Algorithms and combinatorics for geometric graphs (Geomgraphs)

Lecture 7

Drawing graphs embedded on surfaces

november 6, 2024

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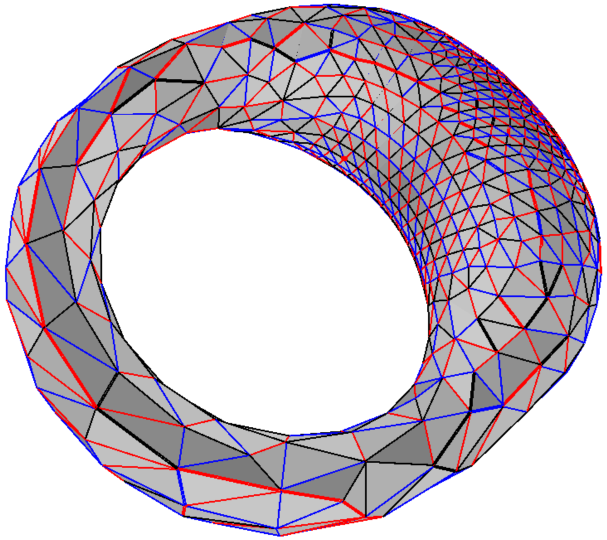
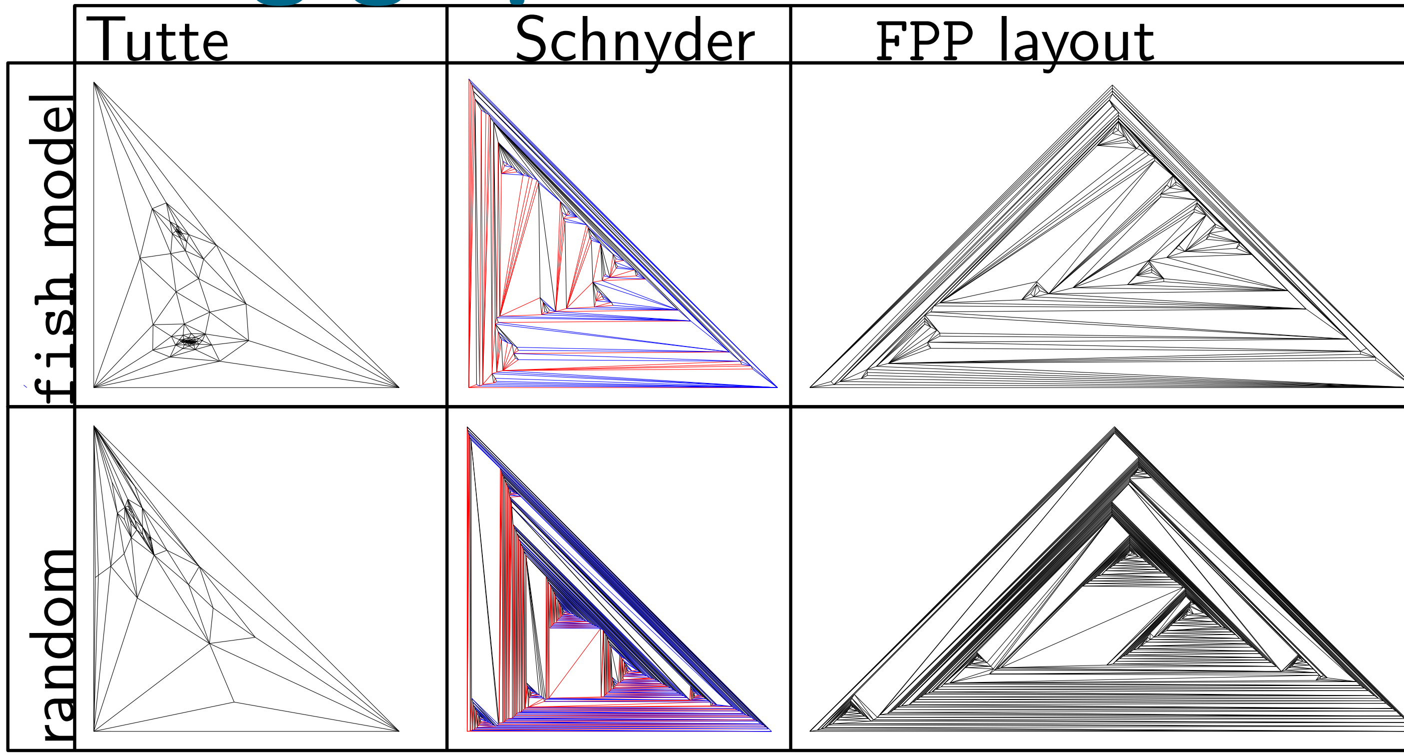
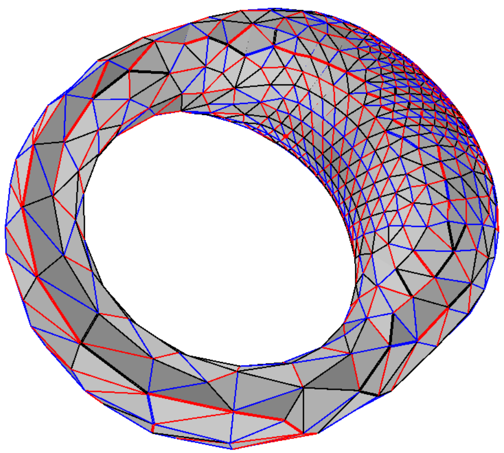


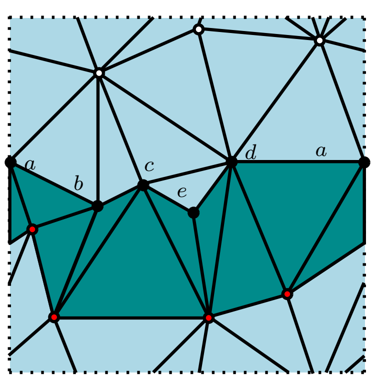
Goal: drawing graphs on surfaces

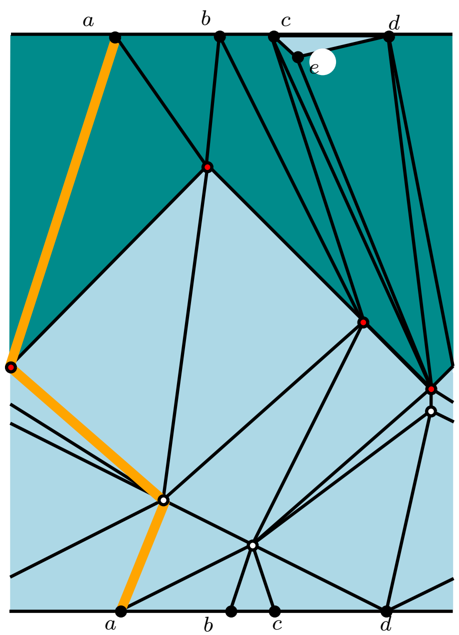
Schnyder woods and canonical orderings for higher genus surfaces



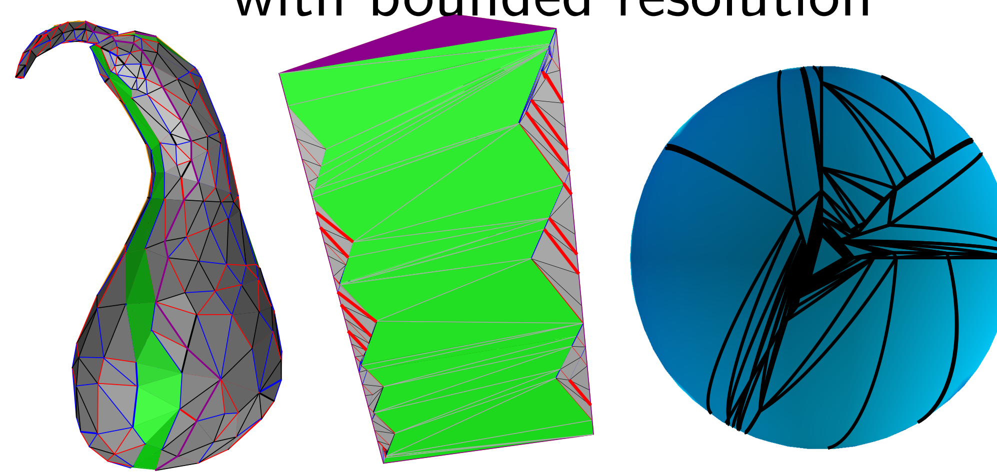


periodic toroidal drawings





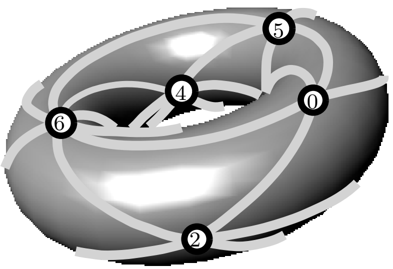
Spherical drawings with bounded resolution



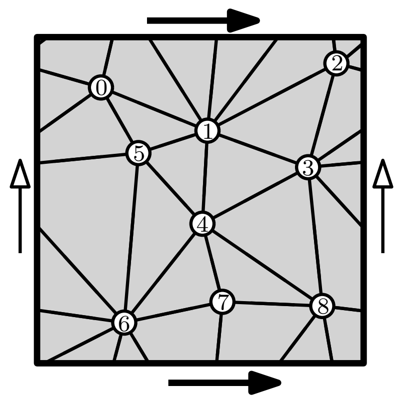
Graphs on surfaces

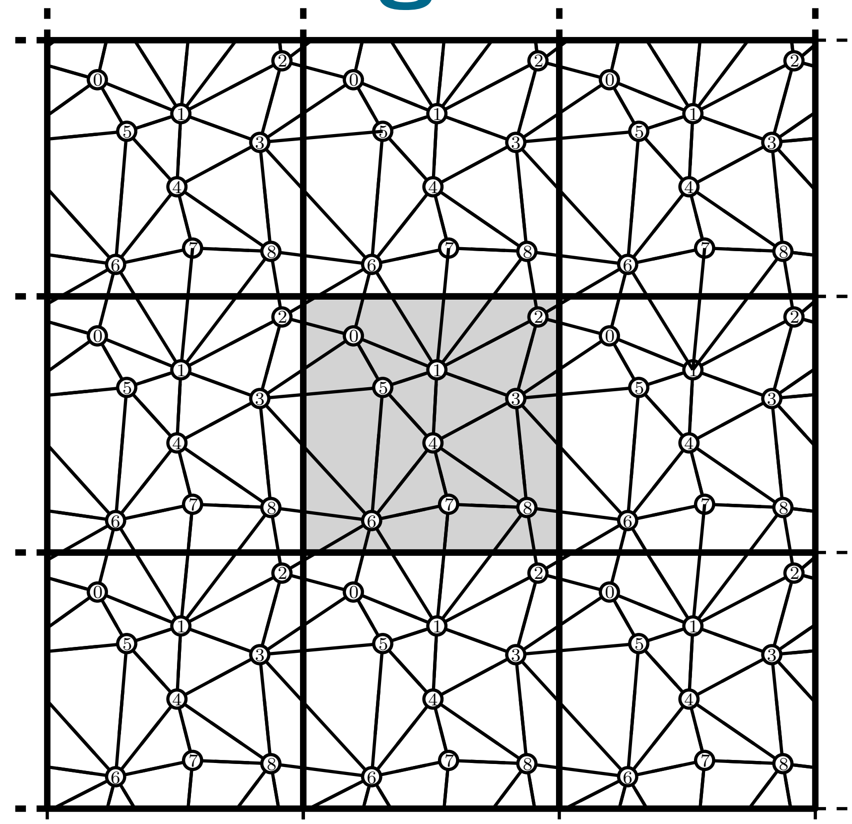
(some definitions and notations)

Periodic (planar) drawings



flat torus

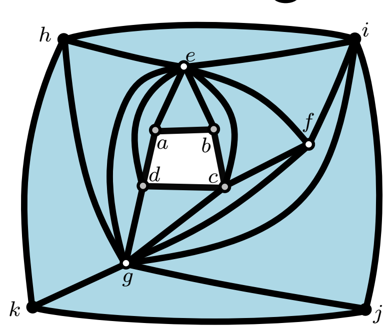




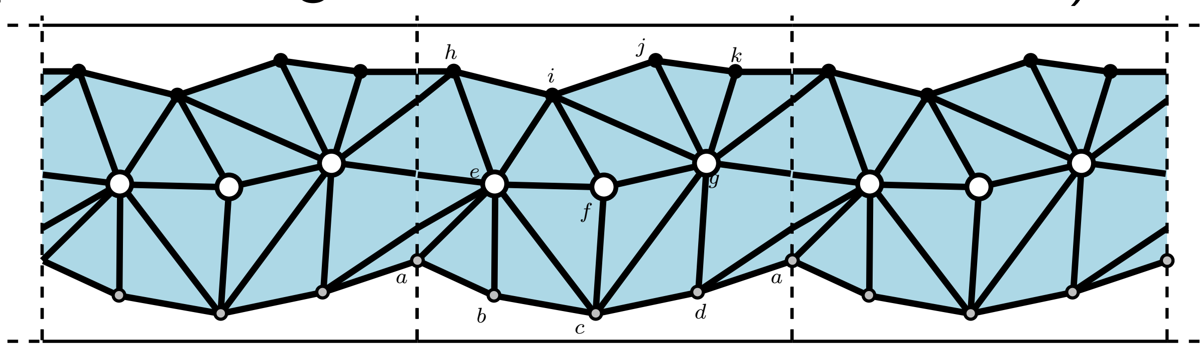
G: (simple) toroidal triangulation

G^{∞} (infinite graph)   universal cover

G: cylindric triangulation (planar triangulation with two boundaries)

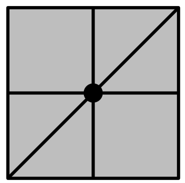


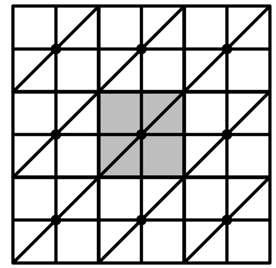
annular representation of G



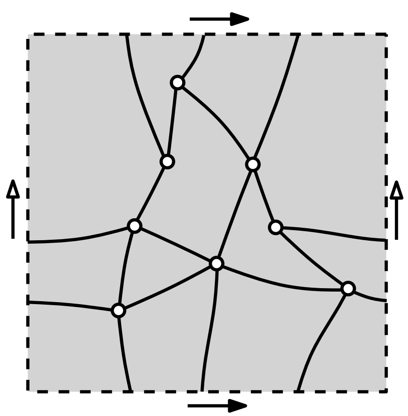
x-periodic drawing of G

Simple and 3-connected graphs on surfaces

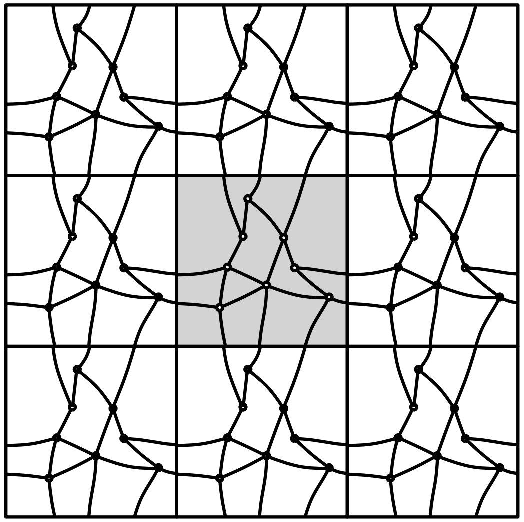




G: essentially simple graph

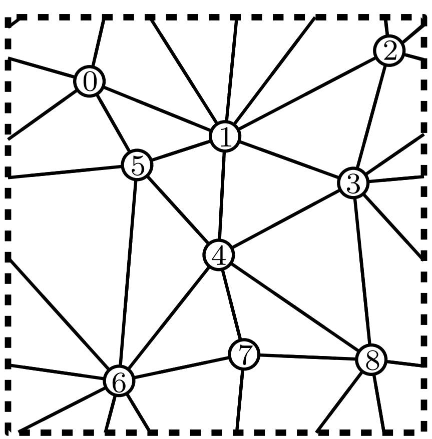


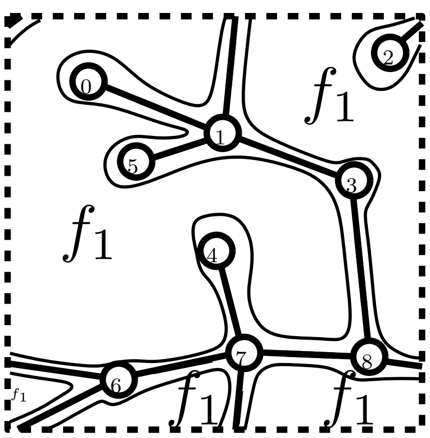
G: essentially 3-connected graph



G^{∞} is 3-connected (in the universal cover)

G





G' (its endowed embedding) has a unique face f_1

$\mathcal{S} \setminus G'$ is homeomorphic to a topological disk

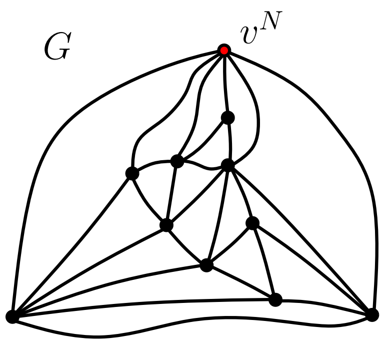
$G' :=$ cut-graph of G (in this example S is a sub-graph spanning all vertices)

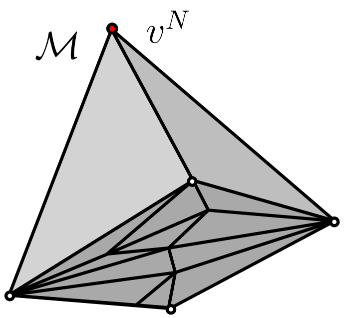
Drawing graphs on surfaces

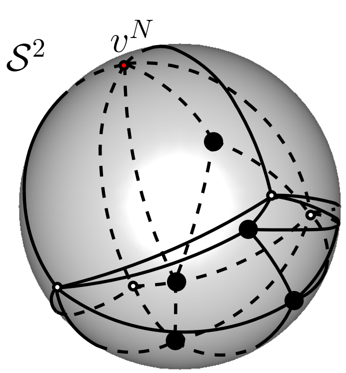
(periodic straight line drawings)

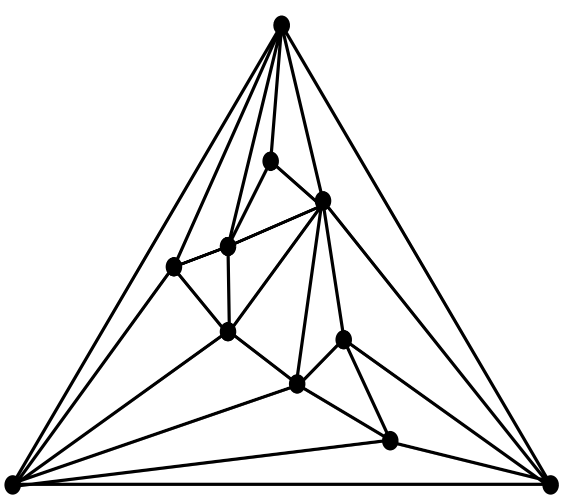
Drawing higher genus graphs

$g = 0$



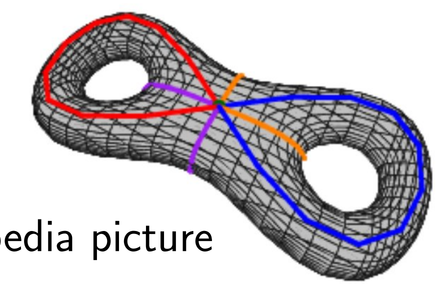




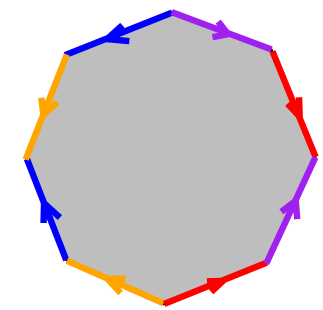


Let us try planarize the graph

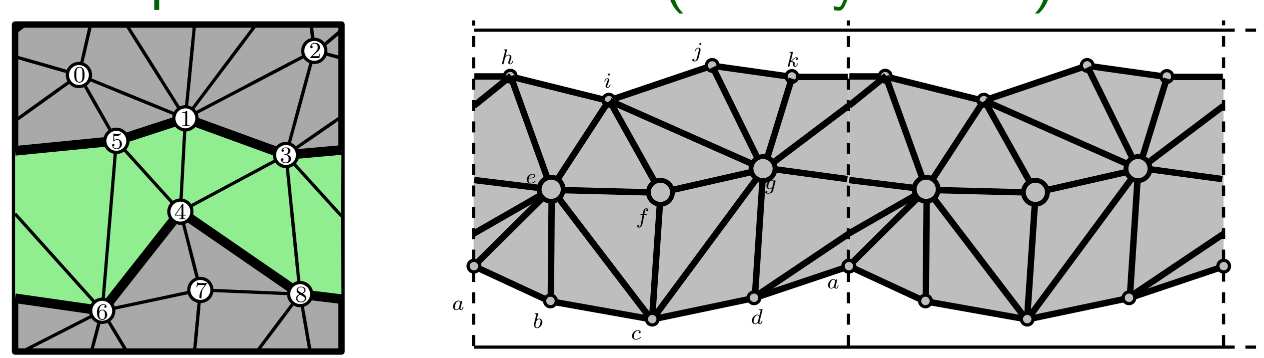
1- compute (canonical) polygonal schemes



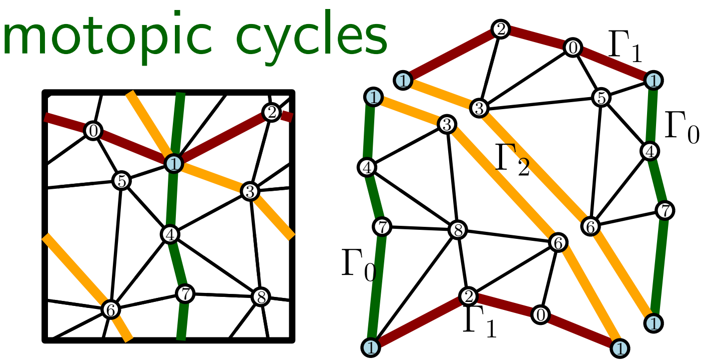
Wikipedia picture



2- compute a tambourine (two cylinders)



3- compute 3 non homotopic cycles

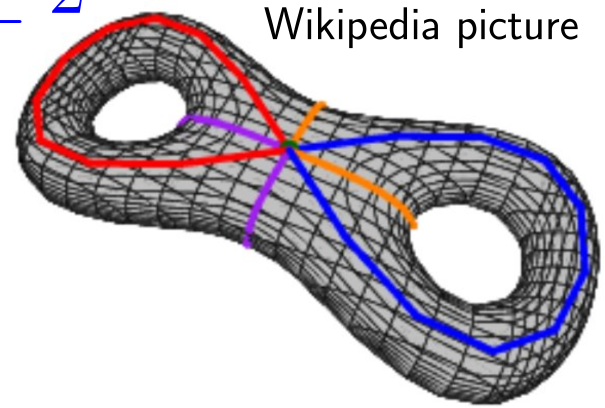


Drawing higher genus graphs

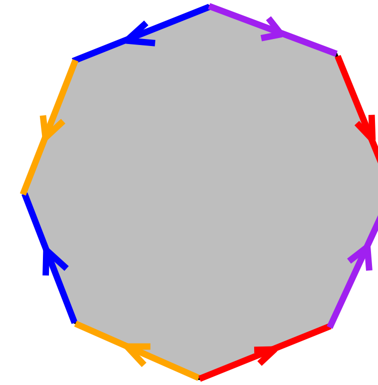
(Palais de la Découverte,Fête de la Science, October 2013)

$g \geq 2$

Wikipedia picture



Polygonal scheme

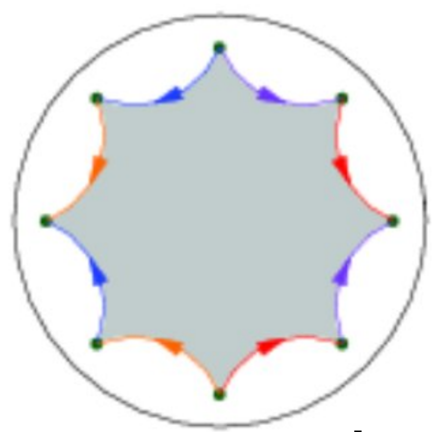


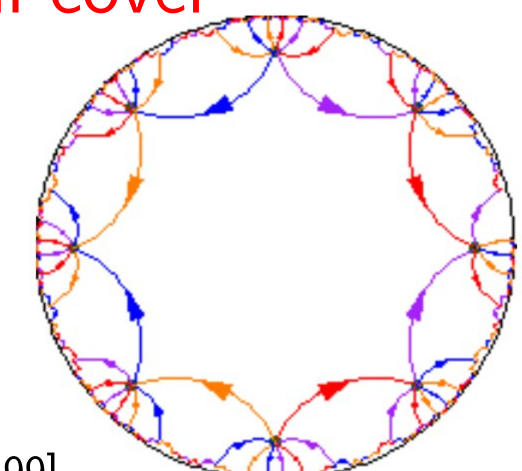
drawing in polynomial area

[Duncan, Goodrich, Kobourov, GD'09]

[Chambers, Eppstein, Goodrich, Löffler, GD'10]

Universal cover





[Mohar'99]

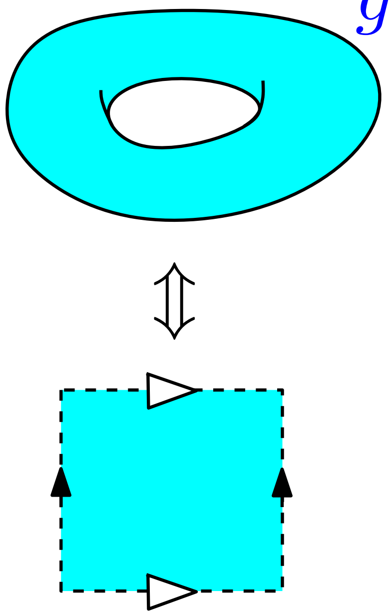
periodic drawing

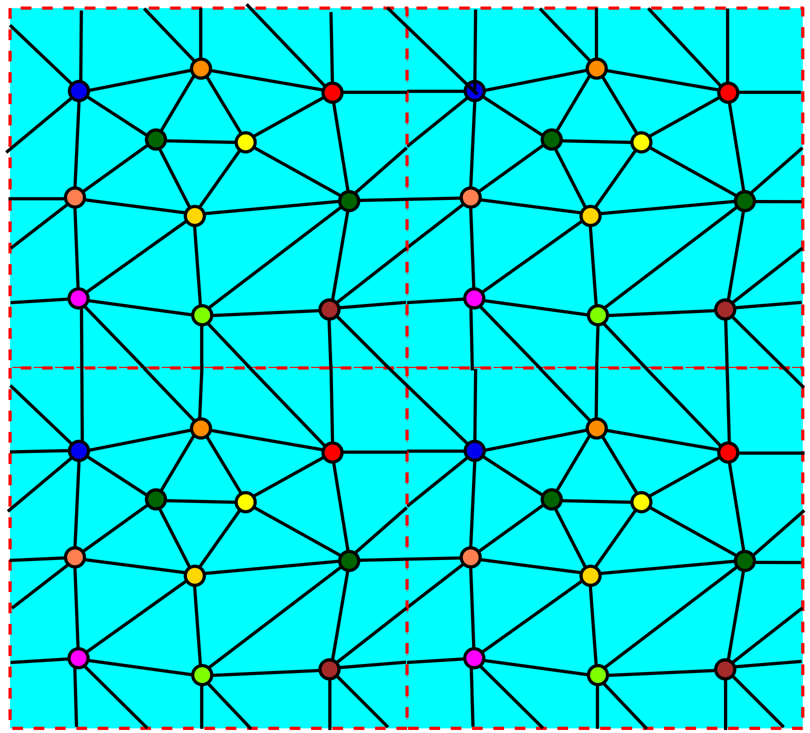
out of circle packing

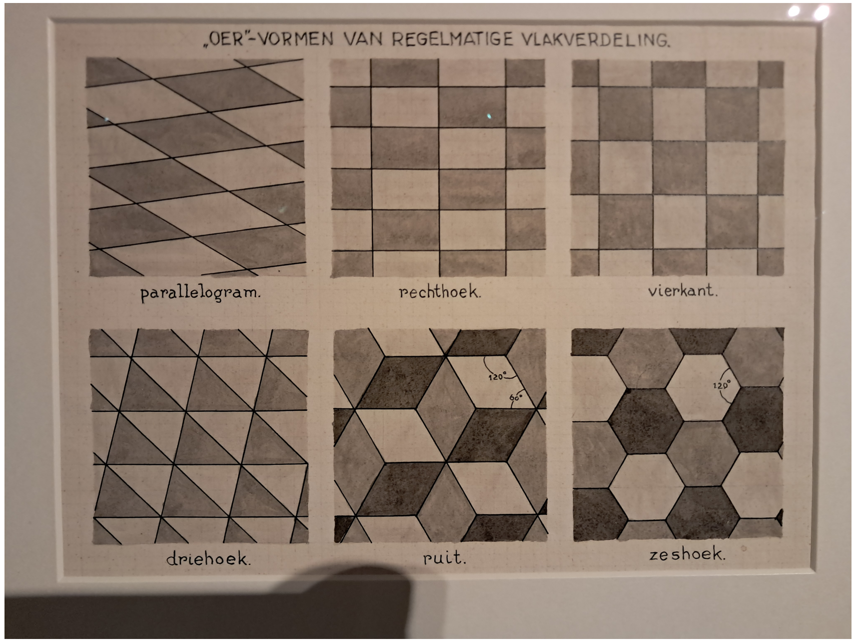
Drawing toroidal graphs

On the torus

$g = 1$





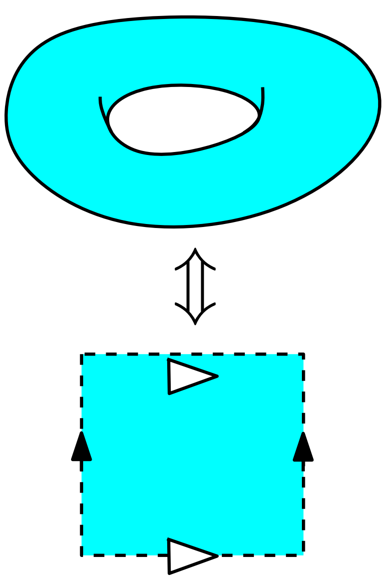


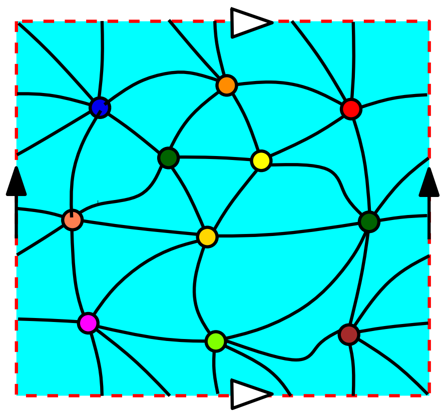
(Escher)

(Palais de la Découverte, Fête de la Science, October 2013)

Periodic straight-line drawings

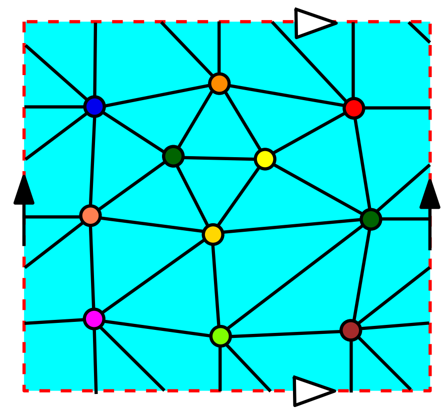
On the torus





drawing on the flat torus

$\Rightarrow$



straight-line drawing

x-periodic and

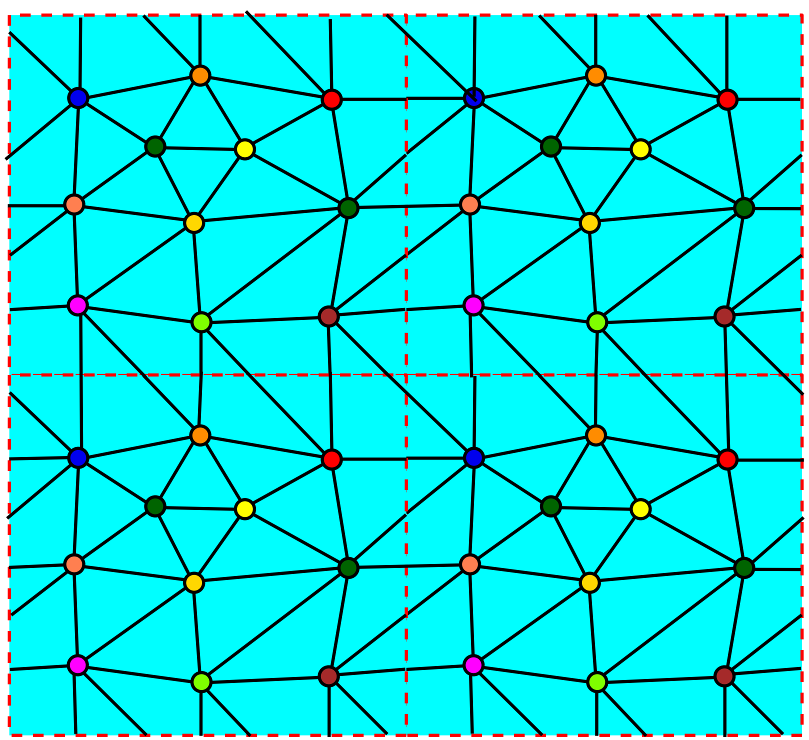
y-periodic drawing

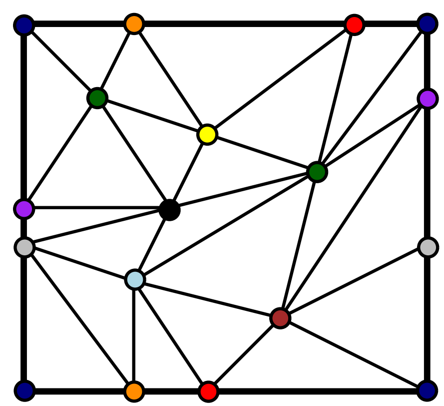
[Castelli Devillers Fusy, GD'12]

$O(n \times n^{\frac{3}{2}})$ **grid**

[Goncalves Lévêque, DCG]

$O(n^2 \times n^2)$ **grid**





straight-line frame

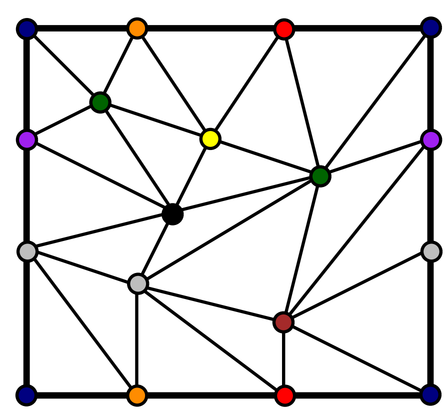
not x-periodic

not y-periodic

[Chambers et al., GD'10]

[Duncan et al., GD'09]

$O(n \times n^2)$ **grid**



straight-line frame

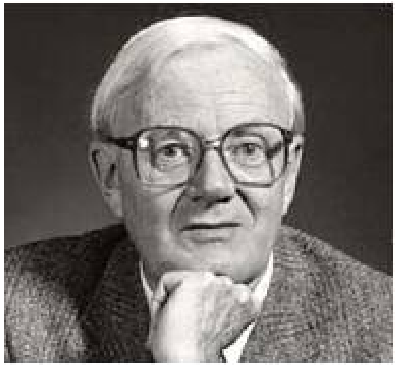
x-periodic and

y-periodic drawing

[Castelli Fusy Kostrygin, Latin'14]

Tutte drawings on surfaces

Tutte drawings (in the plane)



Thm (Tutte barycentric method, 1963)

Every 3-connected planar graph G admits a convex representation ρ in R^2.

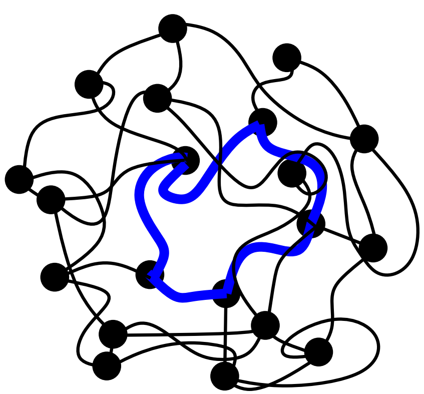
$$\rho : (V_G) \longrightarrow R^2$$

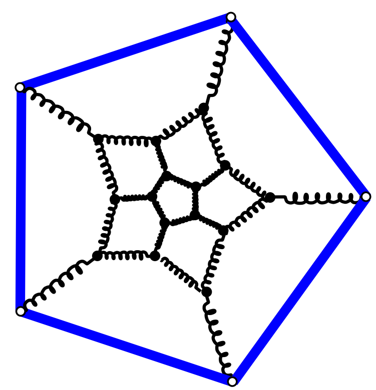
the images of interior vertices are barycenters of their neighbors

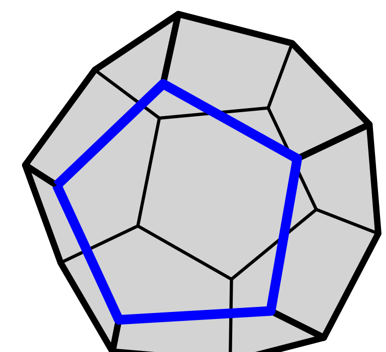
$$\rho(v_i) = \sum_{j \in N(i)} w_{ij}\rho(v_j)$$

where w_{ij} satisfy $\sum_j w_{ij} = 1$, and $w_{ij} > 0$

according to Tutte: $w_{ij} = \frac{1}{deg(v_i)}$



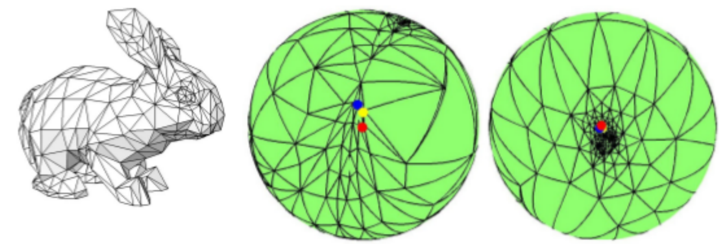




Spherical parameterization (Tutte on the sphere)

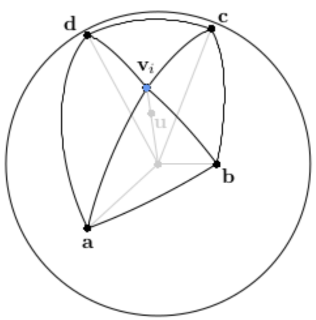
$$\mathbf{v}_i = \frac{\mathbf{u}_i}{||\mathbf{u}_i||}, \quad \text{with} \quad \mathbf{u}_i = \sum_{j=1}^{n} w_{ij}\mathbf{v}_j, \quad i = 1, 2, \ldots, N.$$

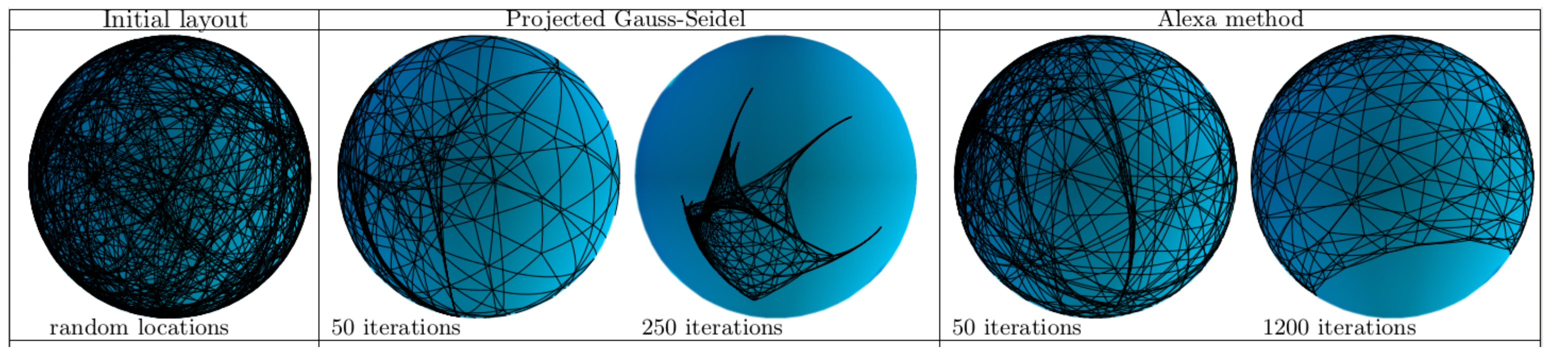
(system of quadratic equations)



(Gotsman Gu Sheffer, 2003)

$$\mathbf{v}_i = \frac{\mathbf{u}}{||\mathbf{u}||}.$$





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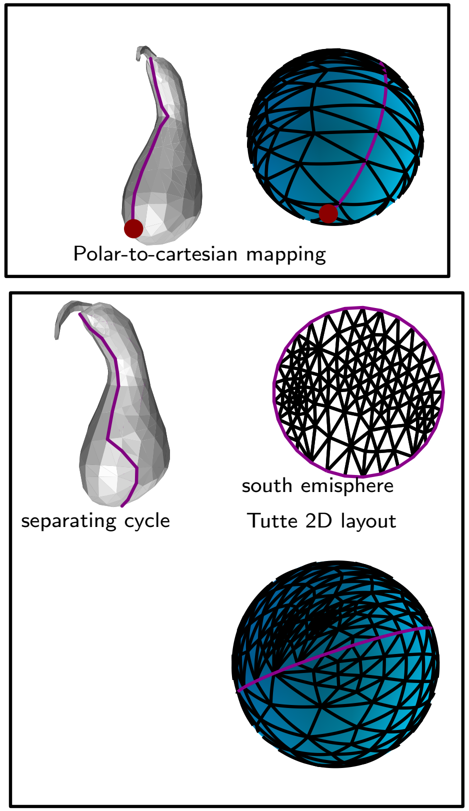
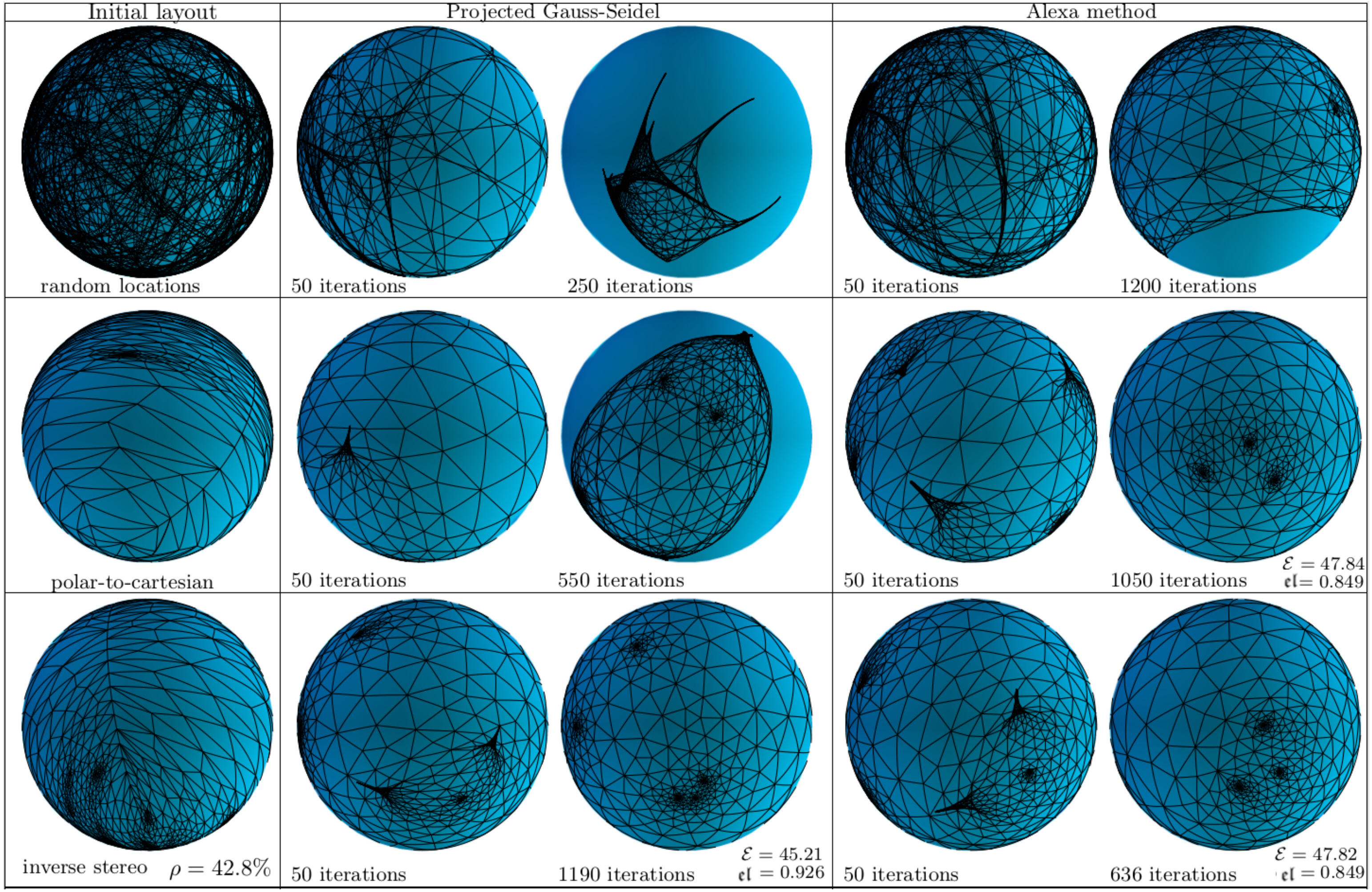
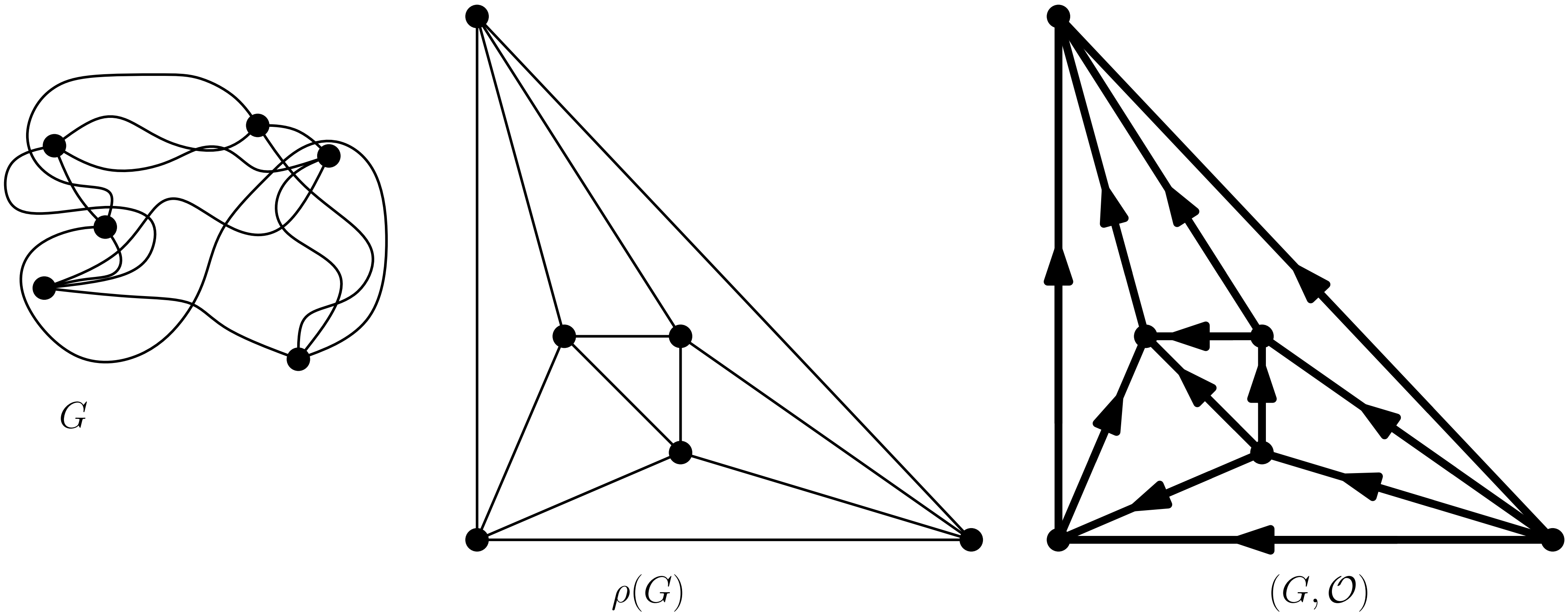
(system of quadratic equations)





Tutte drawings, from another point of view

Planar drawings and edge orientations



planar drawing

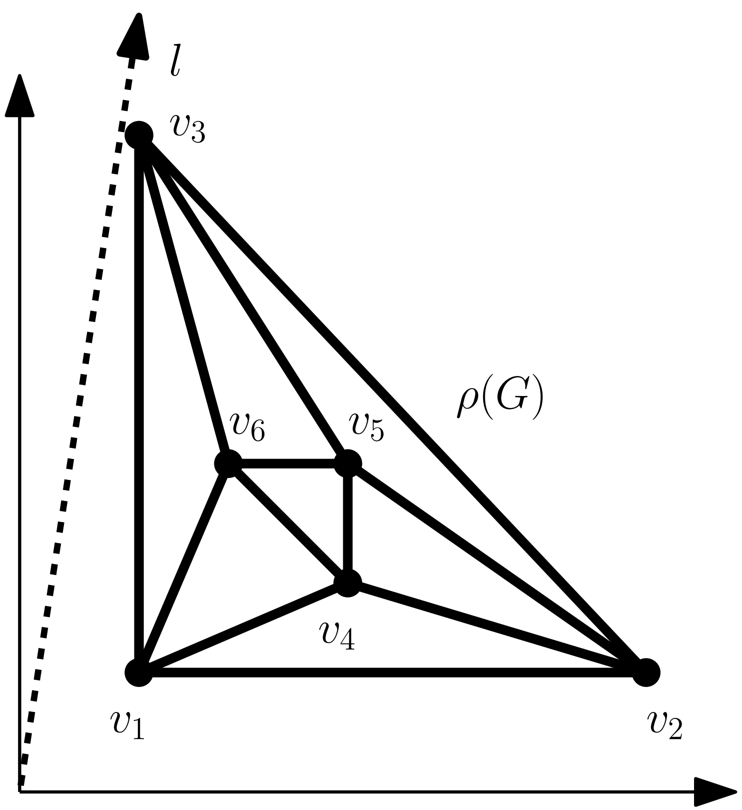
edge orientation

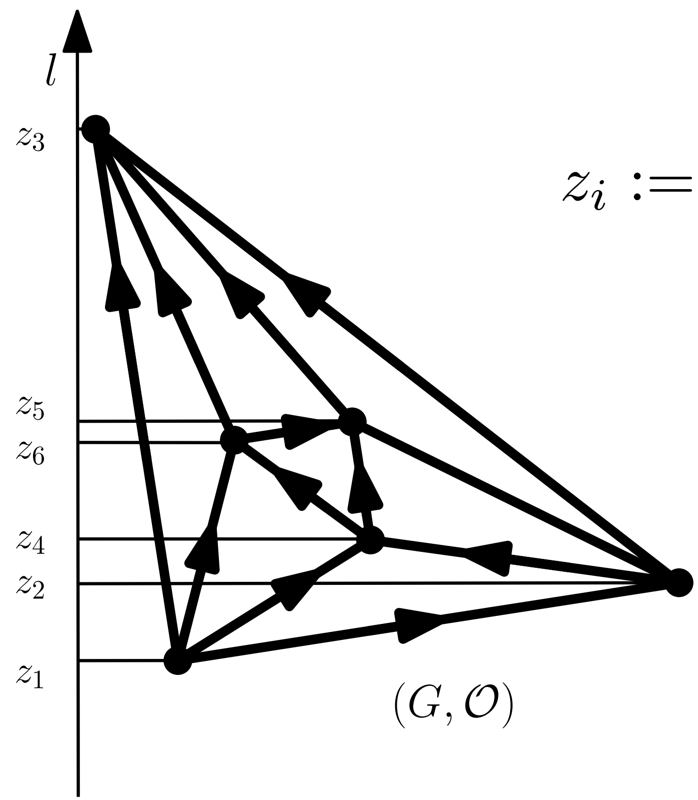
Planar drawings and edge orientations

Choose a generic line l and project all vertices on l (such that the images z_i of vertices are distinct)

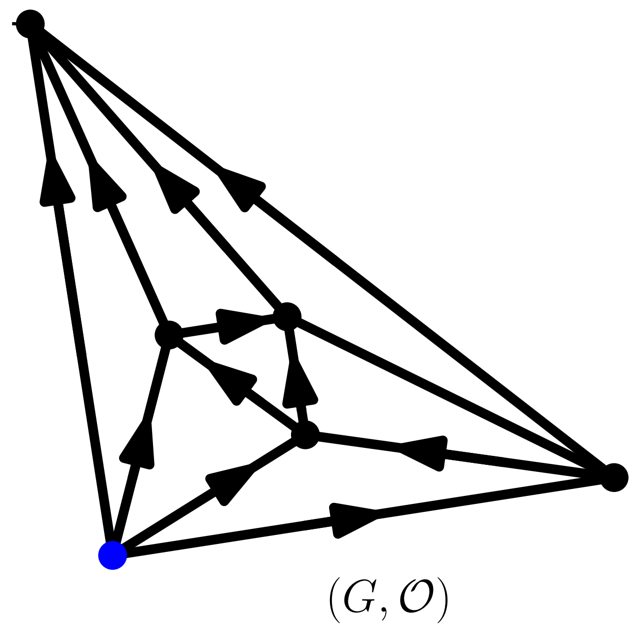
$$R_{\alpha\beta} = \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix}$$

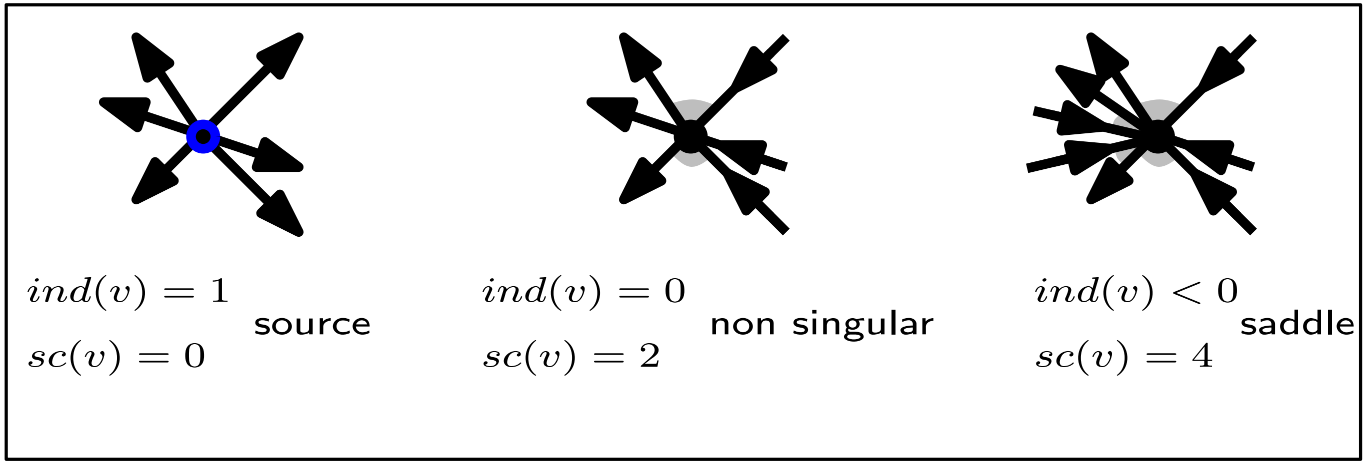
$$z_i := \alpha x_i + \beta y_i$$

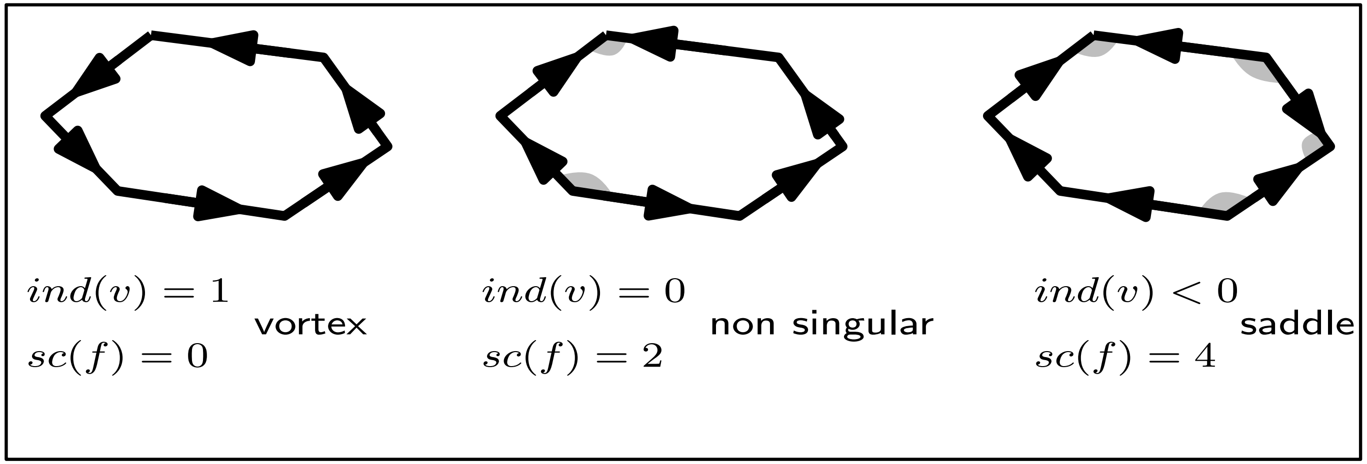




Vertex and face classification







Index of a vertex $ind(v) := \dfrac{(2 - sc(v))}{2}$

$sc(v) :=$ number of sign changes around vertex v

Index of a face $ind(f) := \dfrac{(2 - sc(f))}{2}$

$sc(f) :=$ number of sign changes around face f

Discrete Index Theorem (Poincaré-Hops)

Thm Given a closed manifold mesh of genus g:

$$\sum_{v \in V} ind(v) + \sum_{f \in F} ind(f) = 2 - 2g$$

Proof

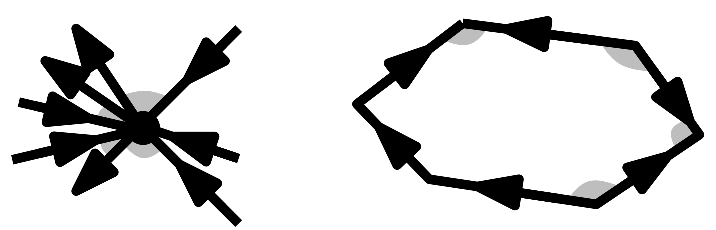
$$\sum_{v \in V} ind(v) + \sum_{f \in F} ind(f) = \frac{1}{2} \sum_{v \in V} (2 - sc(v)) + \frac{1}{2} \sum_{f \in F} (2 - sc(f))$$

$$= V + F - \frac{1}{2} [\sum_{v \in V} sc(v) + \sum_{f \in F} sc(f)]$$

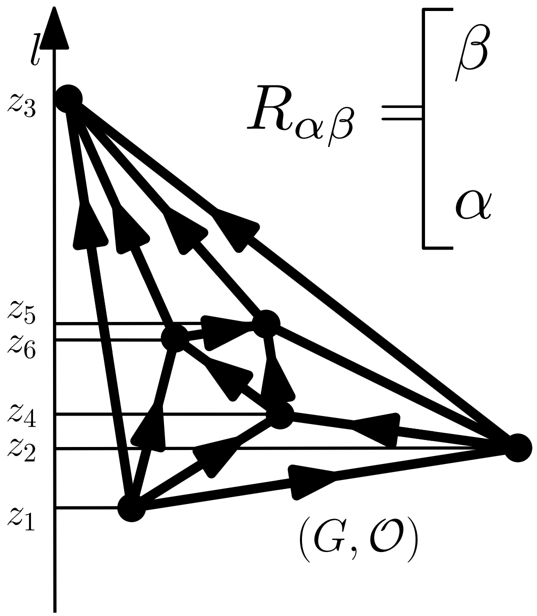
$$= V + F - \frac{1}{2} [2 \cdot E]$$

$$= 2 - 2g$$

Claim: the total number of changes of edge direction (around each vertex and around each face) is equal to the number of half-edges



Discrete one-forms and Tutte equations



$$R_{\alpha\beta} = \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix}$$

$$z_i := \alpha x_i + \beta y_i$$

$$\Delta z_{ij} := z_j - z_i$$

(value of the one-form Δ for edge (i, j))

Remark: $\Delta_{ij} = -\Delta_{ji}$ and $\Delta_{ij} \neq 0$

Fact: Given a Tutte barycentric drawing of a planar graph G, we have:

$\sum_{v_j \in N(i)} \frac{1}{d_i} \Delta z_{ij} = 0$, for each inner vertex v_i

$\sum_{(u,v) \in \partial f} \Delta z_{uv} = 0$ for each face f of G

Discrete one-forms and Tutte equations

$\sum_{v_j \in N(i)} \frac{1}{d_i} \Delta z_{ij} = 0$, for each inner vertex v_i

$z_i := \alpha x_i + \beta y_i$

$\sum_{(u,v) \in \partial f} \Delta z_{uv} = 0$ for each face f of G

$\Delta z_{ij} := z_j - z_i$

Proof: inner vertices are placed at the barycenter of their neighbors

$$v_i = \sum_{j \in N(i)} \frac{1}{d_i} v_j, \quad \text{for any inner vertex } v_i$$

which is equivalent to: $x_i = \sum_{j \in N(i)} \frac{1}{d_i} x_j$ et $y_i = \sum_{j \in N(i)} \frac{1}{d_i} y_j$. By definition we have:

$$z_i := \alpha x_i + \beta y_i = \alpha \sum_j \frac{1}{d_i} x_j + \beta \sum_j \frac{1}{d_i} y_j = \sum_j \frac{1}{d_i} (\alpha x_j + \beta y_j) = \sum_j \frac{1}{d_i} z_j$$

implying:

$$\sum_{v_j \in N(i)} \frac{1}{d_i} \Delta z_{ij} = \sum_{v_j \in N(i)} \frac{1}{d_i} (z_j - z_i) = \Big(\sum_{v_j \in N(i)} \frac{1}{d_i} z_j \Big) - d_i z_i = 0$$

Discrete one-forms and Tutte equations

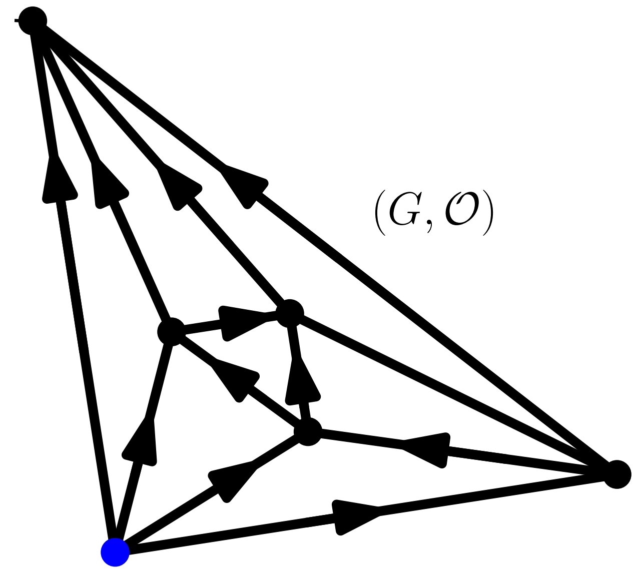
$\sum_{v_j \in N(i)} \frac{1}{d_i} \Delta z_{ij} = 0$, for each inner vertex v_i

$z_i := \alpha x_i + \beta y_i$

$\sum_{(u,v) \in \partial f} \Delta z_{uv} = 0$ for each face f of G

$\Delta z_{ij} := z_j - z_i$

Fact (exercise): In an orientation induced by a Tutte drawing there are no saddle vertices and no saddle faces



Corollary (exercise): all faces in a Tutte drawing are convex

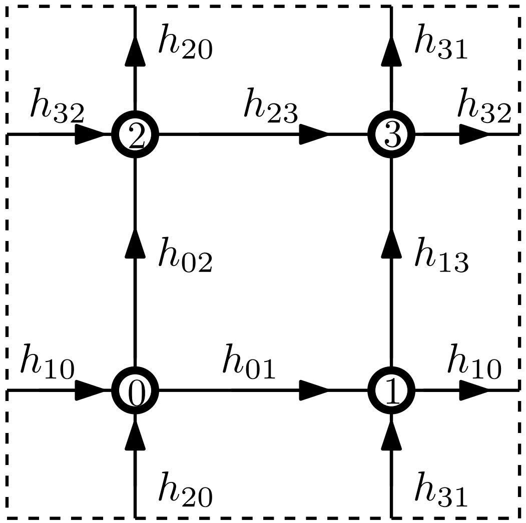
Tutte equations on the torus

$\Delta z_h :=$ one-form associated to half-edge h

$$\begin{cases} \sum_{h\in\partial f} \Delta z_h = 0 \text{ for each face } f \text{ of } G & E \text{ unknowns} \\ \sum_{h\in N(i)} \frac{1}{d_i}\Delta z_h = 0, \text{ for } \textbf{each} \text{ vertex } v_i & \text{F+V equations} \end{cases}$$

The rank of the two systems is $(F-1)$ and $(V-1)$

The solution space has dimension
$E - [(V-1)+(F-1)] = 2g = 2$



$$\begin{cases} h_{01}+h_{02}-h_{10}-h_{20}=0 & \text{for vertex } v_0 \\ h_{10}+h_{13}-h_{01}-h_{31}=0 & \text{for vertex } v_1 \\ h_{23}+h_{20}-h_{32}-h_{02}=0 & \text{for vertex } v_2 \\ h_{32}+h_{31}-h_{23}-h_{13}=0 & \text{for vertex } v_3 \end{cases}$$

$$Eq_3 = -(Eq_0 + Eq_1 + Eq_2)$$

Computing coordinates on the torus

$\Delta z_h :=$ one-form associated to half-edge h

$$\begin{cases} \sum_{h\in\partial f} \Delta z_h = 0 \text{ for each face } f \text{ of } G \\ \sum_{h\in N(i)} \frac{1}{d_i}\Delta z_h = 0, \text{ for \textbf{each} vertex } v_i \end{cases}$$

E unknowns

F+V equations

- Compute the null-space of the matrix
- Choose any pair of linearly indenpendent one-forms Δx and Δy

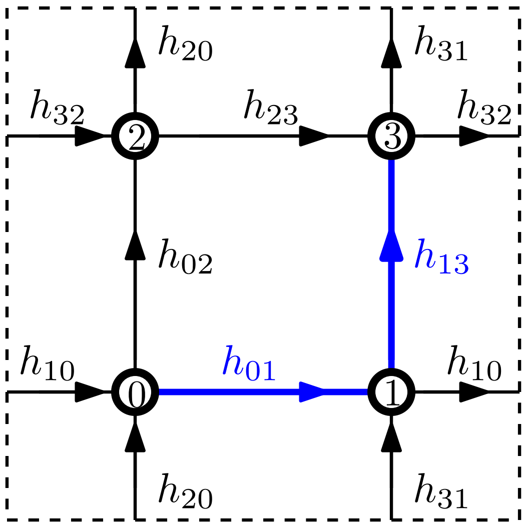
$\Delta x = (1,1,1,1,0,0,0,0)$
$\Delta y = (0,0,0,0,1,1,1,1)$

- Choose a vertex v_0 as origin

$(x_0, y_0) = (0,0)$

- Compute coordinates relatives to v_0 following an arbitrary path $P = \{v_0, v_1, \ldots\}$

$$(x_i, y_i) = \left(\sum_{h\in P(v_0,v_i)} \Delta x_h, \sum_{h\in P(v_0,v_i)} \Delta y_h\right)$$



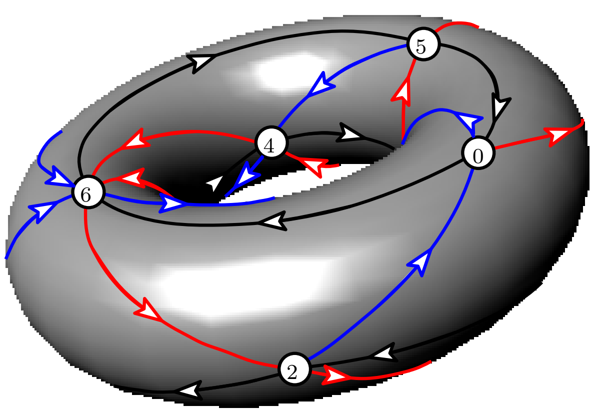
$$P(v_0, v_3) = \{(v_0, v_1), (v_1, v_3)\}$$

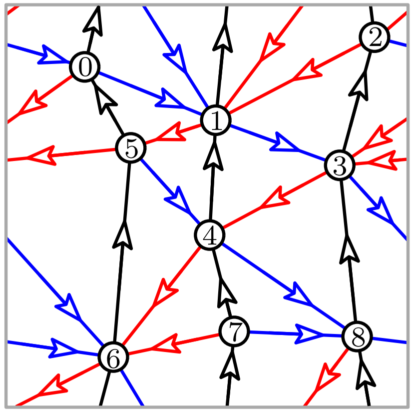
$$(x_3, y_3) = (1,0) + (0,1)$$

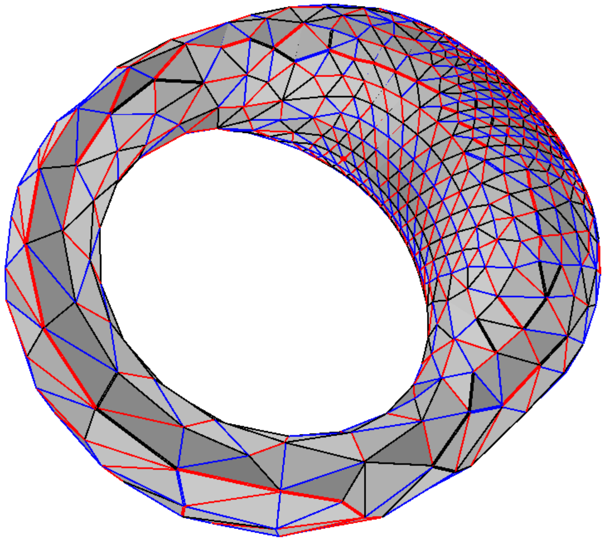
$h_{01} = (0,1)$ $h_{13} = (1,0)$

$$\begin{cases} h_{01} + h_{02} - h_{10} - h_{20} = 0 \\ h_{10} + h_{13} - h_{01} - h_{31} = 0 \\ h_{23} + h_{20} - h_{32} - h_{02} = 0 \\ h_{32} + h_{31} - h_{23} - h_{13} = 0 \end{cases}$$

Toroidal drawings II: toroidal Schnyder woods

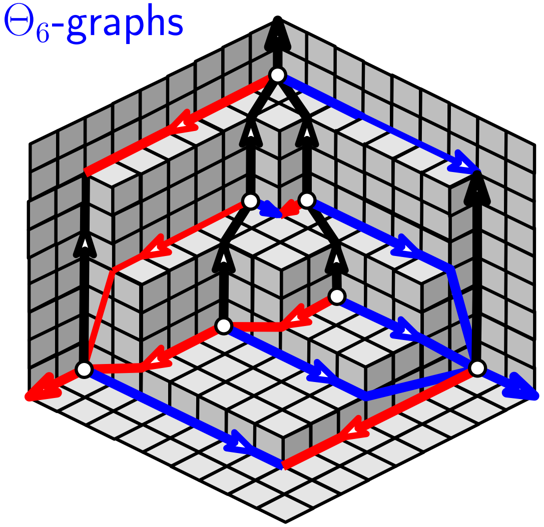






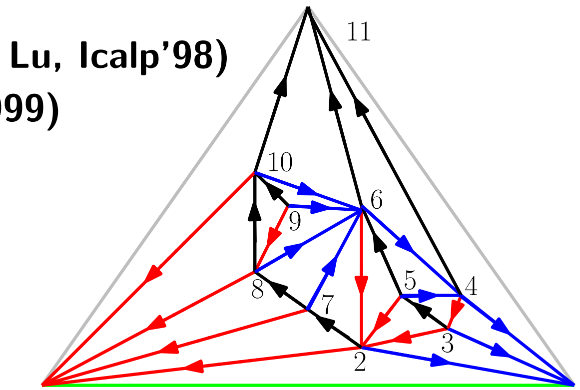
Schnyder woods: some (classical) applications

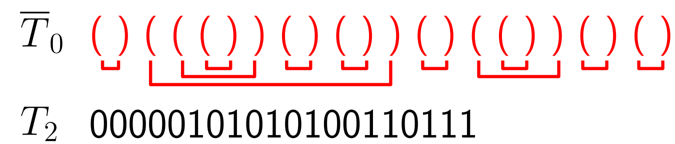
[Felsner, Bonichon et al. '10, ...]
geodesic embeddings on coplanar orthogonal surfaces, TD-Delaunay graphs and Half-Θ_6-graphs



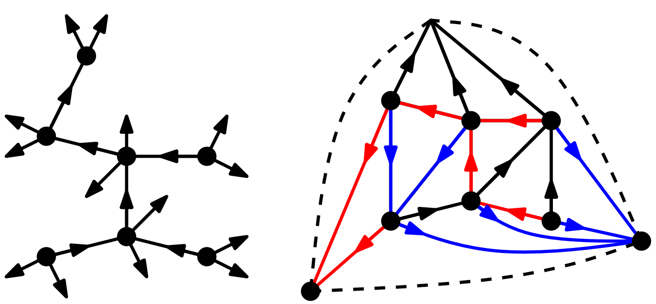
(Chuang, Garg, He, Kao, Lu, Icalp'98)
(He, Kao, Lu, 1999)

Graph encoding





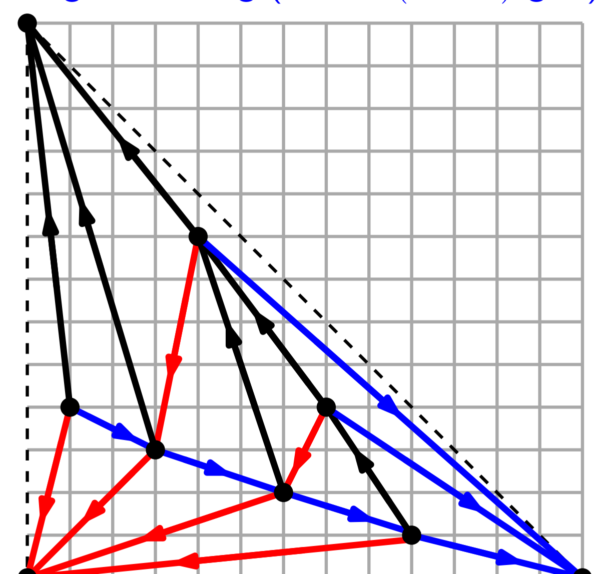
(Poulalhon-Schaeffer, Icalp 03)
bijective counting, random generation



$$c_n = \frac{2(4n+1)!}{(3n+2)!(n+1)!}$$

$\Rightarrow$ optimal encoding ≈ 3.24 bits/vertex

(Schnyder '90)
Planar straight-line grid drawing (on a $O(n \times n)$ grid)



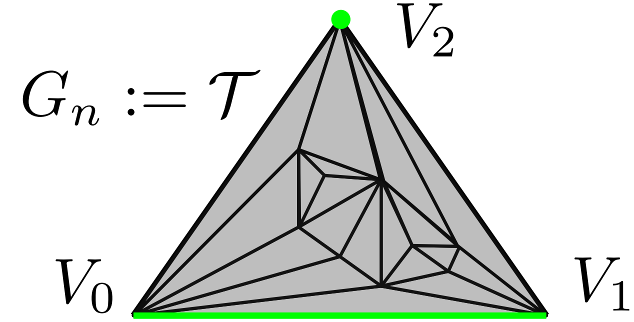
Reminder: linear-time computation of (planar) Schnyder woods

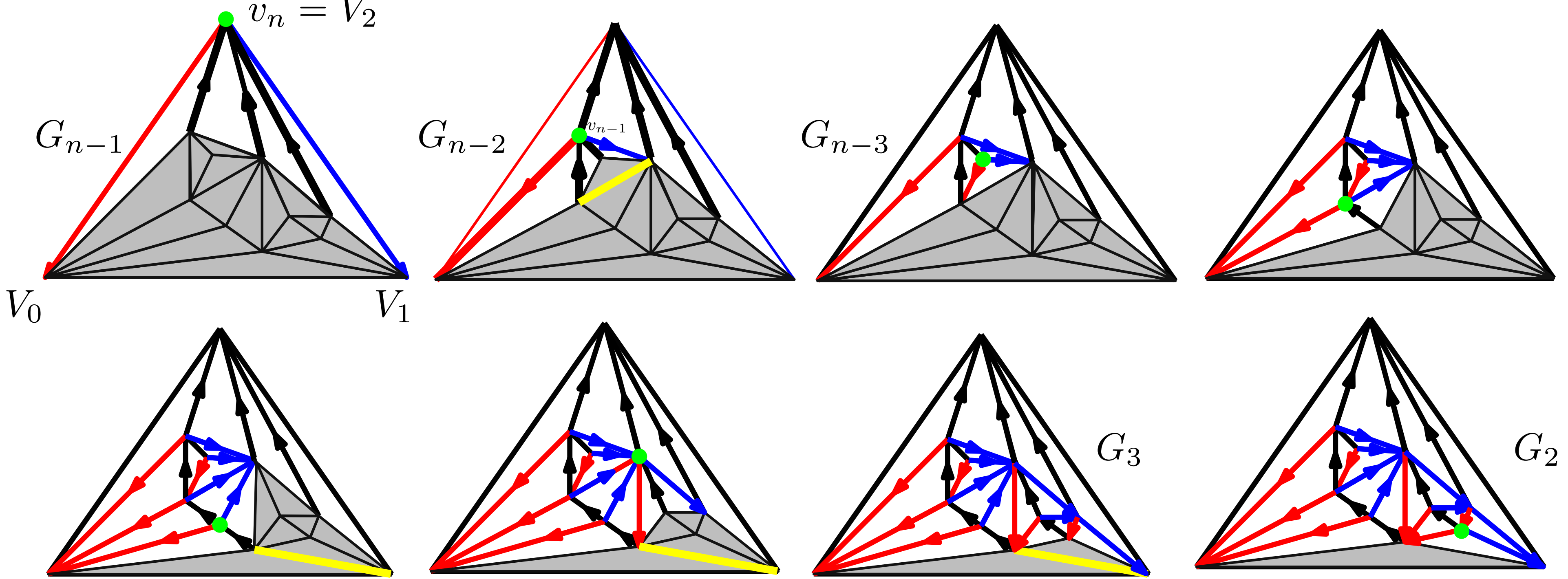
use Canonical Orderings [De Fraysseix, Pach, Pollack '89]

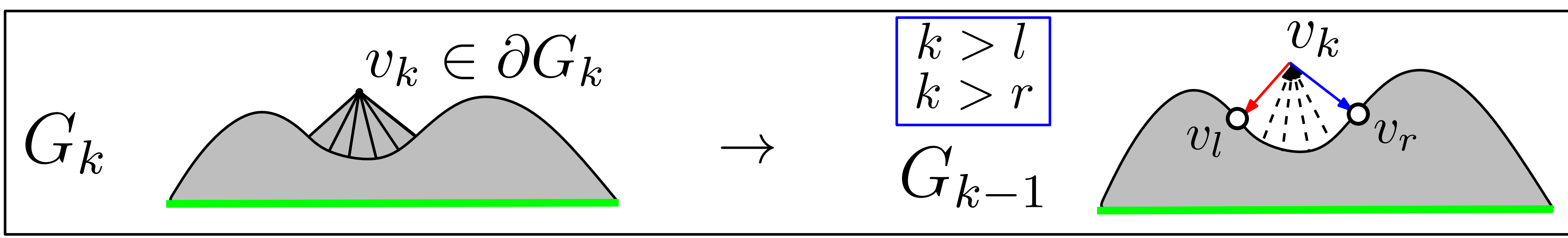
Theorem (Brehm, 2000)

A Schnyder wood can be computed in linear-time
(via a sequence of $n-2$ vertex shellings)

Remove at each step a vertex v on the boundary ∂G_k
(with no incident chordal edges in the gray region)





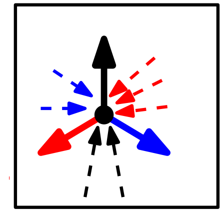


Toroidal Schnyder woods: definition

Toroidal Schnyder woods [Goncalves Lévêque, DCG'14]

$g = 1 \qquad e = 3n$

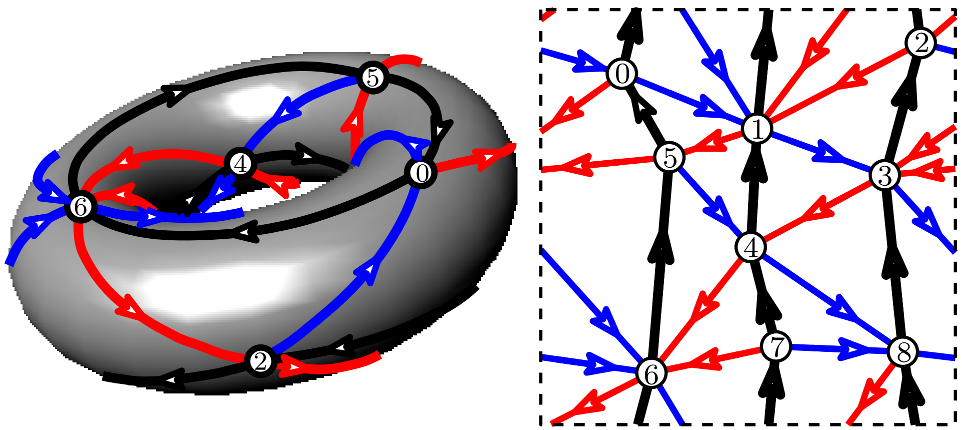
$n - e + f = 2 - 2g$



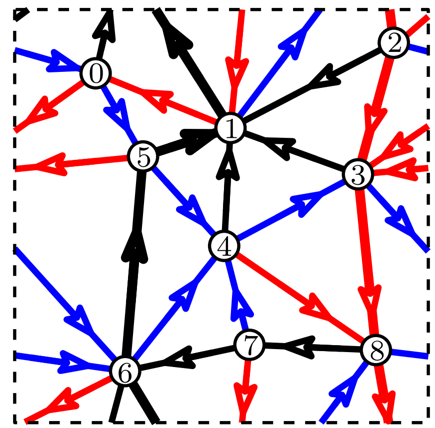
- 3-orientation + Schnyder local rule valid at each vertex

Toroidal Schnyder woods are **crossing** if

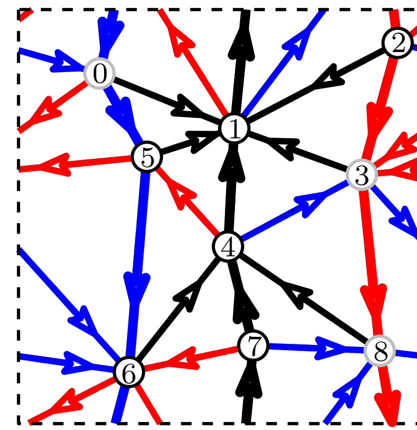
- every monochromatic cycle intersects at least one monochromatic cycle of each color



crossing Schnyder wood



half-crossing Schnyder wood

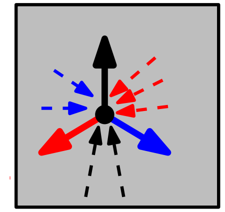


the Schnyder wood is **not half-crossing**

Toroidal Schnyder woods vs. 3-orientations

Toroidal Schnyder woods [Goncalves Lévêque, DCG'14]

- 3-orientation + Schnyder local rule valid at each vertex



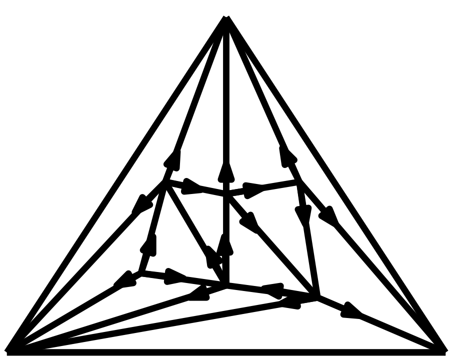
$g = 1 \qquad e = 3n$

$n - e + f = 2 - 2g$

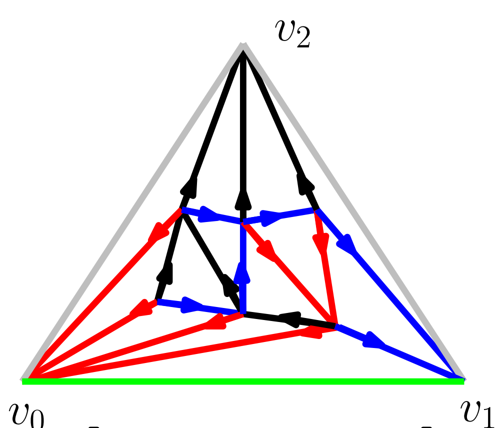
Toroidal Schnyder woods are **crossing** if

- every monochromatic cycle intersects at least one monochromatic cycle of each color

Remark: unlike the planar case, some 3-orientations do not lead to valid Schnyder woods



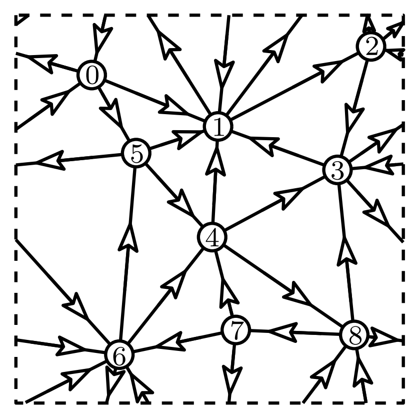
[3-orientation]



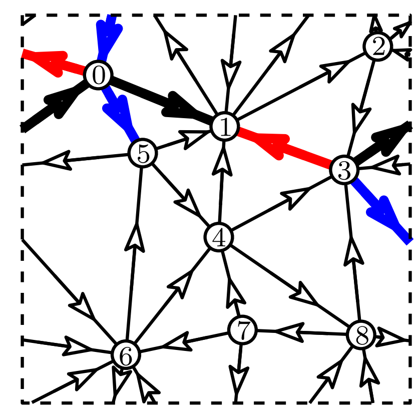
[Schnyder wood]

(in the plane there is a bijection between 3-orientations and Schnyder woods)

simple toroidal triangulation



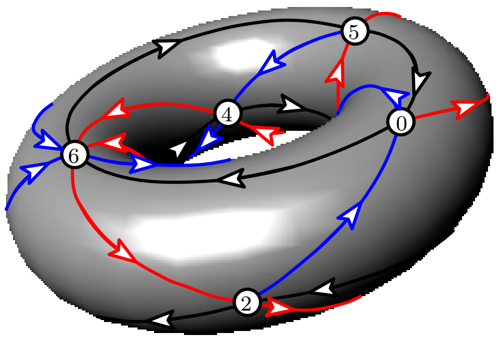
valid 3-orientation

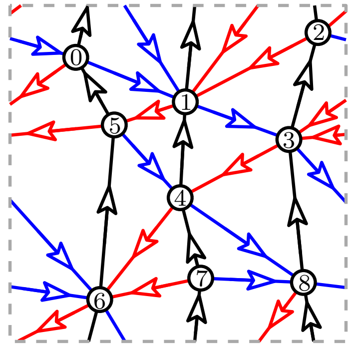


not valid Schnyder wood

(local Schnyder rule cannot be propagated everywhere)

Toroidal Schnyder woods: cycles



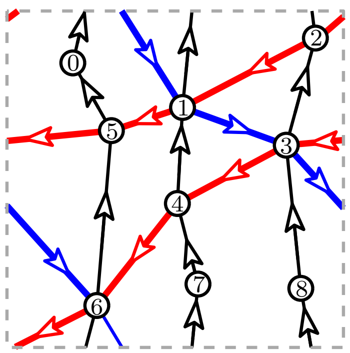


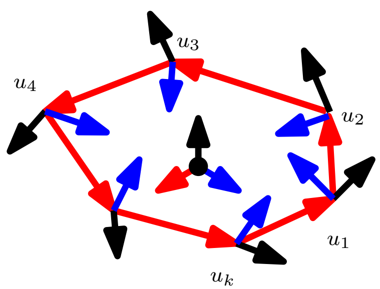
toroidal Schnyder wood

$g = 1$ $\quad e = 3n$

$n - e + f = 2 - 2g$

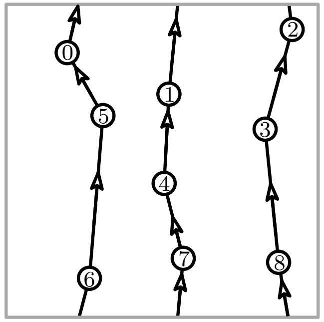
- toroidal Schnyder woods must contain a (mono-chromatic) cycle in each color: $e = 3n$ (n edges in each color)
- mono-chromatic cycles are non-contractibles





Remark: the inner region of a contractible mono-chromatic cycle is a topological disk

- some colors may define disconnected components



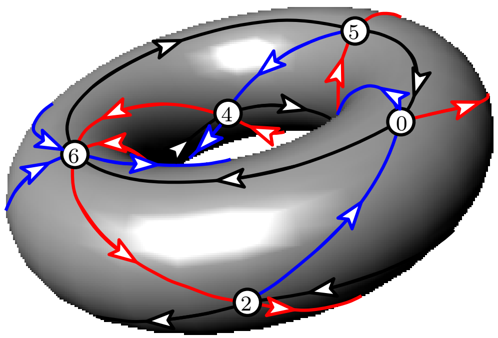
(there are 3 disjoint mono-chromatic cycles of color 2)

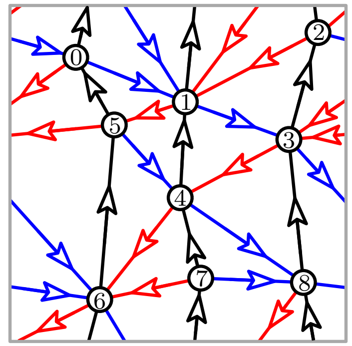
Open problem: is it possible to find (at least) one toroidal Schnyder wood with connected mono-chromatic components

n	# irreducible triangulations	#triangulations ($g = 1$)
7	1	1
8	4	7
9	15	112
10	1	2109
11	–	37867

(true for all triangulations of size at most $n = 11$)

Toroidal Schnyder woods: cycles



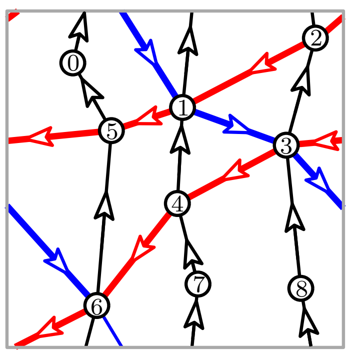


toroidal Schnyder wood

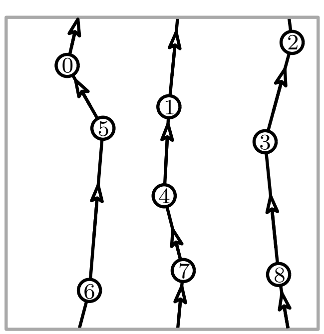
$g = 1$	$e = 3n$
$n - e + f = 2 - 2g$	

- toroidal Schnyder woods must contain a (mono-chromatic) cycle in each color: $e = 3n$

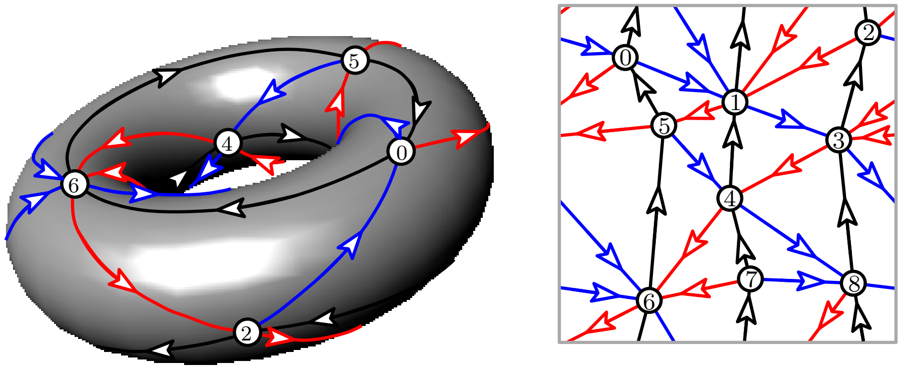
- mono-chromatic cycles are non-contractibles



- all mono-chromatic cycles of the same color are homotopic (parallel) and oriented in one direction



Toroidal Schnyder woods: cycles



toroidal Schnyder wood

$g = 1$	$e = 3n$
$n - e + f = 2 - 2g$	

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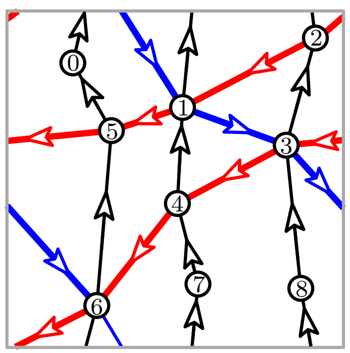
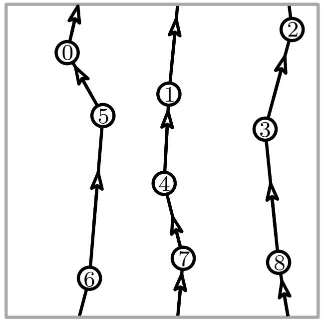
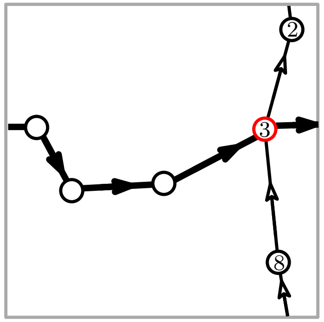
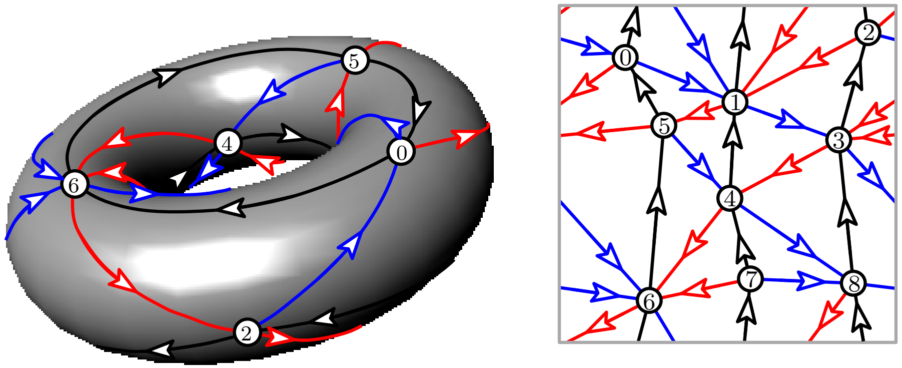


- all mono-chromatic cycles of the same color are:
  homotopic and disjoint (parallel) and oriented in one direction





Toroidal Schnyder woods: cycles



toroidal Schnyder wood

$g = 1$	$e = 3n$
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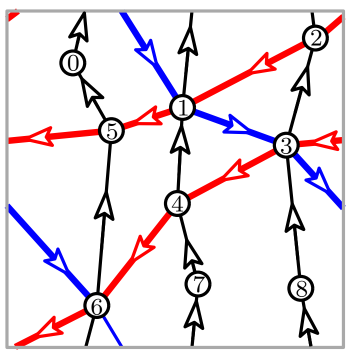
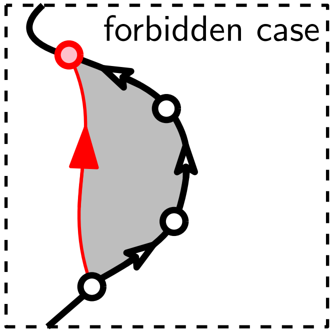
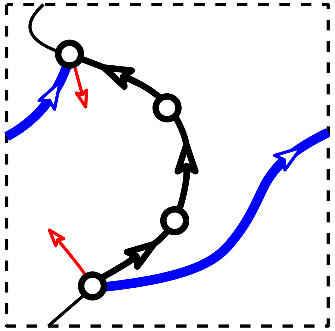
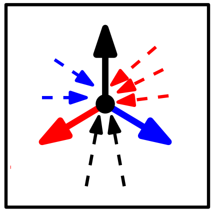


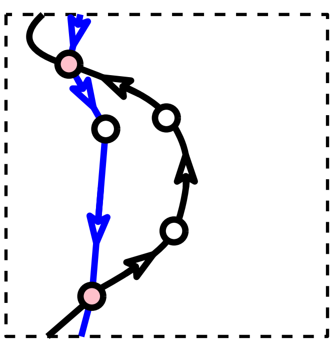


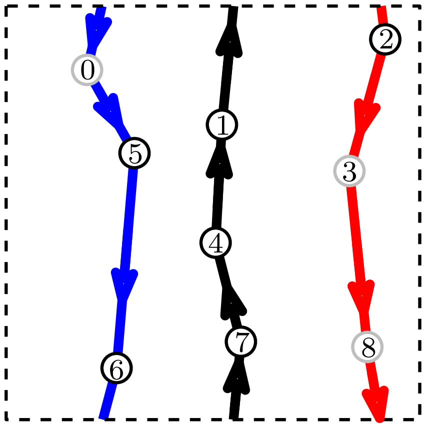


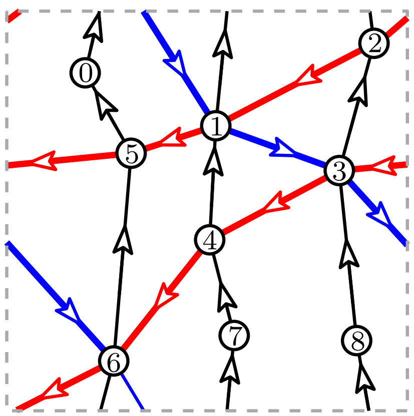
mono-chromatic paths $P_i(v)$ may have incident chords

- all mono-chromatic cycles of different colors are: either homotopic and disjoint (parallel) or crossing









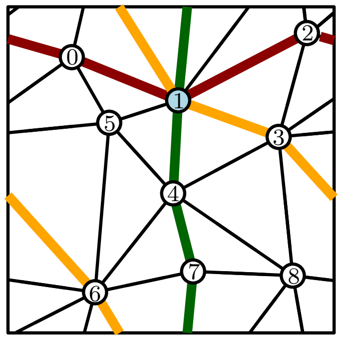
Existence and computation of toroidal Schnyder woods

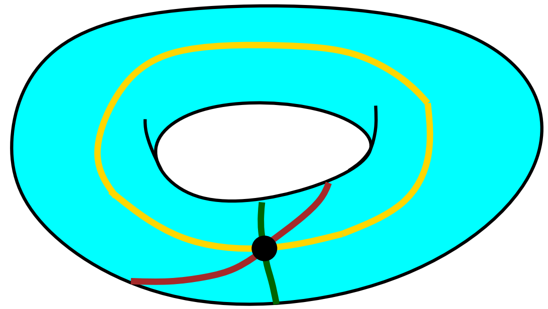
(three different proofs)

Toroidal Schnyder woods: existence I

Thm[Fijavz, unpublished] (for simple toroidal triangulations)

A simple toroidal triangulation contains three non-contractible and non-homotopic cycles that all intersect on one vertex and that are pairwise disjoint otherwise.

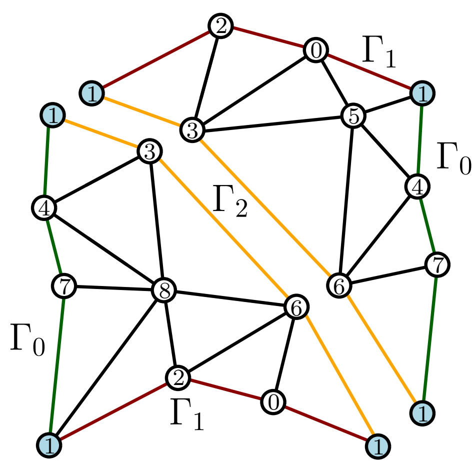




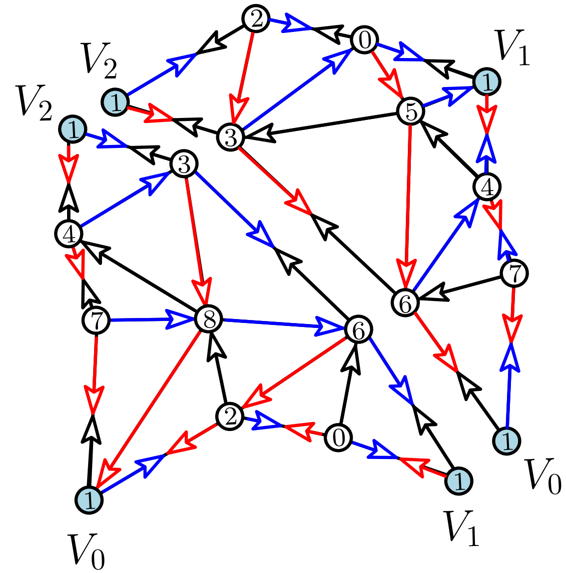
Corollary[Goncalves Lévêque, DCG'14]

Any simple toroidal triangulation admits a toroidal **crossing Schnyder wood**

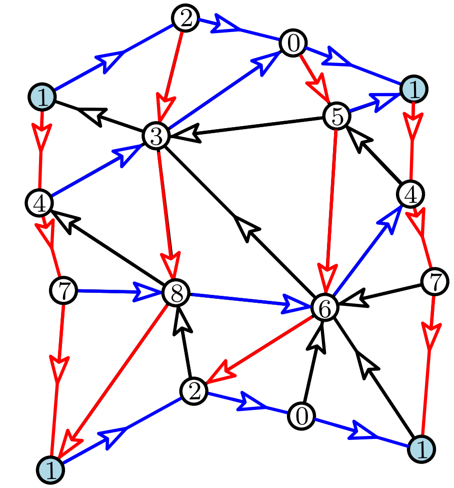
split along Γ_0, Γ_1, Γ_2



(two planar quasi-triangulations)



crossing toroidal Schnyder wood (for simple triangulations)

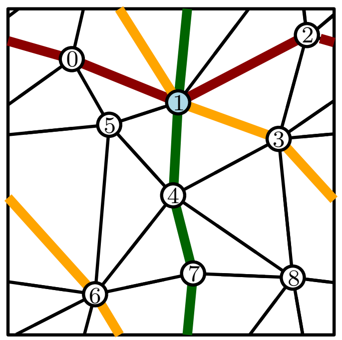


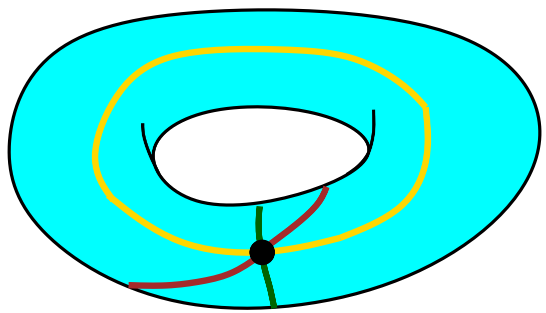
Toroidal Schnyder woods: existence I

(for simple toroidal triangulations)

Thm[Fijavz, unpublished]

A simple toroidal triangulation contains three non-contractible and non-homotopic cycles that all intersect on one vertex and that are pairwise disjoint otherwise.



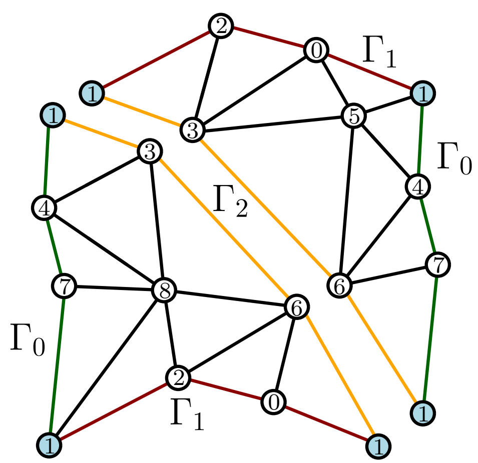


Conjecture: is it possible to find (at least) one toroidal Schnyder wood with connected mono-chromatic components and such the intersection of the three cycles is a single vertex?

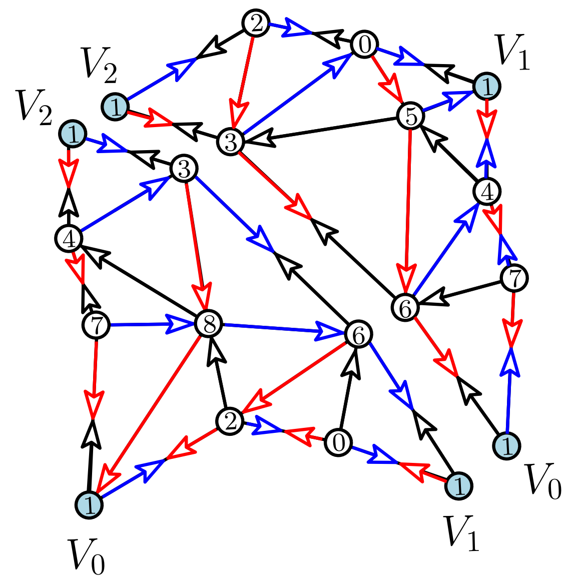
Corollary[Goncalves Lévêque, DCG'14]

Any simple toroidal triangulation admits a toroidal **crossing Schnyder wood**

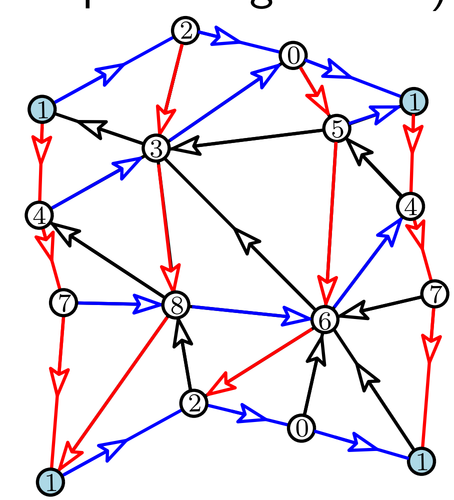
split along Γ_0, Γ_1, Γ_2



(two planar quasi-triangulations)



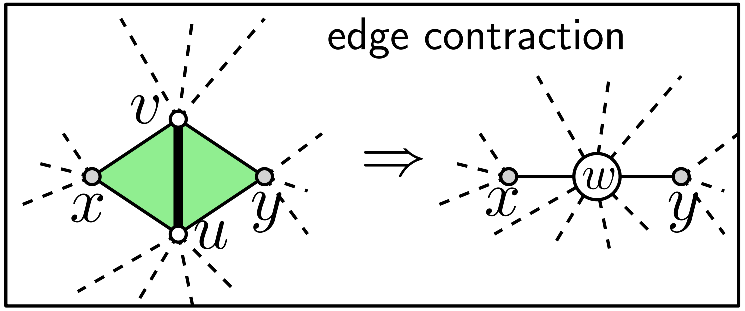
crossing toroidal Schnyder wood (for simple triangulations)



Toroidal Schnyder woods: existence II

Thm[Goncalves Lévêque, DCG'14] (for general toroidal triangulations and maps)

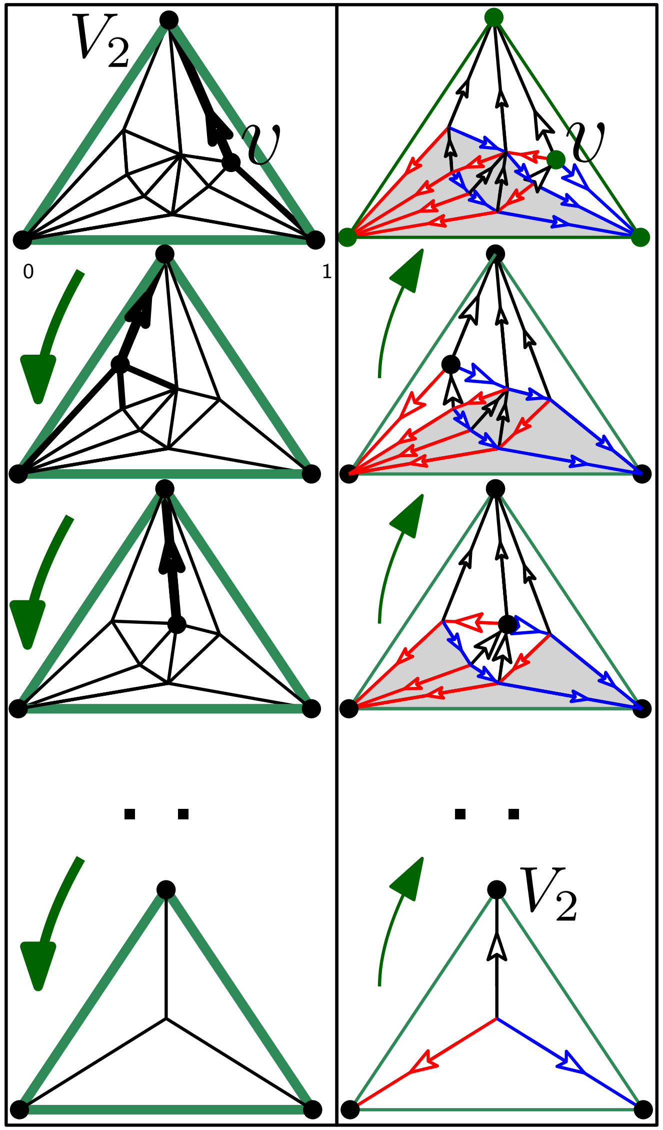
Any toroidal triangulation admits a toroidal **crossing Schnyder wood**



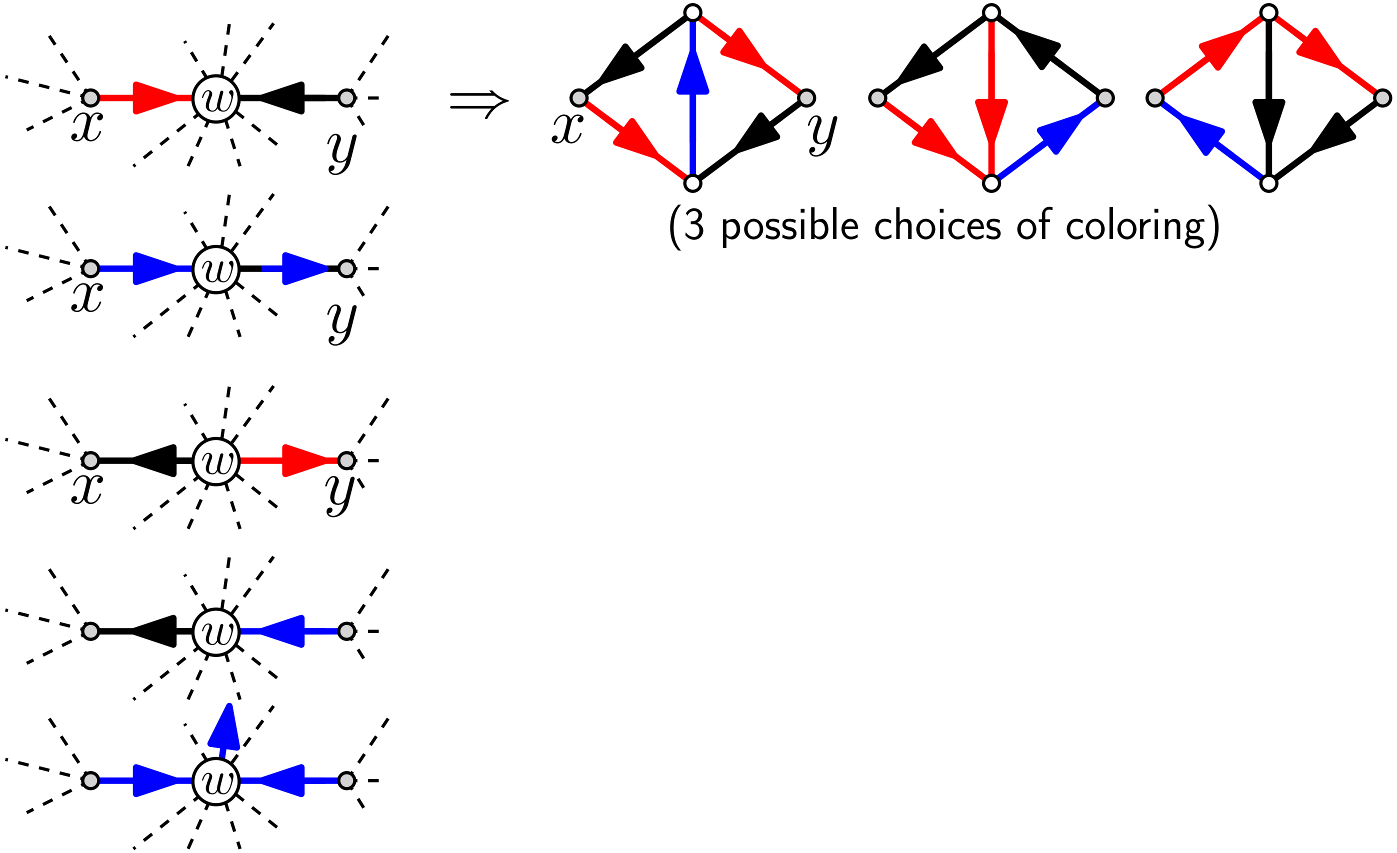
computation of (planar) Schnyder woods

first phase: perform edge contractions

second phase: perform edge expansion+edge coloring



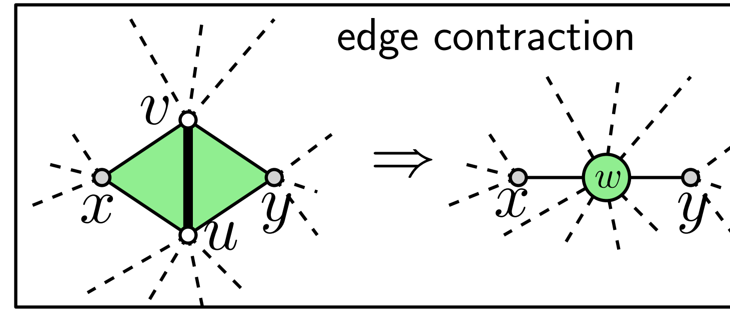
(several cases to distinguish during the decontraction)



(3 possible choices of coloring)

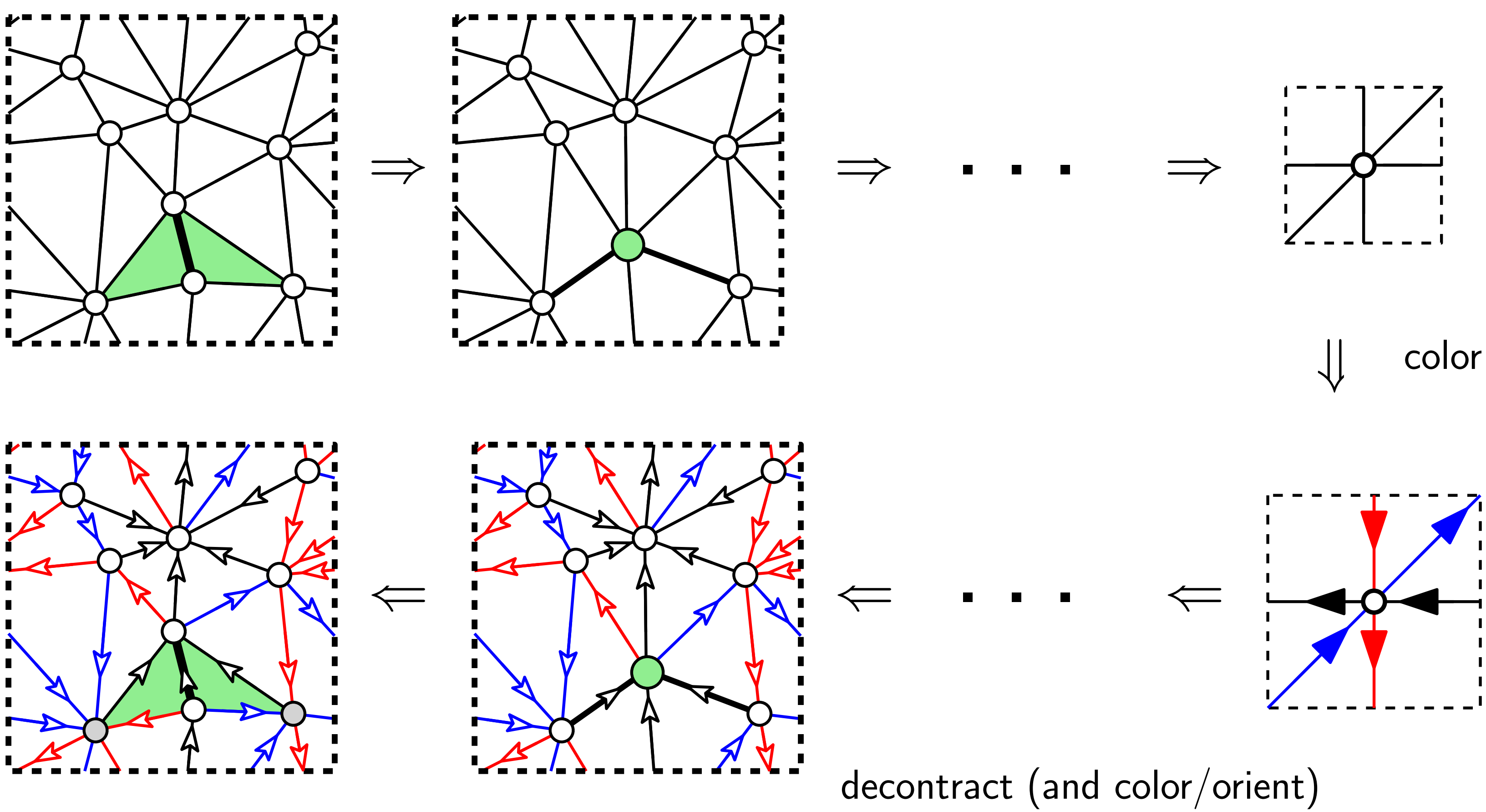
Toroidal Schnyder woods: existence II

Thm[Goncalves Lévêque, DCG'14] (for general toroidal triangulations and maps)
Any toroidal triangulation admits a toroidal **crossing Schnyder wood**



remark: maintaining the crossing property can require quadratic time

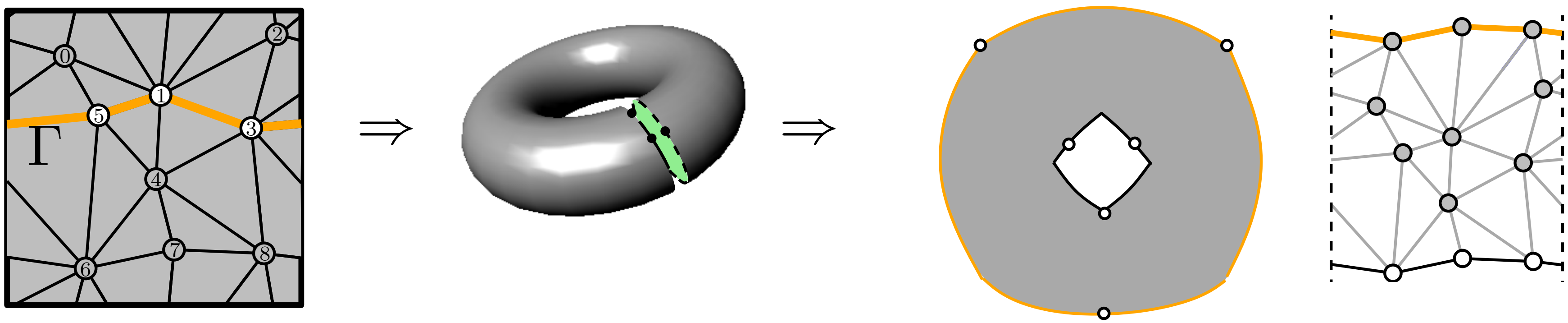
perform (carefully) a sequence of $n-1$ edge contractions



Toroidal Schnyder woods: existence III

Thm[SoCG'25] (for simple toroidal triangulations)

Any toroidal triangulation admits a toroidal **(crossing) Schnyder wood**



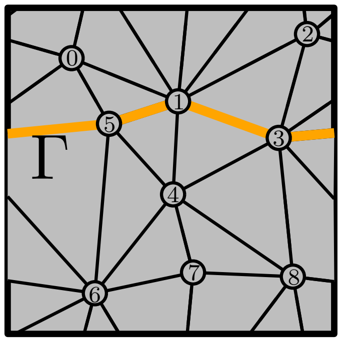
cut along a non-contractible cycle Γ

cylindric triangulation: planar triangulation with two boundaries

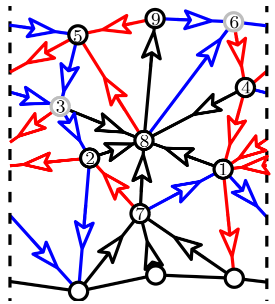
Toroidal Schnyder woods: existence III

Thm[SoCG'25] (for simple toroidal triangulations)

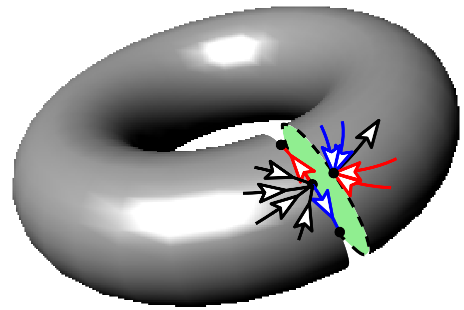
Any toroidal triangulation admits a toroidal **(not necessarily crossing) Schnyder wood**



⇒

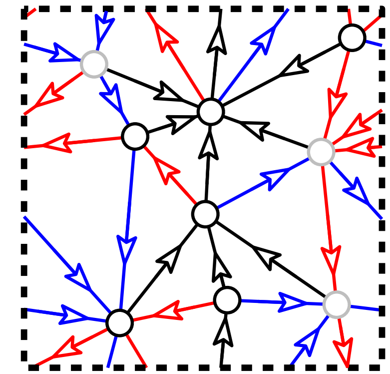


⇒



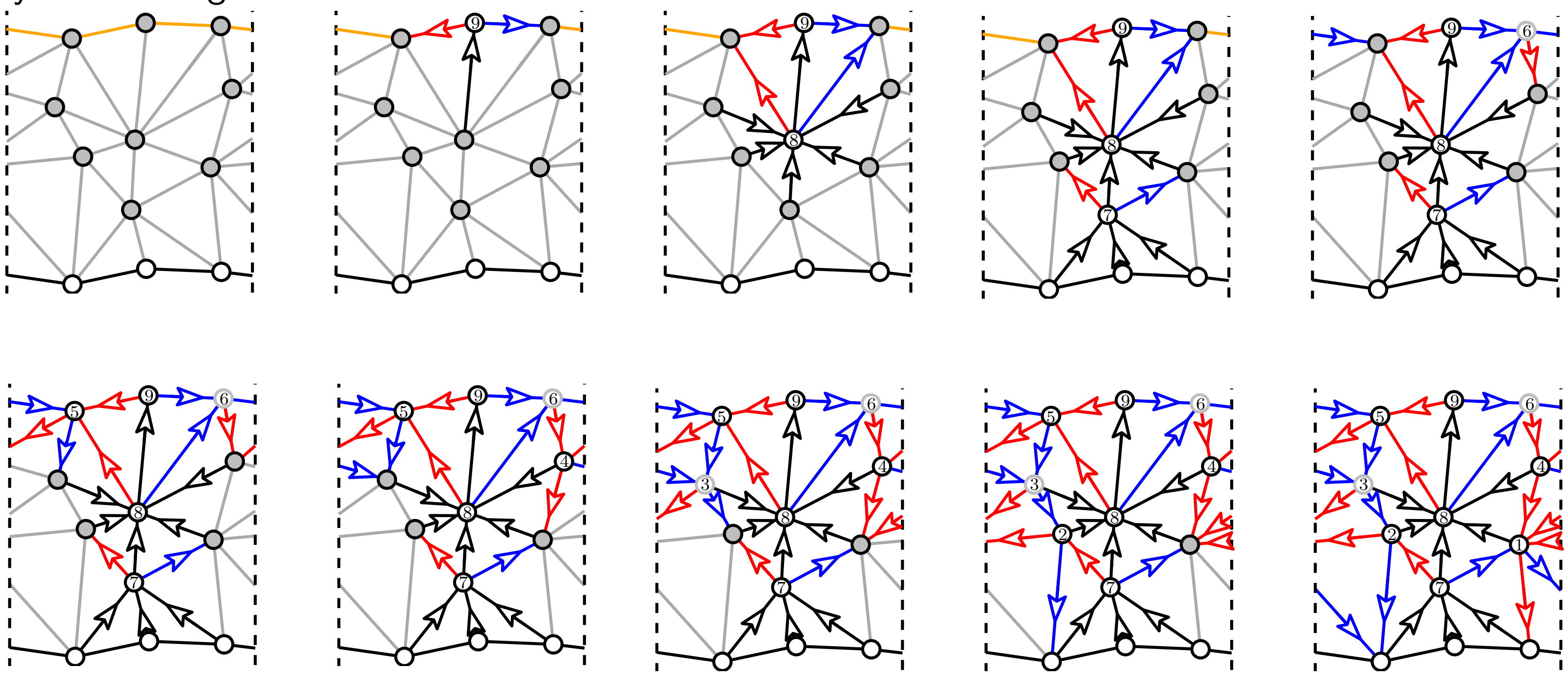
glue together the two boundaries

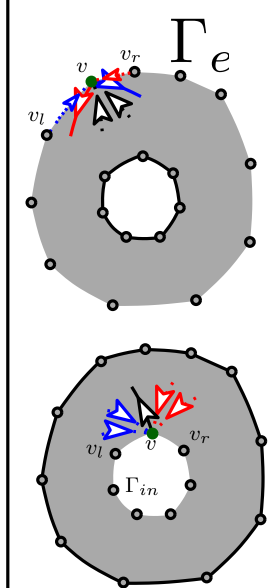
Toroidal Schnyder wood



(the local Schnyder woods remains satisfied on Γ)

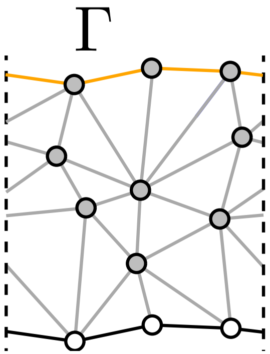
Cylindric triangulation



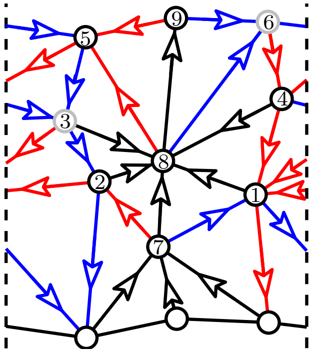


local invariant

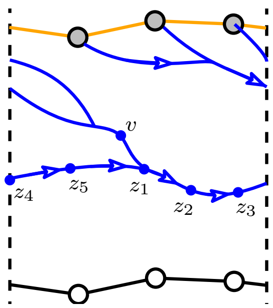
Structural properties of Schnyder woods computed via cylindric Schnyder woods

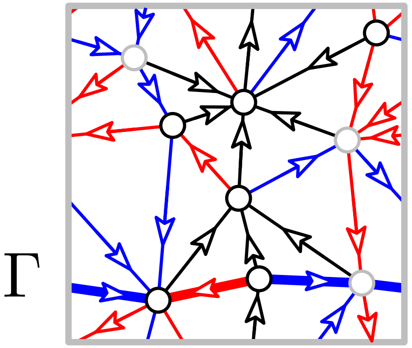


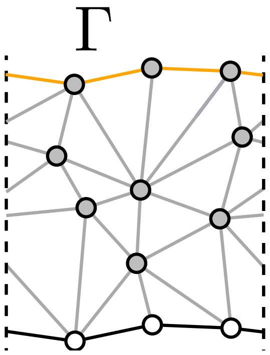
- edges of Γ are either 0 or 1
- 0 and 1-paths are oriented downward
- 2-paths are oriented upward
- 0, 1 and 2-paths cross the cycle Γ



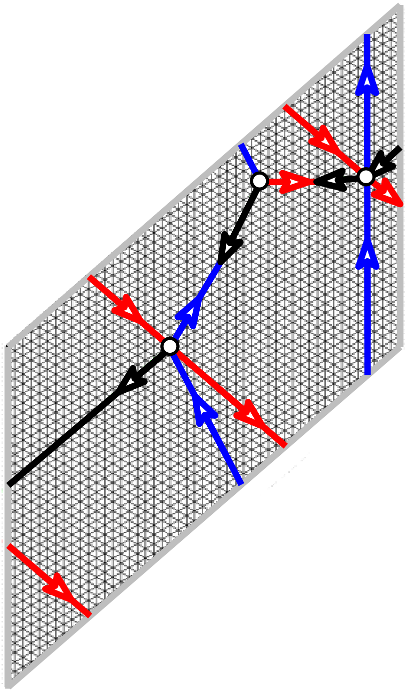
- 0, 1 and 2-cycles are never homotopic to Γ: they must cross Γ





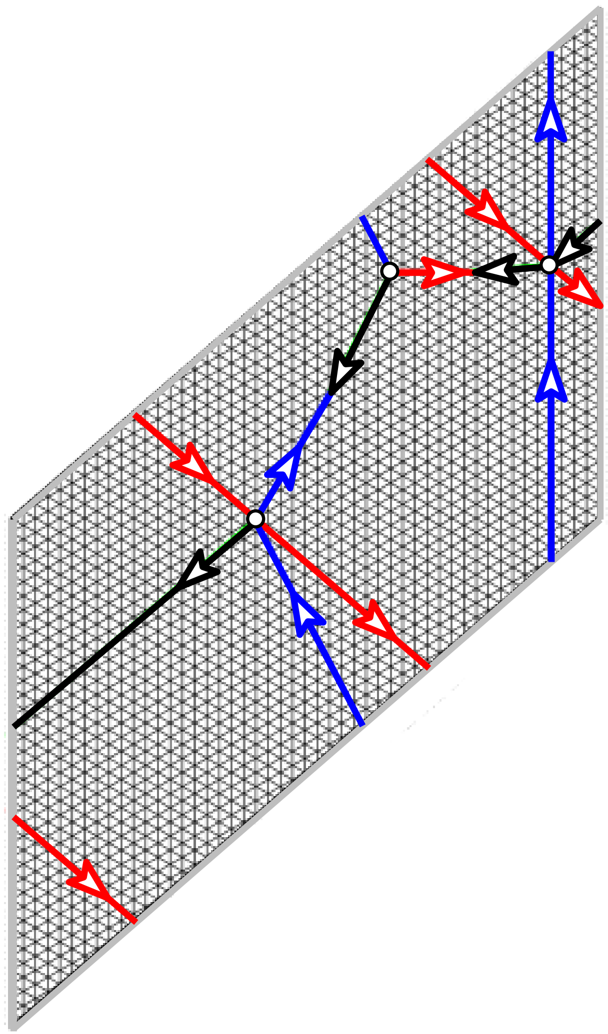


Periodic (planar) Schnyder drawings of toroidal graphs



Toroidal Schnyder (periodic) drawings

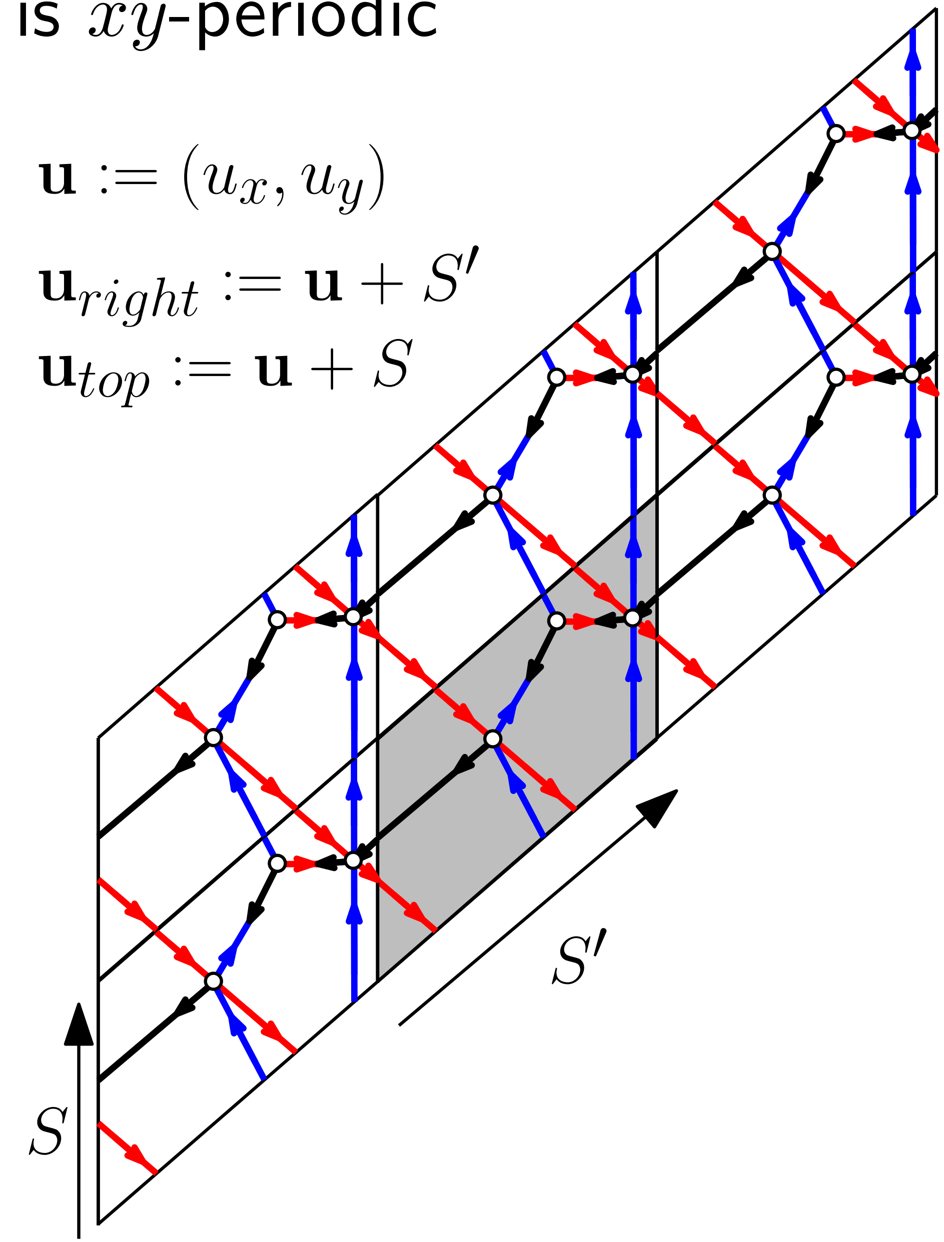
Goal: try to generalize the region counting method to obtain a straight-line grid drawing which is xy-periodic



$\mathbf{u} := (u_x, u_y)$

$\mathbf{u}_{right} := \mathbf{u} + S'$

$\mathbf{u}_{top} := \mathbf{u} + S$



Region counting on the torus

How regions are defined on the torus?

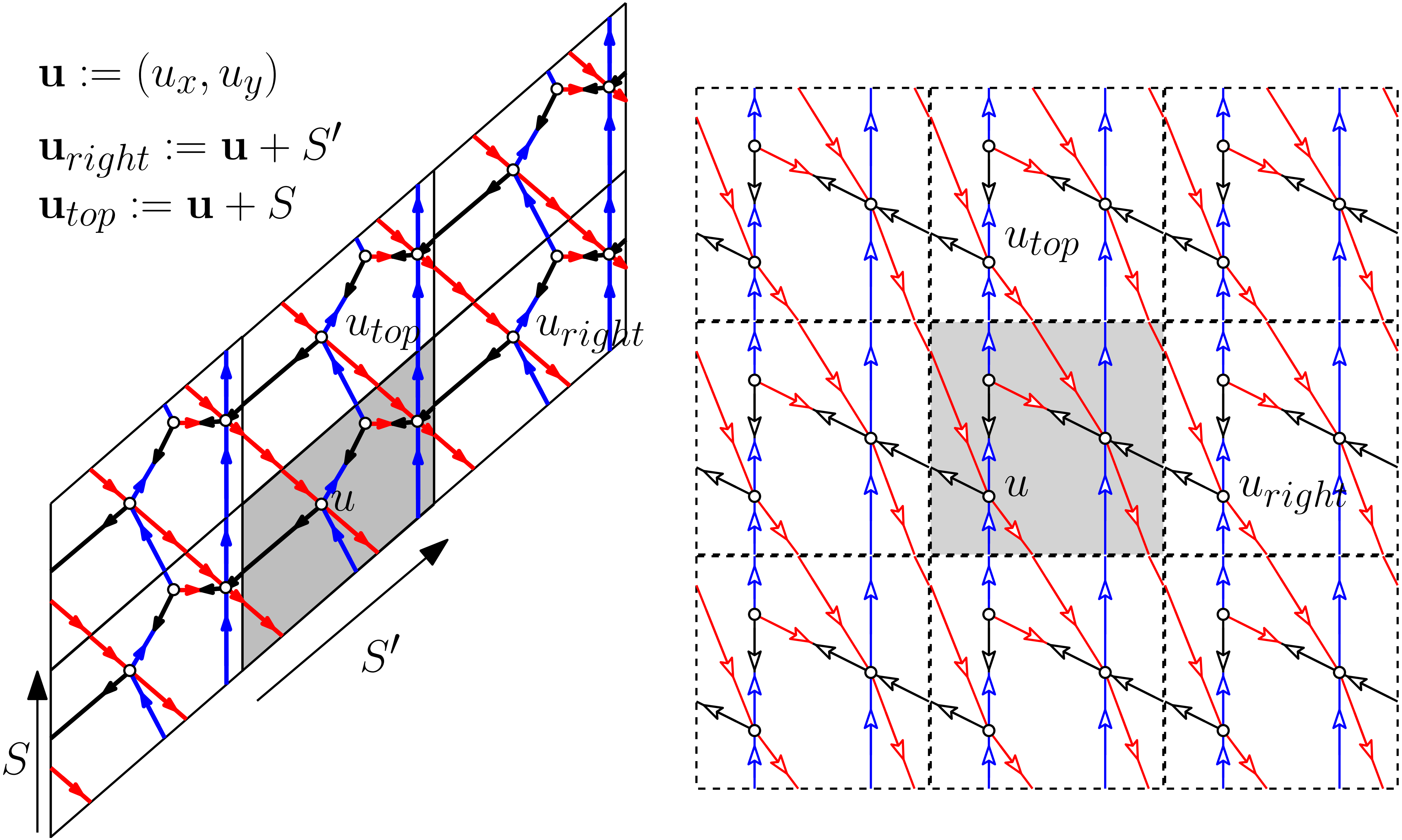
How to assign coordinates to vertices to ensure periodicity?

How periodicity is defined? (how vectors S, S' are defined?)

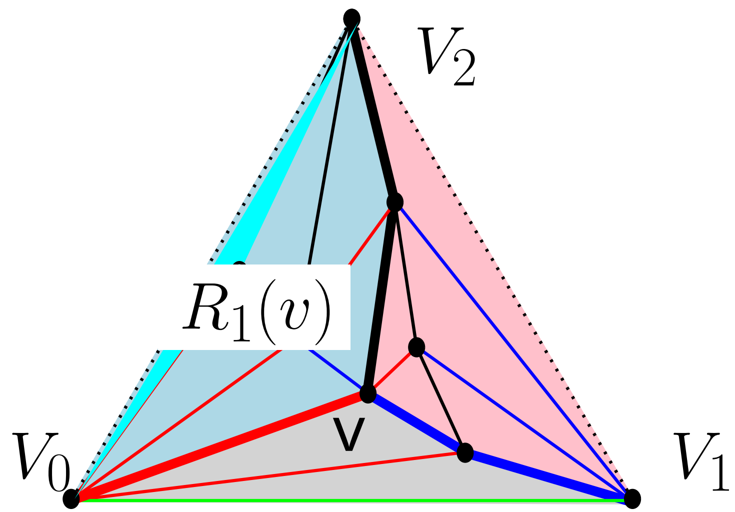
$\mathbf{u} := (u_x, u_y)$

$\mathbf{u}_{right} := \mathbf{u} + S'$

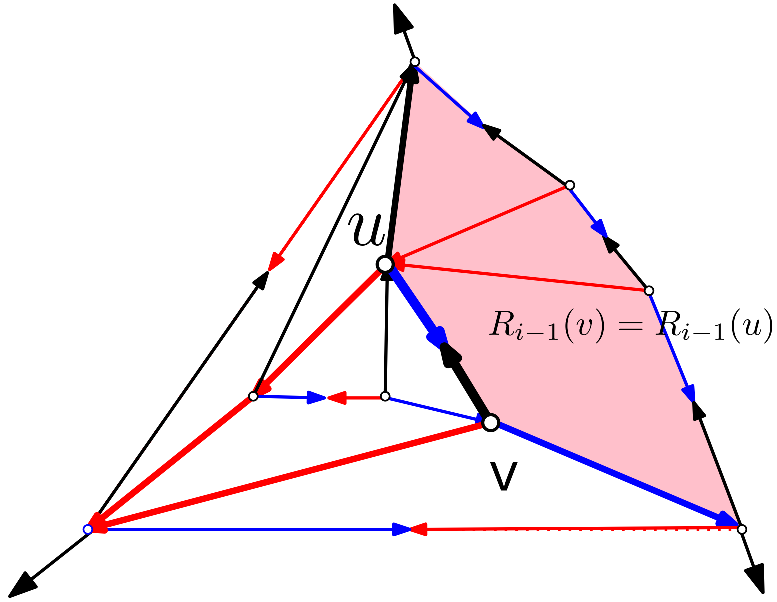
$\mathbf{u}_{top} := \mathbf{u} + S$



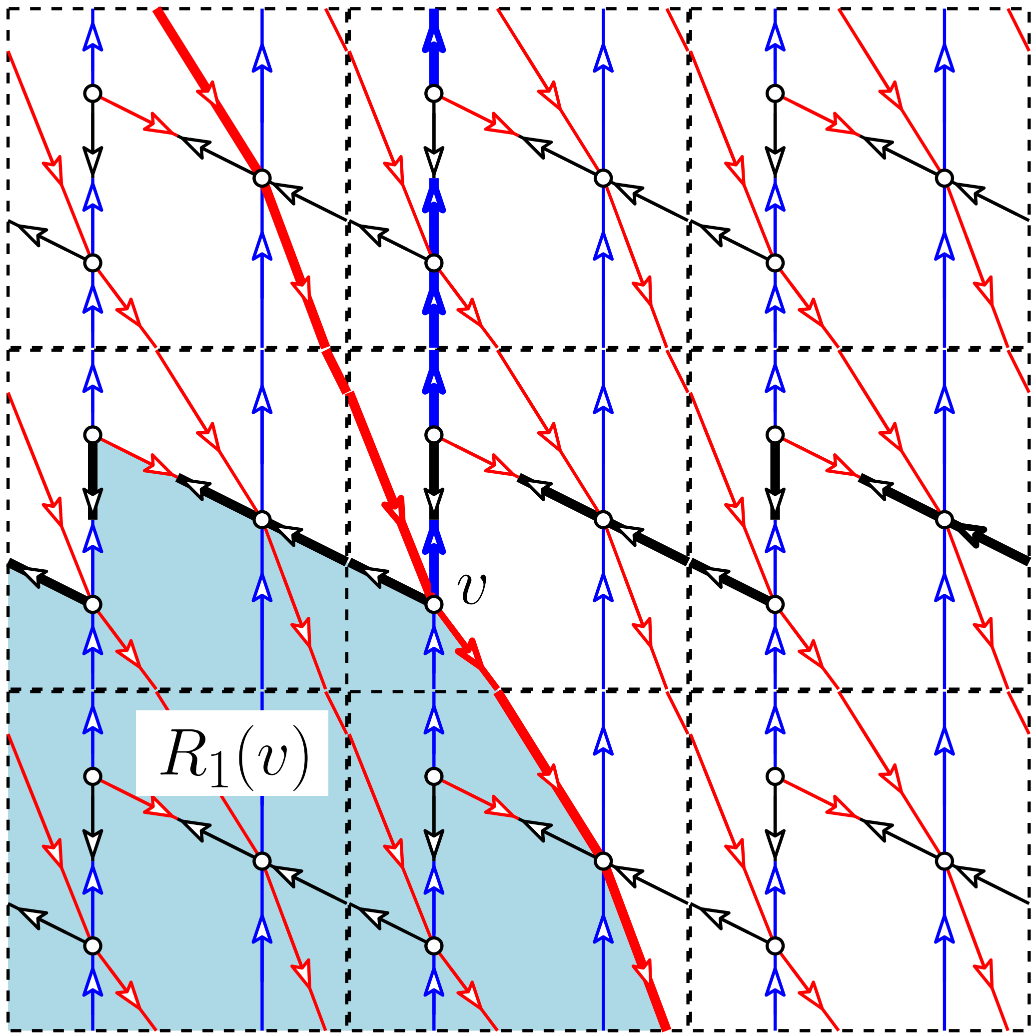
Regions are unbounded



$$v =: \frac{|R_0(v)|}{|F|-1}V_0 + \frac{|R_1(v)|}{|F|-1}V_1 + \frac{|R_2(v)|}{|F|-1}V_2$$



Planar case: bounded number of faces in each region



Toroidal case: unbounded regions

Regions are unbounded

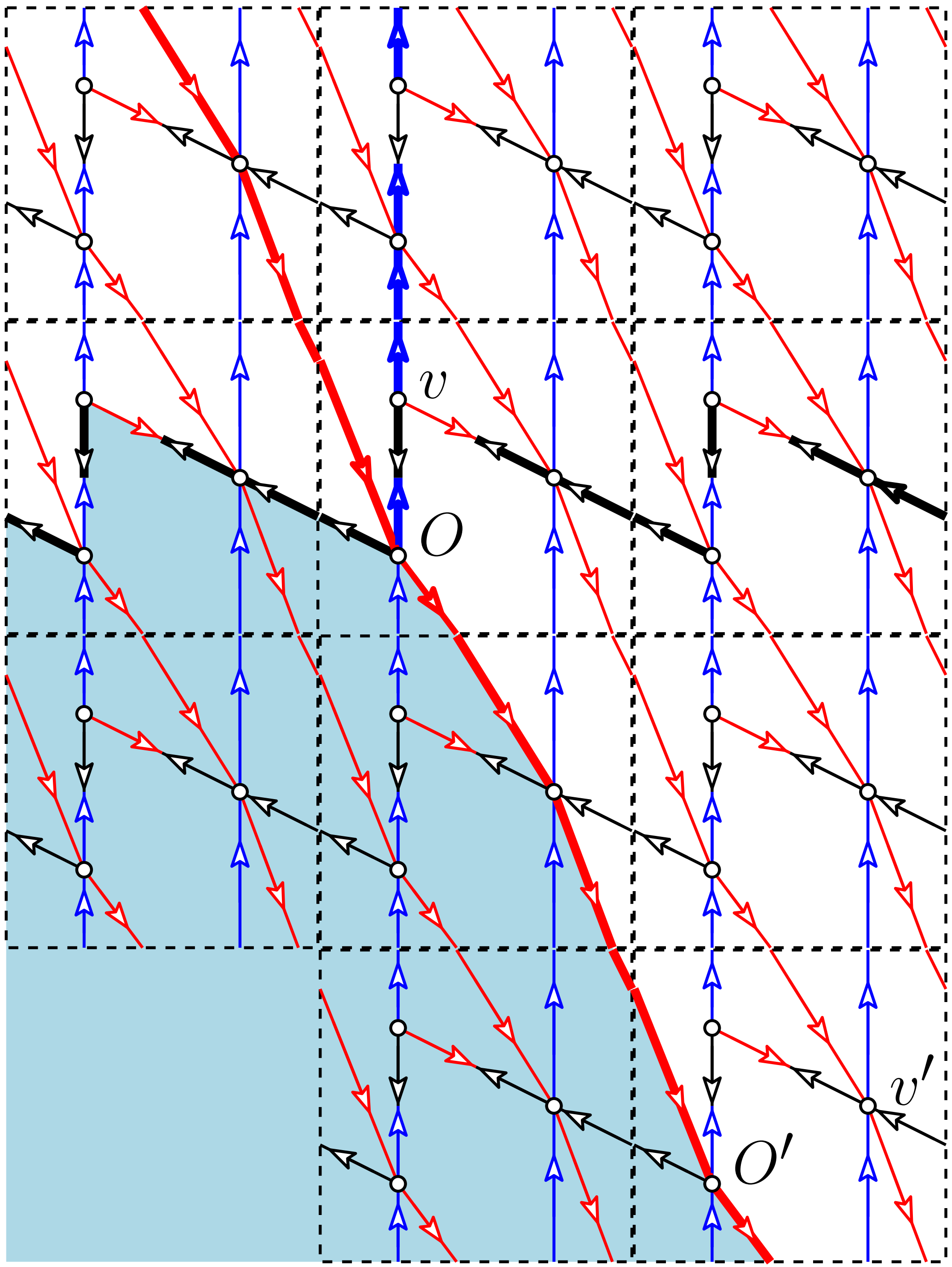
Do not use absolute coordinates

$v =: \frac{|R_0(v)|}{|F|-1} V_0 + \frac{|R_1(v)|}{|F|-1} V_1 + \frac{|R_2(v)|}{|F|-1} V_2$

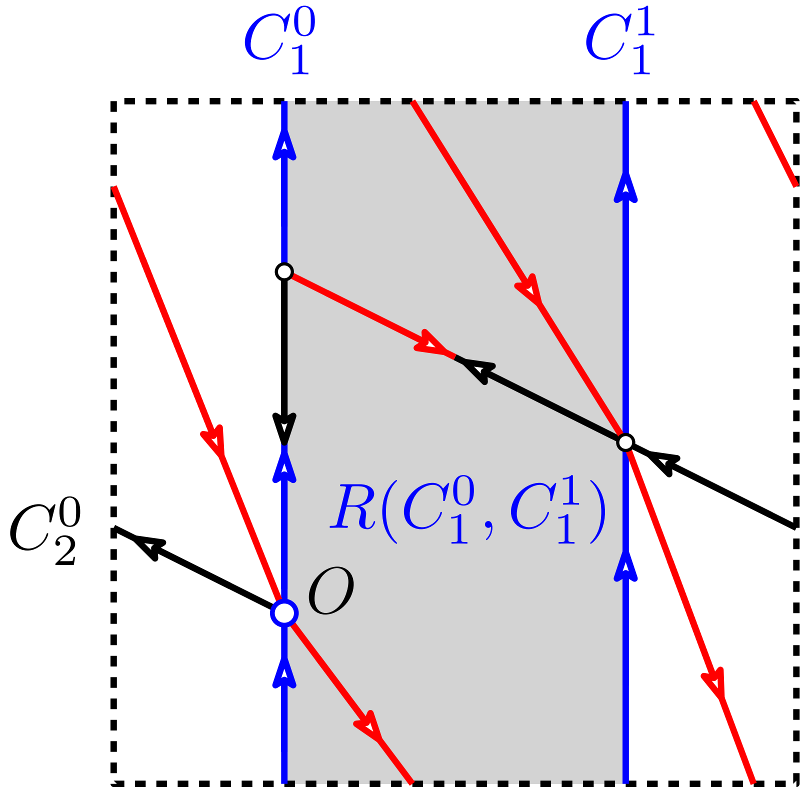
Toroidal case: regions are unbounded but differences between regions is bounded

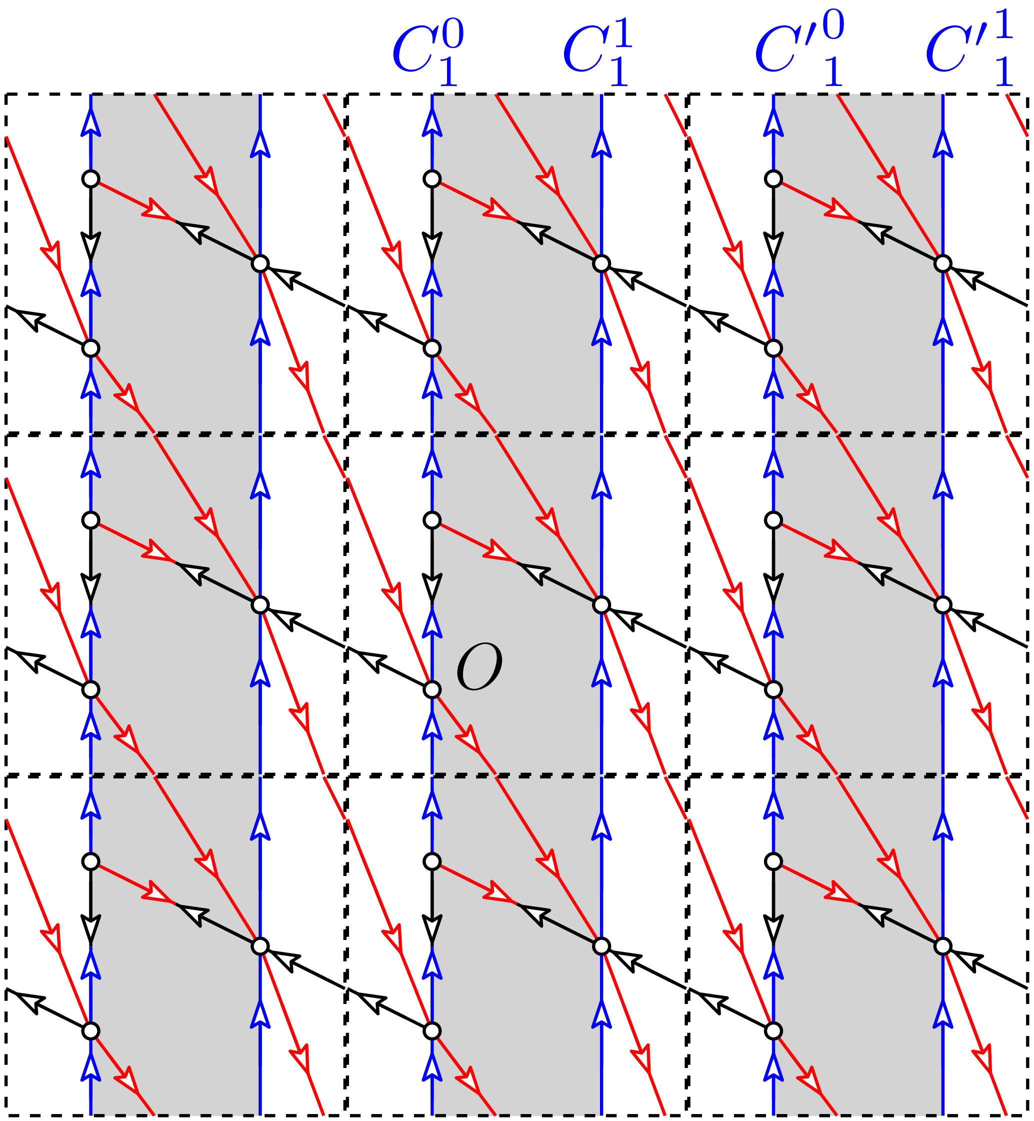
Fix an origin vertex O

Define coordinates of v relatives to O



How to define the size of a region





$\mathcal{C}_1 := \{C_1^0, C_1^1\}$
2 mono-chromatic consecutive blue cycles

$\mathcal{C}_2 := \{C_2^0\}$
1 mono-chromatic black cycle

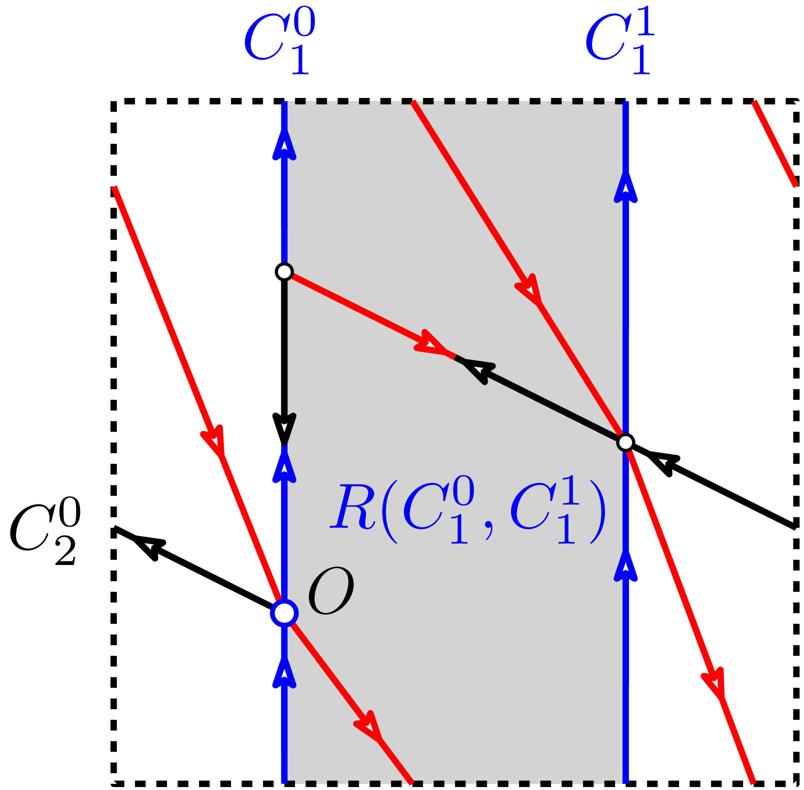
$\mathcal{L}_1^0 := \{C_1^0, C'^0_1, \dots\}$ (lines in the universal cover)

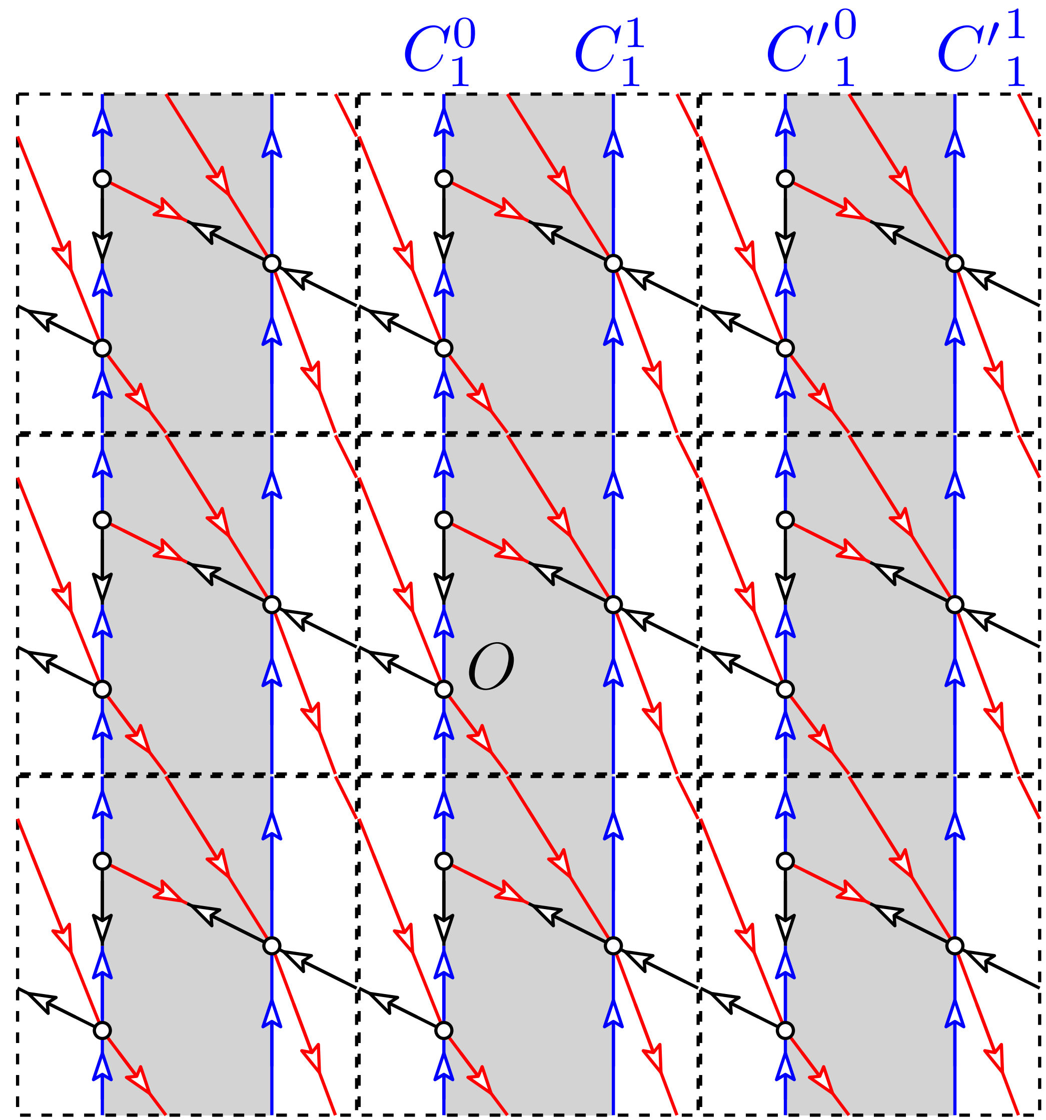
$R(C_1^j, C_1^{j+1}) :=$ region between consecutives 1-cycles

$f_1^j := \|R(C_1^j, C_1^{j+1})\|$ (size of the 1-region: number of faces)

(how many faces in the gray region?)
$\|R(C_1^0, C_1^1)\| = ?$

How to define the size of a region





$\mathcal{C}_1 := \{C_1^0, C_1^1\}$

2 mono-chromatic consecutive blue cycles

$\mathcal{C}_2 := \{C_2^0\}$

1 mono-chromatic black cycle

$$\sum_j \|R(C_i^j, C_i^{j+1})\| = F$$

(for each color $i \in \{0, 1, 2\}$)

where F:= number of faces of G

$\|R(C_0^0, C_0^0)\| = 4$

$\|R(C_2^0, C_2^0)\| = 4$

(2 faces in the gray region)

$\|R(C_1^0, C_1^1)\| = 2$

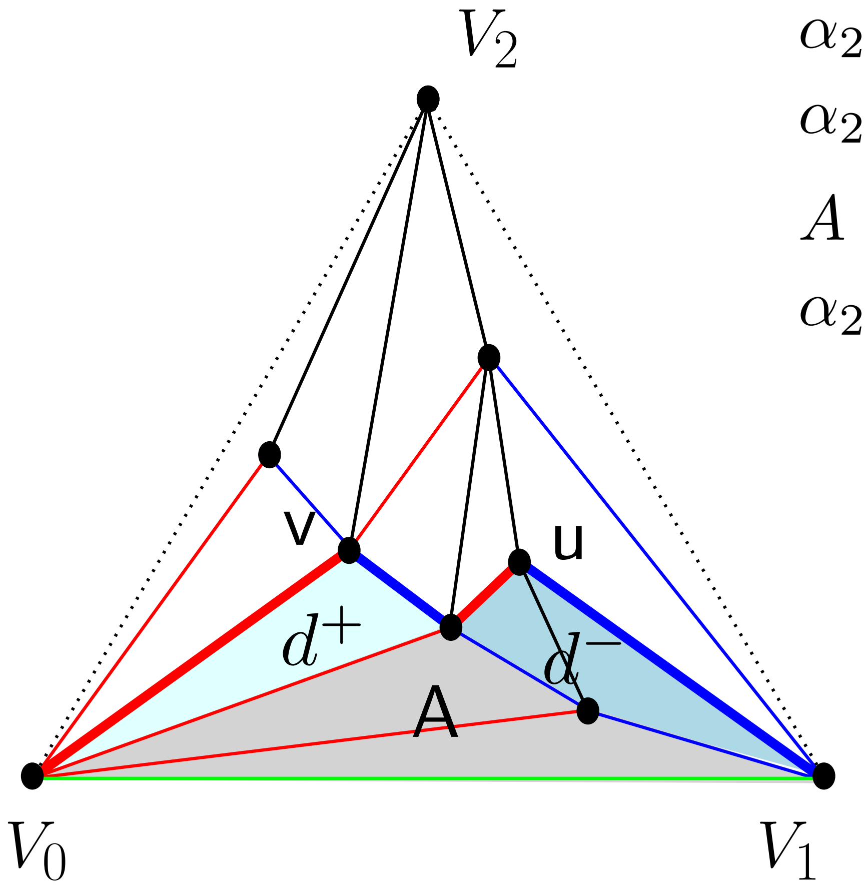
(2 faces in the white region)

$\|R(C_1^1, C_1^0)\| = 2$

How assign to ccordinates to vertices

Goal: assign relative coordinates to vertices

Let us revise the planar case first



$$\alpha_2(v) =: |R_2(v)| = 3$$
$$\alpha_2(u) =: |R_2(u)| = 4$$
$$A =: |R_2(v) \cap R_2(u)| = 2$$
$$\alpha_2(v) = 4 + (1 - 2)$$

$$\alpha_2(v) =: |R_2(v)| = A + d^+$$
$$\alpha_2(u) =: |R_2(u)| = A + d^-$$

$$\boxed{\alpha_2(v) = \alpha_2(u) + (d^+ - d^-)}$$

$$v =: \tfrac{|R_0(v)|}{|F|-1} V_0 + \tfrac{|R_1(v)|}{|F|-1} V_1 + \tfrac{|R_2(v)|}{|F|-1} V_2$$
$$v =: \alpha_0 V_0 + \alpha_1 V_1 + \alpha_2 V_2$$

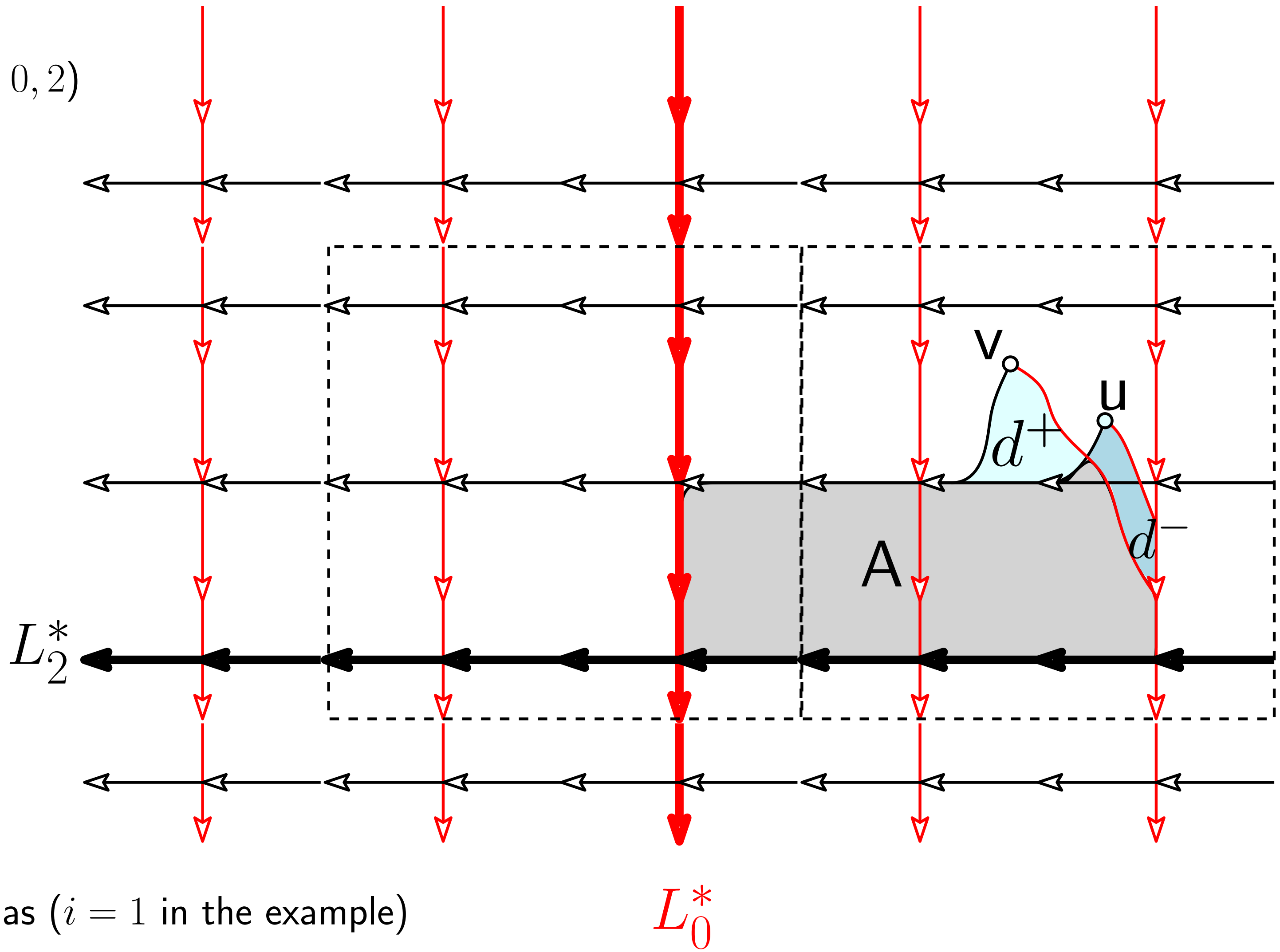
How assign to ccordinates to vertices

Goal: assign relative coordinates to vertices

Let us consider now the toroidal case

Consider two vertices u and v in the same "region" (defined by the same mono-chromatic lines)

Choose two references line L_i^* $(i = 0, 2)$



the i-coordinate of v is expressed as ($i = 1$ in the example)

$$\alpha_i(v) = \alpha_i(u) + (d^+ - d^-)$$

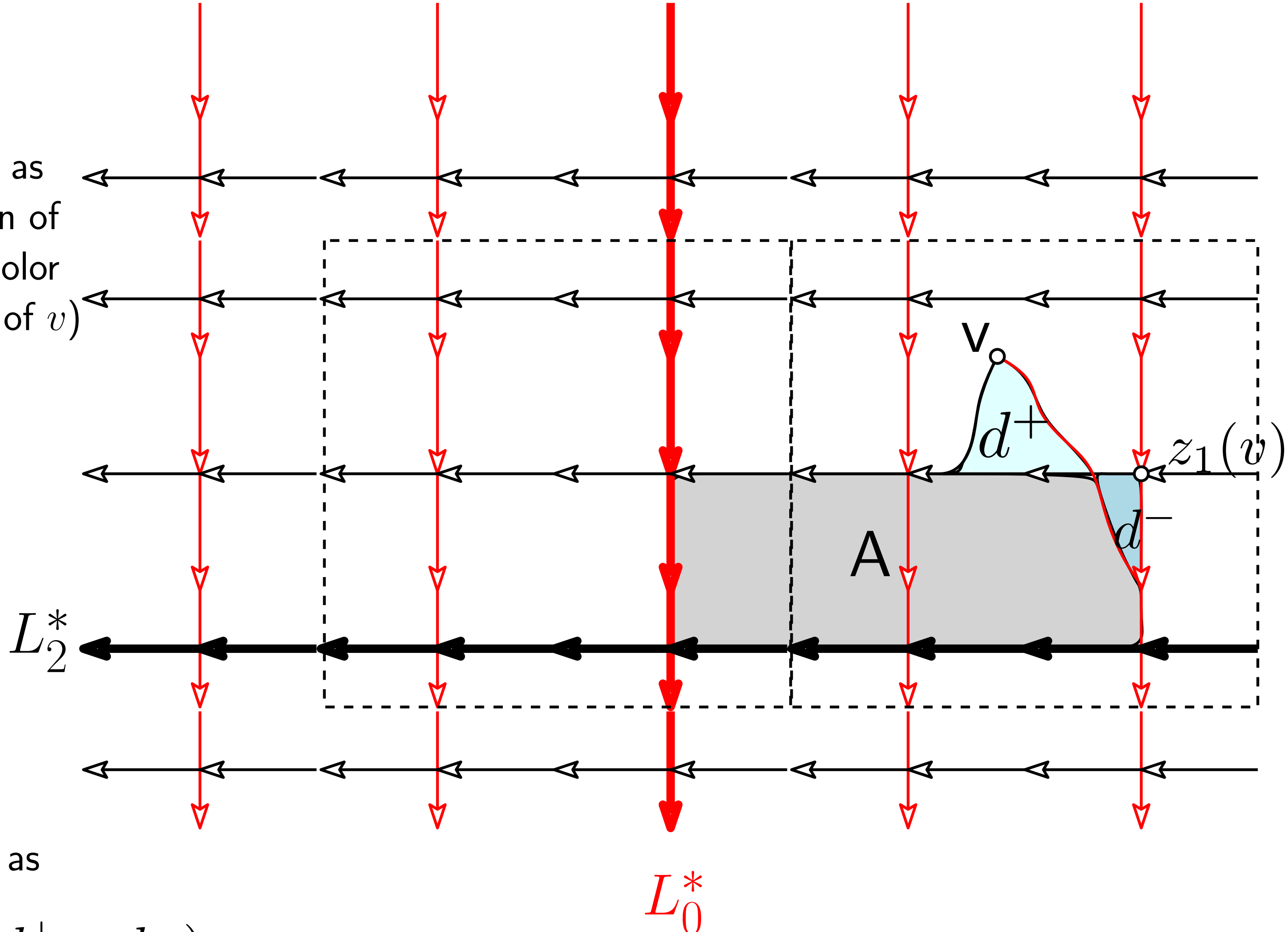
How assign to ccordinates to vertices

Goal: assign relative coordinates to vertices

Let us consider now the toroidal case

Consider two vertices u and v in the same "region" (defined by the same mono-chromatic lines)

Given a vertex v and a color i take as second vertex $z_i(v)$, the intersection of the two mono-chromatic lines (of color $i-1$ and $i+1$ defining the region of v)



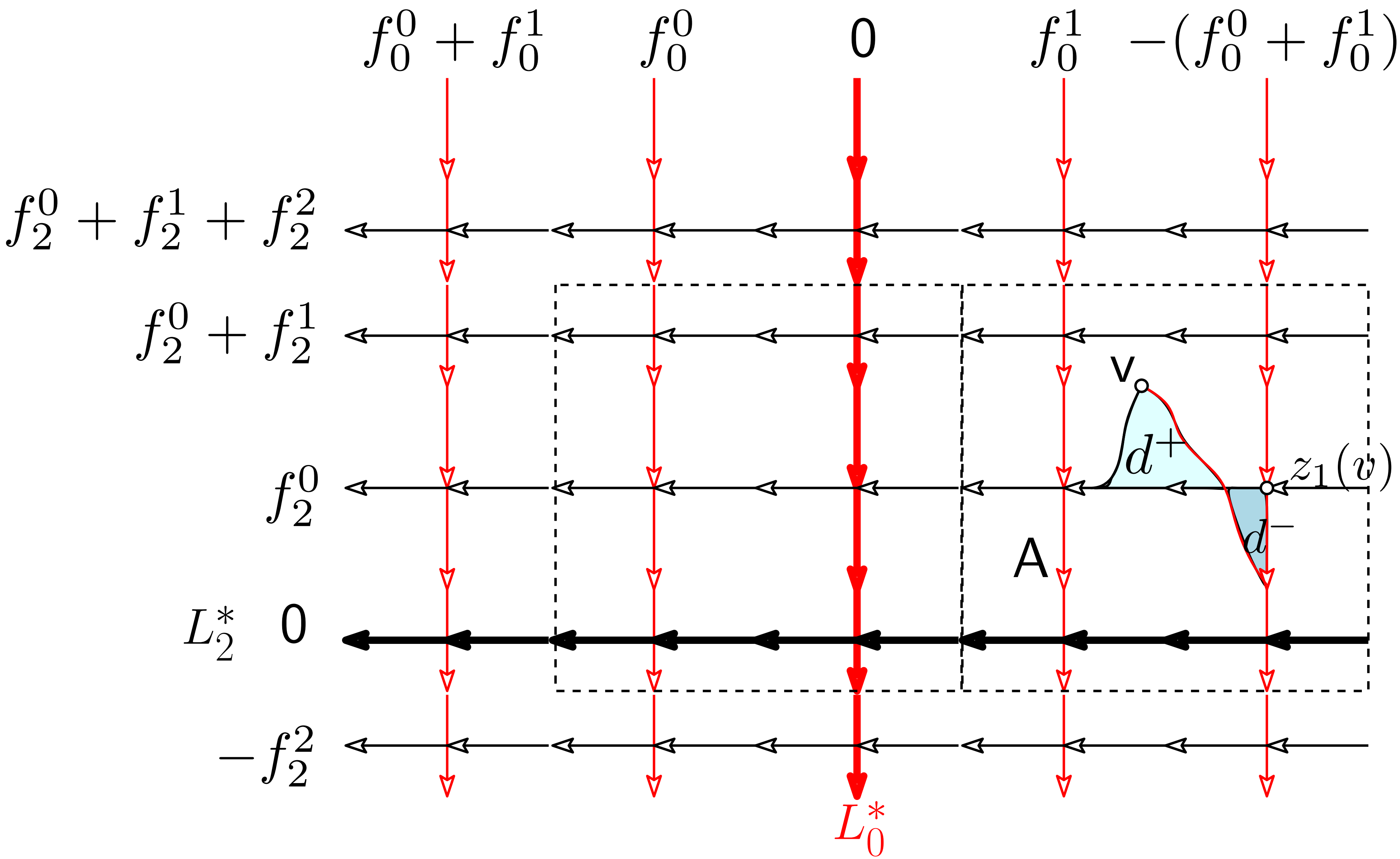
the 2-coordinate of v is expressed as

$$\alpha_2(v) = \alpha_2(z(v)) + (d^+ - d^-)$$

How assign to ccordinates to vertices

Goal: assign relative coordinates to vertices

Assign coordinates to the mono-chromatic lines



Remark: the signs depend on the relative position of the mono-chromatic lines with respect to the reference lines L_i^* (top/bottom, left/right)

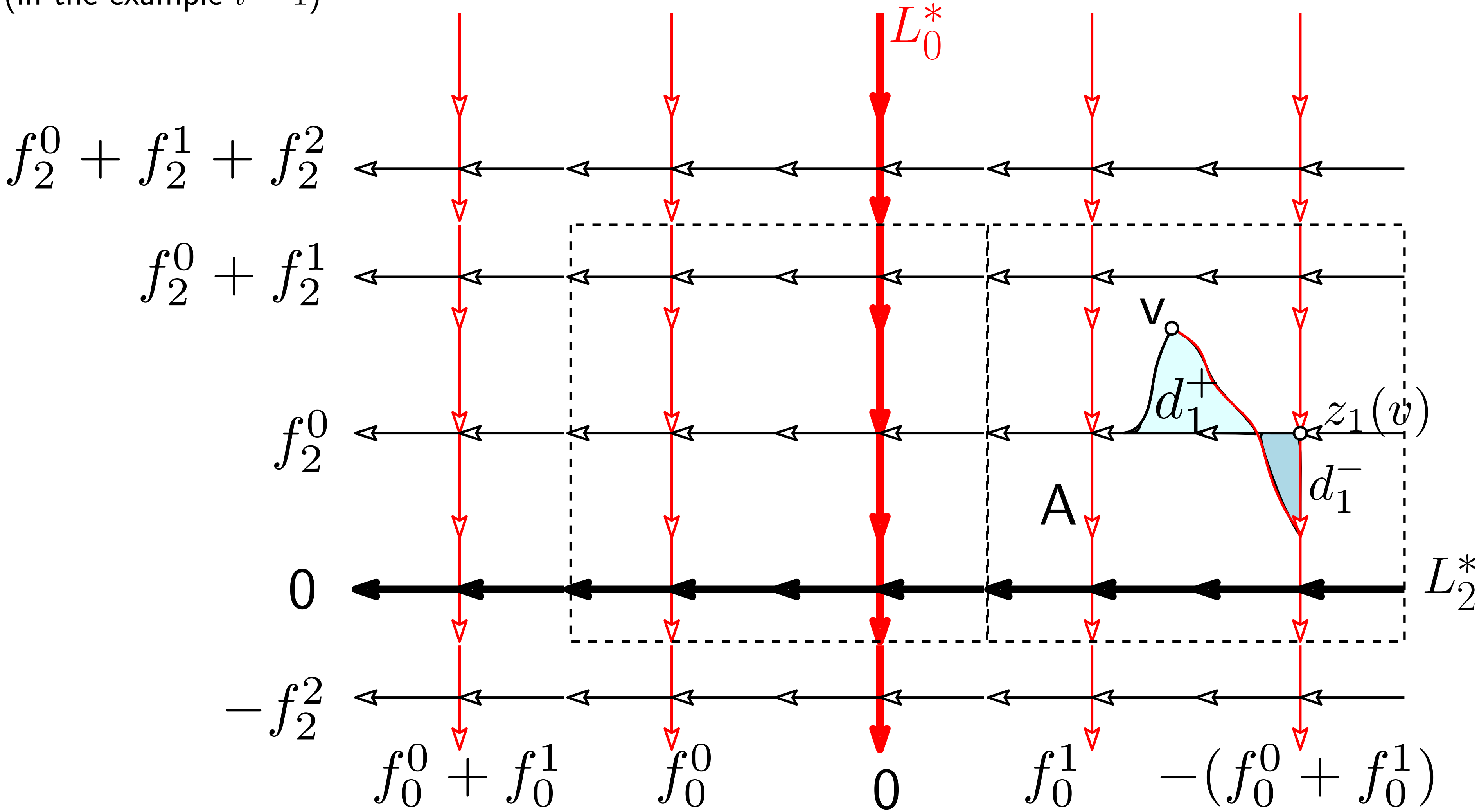
How assign to ccordinates to vertices

We can now define the i coordinate α_i of a vertex v (N constant, appropriately choosen)

$$\alpha_i(v) := d_i(v, z_i(v)) + N \cdot (f_{i+1}(L_{i+1}(v)) - f_{i-1}(L_{i-1}(v)))$$

$$\alpha_1(v) := (d_1^+ - d_1^-) + N \cdot (f_2^0 - (f_0^0 + f_0^1))$$

(in the example $i = 1$)



How assign to ccordinates to vertices

We can now define the i coordinate α_i of a vertex v

$$\alpha_i(v) := d_i(v, z_i(v)) + N \cdot (f_{i+1}(L_{i+1}(v)) - f_{i-1}(L_{i-1}(v)))$$

(set $N = 3$ as constant in this example)

Assign $(0, 0, 0)$ to the origin vertex O

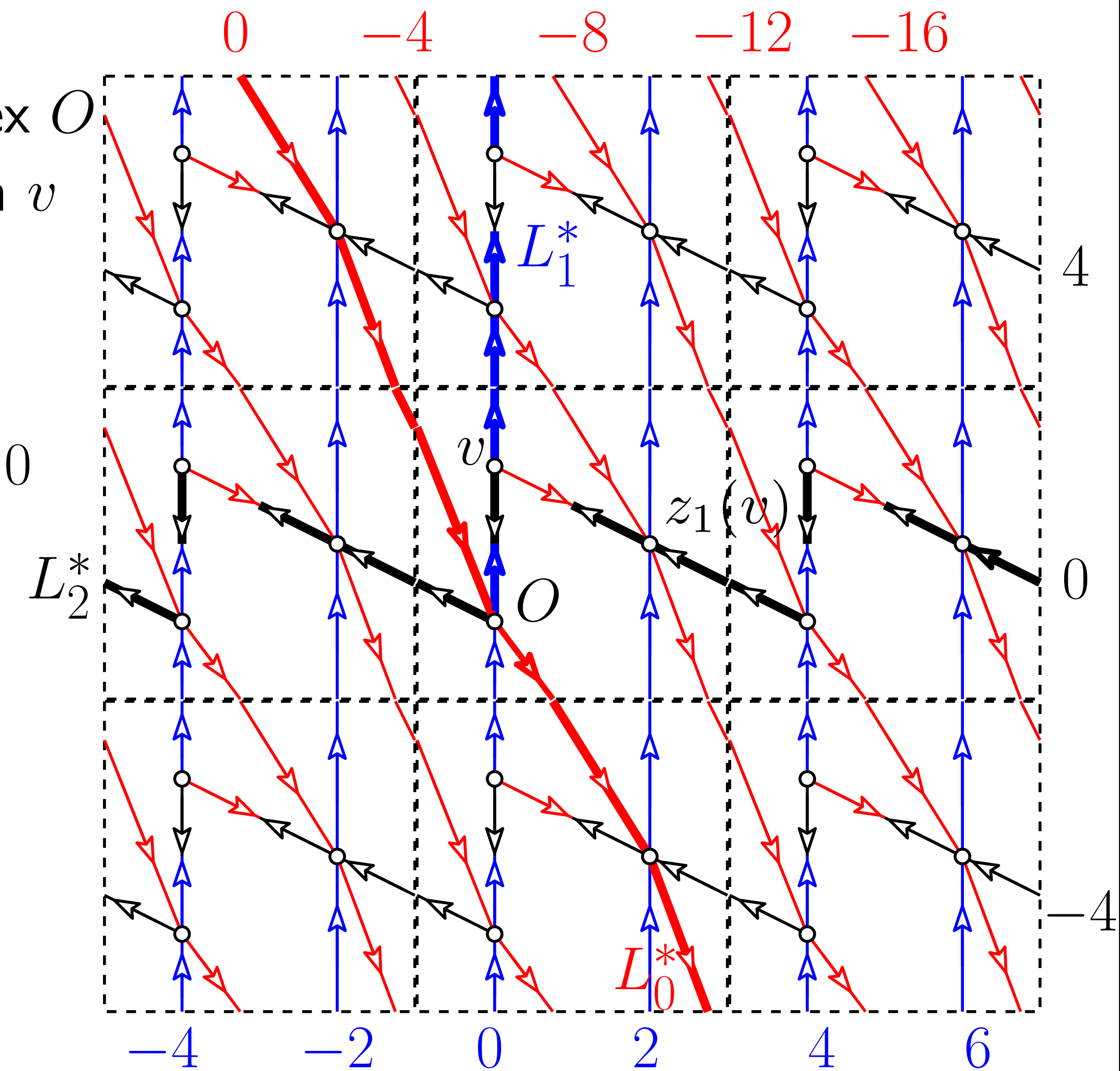
Observe that $z_0(v)$ coincides with v

so: $d_0(v, z_0(v)) = 0$

v lies on L_1^* and L_2^*

so: $f_1(L_1^*(v)) = 0$, $f_2(L_2^*(v)) = 0$

$\alpha_0(v) = 0 + 3 \cdot (0 - 0) = 0$
$\alpha_1(v) = 0 + 3 \cdot (0 - (-4)) = 12$
$\alpha_2(v) = 0 + 3 \cdot (-4 - 0) = -11$



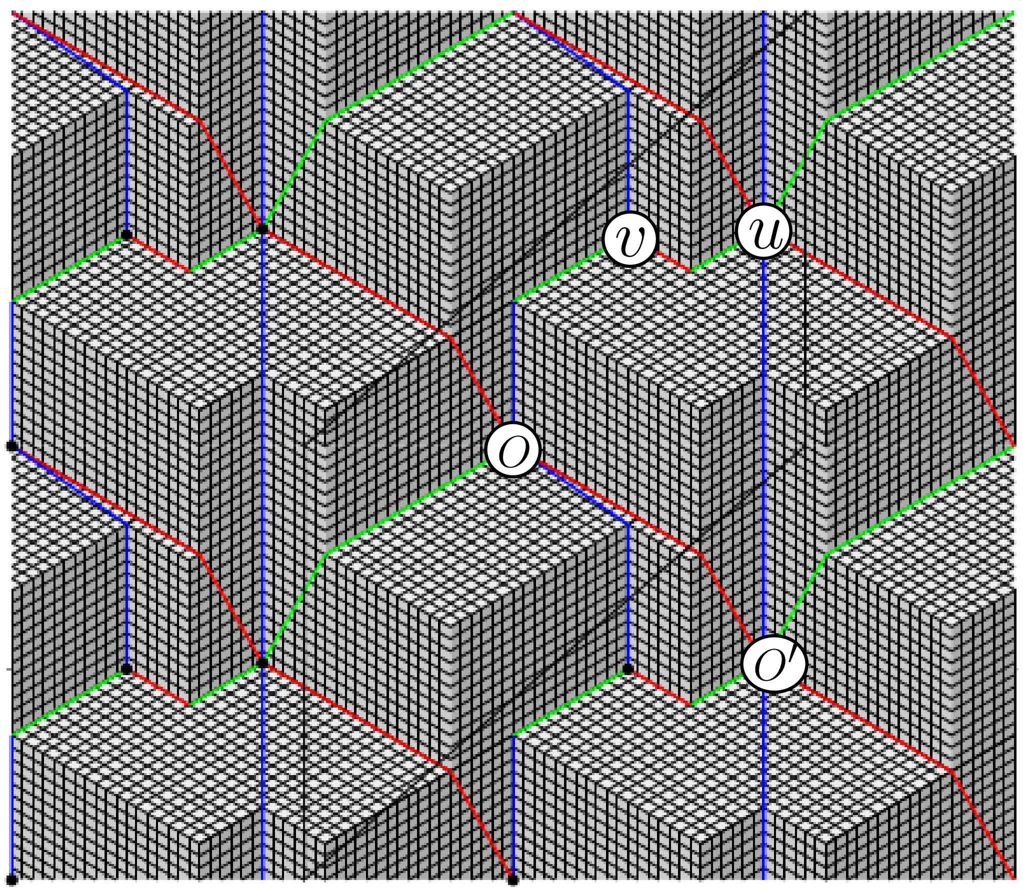
Toroidal Schnyder woods: drawing

Thm[Goncalves Lévêque] (planar simple triangulations)

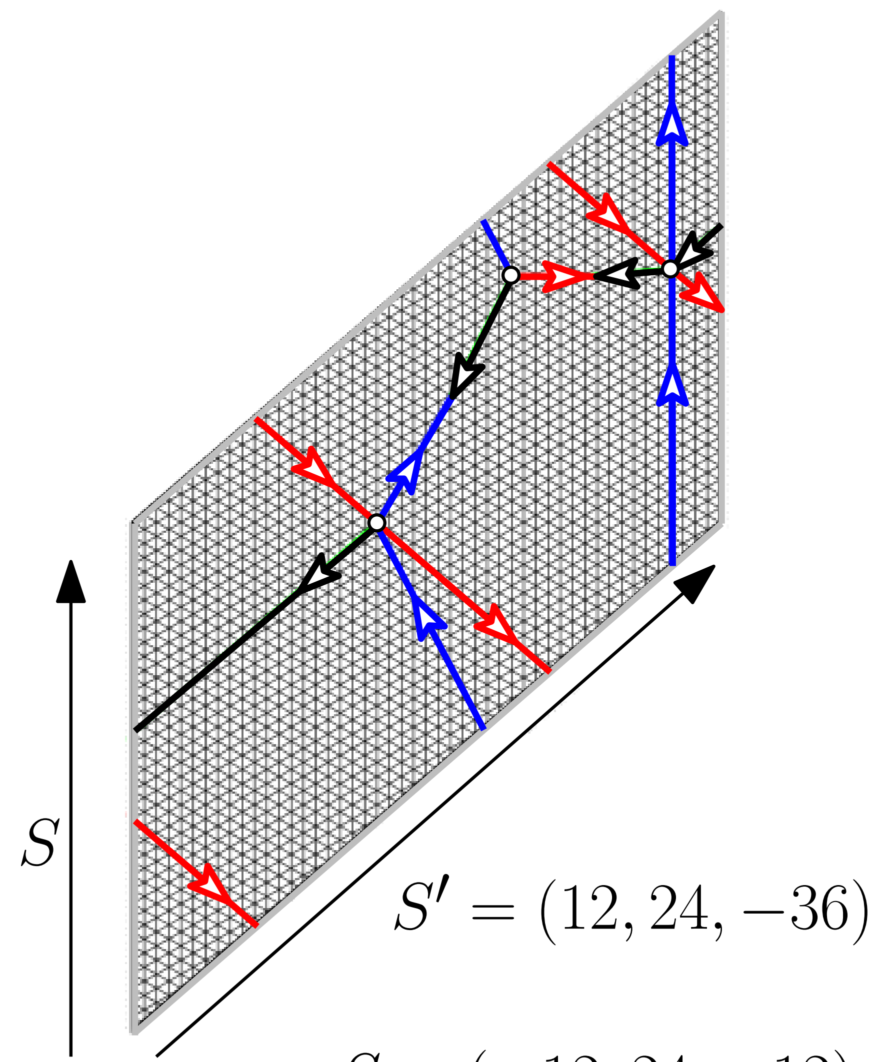
A simple toroidal triangulation admits a straight-line periodic drawing on a grid of size $O(n^2 \times n^2)$

$$O = (0, 0, 0)$$

$$v = (0, 12, -11) \qquad u = (6, 12, -18)$$



$$O' = O + S' = (12, 24, -36)$$



Toroidal Schnyder woods: drawing

Thm[Goncalves Lévêque]

A simple toroidal triangulation admits a straight-line periodic drawing on a grid of size $O(n^2 \times n^2)$

$$O = (0,0,0)$$

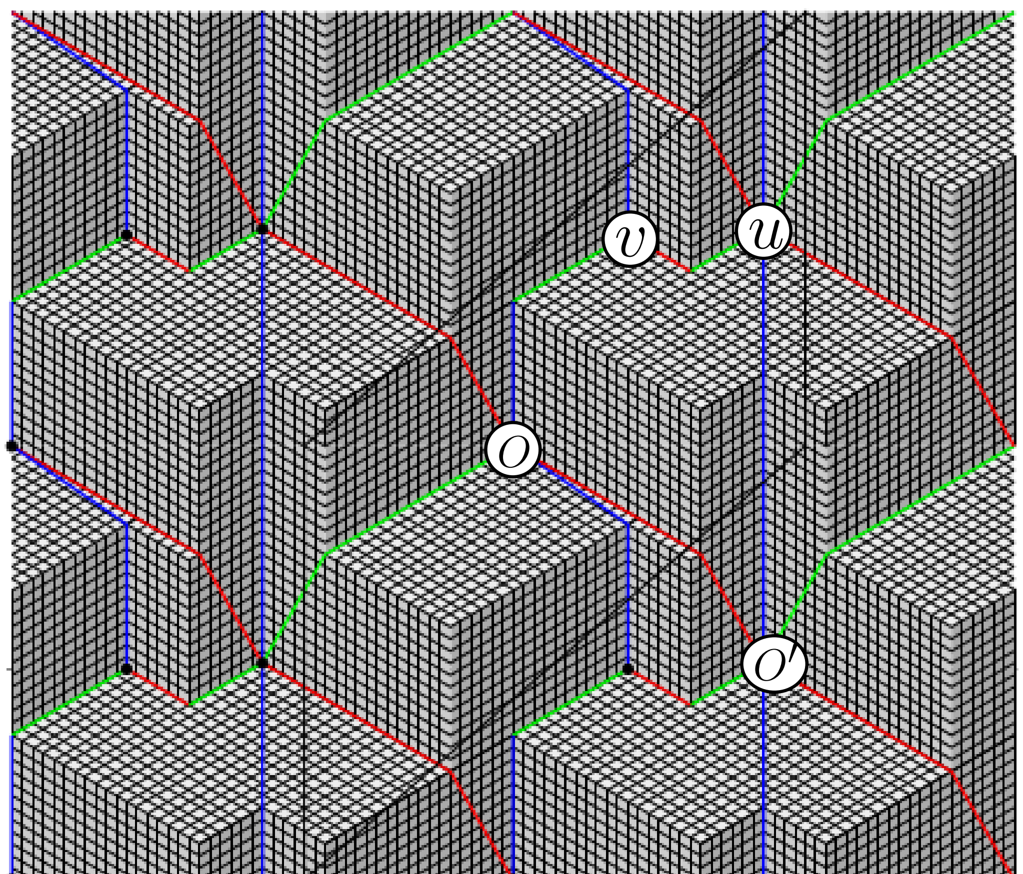
$$v = (0,12,-11) \qquad u = (6,12,-18)$$

Remark:

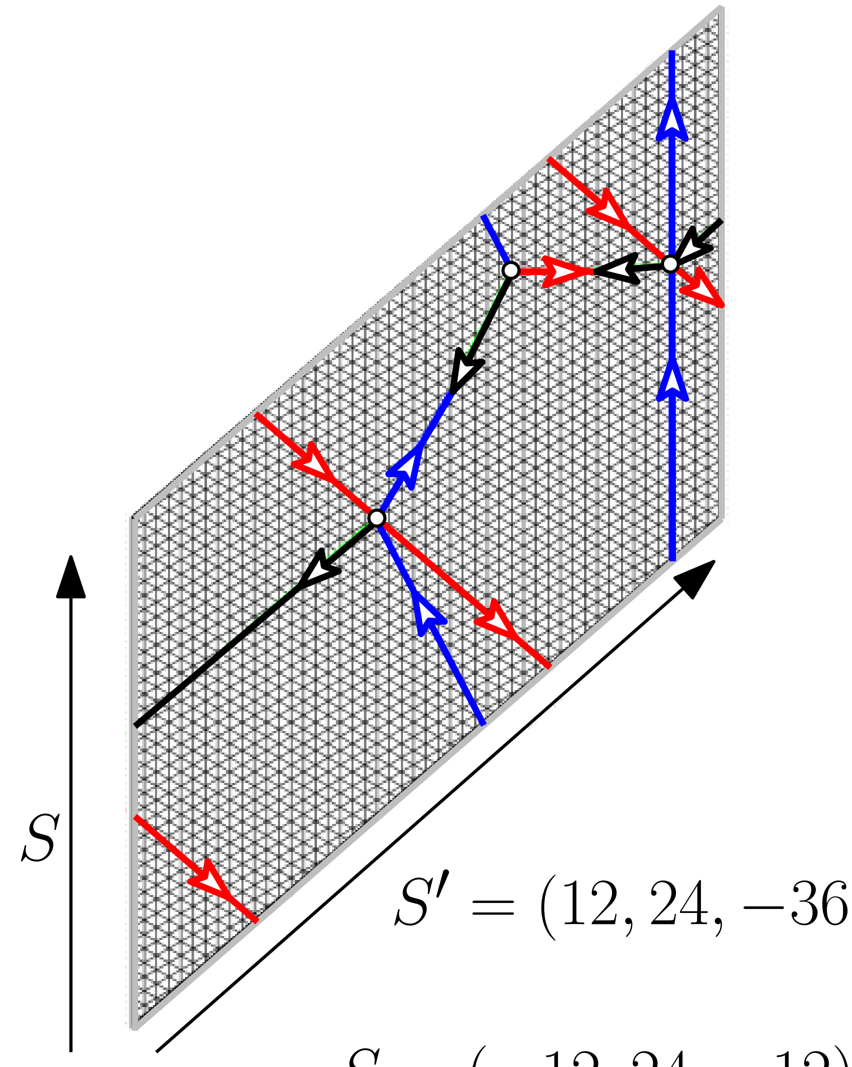
Points are not coplanar

$u \in H_0 : x+y+z=0$

$v \in H_1 : x+y+z=1$



$$O' = O + S' = (12,24,-36)$$



$$S' = (12,24,-36)$$

$$S = (-12,24,-12)$$

Toroidal Schnyder woods: drawing

Thm[Goncalves Lévêque]

A simple toroidal triangulation admits a straight-line periodic drawing on a grid of size $O(n^2 \times n^2)$

Remark:

Points are not coplanar

$u \in H_0 : x + y + z = 0$

$v \in H_1 : x + y + z = 1$

$c_i :=$ number of times the i-cycles cross the boundary of the tile (vertically)

$c'_i :=$ number of times the i-cycles cross the boundary of the tile (horizontally)

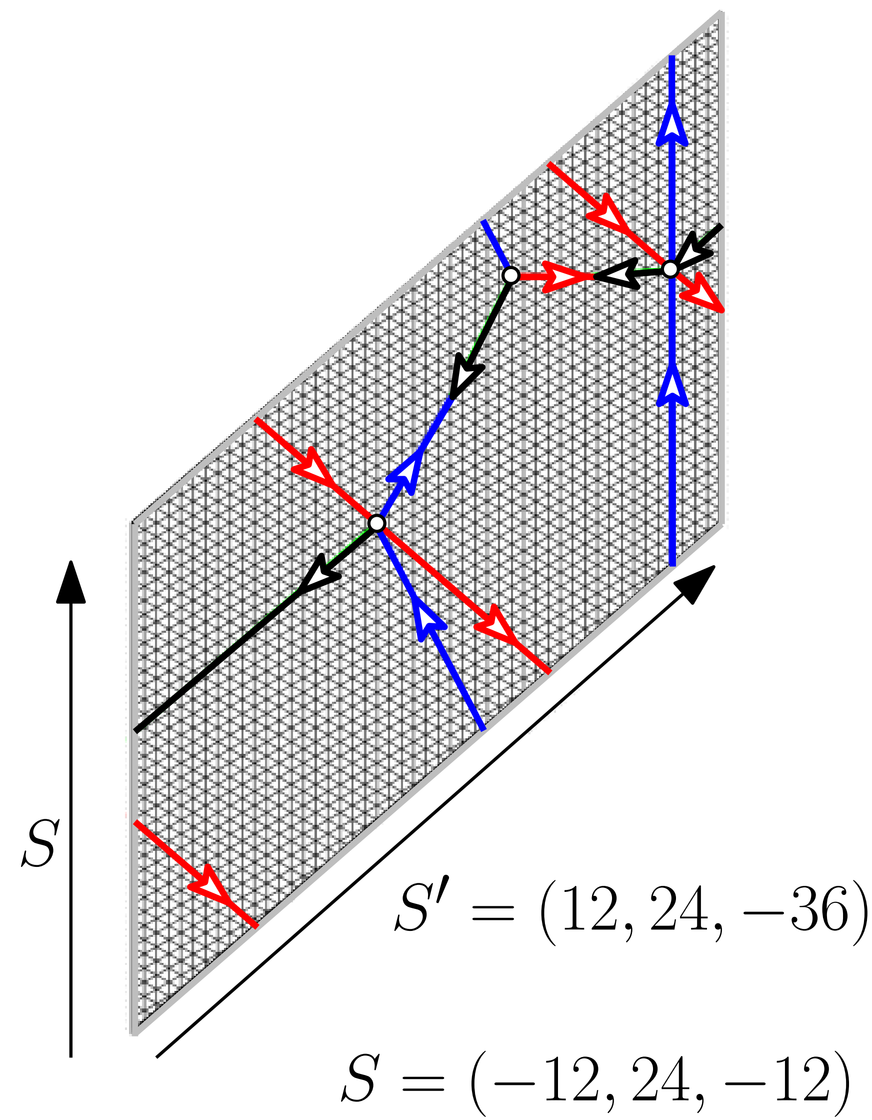
$$S'_i = N \cdot (c_{i+1} - c_{i-1})$$

$$S'_i = N \cdot (c'_{i+1} - c'_{i-1})$$

$c_0 = -1, c'_0 = -2$

$c_1 = -1, c'_1 = 0$

$c_2 = 1, c'_2 = 0$

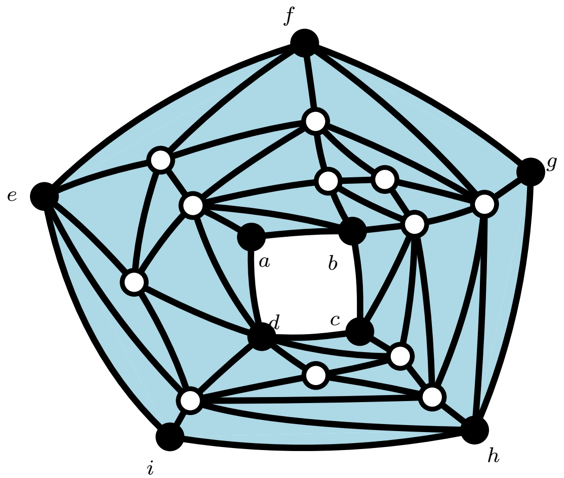


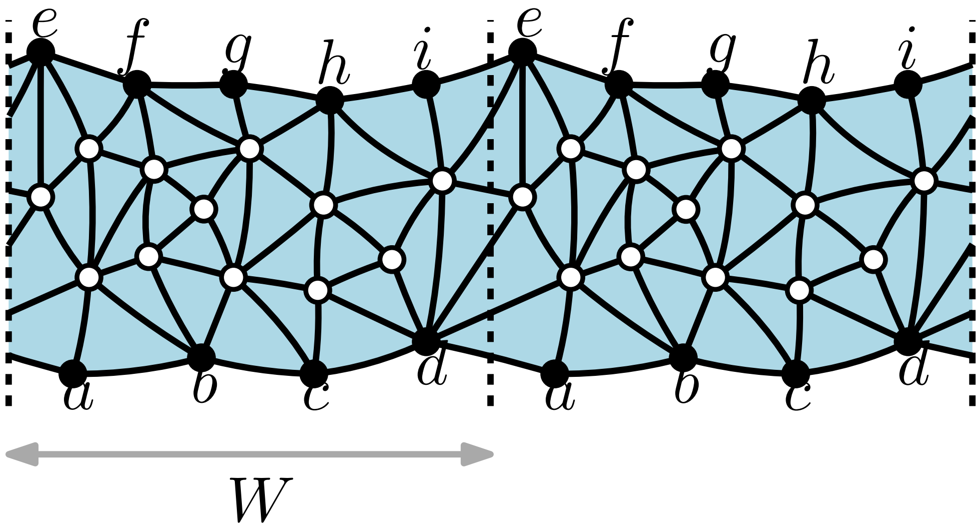
A simple Schyder drawing for cylindic (and toroidal) triangulations

(on a grid of size $O(n) \times O(n)$)

Let us start with cylindric triangulations

Goal: compute a x-periodic grid drawing, with period W



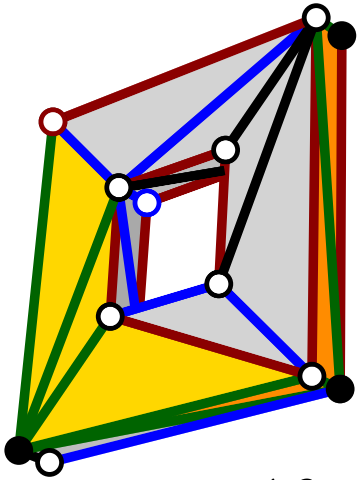


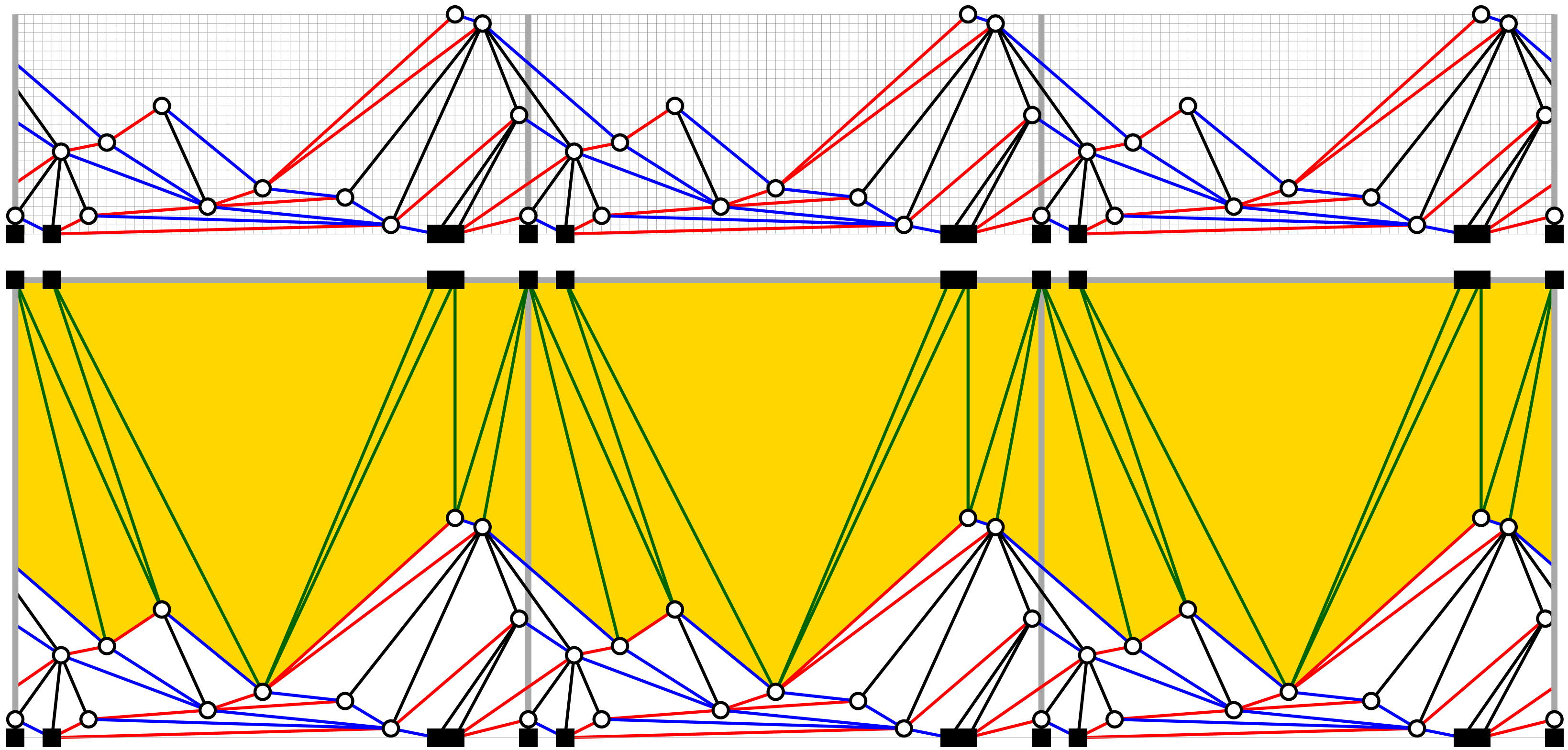
From cylindric to toroidal drawings

$n_w = 1$

height=24

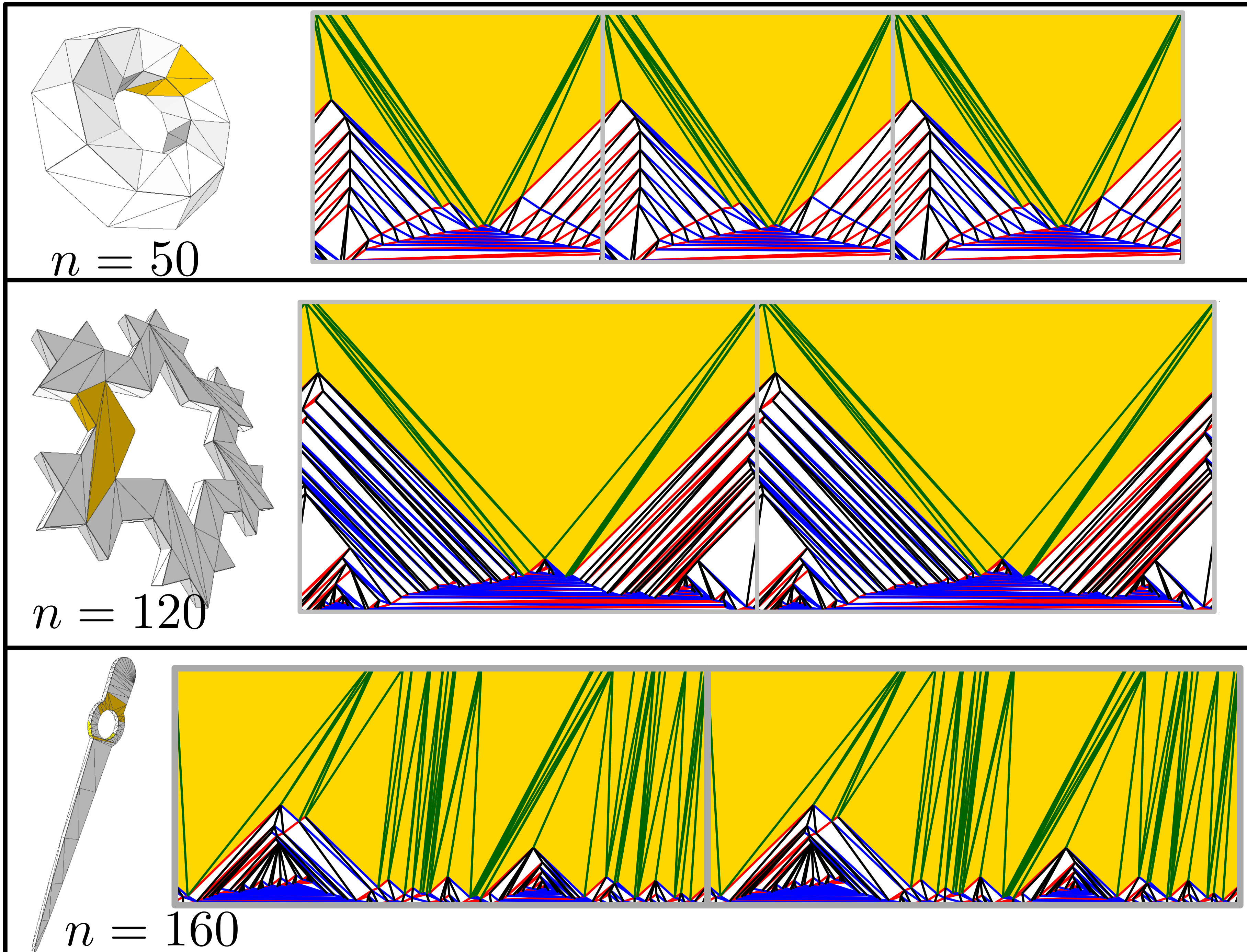
width=56





$n = 16$
$f = 32$

Examples of periodic toroidal drawings

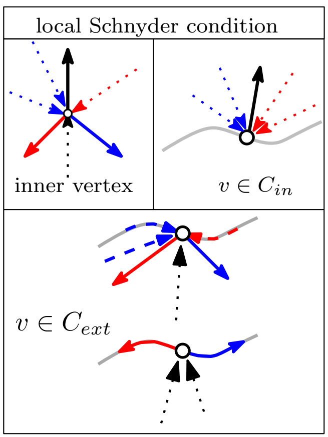


First step: compute a cylindric Schnyder wood

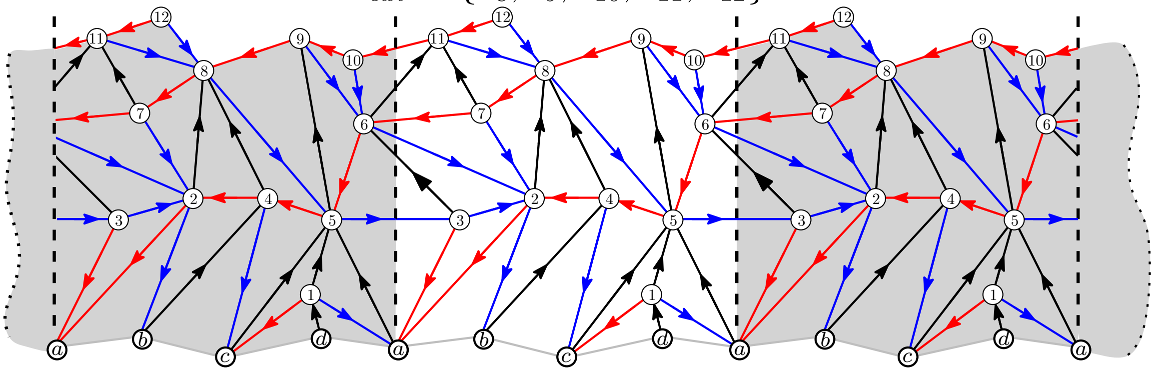
Remark:

The red path $P_0(v)$ and the blue path $P_1(v)$ cross the bottom boundary

The black path $P_2(v)$ crosses the top boundary



$C_{ext} = \{v_8, v_9, v_{10}, v_{11}, v_{12}\}$



$C_{in} = \{a, b, c, d\}$

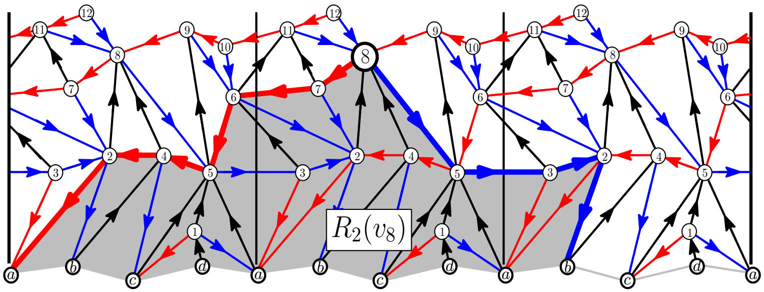
$P_0(v_{12}) = \{v_{12}, v_{11}, v_{10}, v_9, v_8, v_7, v_6, v_5, v_4, v_2, a\}$

$P_1(v_{12}) = \{v_{12}, v_8, v_5, v_3, v_2, b\}$

Compute x-periodic vertex coordinates

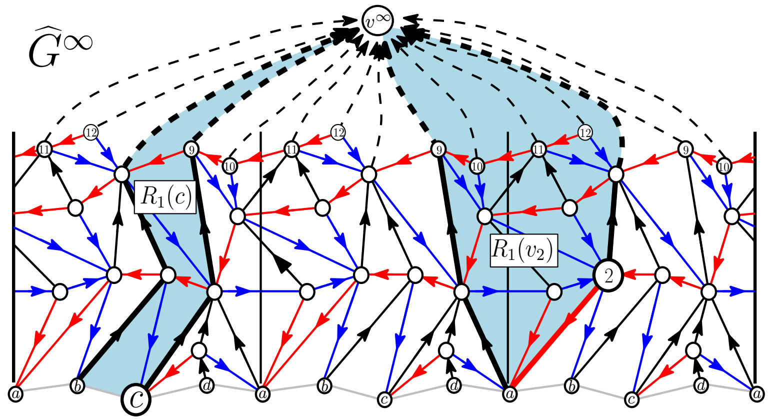
Universal cover (infinite graph)

G^∞



Add a dummy vertex (and dummy edges)

$\widehat{G}^\infty$



Definition of region $R_2(v)$: all faces in the universal cover below the red path $P_0(v)$ and the blue path $P_1(v)$

Remark $R_2(v)$ is a finite set of faces. Each face may appear more then once ($R_2(v)$ has possibly $O(n^2)$ faces)

Definition (vertical coordinates)

$$y(v) := |R_2(v)|$$

Remark vertical coordinates are absolute

Definition of region $R_1(v)$: for a given edge (v, u) of color 0 (red), $R_1(v)$ contains all faces between the black path $R_2(u)$ and the union of $R_2(v)$ and the edge (u, v)

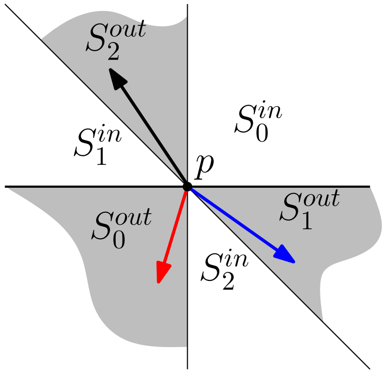
Remark $R_1(v)$ has at most $2n$ faces

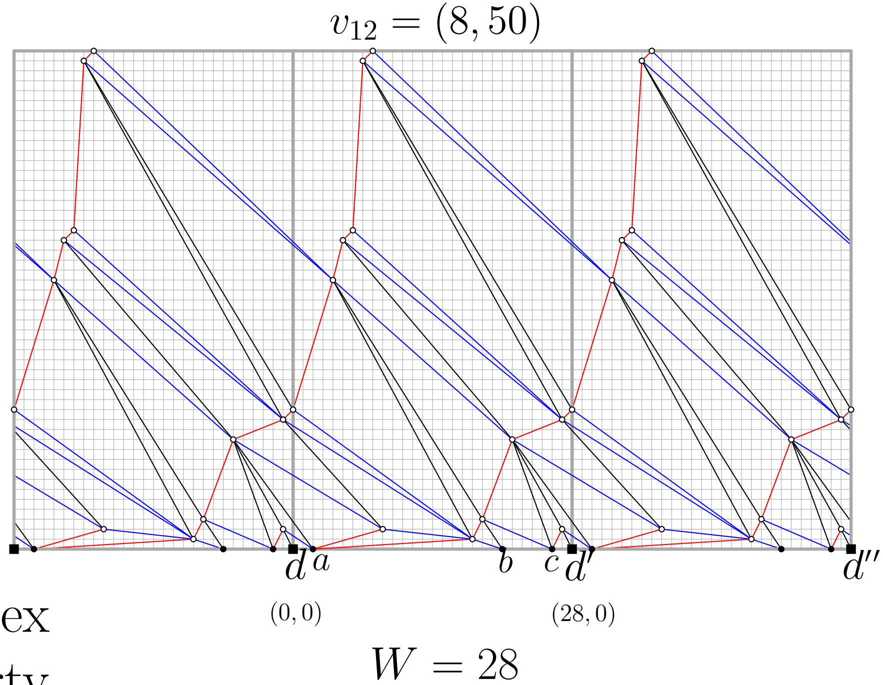
Definition (horizontal coordinates)

$$x(v) = x(u) + |R_1(v)|$$

Remark horizontal coordinates are relative

The drawing is periodic and crossing-free



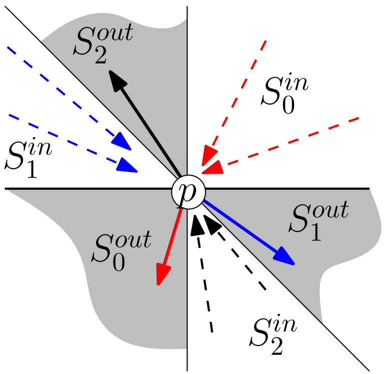


Correctness: each vertex satisfies the sector property (as in the planar case)

The drawing is periodic and crossing-free

Correctness: each vertex satisfies the sector property (as in the planar case)

$p := (x(u), y(u))$



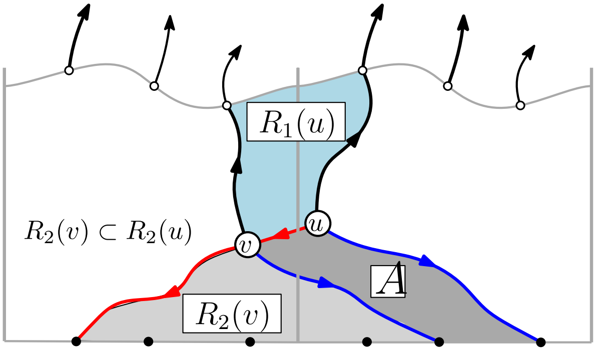
case 1: (u, v) is red

$$x(u) := x(v) + |R_1(u)|$$

$$y(u) = y(v) + |A|$$

$\downarrow$

$$(x(v), y(v)) \in S_0^{out}$$



case 2: (u, v) is black

$$x(u) := x(v) + |B|$$

$$y(v) = y(u) + |B| + |C|$$

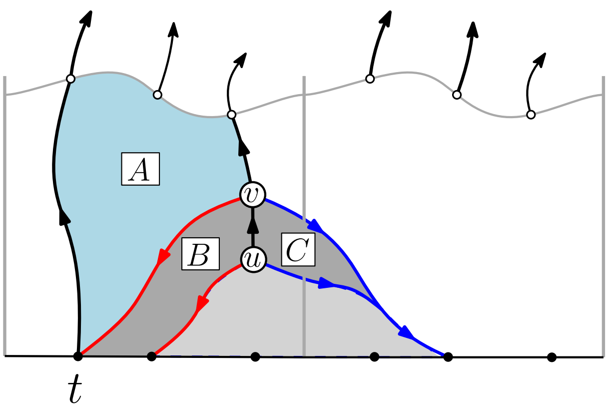
$\rightarrow$

$$x(v) < x(u)$$

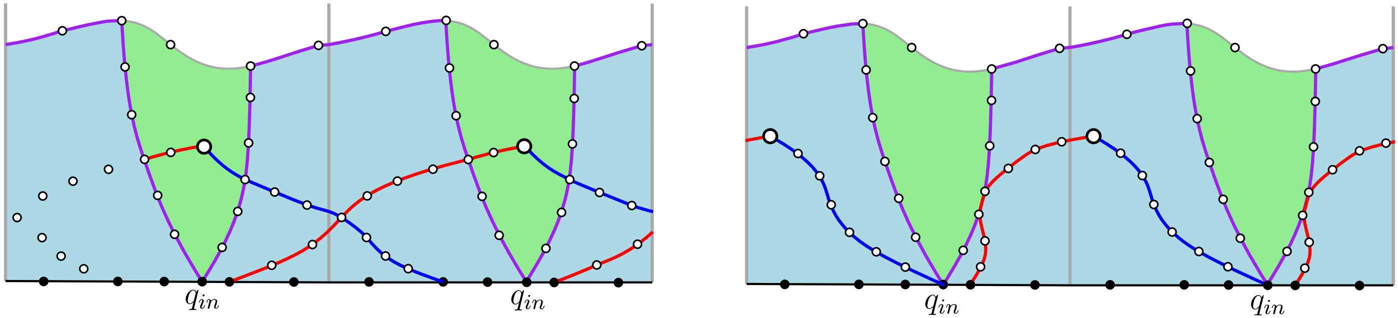
$$\left|\frac{y(v)-y(u)}{x(v)-x(u)}\right| = \frac{|B|+|C|}{|B|} > 1$$

$\downarrow$

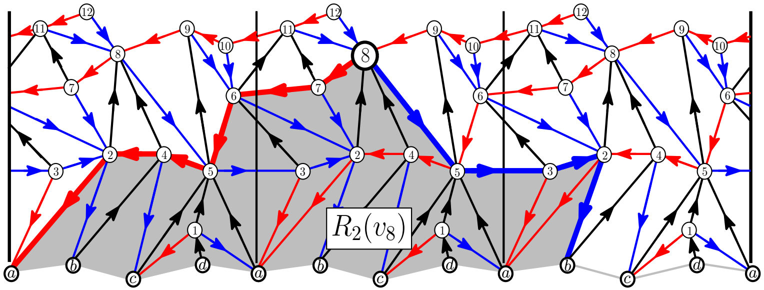
$$(x(v), y(v)) \in S_2^{out}$$



The drawing area can be reduced to $O(n) \times O(n)$



Idea compute a cylindric Schnyder wood such that the red path $P_0(v)$ and the blue path $P_1(v)$ cross at most $O(1)$ times



Remark $R_2(v)$ is large (possibly having $O(n^2)$ faces) only if the red and blue path crosses $\Omega(n)$ times

A modified shelling algorithm on the cylinder

(red and blue paths can cross at most only once)