Lecture 6
Schnyder woods and applications

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M PRI  2-38-1: Algorithms and combinatorics for geometric graphs

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Some facts about planar graphs

(“As I have known them”)
Some facts about planar graphs

**Thm (Schnyder, Trotter, Felsner)**

\( G \) planar if and only if \( \dim(G) \leq 3 \)

**Thm (Koebe-Andreev-Thurston)**

Every planar graph with \( n \) vertices is isomorphic to the intersection graph of \( n \) disks in the plane.

**Thm (Kuratowski, excluded minors)**

\( G \) planar if and only if \( G \) contains neither \( K_5 \) nor \( K_{3,3} \) as minors

**Thm (Tutte)**

\[
E(\rho) := \sum_{(i,j) \in E} |x(v_i) - x(v_j)|^2 = \sum_{(i,j) \in E} (x_i - x_j)^2 + (y_i - y_j)^2
\]

\[
x(v_i) = \sum_{j \in N(i)} \frac{1}{\deg(v_i)} x(v_j)
\]
Straight-line planar drawings of planar graphs

Thm (Schnyder 1990)

face counting via Schnyder woods

Thm (De Fraysseix, Pach, Pollack 1989)

FPP algorithm

shift algorithm via Canonical orderings

linear time algorithms

$O(n) \times O(n)$ grid drawings

not trivial to implement

to be extremely fast: they can process millions of vertices per second

Spring embedder (Eades, 1984)

force-directed paradigm

easy to implement

pretty slow: $O(n^2)$ or $O(n \log n)$ time per iteration

$F_a(v) = c_1 \cdot \sum_{(u,v) \in E} \frac{1}{\text{dist}(u,v)}$ 

$F_r(v) = c_3 \cdot \sum_{u \in V} \frac{1}{\sqrt{\text{dist}(u,v)}}$ 

[Tutte’63] Tutte barycentric embedding

minimize the spring energy

$E(\rho) := \sum_{(i,j) \in E} |x(v_i) - x(v_j)|^2 = \sum_{(i,j) \in E} (x_i - x_j)^2 + (y_i - y_j)^2$

solve large sparse linear systems

$x(v_i) = \sum_{j \in N(i)} \frac{1}{\text{deg}(v_i)} x(v_j)$

easy to implement

not very fast: they can process $\approx 10^4$

vertices per second
## Straight-line planar drawings of planar graphs

**Thm (Schnyder 1990)**

- Face counting via Schnyder woods

**Thm (De Fraysseix, Pack Pollack 1989)**

- FPP algorithm
  - Shift algorithm via Canonical orderings

**[Tutte’63]**

- Tutte barycentric embedding
- Face counting via Schnyder woods
- Shift algorithm via Canonical orderings
- Linear time algorithms

### Timing performances

**Schnyder drawing or FPP algorithm:**

- Less than 1 second (Java, 2.66GHz Intel i7 CPU)

**Chinese dragon (655k vert.)**

- Solve sparse linear systems with the conjugate gradient solver of MTJ (Java) library
  - Numeric precision $10^{-6}$

### Linear time algorithms

- $O(n) \times O(n)$ grid drawings
- Not trivial to implement
- Extremely fast: they can process millions of vertices per second

### Equation

**Tutte barycentric embedding**

$$E(\rho) := \sum_{(i,j) \in E} |x(v_i) - x(v_j)|^2 = \sum_{(i,j) \in E} (x_i - x_j)^2 + (y_i - y_j)^2$$

**Minimize the spring energy**

**Easy to implement**

**Not very fast:** they can process $\approx 10^4$ vertices per second

### Graphs

- ISP layout
- PC layout
- (Numeric precision $10^{-6}$)

### Chinese dragon (655k vert.)
Using circles to measure distances

**Thm (Koebe-Andreev-Thurston)**

Every planar graph with \( n \) vertices is isomorphic to the intersection graph of \( n \) disks in the plane.

\[
\text{Voronoi cell: } C(s_i) = \{ x / d(s_i, x) \leq d(s_j, x) \forall i \neq j \}
\]

**Delaunay Triangulation:**

- \( s_i \) is a neighbour \( s_j \) if \( f \) \( C(s_i) \cap C(s_j) \neq \emptyset \)
- General Position:
  - No 3 points collinear
  - No 4 points co-circular.

Alternative def: There is an edge \( (s_i, s_j) \) if there is an empty circle supporting \( s_i \) and \( s_j \).

\[ \Rightarrow \text{ each face is supported by an empty circle.} \]
Using triangles to measure distances

Thm (de Fraysseix, Ossona de Mendez, Rosenstiehl, ’94)

Every planar triangulation is TD-Delaunay realizable

Chew, ’89

TD-Delaunay: triangular distance Delaunay triangulations

- Distance triangulaire :
  \[ d(u,v) = \text{taille du plus petit triangle équilatérale à base horizontale centré en } u \text{ contenant } v. \]

- Rq : \( d(u,v) \neq d(v,u) \) en général

(images by S. Felsner)

(images by N. Bonichon)
Schnyder woods and canonical orderings: overview of applications

(graph drawing, graph encoding, succinct representations, compact data structures, exhaustive graph enumeration, bijective counting, greedy drawings, spanners, contact representations, planarity testing, untangling of planar graphs, Steinitz representations of polyhedra, . . .)
Some (classical) applications

(Chuang, Garg, He, Kao, Lu, Icalp’98)
(He, Kao, Lu, 1999)

Graph encoding ($4n$ nits)

(Poulalhon-Schaeffer, Icalp 03)

bijective counting, random generation

$C_n = \frac{2(4n+1)!}{(3n+2)!(n+1)!}$

$\Rightarrow$ optimal encoding $\approx 3.24$ bits/vertex

Thm (Schnyder ’90)

Planar straight-line grid drawing (on a $O(n \times n)$ grid)
More ("recent") applications

Schnyder woods, TD-Delaunay graphs, orthogonal surfaces and Half-Θ₆-graphs

[ Bonichon et al., WG'10, Icalp '10, ...]

Every planar triangulation admits a greedy drawing (Dhandapani, Soda08)
(conjectured by Papadimitriou and Ratajczak for 3-connected planar graphs)
Schnyder woods

(definitions)
Schnyder woods (for triangulations): definition

A Schnyder wood of a (rooted) planar triangulation is partition of all inner edges into three sets $T_0$, $T_1$ and $T_2$ such that

i) edge are colored and oriented in such a way that each inner node has exactly one outgoing edge of each color

ii) colors and orientations around each inner node must respect the local Schnyder condition
Schnyder woods: equivalent formulation

[Schnyder labeling]

[3-orientation]
A1) the angles at $a_i$ have labels $i + 1$, $i - 1$

A2) rule for vertices: at each vertex there are non-empty intervals of labels 0, 1 and 2 (listed counter-clockwise)

A3) rule for faces: at each inner faces the angles define three non-empty intervals of labels 0, 1 and 2 in ccw order. For the outer face the angles are listed clockwise.
Schnyder woods (3-connected maps): definition

3-connected graphs [Felsner]

W1) edges have one or two (opposite) orientations. If an edge is bi-oriented than the two direction have distinct colors

W2) the edges at $a_i$ are outgoing of color $i$

W3) **local rule for vertices:** at each vertex there are three outgoing edges (one in each color) satisfying the local Schnyder rule

W4) there is no interior face whose boundary is a directed cycle in one color
Lemma
Given a Schnyder labeling of $M^\sigma$, the angles of each edge have colors 0, 1, 2 and are of the following 2 types:

\begin{align*}
\begin{array}{c}
\text{proof:}
\end{array}
\end{align*}
Lemma
Given a Schnyder labeling of $M^\sigma$, the angles of each edges have colors 0, 1, 2 and are of the following 2 types:

\[ i \quad i + 1 \quad i - 1 \quad i \]

proof:
possibly valid configurations
\[ i \quad i + 1 \quad i - 1 \quad i \]
\[ i \quad i \]

forbidden configurations
\[ 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \]
\[ 0 \quad 2 \quad 0 \quad 0 \quad 2 \quad 0 \quad 2 \quad 2 \]

use a counting argument (double counts the angles)
\[
\sum_v d(v) + \sum_f d(f) = 3n + 3|f| = 3|E| + 6
\]

\[
\epsilon(e) = \begin{cases} 
0, & \text{at vertex } a_i \text{ there are two label changes} \\
3, & \text{for all (normal) edges}
\end{cases}
\]

$\alpha_1 \quad e \quad \alpha_2$
$\alpha_3 \quad \alpha_4$

\[
d(v) := \text{number of label changes for the angles around } v \\
d(f) := \text{number of label changes for the angles in face } f
\]
Schnyder labelings: angles at exterior vertices

Corollary

Given a Schnyder labeling of $M^{\sigma}$, all interior angles at a vertex $a_i$ have label $i$

$$
\begin{align*}
\text{angle at vertex } a_i &\rightarrow i \\
\text{angle at vertex } a_{i+1} &\rightarrow i + 1 \\
\text{angle at vertex } a_{i-1} &\rightarrow i - 1
\end{align*}
$$
Correspondence between Schnyder labelings and Schnyder woods

Theorem
There is a correspondence between the Schnyder labelings of $M^\sigma$ and the Schnyder woods of $M^\sigma$. 

Schnyder wood + Schnyder labeling of $M^\sigma$
Theorem
There is a correspondence between the Schnyder labelings of $M^\sigma$ and the Schnyder woods of $M^\sigma$.

**proof:** Assume $M^\sigma$ is endowed with a Schnyder labeling.

Assume (W4) is violated: there is a cycle in one color

Then the coloring rule of bi-oriented edges implies that all angles have the same color.

Rule of vertices (A2)  
Rule of faces (A3)  
local Schnyder rule (W3)  
no directed cycles in one color (W4)
Correspondence between Schnyder labelings and Schnyder woods

**Theorem**
There is a correspondence between the Schnyder labelings of $M^\sigma$ and the Schnyder woods of $M^\sigma$

**proof:** Assume $M^\sigma$ is endowed with a Schnyder wood

use a counting argument (double counts the angles around vertices/faces/edges)

$$d(v) = 3 \quad d(e) = \begin{cases} 3 & \text{for all (normal) edges} \\ 2 & \text{for the three half-edges} \end{cases}$$

**Remark:**
Turning around a face in ccw direction the angle will be $i$ or $i + 1$ (otherwise there is a directed cycle of edges in one color)

$$\sum_v d(v) + \sum_f d(f) = \sum_e d(e) = 3n + \sum_f d(f) = 3|E| + 6$$

Euler formula implies $\sum_f d(f) = 3|F|$ $d(f) = 3$ for all faces
Correspondence between Schnyder labelings and Schnyder woods

Remark:
The condition (W4) of Schnyder woods is important

valid Schnyder labeling
conditions (W1)-(W4) of Schnyder woods are satisfied

not valid Schnyder labeling
condition (W4) of Schnyder woods is not satisfied
Theorem [Schnyder '90] \( T_i := \) digraph defined by directed edges of color \( i \)

The three sets \( T_0, T_1, T_2 \) are spanning trees of the inner vertices of \( T \) (each rooted at vertex \( v_i \))
Spanning property for triangulations

**Theorem** [Schnyder ‘90]
The three sets $T_0$, $T_1$, $T_2$ are spanning trees of the inner vertices of $\mathcal{T}$ (each rooted at vertex $v_i$).

**proof** (use a counting argument)

**Claim 1:** $T_i$ does not contain cycles
(assume there are monochromatic cycles, by contradiction)

There is a vertex $u$ violating Schnyder rule.

**Case 1:** $C$: non oriented monochromatic cycle of size $k$

Schnyder local rule implies:
(count edges in the triangulation bounded by the cycle)

$e_i := 3n_i + k$

1 outgoing edges for boundary vertices
3 outgoing edges for inner vertices

**Case 2:**
$C$: monochromatic cycle of size $k$ (cw or ccw) oriented

$C := \text{monochromatic cycle of size } k$ (cw or ccw) oriented

$C := \text{non oriented monochromatic cycle of size } k$

$T_0, T_1, T_2$ are spanning trees of the inner vertices of $\mathcal{T}$ (each rooted at vertex $v_i$)

Local Schnyder rule

$e_i = 3n_i + (k - 3)$

$n_i := \# \text{inner vertices}$

$k := \# \text{boundary edges} = \# \text{boundary vertices}$

Proof (use a counting argument)

Three sets $T_0, T_1, T_2$ are spanning trees of the inner vertices of $\mathcal{T}$ (each rooted at vertex $v_i$).

Local Schnyder rule implies:
(count edges in the triangulation bounded by the cycle)

$e_i = 3n_i + k$

1 outgoing edges for boundary vertices
3 outgoing edges for inner vertices

There is a vertex $u$ violating Schnyder rule.

**Case 1:**
$C$: non oriented monochromatic cycle of size $k$

$C := \text{non oriented monochromatic cycle of size } k$

$C := \text{monochromatic cycle of size } k$ (cw or ccw) oriented

$C := \text{non oriented monochromatic cycle of size } k$

Three sets $T_0, T_1, T_2$ are spanning trees of the inner vertices of $\mathcal{T}$ (each rooted at vertex $v_i$).

Local Schnyder rule implies:
(count edges in the triangulation bounded by the cycle)

$e_i = 3n_i + k$

1 outgoing edges for boundary vertices
3 outgoing edges for inner vertices

There is a vertex $u$ violating Schnyder rule.
Spanning property for triangulations

**Theorem** [Schnyder '90]

The three sets $T_0$, $T_1$, $T_2$ are spanning trees of the inner vertices of $\mathcal{T}$ (each rooted at vertex $v_i$)

**proof** (use a counting argument)

Claim 2: $T_i$ is connected
(by contradiction, assume there are several disjoint components)

Let $G$ be a connected component not containing $v_i$

- $G$ is connected and without cycles
- all vertices of $G$ are inner vertices (distinct from $v_0$, $v_1$ and $v_2$)

there is a vertex $u \in G$ violating Schnyder rule: no outgoing edge of color $i$
Spanning property for 3-connected maps

Theorem Let \((T_0, T_1, T_2)\) a Schnyder wood of \(M\). Then each digraph \(D_i := T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}\) is acyclic.

proof: Let \(Z\) a directed cycle enclosing a region \(F\) of minimal size.

Claim 1: \(F\) is a single face

- **Case a:** \(x \in F\)
  - \(F'\) is a smaller directed cycle

- **Case b:** \(F\) is empty of vertices
  - there is an edge inside \(F\)

Claim 2: there is no face \(F\) whose boundary is a directed cycle

Visit \(F\) in ccw order starting from \(v\) and propagate colors (first color is \(i\)): there is no angle with label \(i - 1\).

The coloring rule for faces is violated.

Corollary: Each sets \(T_i\) is spanning tree \(M\) (rooted at vertex \(a_i\))
Non crossing paths

Corollary:
Each sets $T_i$ is spanning tree $\mathcal{M}$ (rooted at vertex $a_i$)

Corollary:
For each inner vertex $v$ the three monochromatic paths $P_0$, $P_1$, $P_2$ directed from $v$ toward each vertex $a_i$ are vertex disjoint (except at $v$) and partition the inner faces into three sets $R_0(v)$, $R_1(v)$, $R_2(v)$

**proof**: the existence of two paths $P_i(v)$ and $P_{i+1}(v)$ which are crossing would contradicts previous theorem
Consequences

Efficient graph data structure for planar graphs
There exist a (simple) data structure of size $O(n \log n)$ bits supporting constant time adjacency test between vertices

Adjacency matrix

$A_G[i, j] = \begin{cases} 1 & \text{if } v_i \text{ adjacent } v_j \\ 0 & \text{otherwise} \end{cases}$

space: $O(n^2)$ bits
adjacency: $O(1)$ time

Menger theorem for planar triangulations
Schnyder woods allows us to compute in linear time, for any pair of vertices $(u, v)$, 3 vertex disjoint paths between $u$ and $v$

Adjacency lists

space: $O(n \log n)$ bits
adjacency: $O(n)$ time

Thm (Menger)
If $G$ is $k$-connected, then for each pair $u, v$ there exist $k$ disjoint path from $u$ to $v$
Consequences

Efficient graph data structure for planar graphs

There exist a (simple) data structure of size $O(n \log n)$ bits supporting constant time adjacency test between vertices.

Truncated adjacency lists: store only 3 successors

$\begin{bmatrix}
v_0 & 0 & 6 & 8 \\
v_7 & 0 & 6 & 10 \\
v_8 & 6 & 8 & 10 \\
\vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}$

Menger theorem for planar triangulations

Schnyder woods allows us to compute in linear time, for any pair of vertices $(u, v)$, 3 vertex disjoint paths between $u$ and $v$.
Consequences

Efficient graph data structure for planar graphs
There exist a (simple) data structure of size $O(n \log n)$ bits supporting constant time adjacency test between vertices

Truncated adjacency lists: store only 3 successors

---

Menger theorem for planar triangulations
Schnyder woods allows us to compute in linear time, for any pair of vertices $(u, v)$, 3 vertex disjoint paths between $u$ and $v$

Case 1:

Case 2:
Consequences

Efficient graph data structure for planar graphs
There exist a (simple) data structure of size $O(n \log n)$ bits supporting constant time adjacency test between vertices

Truncated adjacency lists: store only 3 successors

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Menger theorem for planar triangulations
Schnyder woods allows us to compute in linear time, for any pair of vertices $(u, v)$, 3 vertex disjoint paths between $u$ and $v$

Case 1:

Case 2:
Number and structure of Schnyder woods

**Counting Schnyder woods:** (there are graphs admitting an exponential number)

[Bonichon '05]

# Schnyder woods of triangulations of size $n$: $\approx 16^n$
(all Schnyder woods over all distinct triangulations of size $n$)

[Felsner Zickfeld '08]

(count of Schnyder woods of a fixed triangulation)

$$2.37^n \leq \max_{T \in \mathcal{T}_n} |SW(T)| \leq 3.56^n$$

$\mathcal{T}_n :=$ class of planar triangulations of size $n$

$SW(T) :=$ set of all Schnyder woods of the triangulation $T$

---

reversal of oriented triangles
**Structure of Schnyder woods: distributive lattice**

**Thm:** [Ossona de Mendez’94], [Felsner’03]
The set $S(T)$ of all distinct Schnyder woods of a given triangulation $T$ is a partial order set with respect to the flip operation (a lattice): for every pair of Schnyder woods of $T$ there is a unique supremum and infimum.

The min is the unique $S \in S(T)$ with **no clockwise circuit**

**Flip:**

- minimal Schnyder wood
- maximal Schnyder wood

- reversal of directed cycles (cycles could bound several faces)
Schnyder woods: existence (algorithm I)

Via **Canonical orderings** (see Lecture 2)

The traversal starts from the root face

**Theorem**

Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

[incremental vertex shelling, Brehm’s thesis]
Theorem

Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

The traversal starts from the root face $v_0$ and moves to $v_1$. Via canonical orderings (see Lecture 2), perform a vertex conquest at each step:

From $G_k$ to $G_{k-1}$.
Theorem

Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

Via Canonical orderings (see Lecture 2)

The traversal starts from the root face
The traversal starts from the root face.

**Theorem**

Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

perform a vertex conquest at each step

\[ G_k \quad \Downarrow \quad G_{k-1} \]
The traversal starts from the root face

**Theorem**

Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

perform a vertex conquest at each step
The traversal starts from the root face

**Theorem**
Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

**Invariant:**
The traversal starts from the root face $v_0$.

**Theorem**
Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

**Invariant:**
The traversal starts from the root face

Theorem
Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

Invariant:
Schnyder woods: existence (algorithm I)

[incremental vertex shelling, Brehm’s thesis]

**Theorem**

Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

The traversal starts from the root face.

**Invariant:**
Planar straight-line drawings
(of planar graphs)
Planar straight-line drawings

[Wagner’36]
[Fary’48]
Planar straight-line drawings

existence of straight-line drawing ⇒

Classical algorithms:

[Tutte’63] spring-embedding

[De Fraysseix, Pach, Pollack 89] incremental (Shift-algorithm)

[Schnyder’90] face-counting principle

[Wagner’36]  
[Fary’48]  
[Stein’51]
Face counting algorithm
(Schnyder algorithm, 1990)
Face counting algorithm

Geometric interpretation

\[ v = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 \]
where \( \alpha_i \) is the normalized area.

Theorem

For a 3-connected planar map \( M \) having \( f \) vertices, there is drawing on a grid of size \( (f - 1) \times (f - 1) \).

Theorem (Schnyder, Soda ’90)

For a triangulation \( T \) having \( n \) vertices, we can draw it on a grid of size \( (2n - 5) \times (2n - 5) \), by setting \( x_0 = (2n - 5, 0) \), \( x_1 = (0, 0) \) and \( x_2 = (0, 2n - 5) \).

\[ v \rightarrow (5, 6, 2) := (v_0, v_1, v_2) \]
\[ u \rightarrow (7, 3, 3) := (u_0, u_1, u_2) \]
Face counting algorithm: example

Input: $\mathcal{T}$

$\mathcal{T}$ endowed with a Schnyder wood

Face counting algorithm: example

$$
\begin{align*}
    a & \rightarrow (0, 0) & b & \rightarrow (0, 1) & i & \rightarrow (1, 0) \\
    c & \rightarrow \left(\frac{9}{13}, \frac{1}{13}\right) & d & \rightarrow \left(\frac{5}{13}, \frac{6}{13}\right) \\
    e & \rightarrow \left(\frac{7}{13}, \frac{4}{13}\right) & f & \rightarrow \left(\frac{3}{13}, \frac{3}{13}\right) \\
    g & \rightarrow \left(\frac{4}{13}, \frac{8}{13}\right) & h & \rightarrow \left(\frac{1}{13}, \frac{4}{13}\right)
\end{align*}
$$
Face counting algorithm: example

Input: $\mathcal{T}$

$\mathcal{T}$ endowed with a Schnyder wood

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>(13, 0, 0)</td>
</tr>
<tr>
<td>b</td>
<td>(0, 13, 0)</td>
</tr>
<tr>
<td>c</td>
<td>(9, 3, 1)</td>
</tr>
<tr>
<td>d</td>
<td>(5, 6, 2)</td>
</tr>
<tr>
<td>e</td>
<td>(2, 7, 4)</td>
</tr>
<tr>
<td>f</td>
<td>(7, 3, 3)</td>
</tr>
<tr>
<td>g</td>
<td>(1, 4, 8)</td>
</tr>
<tr>
<td>h</td>
<td>(8, 1, 4)</td>
</tr>
<tr>
<td>i</td>
<td>(0, 0, 13)</td>
</tr>
</tbody>
</table>
Face counting algorithm: example

Input: $\mathcal{T}$

$\mathcal{T}$ endowed with a Schnyder wood

\[ x + y + z = 2n - 5 \]

- $a \rightarrow (13, 0, 0)$
- $b \rightarrow (0, 13, 0)$
- $c \rightarrow (9, 3, 1)$
- $d \rightarrow (5, 6, 2)$
- $e \rightarrow (2, 7, 4)$
- $f \rightarrow (7, 3, 3)$
- $g \rightarrow (1, 4, 8)$
- $h \rightarrow (8, 1, 4)$
- $i \rightarrow (0, 0, 13)$
Lemma Let \((T_0, T_1, T_2)\) a Schnyder wood of \(\mathcal{M}\).
If \(u \in R_i(v)\) then \(R_i(u) \subseteq R_i(v)\)
If \(u \in R_i^{int}(v)\) then \(R_i(u) \subset R_i(v)\)

**proof:**

**Case 1:** \(u \in R_i^{int}(v)\)

**first step:** compute the paths \(P_{i+1}(u)\) and \(P_{i-1}(u)\)
They must intersect the boundary of \(R_i(v)\) at \(x\) and \(y\)
Remark: \(x\) and \(y\) are different from \(v\)
and we have \(y \in P_{i+1}(u)\) and \(x \in P_{i-1}(u)\)
(because of Schnyder rule)
so we have: \(R_i(u) \subset R_i(v)\)

**Case 2a:** \(u \in P_{i-1}(v)\)

**Case 2b:** \(u \in P_{i-1}(v)\)
\((u, u')\) is bi-oriented
Proceed by induction on the path \(P_{i-1}(v)\)
\(R_i(u) \subseteq R_i(v)\)
Remarks: Let \((u, v)\) of color \(i\) oriented from \(u\) to \(v\)

\[
v \in P_i(u) \quad \begin{cases} 
  v \in R_{i+1}(u) \\
  v \in R_{i-1}(u) \\
  u \in R_i(v)
\end{cases}
\]

Case 1: \((u, v)\) is unidirectional

\[
R_i(u) \subset R_i(v) \quad R_{i+1}(v) \subset R_{i+1}(u) \quad R_{i-1}(v) \subset R_{i-1}(u)
\]

Case 2: \((u, v)\) is bidirectional

\[
R_i(u) \subset R_i(v) \quad R_{i-1}(v) \subseteq R_{i-1}(u) \quad R_{i+1}(v) \subseteq R_{i+1}(u)
\]
**Regions and coordinates**

**Remarks:** Let $(u, v)$ of color $i$ oriented from $u$ to $v$

\[ v =: \frac{|R_0(v)|}{|F|-1} x_0 + \frac{|R_1(v)|}{|F|-1} x_1 + \frac{|R_2(v)|}{|F|-1} x_2 = \]
\[ = \frac{v_0}{|F|-1} x_0 + \frac{v_1}{|F|-1} x_1 + \frac{v_2}{|F|-1} x_2 \]

- $R_i(u) \subseteq R_i(v)$ \[ \implies |R_i(u)| \leq |R_i(v)| \implies u_i \leq v_i \]

- $v_0 + v_1 + v_2 = f - 1$

- $R_i(u) \subset R_i(v)$
- $R_{i+1}(v) \subset R_{i+1}(u)$
- $R_{i-1}(v) \subset R_{i-1}(u)$

- For every edge $(u, v)$ there are some indices $i, j \in \{0, 1, 2\}$ s.t.

\[
\begin{align*}
  u_i &< v_i \\
  u_j &> v_j
\end{align*}
\]
Regions and coordinates

**Remarks:** Let \( (u, v) \) of color \( i \) oriented from \( u \) to \( v \)

\[
v =: \frac{|R_0(v)|}{|F|-1} x_0 + \frac{|R_1(v)|}{|F|-1} x_1 + \frac{|R_2(v)|}{|F|-1} x_2 = \frac{v_0}{|F|-1} x_0 + \frac{v_1}{|F|-1} x_1 + \frac{v_2}{|F|-1} x_2
\]

- \( R_i(u) \subseteq R_i(v) \rightarrow |R_i(u)| \leq |R_i(v)| \rightarrow u_i \leq v_i \)

- \( v_0 + v_1 + v_2 = f - 1 \)

- \( R_i(u) \subset R_i(v) \)
- \( R_{i+1}(v) \subset R_{i+1}(u) \)
- \( R_{i-1}(v) \subset R_{i-1}(u) \)

\[
\begin{align*}
\{ & u_i < v_i \\
& u_{i+1} > v_{i+1} \\
& u_{i-1} > v_{i-1} \}
\end{align*}
\]

**Remark:**

is \( u_i < v_i \) the \( u \) lies in the white sector

the outgoing edges \( (v, w) \) lie in the gray sectors
Barycentric representation of a planar graph

(validity of the Schnyder drawing)
Barycentric representation of a planar graph

Definition: A barycentric representation of a graph $G$ is defined by a mapping $f(v) \rightarrow (v_0, v_1, v_2) \in \mathbb{R}^3$ satisfying:

- $v_0 + v_1 + v_2 = 1$, for each vertex $v$
- for each edge $(x, y) \in E$ and each vertex $z \notin \{x, y\}$ there is an index $k \in \{0, 1, 2\}$ such that $x_k < z_k$ and $y_k < z_k$

Remark: The Schnyder drawing of a planar triangulation $T$ is a barycentric mapping.

proof: it follows from previous inclusion properties of regions $R_i$.
**Barycentric representation of a planar graph**

**Definition:** A **barycentric representation** of a graph $G$ is defined by a mapping $f(v) \rightarrow (v_0, v_1, v_2) \in \mathbb{R}^3$ satisfying:

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**Remark:** The Schnyder drawing of a planar triangulation $T$ is a barycentric mapping.

**Proof:** It follows from previous inclusion properties of regions $R_i$.

**Intuition:** No vertex $z$ in the gray triangle defined by $f(x), f(y)$.

\[ x_0 < z_0 \]
\[ y_0 < z_0 \]
Barycentric representation of a planar graph

Theorem
A barycentric representation defines a planar straight-line drawing of $G$, in the plane spanned by $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$.

proof:

Claim 1: for each edge $(x, y) \in E$ and each vertex $z \notin \{x, y\}$, $f(z)$ cannot lie on $(f(x), f(y))$

$$f(z) = tf(x) + (1 - t)f(y), \text{ for } t \in [0, 1]$$

$$z_k = tx_k + (1 - t)y_k < tz_k + (1 - t)z_k = z_k$$

for $k \in \{0, 1, 2\}$
Barycentric representation of a planar graph

Theorem
A barycentric representation defines a planar straight-line drawing of $G$, in the plane spanned by $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$.

**proof:**

**Claim 2:** given two edges $(x, y), (u, v)$ of $G$ they cannot cross by definition there are four indices $i, j, k, l \in \{0, 1, 2\}$

$$
egin{align*}
    u_i, v_i &< x_i & x_k, y_k &< u_k \\
    u_j, v_j &< y_j & x_l, y_l &< v_l
\end{align*}
$$

**Fact:** $i \neq k$

if $i = k$ we would have

$$
    u_k < x_k \\
    v_k < x_k
$$

contradicting $x_k, y_k < u_k$

In the example above we have $i = j = 2$

$\therefore$ there exists a separating line $l$ parallel to one of the sides of the outer triangle, that separates $(u, v)$ and $(x, y)$

the line $l$ parallel to $[(1, 0, 0), (0, 1, 0)]$ separates $(u, v)$ and $(x, y)$
**Linear-time implementation**

**Problem:** how to efficiently compute $|R_i(v)|$ (for all $v \in V$)?

**Remark:** the number of faces $|R_i(v)|$ can be retrieved from: the number of inner vertices and the number of vertices on the path $P_{i+1}(v)$ and $P_{i-1}(v)$
**Problem:** how to efficiently compute $|R_i(v)|$ (for all $v \in V$)?

**Remark:** the number of faces $|R_i(v)|$ can be retrieved from: the number of inner vertices and the number of vertices on the path $P_{i+1}(v)$ and $P_{i-1}(v)$

$$R_i(v) = 4$$

(inner faces)

$$f_i = 2n_i + k - 2$$

$$\partial R_i(v) := (P_{i+1}(v) + P_{i-1}(v)) - 1 = 4$$

(outer vertices)

$$\sum_{w \in P_{i+1}} |t_w| + \sum_{u} |t_u| = 1$$

(inner vertices)
**Problem:** how to efficiently compute $|R_i(v)|$ (for all $v \in V$)?

```cpp
/* computes number of nodes in tree */
int size(Node node)
{
    if (node == null)
        return 0;
    else
        return (size(node.left) + 1 + size(node.right));
}
```

- Compute and store for each vertex $v$ the subtree size of $T_0(v), T_1(v), T_2(v)$
- Compute the length of the paths $P_0(v), P_1(v), P_2(v)$
- Cumulate the size of sub-trees for all vertices $w_k, u_j$ on the paths $P_{i+1}(v), P_{i-1}(v)$

\[ f_i = 2n_i + k - 2 \]
Practical performances

average timings (over 100 executions)

Timing performances (pure Java, on a core i7-5600 U, 2.60GHz, 1GB Ram):
Schnyder woods can process $\approx 1.43M − 1.92M$ vertices/seconds

Two Schnyder drawings of a sphere graph
Practical performances

<table>
<thead>
<tr>
<th>Tutte</th>
<th>Schnyder</th>
<th>FPP layout</th>
</tr>
</thead>
<tbody>
<tr>
<td>fish model</td>
<td></td>
<td></td>
</tr>
<tr>
<td>random</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Schnyder woods and orthogonal surfaces
Orthogonal surfaces and elbow geodesics

### Dominance order

For $u, v \in \mathbb{Z}^3$:

\[
u \leq v \iff u_i \leq v_i, \forall i = 0, 1, 2\]

\[\Delta_p := \text{cone dominating } p \in \mathbb{R}^3\]

\[\nabla_p := \text{cone dominated by } p \in \mathbb{R}^3\]

Let $V \subset \mathbb{Z}^3$ be an antichain

$\langle V \rangle := \{ \alpha \in \mathbb{R}^3 | \alpha \geq v, \text{ for some } v \in V \} = \bigcup_v \Delta_v$

Orthogonal surface $S_V := \text{boundary of } \langle V \rangle$

\[x + y + z = 1\]
Orthogonal surfaces and elbow geodesics

**Dominance order** \((u, v \in \mathbb{Z}^3)\)

\[ u \leq v \quad \text{iff} \quad u_i \leq v_i, \quad \forall i = 0, 1, 2 \]

**join** \(u \vee v := \text{maximum component-wise join}\)

**meet** \(u \wedge v := \text{minimum component-wise meet}\)

\[
(4, 2, 1) \vee (2, 1, 4) = (4, 2, 4) \\
(0, 7, 0) \vee (0, 3, 5) = (0, 7, 5)
\]

**\(\langle V \rangle := \{ \alpha \in \mathbb{R}^3 | \alpha \geq v, \text{ for some } v \in V \} \)**

Orthogonal surface \(S_V := \text{boundary of } \langle V \rangle\)

Let \(V \subset \mathbb{Z}^3\) be an antichain

(elements are pairwise incomparable)
Orthogonal surfaces and elbow geodesics

\[(4, 2, 1) \land (2, 1, 4) = (4, 2, 4)\]
\[(0, 7, 0) \land (0, 3, 5) = (0, 7, 5)\]

**Elbow geodesic** of \(u\) and \(v\):
the union of the two line segments 
\((u, u \lor v)\) and \((u \lor v, v)\)

- every \(v \in S_V\) has three orthogonal arcs (parallel to each axis)
- every elbow geodesic contains at least one
  bounded orthogonal arc

Orthogonal arcs are parallel to the 3 axis
Orthogonal surfaces and elbow geodesics

A **geodesic embedding** of a planar map $G$: a drawing of $G$ on $S_V$ s.t.

1. **(G1)*** The vertices of $G$ correspond to the points of $S_V$.
2. **(G2)*** Every edge of $G$ is drawn as an elbow geodesic on $S_V$.
   - Every bounded orthogonal arc of $S_V$ is part of an edge of $G$.
3. **(G3)*** There are no edge crossings on $S_V$. 

![Diagram showing geodesic embedding](image)
From geodesic embeddings to Schnyder woods

**Thm:** Consider a Schnyder wood of a planar map $G$ and the corresponding set of vertex coordinates $V$ (region vectors). The resulting drawing of $G$ on $S_V$ is a geodesic embedding (no crossings).
From geodesic embeddings to Schnyder woods

**Thm:** Consider a Schnyder wood of a planar map $G$ and the corresponding set of vertex coordinates $V$ (region vectors). The resulting drawing of $G$ on $S_V$ is a geodesic embedding (no crossings)

Orthogonal arcs parallel to the $x$-axis are red (color 0)
Orthogonal arcs parallel to the $y$-axis are blue (color 1)
Orthogonal arcs parallel to the $z$-axis are black (color 2)
From geodesic embeddings to Schnyder woods

**Thm:** Consider a Schnyder wood of a planar map $G$ and the corresponding set of vertex coordinates $\mathcal{V}$ (region vectors). The resulting drawing of $G$ on $S_{\mathcal{V}}$ is a geodesic embedding (no crossings).

**Claim 1:** The local Schnyder condition (W3) is valid

- Every vertex has 3 outgoing edges (one for each color): the three orthogonal arcs (by construction)
- Let us consider an edge $\{u = (u_0, u_1, u_2), v = (v_0, v_1, v_2)\}$ incident at $v$ in the sector parallel to the vertical $yz$-plane

The edge $\{u, v\}$ contains the orthogonal arc $(u \vee v, u)$ parallel to the $x$-direction and lying in the same horizontal plane of $u$: its color must be red (color $0$), and its orientation is outgoing from $u$.

**Claim 2:** condition (W4) of the definition is valid

Remark: a path of edges of color $i$ lead to increasing coordinates in $i$-direction.

(W4) no cycles
Thm: Consider a Schnyder wood of a planar map $G$ and the corresponding set of vertex coordinates $V$ (region vectors). The resulting drawing of $G$ on $S_V$ is a geodesic embedding (no crossings).

**proof** (assume there are edge crossings)

Claim 1: edge crossing are of the form (as orthogonal arcs cannot cross)

Claim 2: edges $(u, v)$ and $(z, y)$ are of same color, lying on the same plane: $u_2 = y_2$ (in the example)

Claim 3: vertices $u$ and $y$ have the same $z$-coordinate thus there is a bi-directed path between $P^* u$ and $y$
Geodesic embeddings are planar drawings

**Thm:** Consider a Schnyder wood of a planar map $G$ and the corresponding set of vertex coordinates $V$ (region vectors). The resulting drawing of $G$ on $S_V$ is a geodesic embedding (no crossings)

**proof** (assume there are edge crossings)

Let $s :=$ first vertex at the crossing of $P_1(u)$ and $P_1(v)$

**Claim 4:** $s$ cannot belong to the path $P^*$ and $s \neq y$

(there is a cycle in $T_2 \cup T_{-1}^0 \cup T_{-1}^1$: violates previous theorem)
**Thm:** Consider a Schnyder wood of a planar map \( G \) and the corresponding set of vertex coordinates \( \mathcal{V} \) (region vectors). The resulting drawing of \( G \) on \( S_\mathcal{V} \) is a geodesic embedding (no crossings)

**proof (assume there are edge crossings)**

Let \( s := \) first vertex at the crossing of \( P_1(u) \) and \( P_1(v) \)

Remark: \( y \) is an inner vertex in the region \( R_0(v) \)

by assumption \( (z, y) \) is an edge of \( G \) \( \rightarrow \) \( (z, y) \) belong to \( R_0(v) \)

Since \( (z, y) \) belong to \( R_0(v) \) \( v_0 \geq z_0 \) (contradiction)
From geodesic embeddings to straight-line planar drawings

**Thm:** Given a planar (3-connected) map \( G \), the region counting algorithm leads to a planar straight-line drawing of \( G \) (no edge crossings). Moreover, the faces of \( G \) are convex.
Schnyder woods: applications
Graph encoding
Geometric v.s combinatorial information

"Connectivity": the underlying triangulation
(incidence relations between triangles, vertices, edges)

3 × h + n = 19n references

\[
\#\{\text{triangulations}\} = \frac{2(4n+1)!}{(3n+2)!(n+1)!} \approx \frac{16}{27} \sqrt{\frac{3}{2\pi}} n^{-5/2} \left(\frac{256}{27}\right)^n
\]

⇒ entropy = \log_2 \frac{256}{27} \approx 3.24 \text{ bit/vertex.}

David statue (Stanford's Digital Michelangelo Project, 2000)

2 billions polygons
32 Giga bytes (without compression)

2(4n + 1)! / (3n + 2)! (n + 1)! \approx 16/27 \sqrt{3/(2\pi)} n^{-5/2} \left(\frac{256}{27}\right)^n

⇒ entropy = \log_2 \frac{256}{27} \approx 3.24 \text{ bit/vertex.}
A simple encoding scheme

Turan encoding of planar map (1984)

\[ G = (V, E) \quad |V| = n \quad |E| = e \]

\[ T := \text{(any) vertex spanning tree of } G \]

\[ (\ldots(())(())(())) \ldots \]

\[ S(G) \quad \text{parenthesis word of size } 2n \]

\[ (2 \log_2 4)e = 4e = 12n \text{ bits} \]
A more efficient encoding

Canonical orderings - Schnyder woods (He, Kao, Lu '99)

\( T_1 \) is redundant: reconstruct from \( T_0, T_2 \)

(unique way to triangulate each face of \( \overline{T_0} \cup T_2 \))
A more efficient encoding

Canonical orderings - Schnyder woods (He, Kao, Lu ’99)

$4n$ bits (for triangulations)

$T_2$ can be reconstructed from $T_0$ and the number of ingoing edges (for each node)

$T_0$  \[
\emptyset \quad \emptyset \quad \left( \left( \emptyset \right) \quad \left( \emptyset \right) \right) \quad \emptyset \quad \left( \emptyset \right) \quad \emptyset \quad \emptyset
\]

$T_2$  00000101010100110111

$2(n - 1)$ symbols $= 2(n - 1)$ bits

$(n - 1) + (n - 3) = 2n - 4$ bits

$\approx 4n$ bits
A more efficient encoding

Canonical orderings - Schnyder woods (He, Kao, Lu '99)

4n bits (for triangulations)

\[ 2(n-1) \text{ symbols} = 2(n-1) \text{ bits} \]

\[ (n-1) + (n-3) = 2n-4 \text{ bits} \]
Appendix: Schnyder woods for toroidal graphs
**Toroidal Schnyder woods** [Goncalves Lévêque, DCG’14]

- 3-orientation + Schnyder local rule valid at each vertex
- every monochromatic cycle intersects at least one monochromatic cycle of each color

Toroidal Schnyder woods are **crossing** if

\[ n - e + f = 2 - 2g \]

\[ g = 1 \]
\[ e = 3n \]

- **not valid Schnyder wood**: 3-orientation
- **valid Schnyder woods**
- **crossing Schnyder wood**
  - (there are 3 disjoint monochromatic cycles of color 2)

(Local Schnyder rule cannot be propagated everywhere)
**Thm** [Fijavz, unpublished]

A simple toroidal triangulation contains three non-contractible and non-homotopic cycles that all intersect on one vertex and that are pairwise disjoint otherwise.

**split** along $\Gamma_0$, $\Gamma_1$, $\Gamma_2$

(two planar quasi-triangulations)

**crossing** toroidal Schnyder wood (for simple triangulations)
A simple toroidal triangulation admits a straight-line periodic drawing on a grid of size $O(n^2 \times n^2)$.
Open problems: Schyder woods for $g \geq 2$

**Thm** (3-orientations for graphs on surfaces, of arbitrary genus)
[Albar Goncalves Knauer, 2014]

Any triangulation of a surface (the sphere and the projective plane) admits a '3-orientation': orientation without sinks s.t. every vertex has outdegree divisible by three.

**Conjecture** (Existence of Schnyder woods for higher genus triangulations)
[Goncalves Knauer Lévêque, 2016]

Multiple local Schnyder condition: the outdegree of every vertex is a positive multiple of 3.

(there are no sinks)

**Thm** [Suagee, 2021]

Simple triangulations of genus $g \geq 1$ having large **edgewidth** do admit Schnyder woods

**edgewidth** $\geq 40(2^g - 1)$
(size of the smallest non contractible cycle)

**Experimental confirmation**

Exaustive generation of all 3-orientations for all triangulations with $g = 2$, $n \leq 11$

All simple triangulations of genus $g = 2$ and size $\leq 11$ admit Schnyder woods

<table>
<thead>
<tr>
<th>$n$</th>
<th># irreducible triangulations</th>
<th>#triangulations $(g = 2)$</th>
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<tr>
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</tr>
</tbody>
</table>

**surftri** software [Sulanke, 2006]