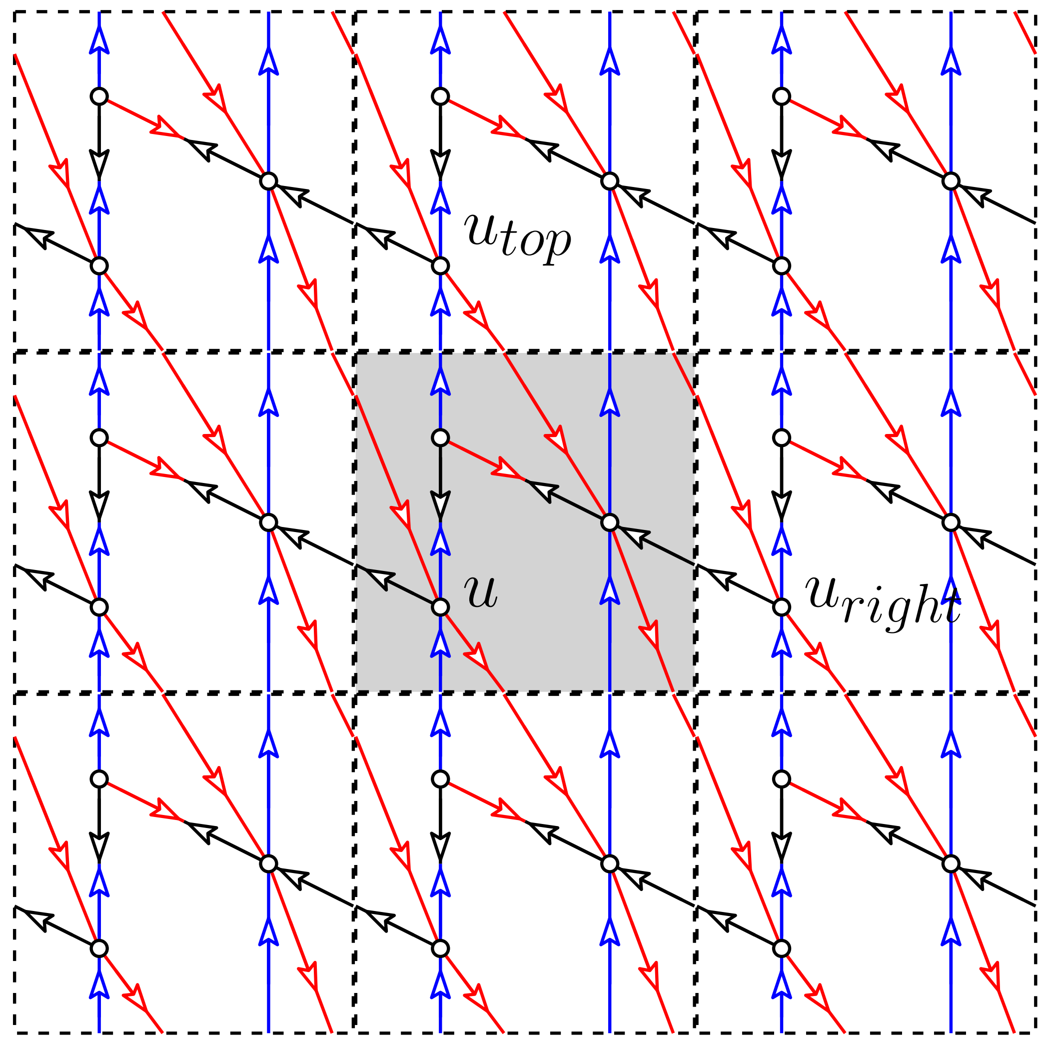
How periodicity is defined? (how vectors S, S' are defined?)

$$\mathbf{u} := (u_x, u_y)$$

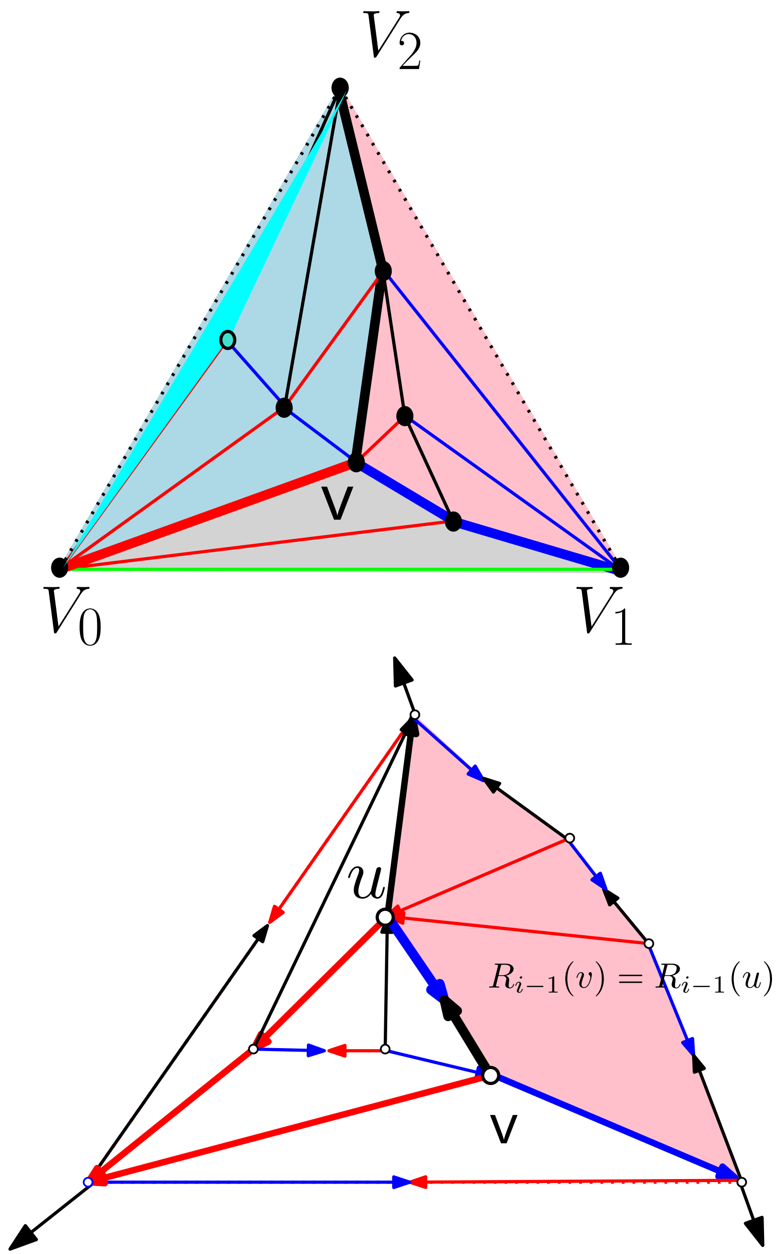
$$\mathbf{u}_{right} := \mathbf{u} + S'$$

$$\mathbf{u}_{top} := \mathbf{u} + S$$

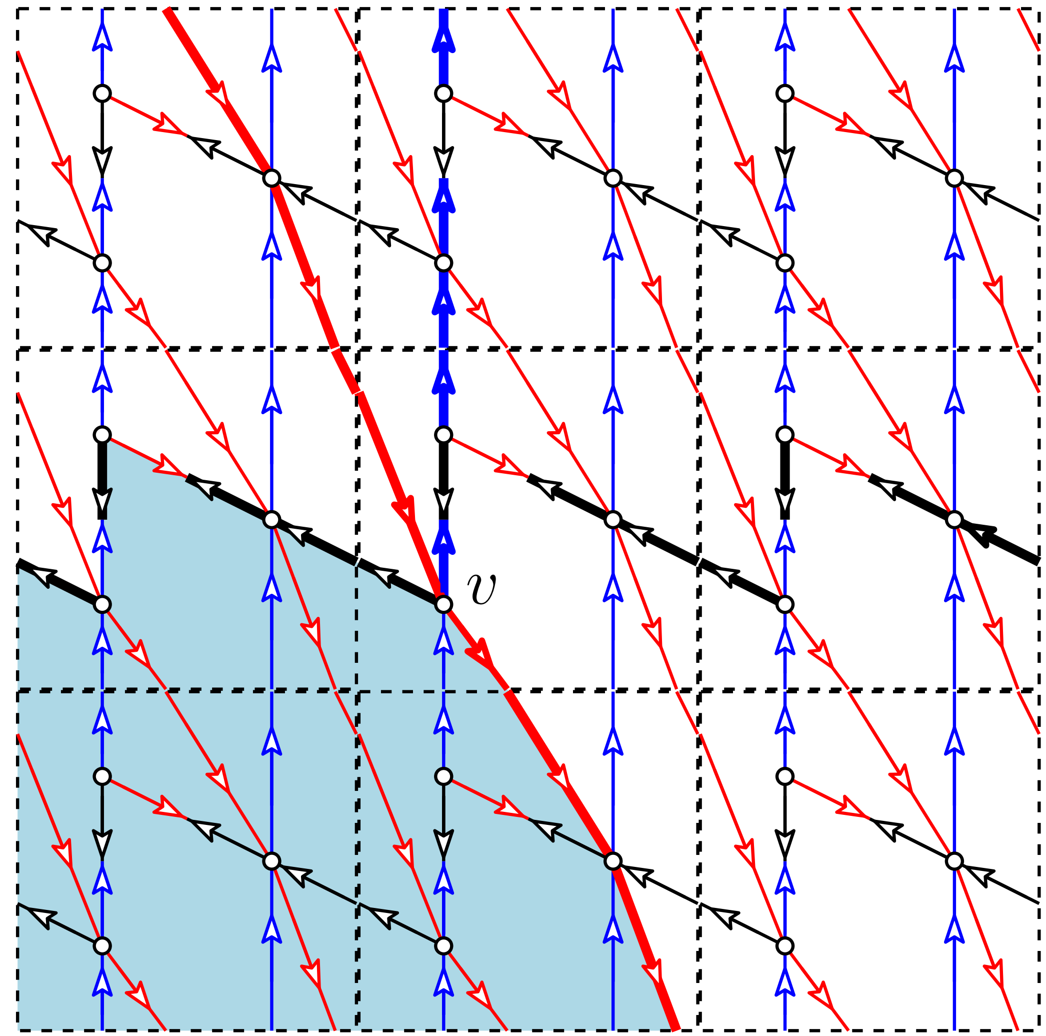




Regions are unbounded



Planar case: bounded number of faces in each region



Toroidal case: unbounded regions

$$v =: \frac{|R_0(v)|}{|F|-1} V_0 + \frac{|R_1(v)|}{|F|-1} V_1 + \frac{|R_2(v)|}{|F|-1} V_2$$

Regions are unbounded

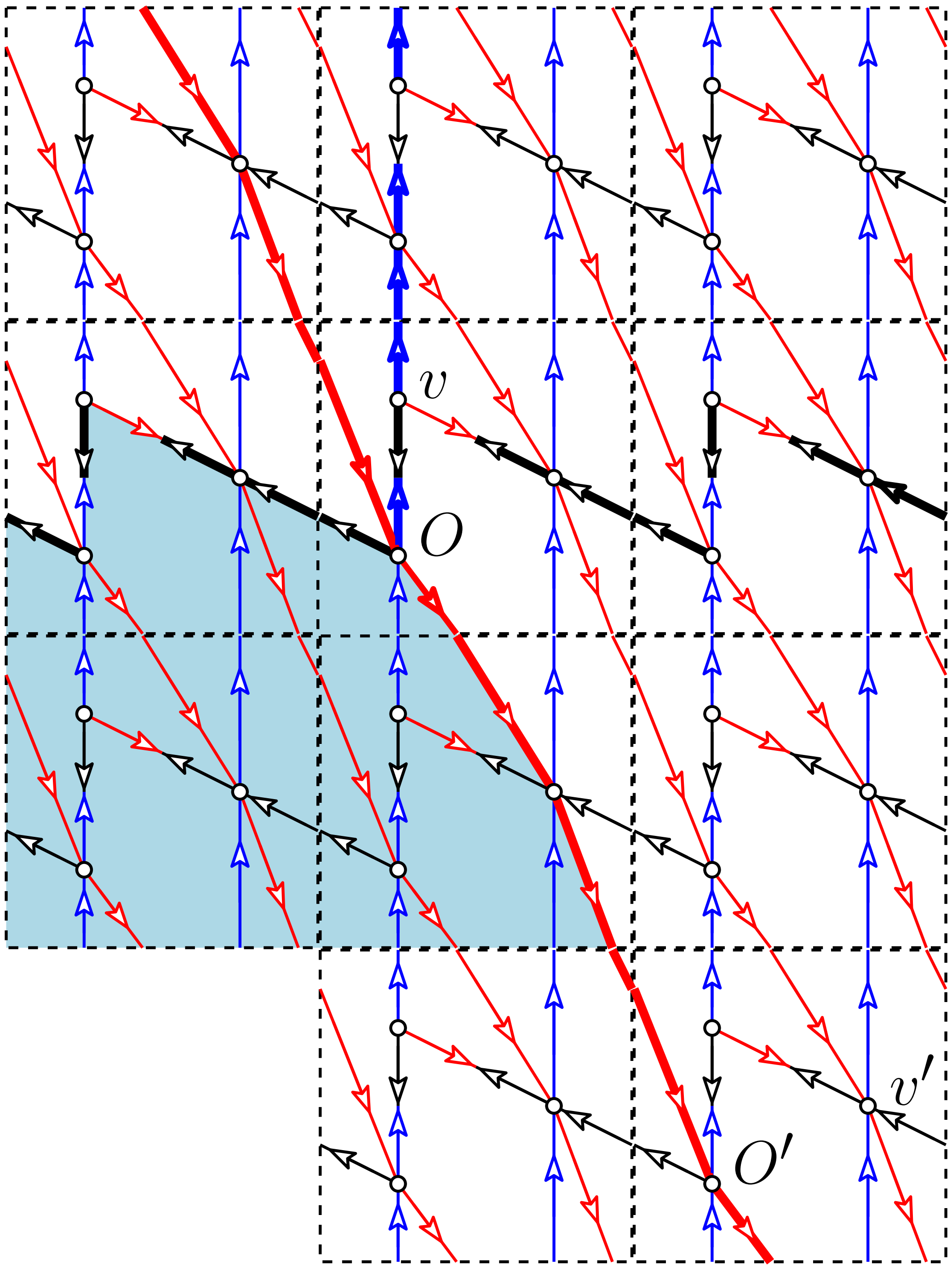
Do not use absolute coordinates

$v =: \frac{|R_0(v)|}{|F|-1} V_0 + \frac{|R_1(v)|}{|F|-1} V_1 + \frac{|R_2(v)|}{|F|-1} V_2$

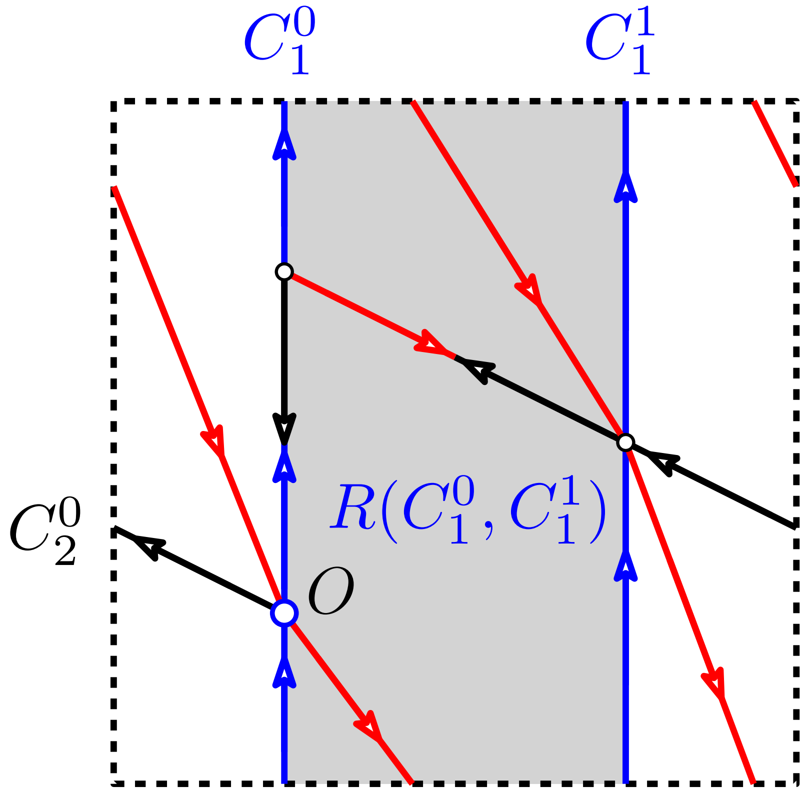
Toroidal case: regions are unbounded but differences between regions is bounded

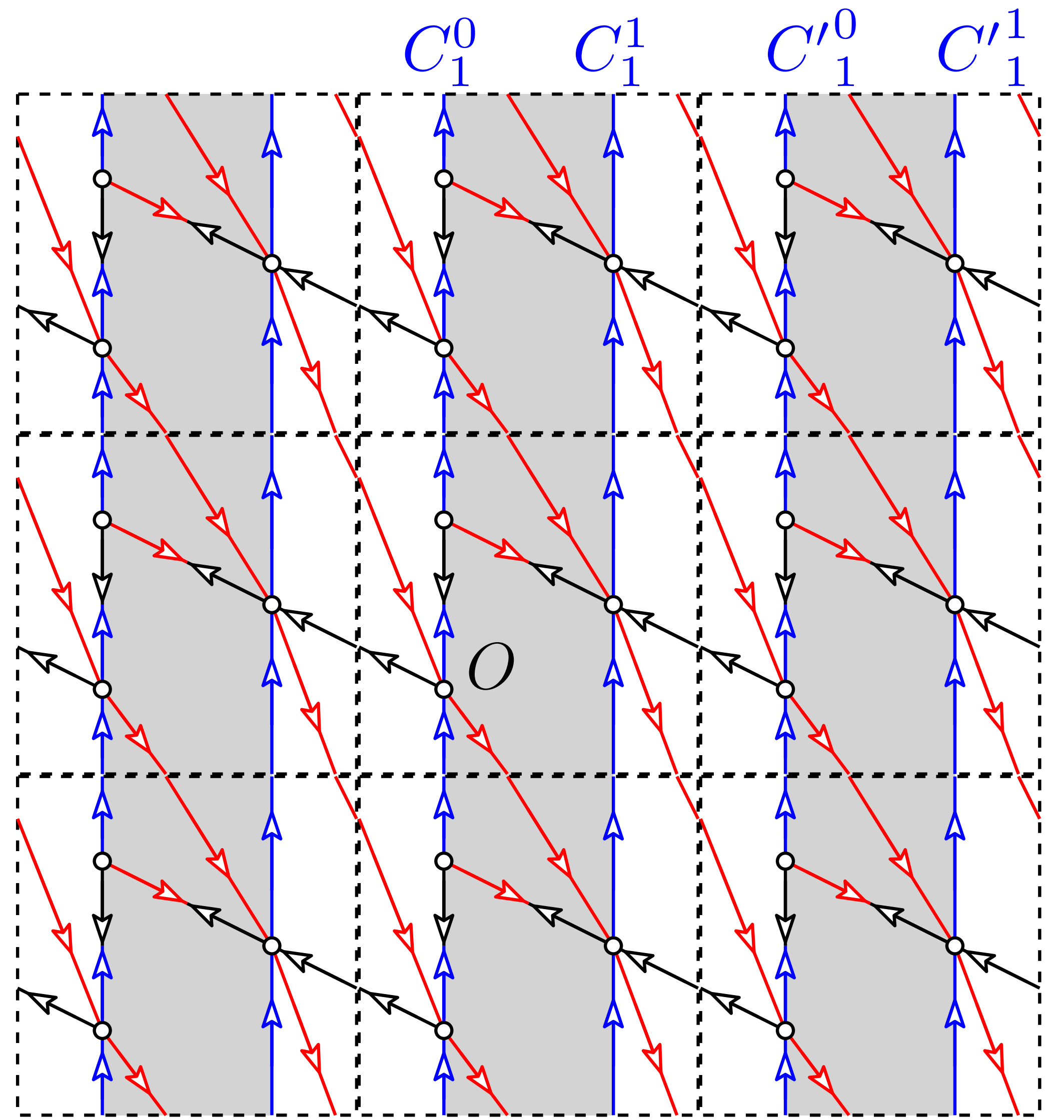
Fix an origin vertex O

Define coordinates of v relatives to O



How to define the size of a region





$\mathcal{C}_1 := \{C_1^0, C_1^1\}$

2 mono-chromatic consecutive blue cycles

$\mathcal{C}_2 := \{C_2^0\}$

1 mono-chromatic black cycle

$\mathcal{L}_1^0 := \{C_1^0, {C'}_1^0, \ldots\}$ (lines in the universal cover)

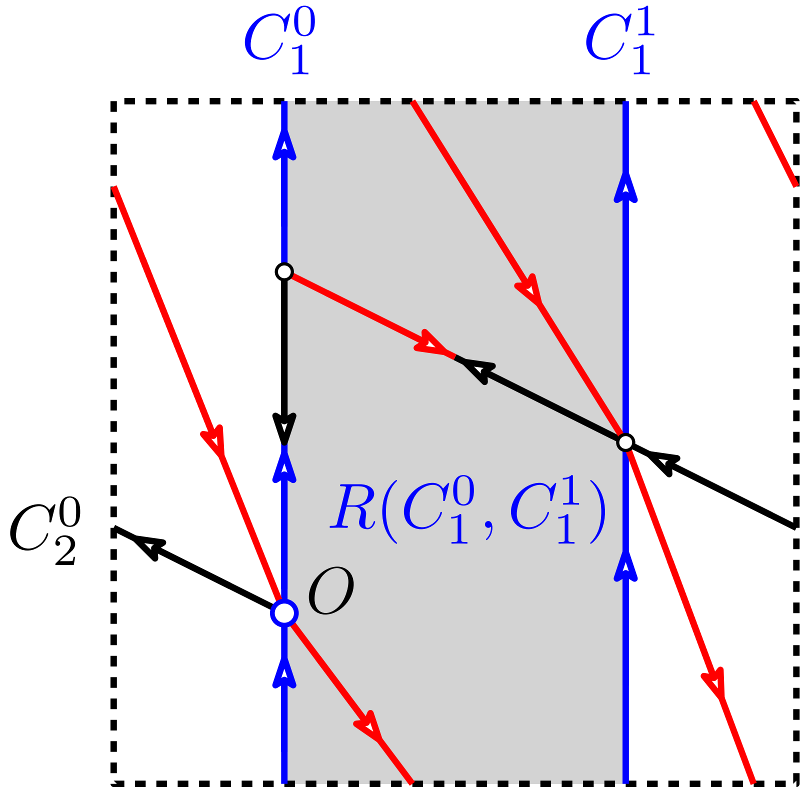
$R(C_1^j, C_1^{j+1}) :=$ region between consecutives 1-cycles

$f_1^j := \|R(C_1^j, C_1^{j+1})\|$ (size of the 1-region: number of faces)

(how many faces in the gray region?)

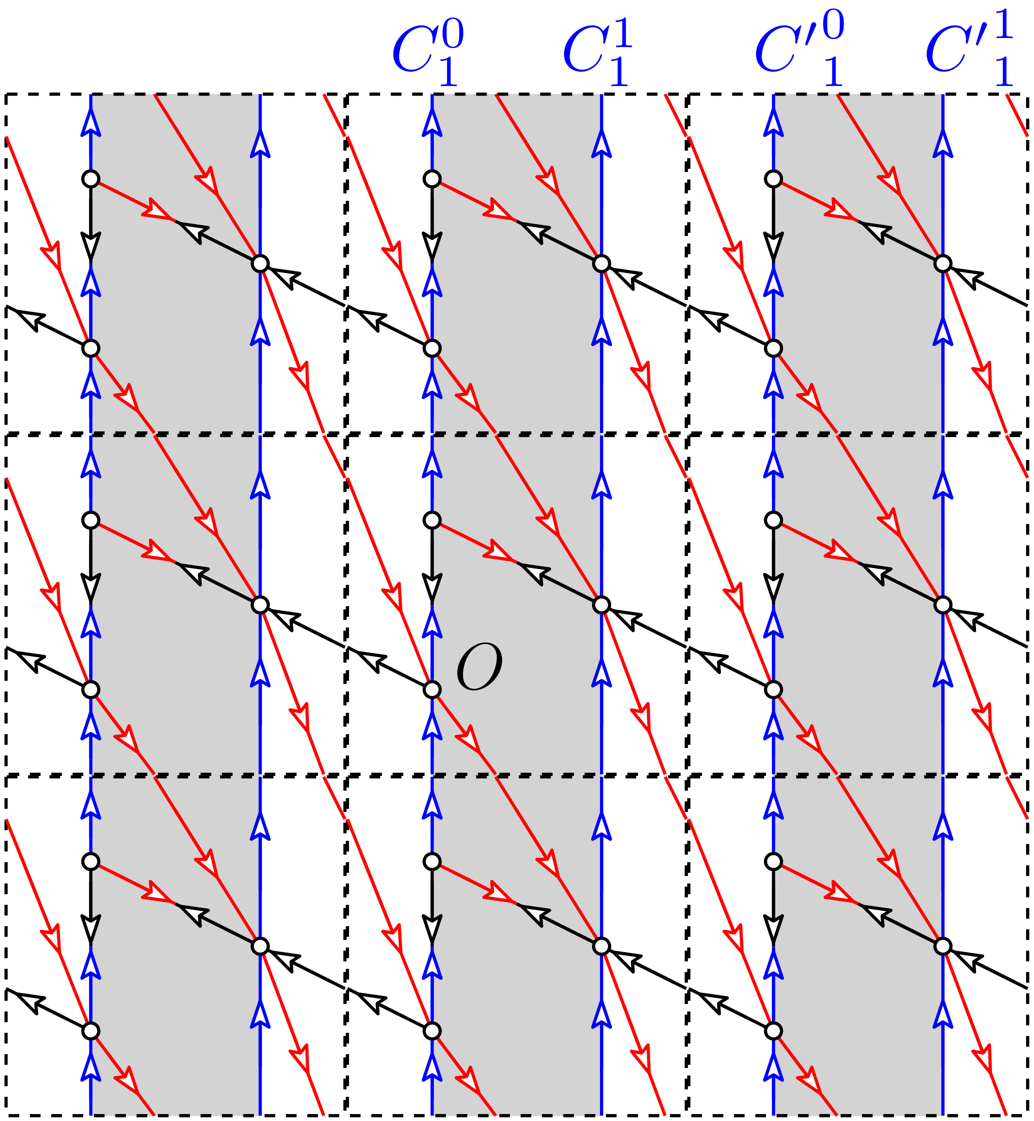
$\|R(C_1^0, C_1^1)\| = ?$

How to define the size of a region



$\mathcal{C}_1 := \{C_1^0, C_1^1\}$
2 mono-chromatic consecutive blue cycles

$\mathcal{C}_2 := \{C_2^0\}$
1 mono-chromatic black cycle



$$\sum_j \|R(C_i^j, C_i^{j+1})\| = F$$

(for each color $i \in \{0,1,2\}$)

where $F{:=}$ number of faces of G

$\|R(C_0^0, C_0^0)\| = 4$

$\|R(C_2^0, C_2^0)\| = 4$

(2 faces in the gray region)

$\|R(C_1^0, C_1^1)\| = 2$

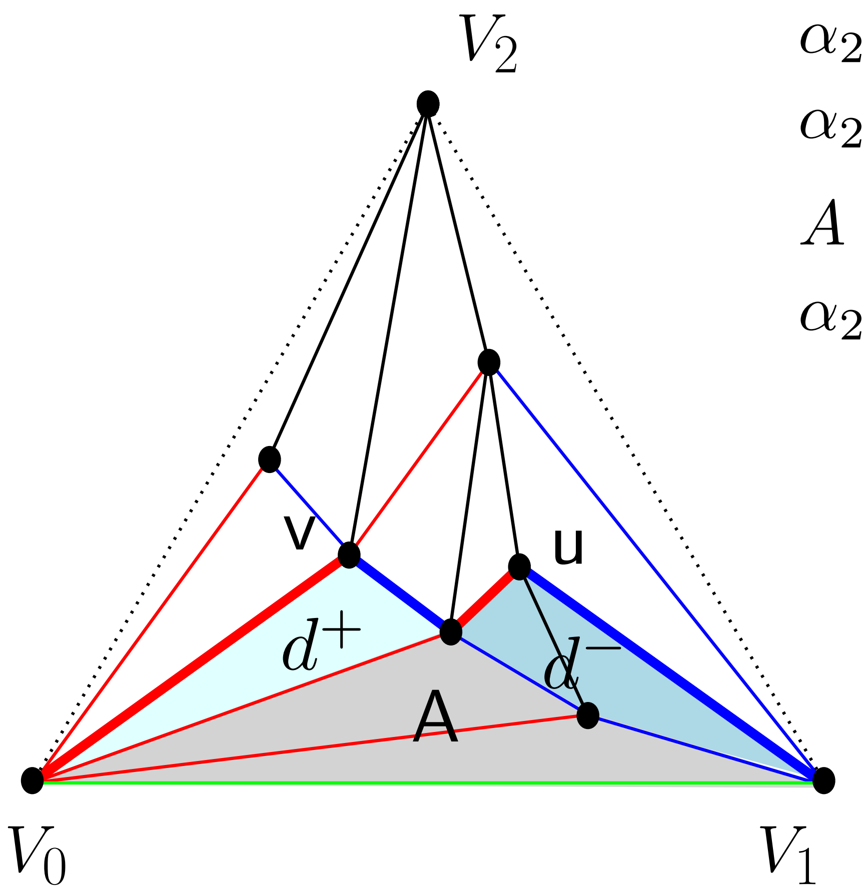
(2 faces in the white region)

$\|R(C_1^1, C_1^0)\| = 2$

How assign to ccordinates to vertices

Goal: assign relative coordinates to vertices

Let us revise the planar case first



$$\alpha_2(v) =: |R_2(v)| = 3$$
$$\alpha_2(u) =: |R_2(u)| = 4$$
$$A =: |R_2(v) \cap R_2(u)| = 2$$
$$\alpha_2(v) = 4 + (1 - 2)$$

$$\alpha_2(v) =: |R_2(v)| = A + d^+$$
$$\alpha_2(u) =: |R_2(u)| = A + d^-$$

$$\boxed{\alpha_2(v) = \alpha_2(u) + (d^+ - d^-)}$$

$$v =: \frac{|R_0(v)|}{|F|-1} V_0 + \frac{|R_1(v)|}{|F|-1} V_1 + \frac{|R_2(v)|}{|F|-1} V_2$$
$$v =: \alpha_0 V_0 + \alpha_1 V_1 + \alpha_2 V_2$$

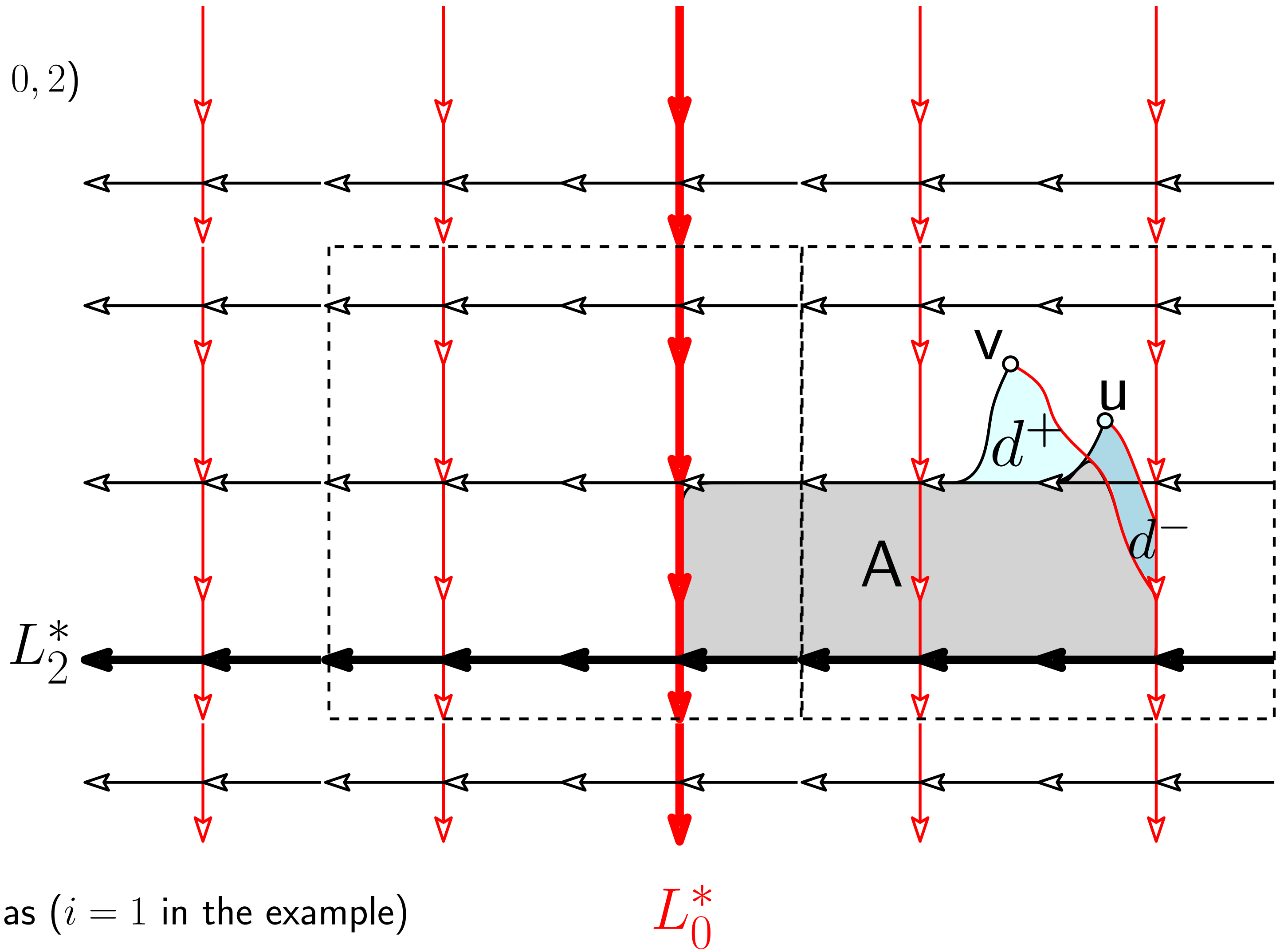
How assign to ccordinates to vertices

Goal: assign relative coordinates to vertices

Let us consider now the toroidal case

Consider two vertices u and v in the same "region" (defined by the same mono-chromatic lines)

Choose two references line L_i^* $(i = 0, 2)$



the i-coordinate of v is expressed as $(i = 1$ in the example)

$$\alpha_i(v) = \alpha_i(u) + (d^+ - d^-)$$

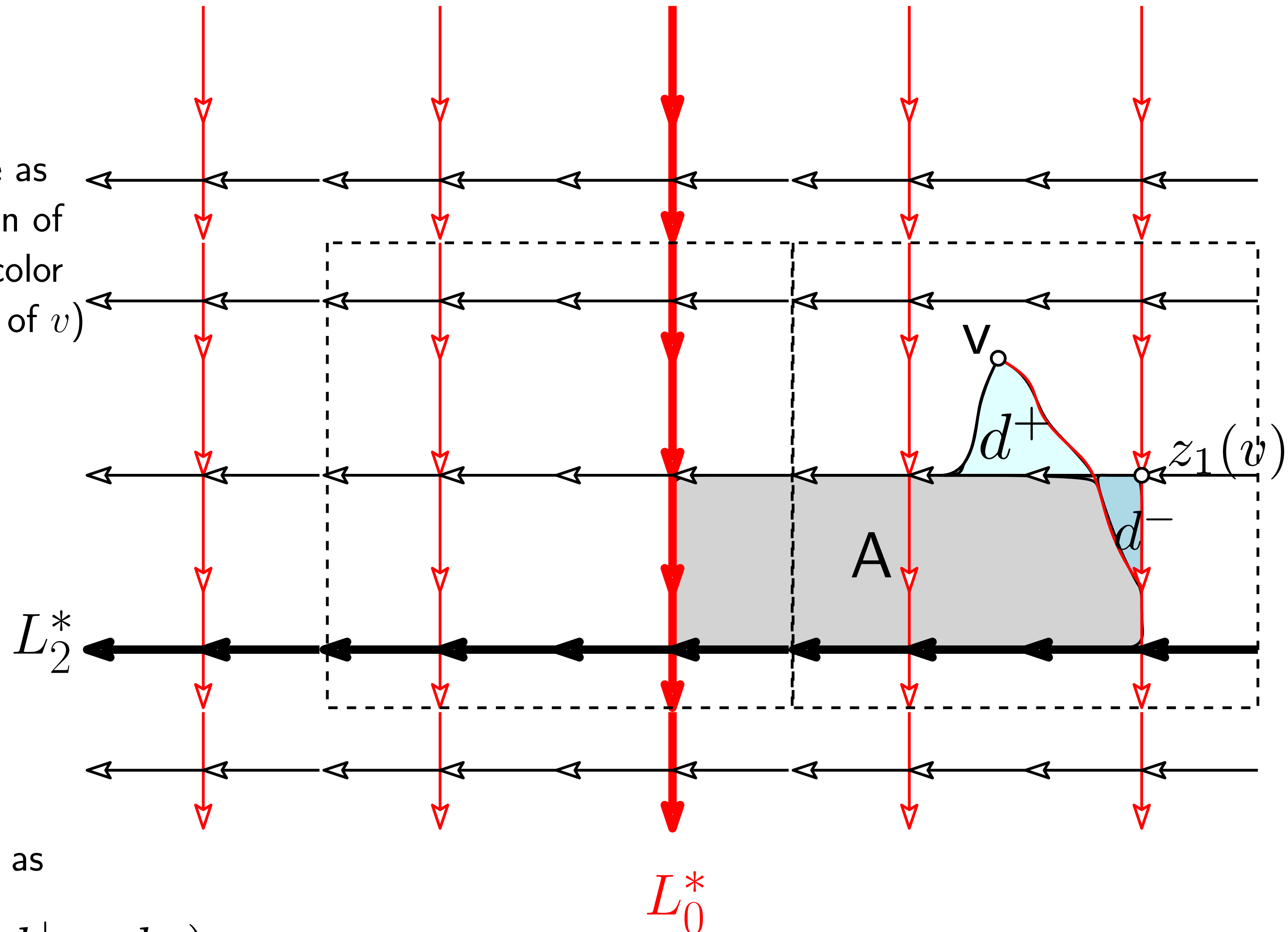
How assign to ccordinates to vertices

Goal: assign relative coordinates to vertices

Let us consider now the toroidal case

Consider two vertices u and v in the same "region" (defined by the same mono-chromatic lines)

Given a vertex v and a color i take as second vertex $z_i(v)$, the intersection of the two mono-chromatic lines (of color $i-1$ and $i+1$ defining the region of v)



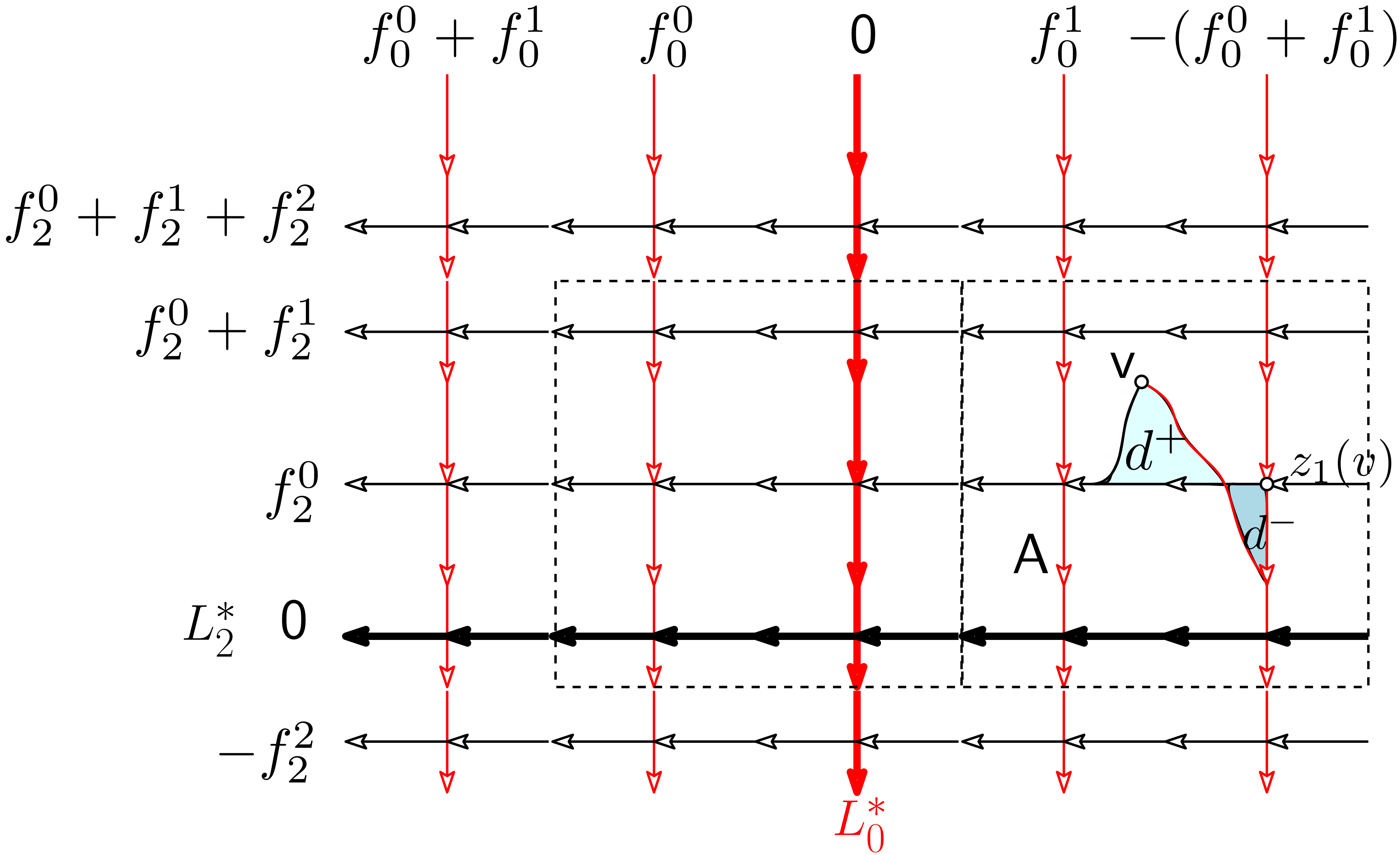
the 2-coordinate of v is expressed as

$$\alpha_2(v) = \alpha_2(z(v)) + (d^+ - d^-)$$

How assign to ccordinates to vertices

Goal: assign relative coordinates to vertices

Assign coordinates to the mono-chromatic lines



Remark: the signs depend on the relative position of the mono-chromatic lines with respect to the reference lines L_i^* (top/bottom, left/right)

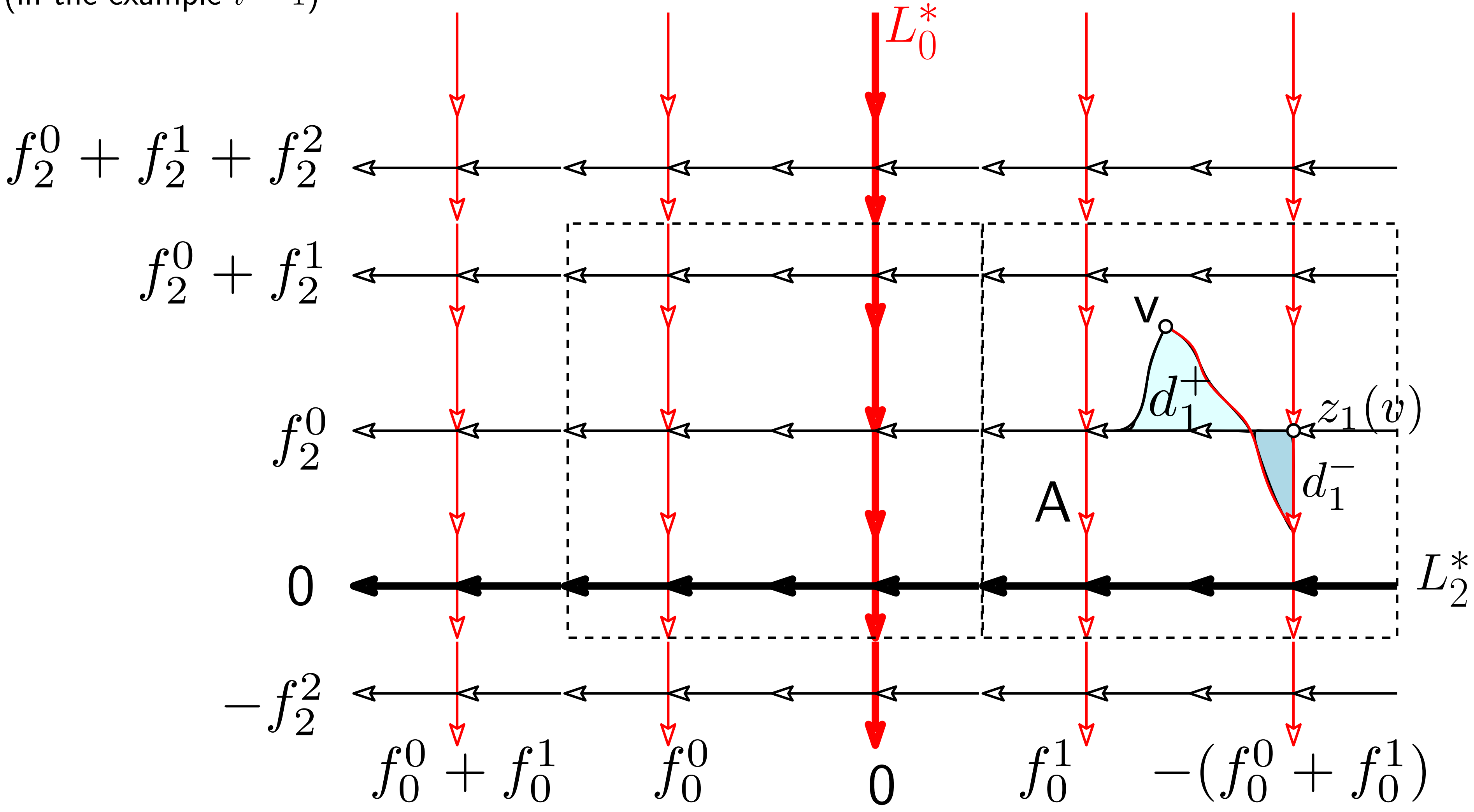
How assign to ccordinates to vertices

We can now define the i coordinate α_i of a vertex v (N constant, appropriately choosen)

$$\boxed{\alpha_i(v) := d_i(v, z_i(v)) + N \cdot (f_{i+1}(L_{i+1}(v)) - f_{i-1}(L_{i-1}(v)))}$$

$$\alpha_1(v) := (d_1^+ - d_1^-) + N \cdot (f_2^0 - (f_0^0 + f_0^1))$$

(in the example $i = 1$)



How assign to ccordinates to vertices

We can now define the i coordinate α_i of a vertex v

$$\boxed{\alpha_i(v) := d_i(v, z_i(v)) + N \cdot (f_{i+1}(L_{i+1}(v)) - f_{i-1}(L_{i-1}(v)))}$$

(set $N = 3$ as constant in this example)

Assign $(0,0,0)$ to the origin vertex O

Observe that $z_0(v)$ coincides with v

so: $d_0(v, z_0(v)) = 0$

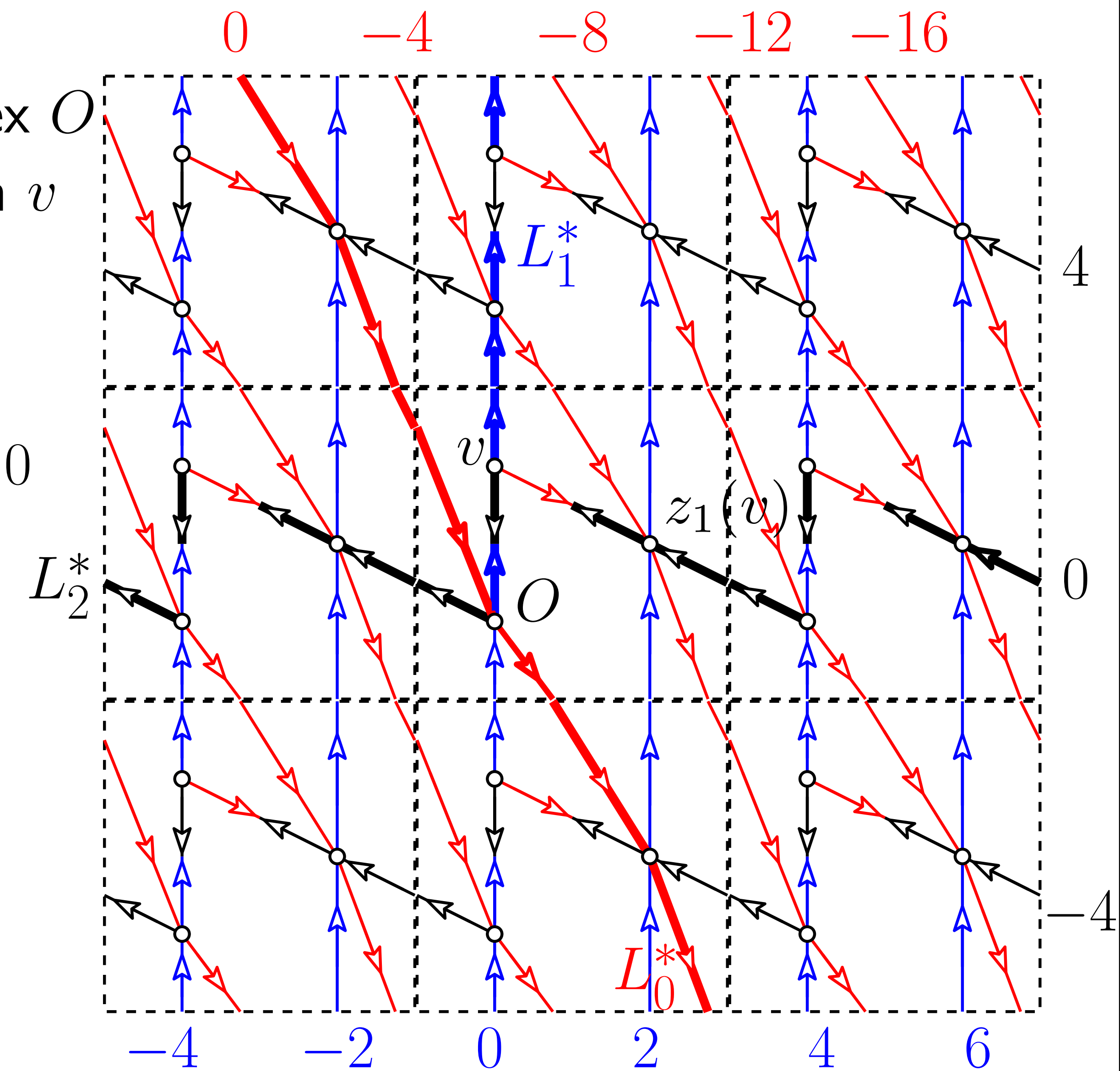
v lies on L_1^* and L_2^*

so: $f_1(L_1^*(v)) = 0$, $f_2(L_2^*(v)) = 0$

$\alpha_0(v) = 0 + 3 \cdot (0 - 0) = 0$

$\alpha_1(v) = 0 + 3 \cdot (0 - (-4)) = 12$

$\alpha_2(v) = 0 + 3 \cdot (-4 - 0) = -11$



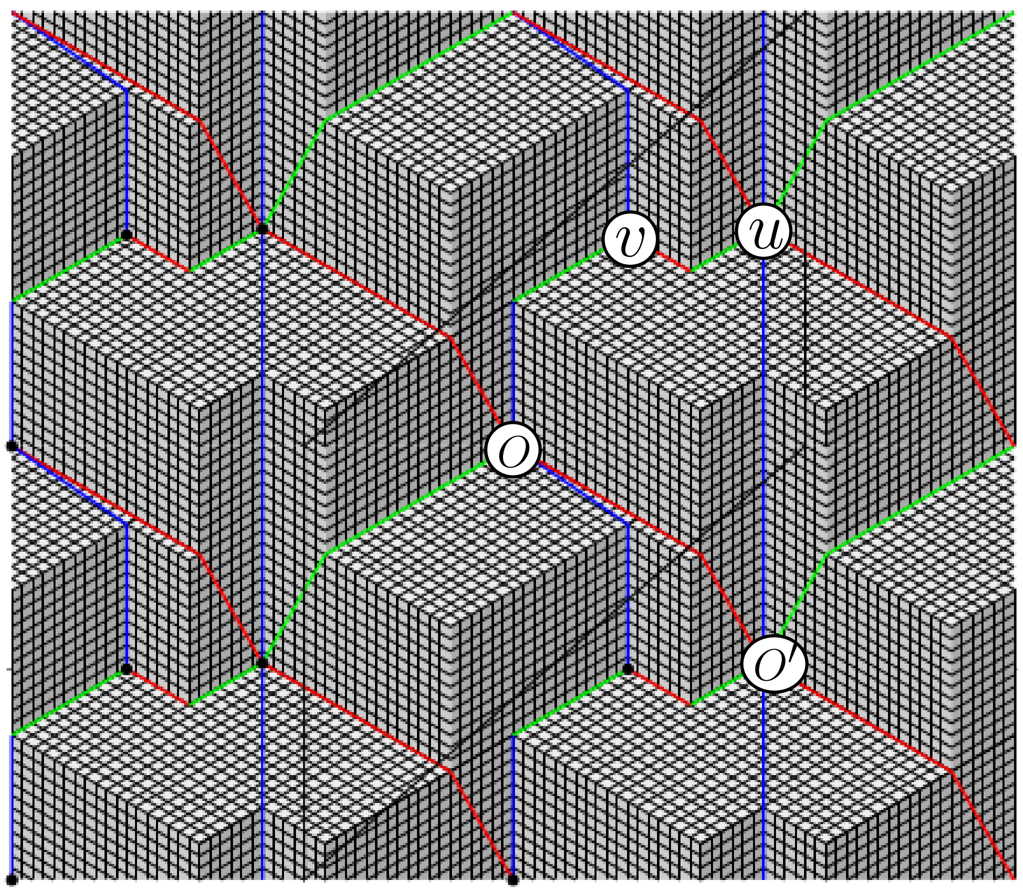
Toroidal Schnyder woods: drawing

Thm[Goncalves Lévêque] (planar simple triangulations)

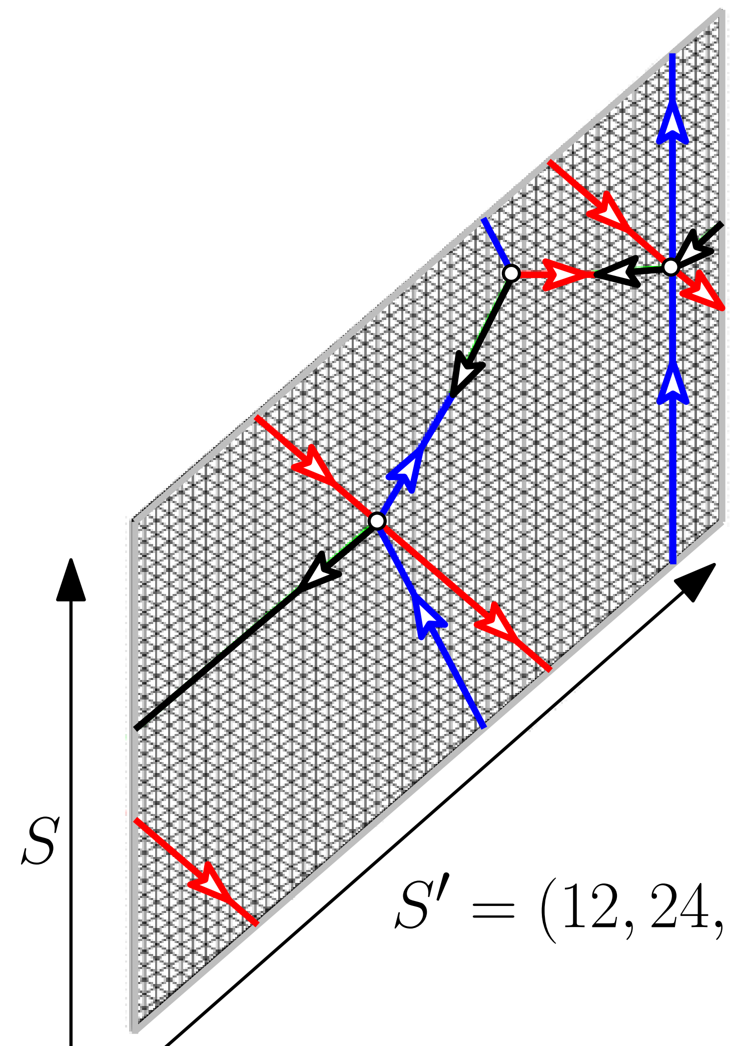
A simple toroidal triangulation admits a straight-line periodic drawing on a grid of size $O(n^2 \times n^2)$

$$O = (0,0,0)$$

$$v = (0,12,-11) \qquad u = (6,12,-18)$$



$$O' = O + S' = (12,24,-36)$$



$$S' = (12,24,-36)$$

$$S = (-12,24,-12)$$

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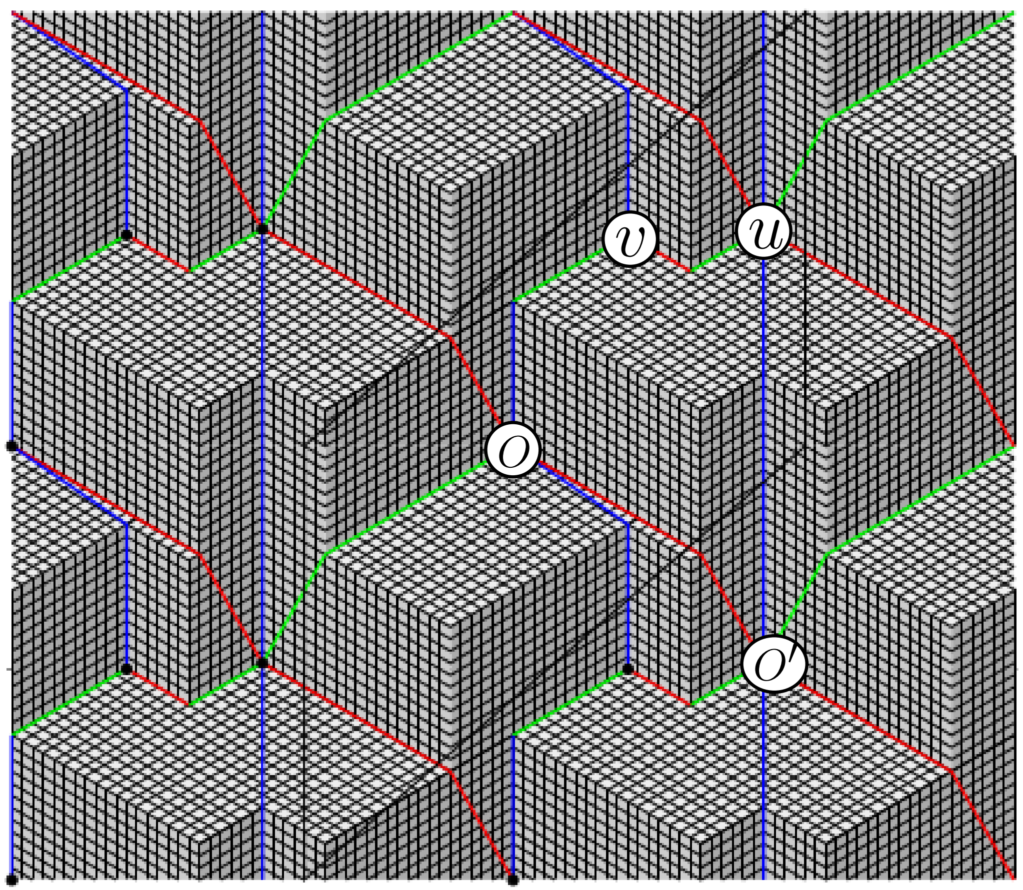
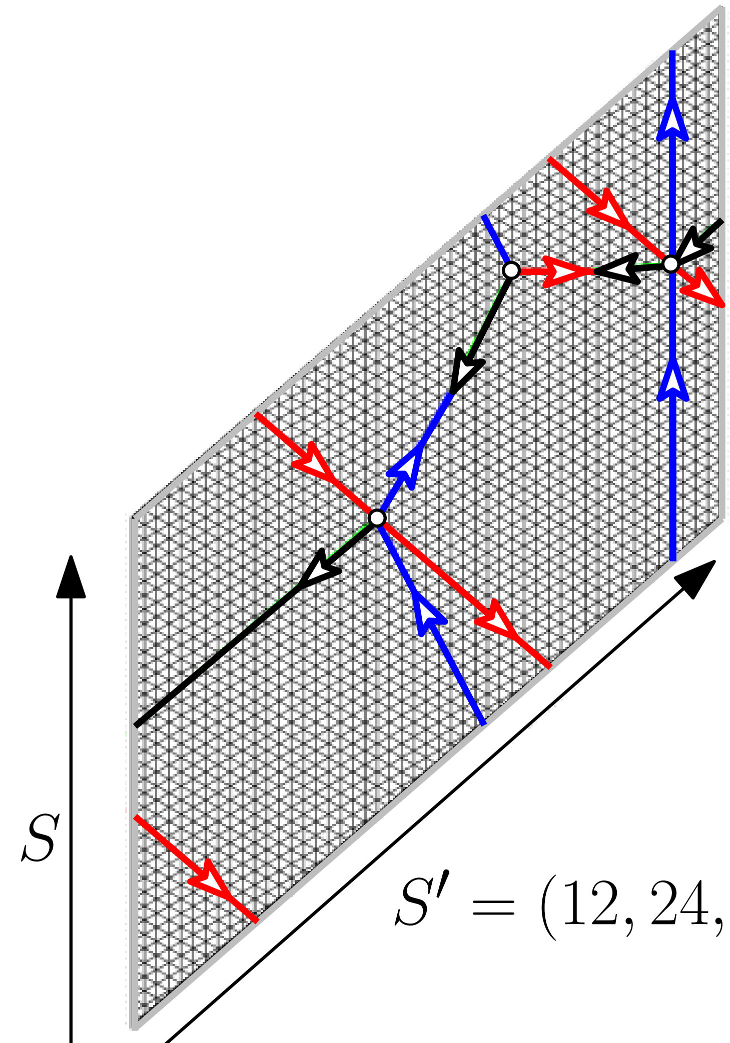


$$O' = O + S' = (12,24,-36)$$

Remark:
Points are not coplanar

$u \in H_0 : x + y + z = 0$

$v \in H_1 : x + y + z = 1$



$$S' = (12,24,-36)$$

$$S = (-12,24,-12)$$

Toroidal Schnyder woods: drawing

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A simple toroidal triangulation admits a straight-line periodic drawing on a grid of size $O(n^2 \times n^2)$

Remark:
Points are not coplanar

$u \in H_0 : x + y + z = 0$

$v \in H_1 : x + y + z = 1$

$c_i :=$ number of times the i-cycles cross the boundary of the tile (vertically)

$c'_i :=$ number of times the i-cycles cross the boundary of the tile (horizontally)

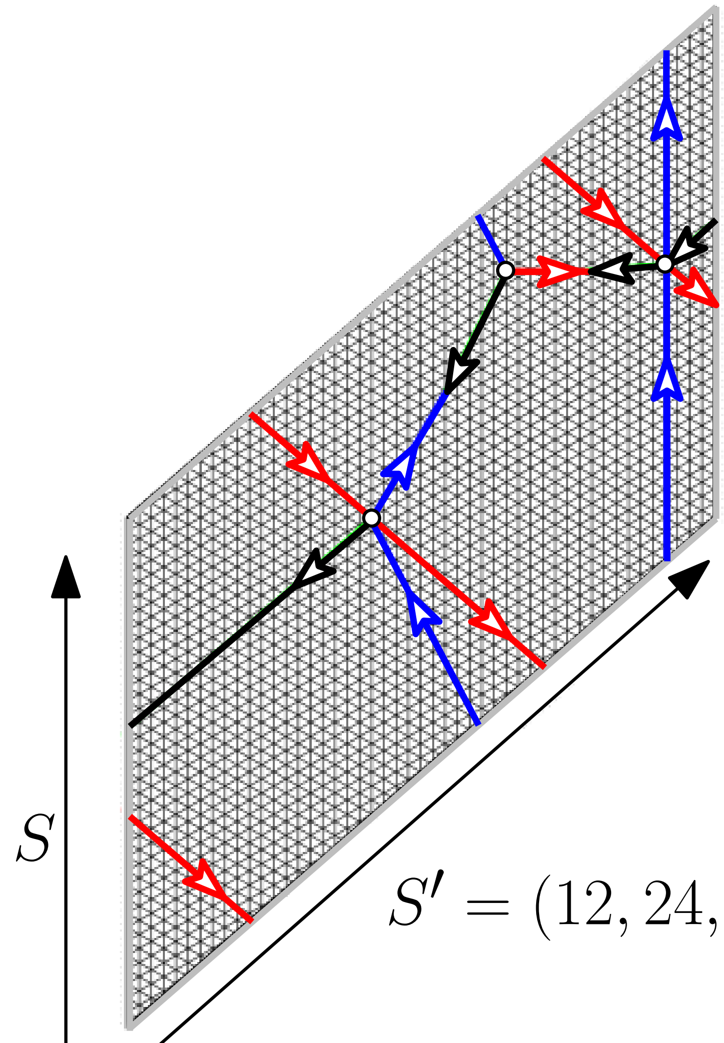
$$S'_i = N \cdot (c_{i+1} - c_{i-1})$$

$$S'_i = N \cdot (c'_{i+1} - c'_{i-1})$$

$c_0 = -1, c'_0 = -2$

$c_1 = -1, c'_1 = 0$

$c_2 = 1, c'_2 = 0$



$S' = (12, 24, -36)$

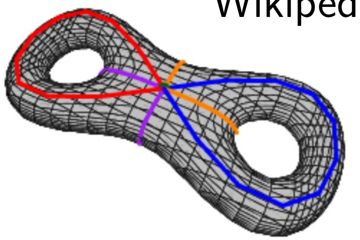
$S = (-12, 24, -12)$

Schyder woods for $g \geq 2$

Thm (3-orientations for graphs on surfaces, of arbitrary genus)
[Albar Goncalves Knauer, 2014]

Any triangulation of a surface (the sphere and the projective plane) admits a '3-orientation': orientation without sinks s.t. every vertex has outdegree divisible by three

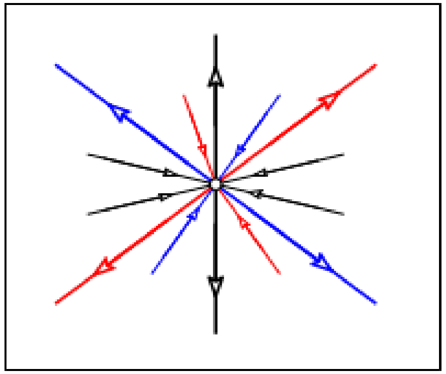
Wikipedia picture



Open problem [Goncalves Knauer Lévêque, 2016]
Existence of Schnyder woods for higher genus triangulations

Multiple local Schnyder condition: the outdegree of every vertex is a **positive** multiple of 3.

(there are no **sinks**)



Thm [Suagee, 2021]

Simple triangulations of genus $g \geq 1$ having "large" **edgewidth** do admit Schnyder woods

edgewidth $\geq 40(2^g - 1)$

(size of the smallest non contractible cycle)

Experimental confirmation $(g = 2)$

exaustive generation of all 3-orientations for all triangulations with $g = 2$, $n \leq 11$

All simple triangulations of genus $g = 2$ **and size** ≤ 11 **admit Schnyder woods**

n	# irreducible triangulations	#triangulations (g = 2)
7	–	–
8	–	–
9	–	–
10	865	865
11	26276	113506

surftri software [Sulanke, 2006]