MPRI 2-38-1: Algorithms and combinatorics for geometric graphs

Lecture 6

Schnyder woods for 3-connected plane graphs

october 23, 2024

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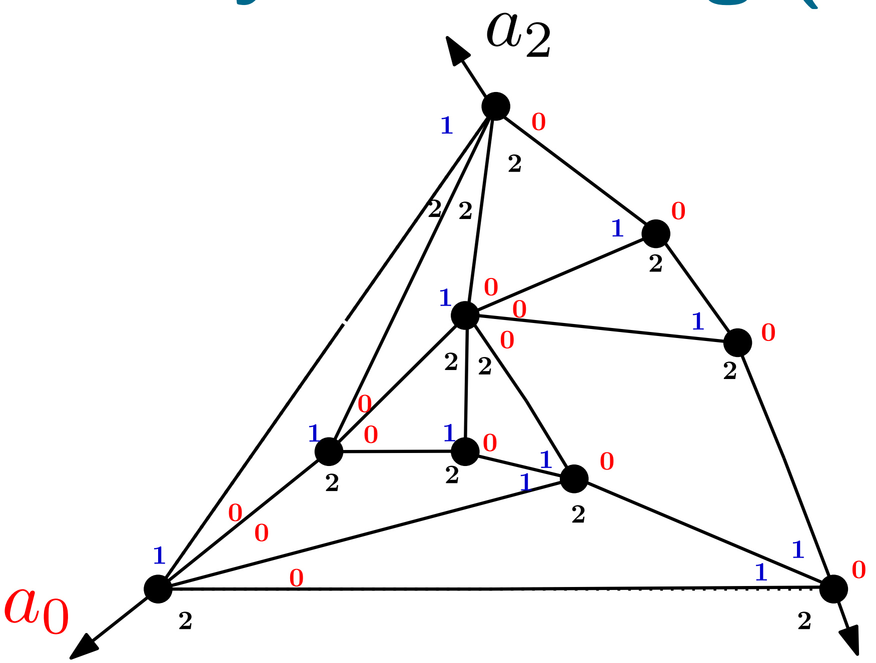
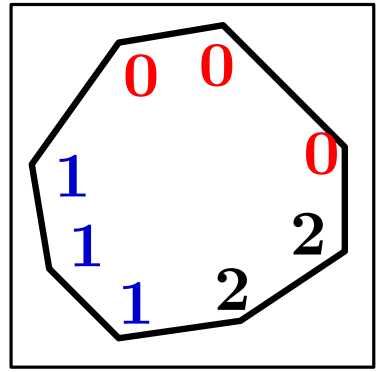
Schnyder woods

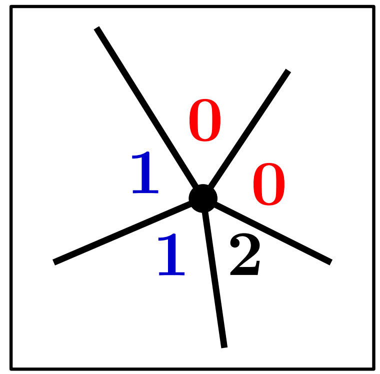
(definitions)

Schnyder labeling (3-connected maps): definition

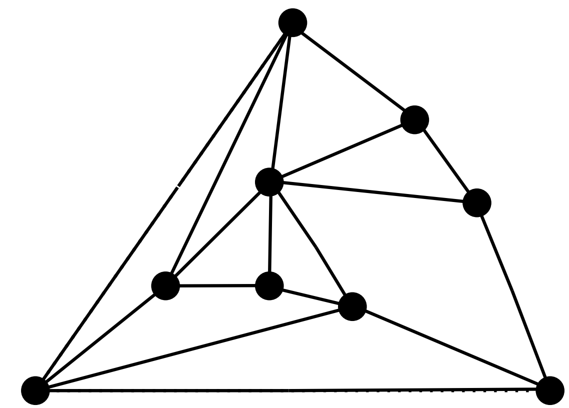


suspension M^σ





3-connected graphs [Felsner]



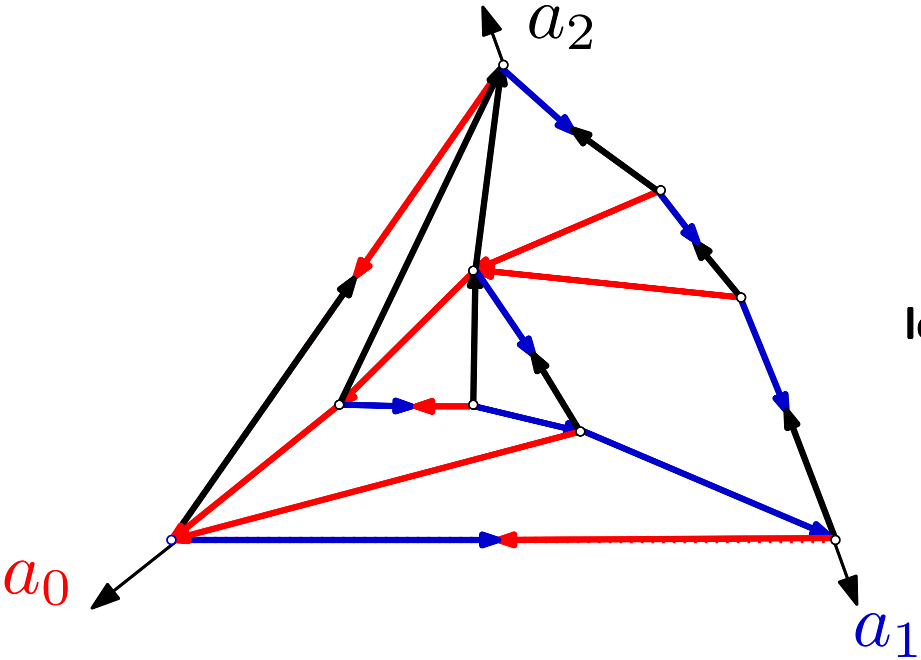
3-connect. map M

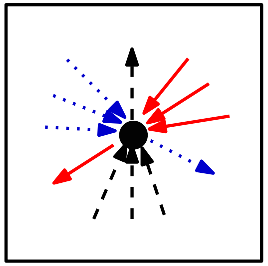
A1) the angles at a_i have labels $i+1$, $i-1$

A2) **rule for vertices:** at each vertex there are non-empty intervals of labels $0, 1$ and 2 (listed counter-clockwise)

A3) **rule for faces:** at each inner faces the angles define three non-empty intervals of labels $0, 1$ and 2 in ccw order. For the outer face the angles are listed clockwise.

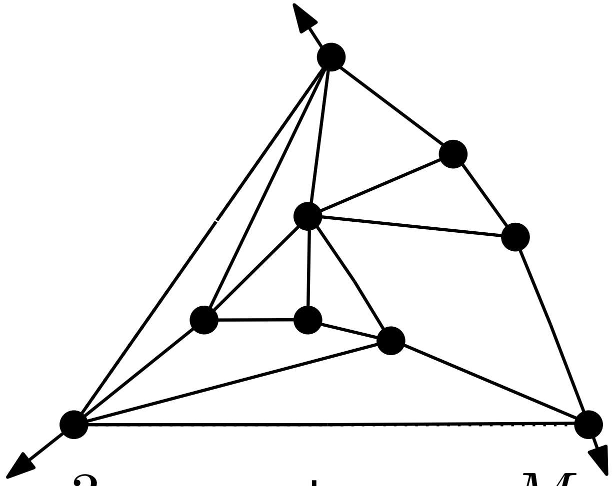
Schnyder woods (3-connected maps): definition





local Schnyder rule

3**-connected graphs [Felsner]**



3-connect. map M

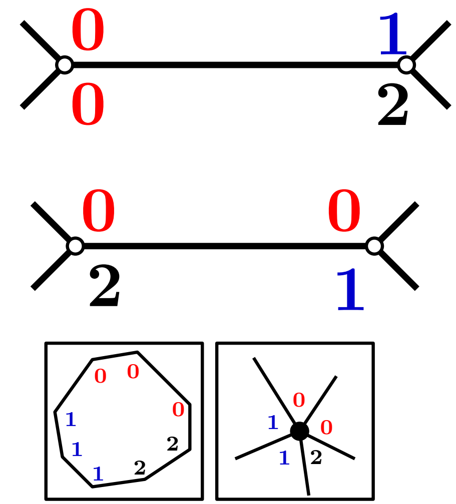
W1) edges have one or two (opposite) orientations. If an edge 3 is bo-oriented than the two direction have distinct colors

W2) the edges at a_i are outgoing of color i

W3) **local rule for vertices:** at each vertex there are three outgoing edges (one in each color) satisfying the local Schnyder rule

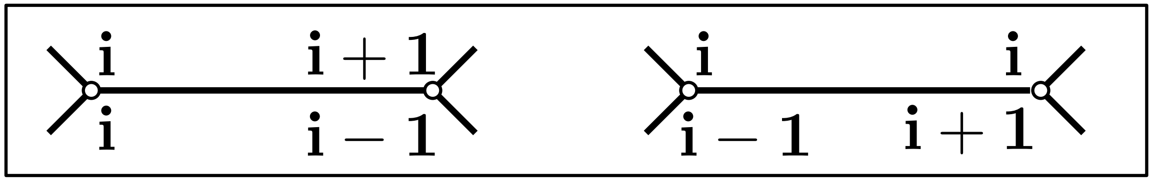
W4) there is no interior face whose boundary is a directed cycle in one color

Schnyder labelings: angles around edges



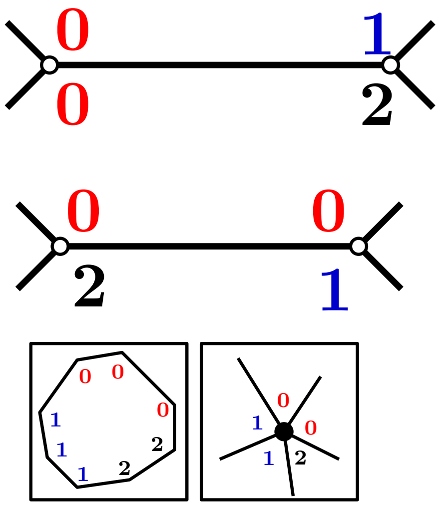
Lemma

Given a Schnyder labeling of M^σ, the angles of each edges have colors $0, 1, 2$ and are of the following 2 types:



proof:

Schnyder labelings: angles around edges



Lemma

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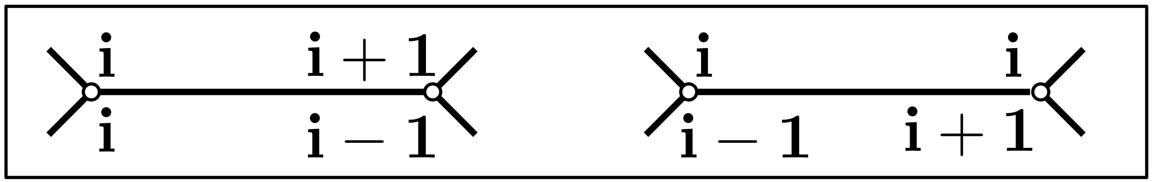
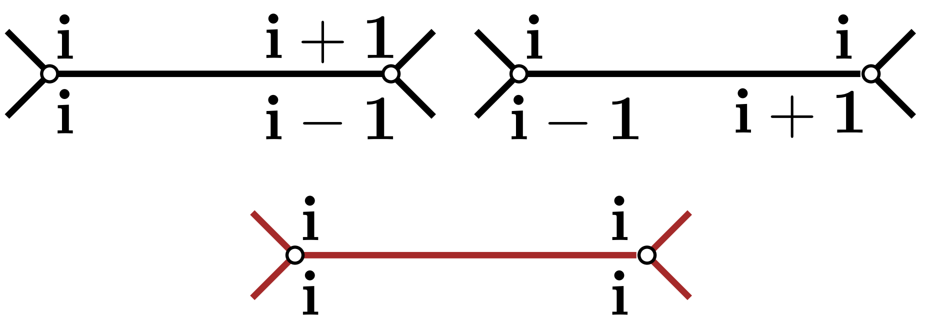
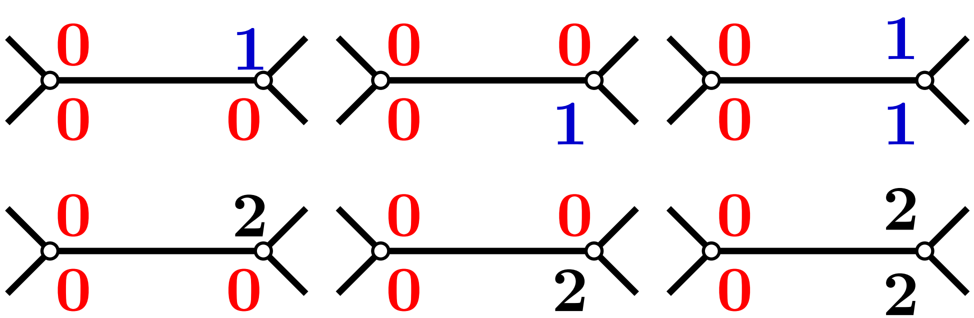


proof:

possibly valid configurations



forbidden configurations

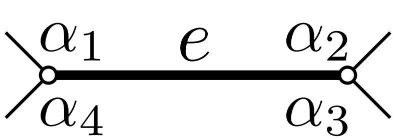


use a counting argument (double counts the angles)

$d(v)$:= number of label changes for the angles around v
$d(f)$:= number of label changes for the angles in face f

$$\sum_v d(v) + \sum_f d(f) = 3n + 3|f| = 3|E| + 6$$

use Euler formula: 3n+3(2+—E—-n)



at vertex a_i there are two label changes

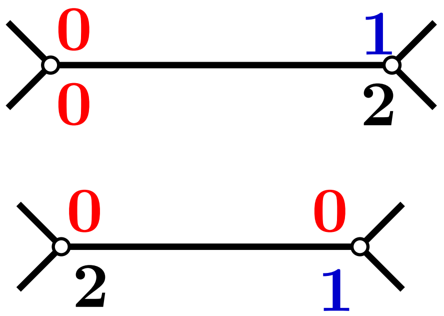
$\epsilon(e)$ = number of label changes at the angles around e

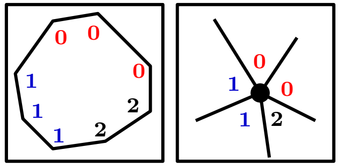
$$\epsilon(e) = \begin{cases} 0 \\ 3 \end{cases}$$

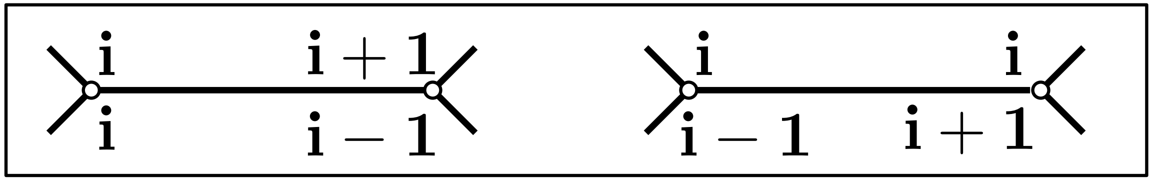
→ $\epsilon(e) = 3$ for all (normal) edges

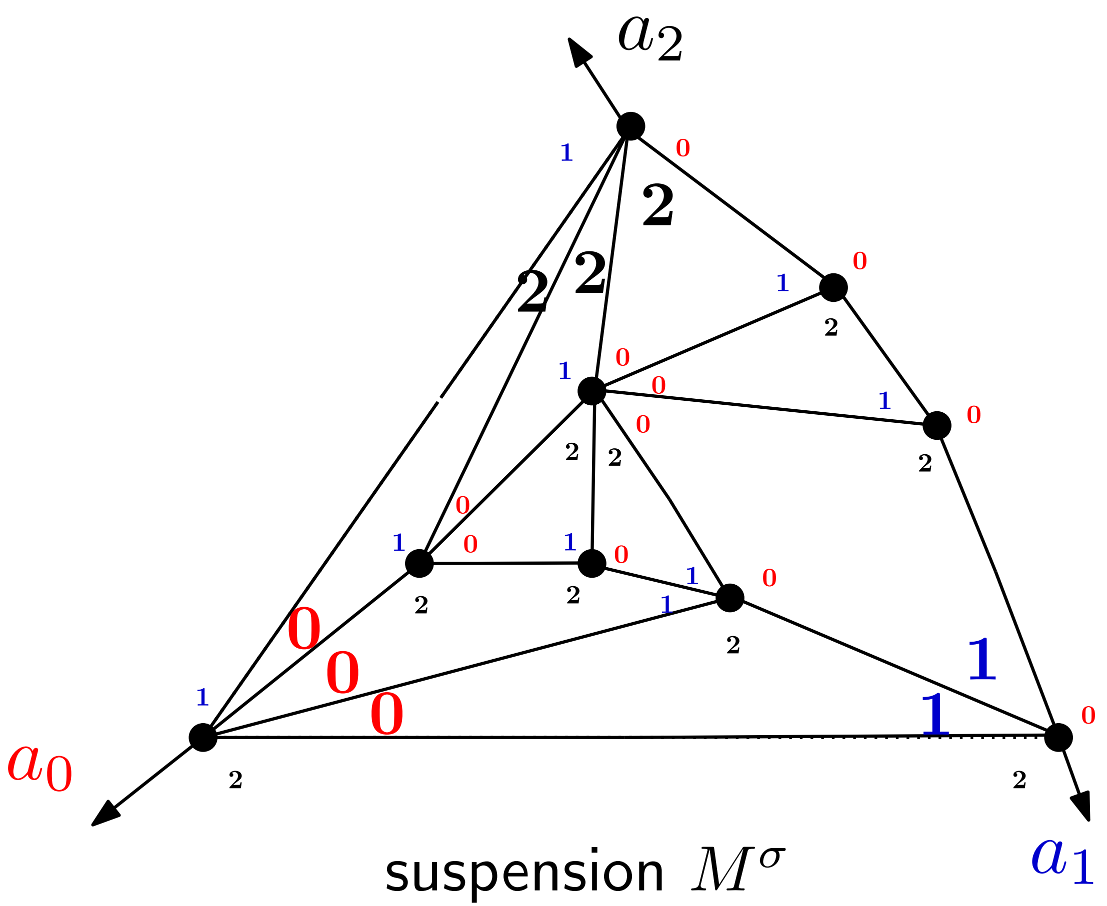
Schnyder labelings: angles at exterior vertices

Corollary
Given a Schnyder labeling of M^σ, all interior angles at a vertex a_i have label i



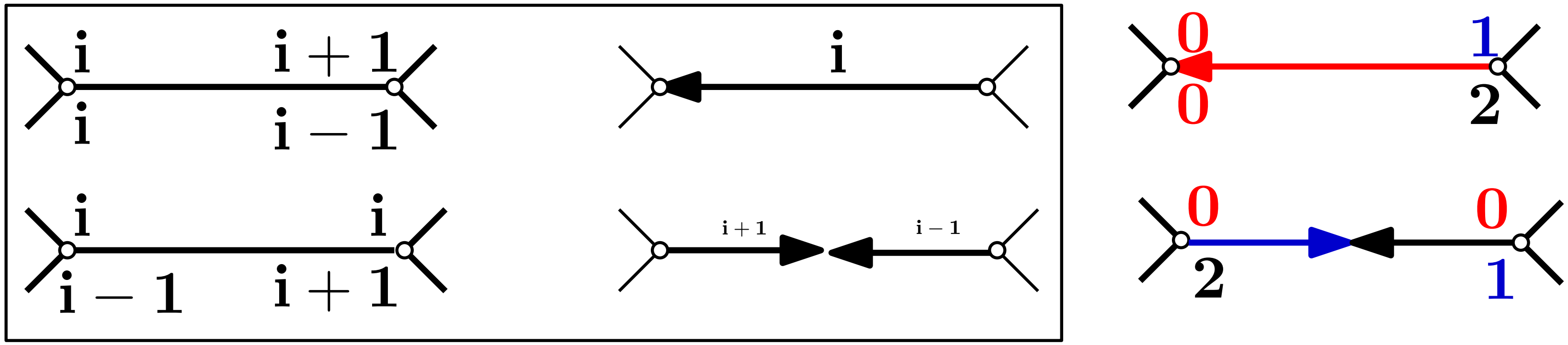






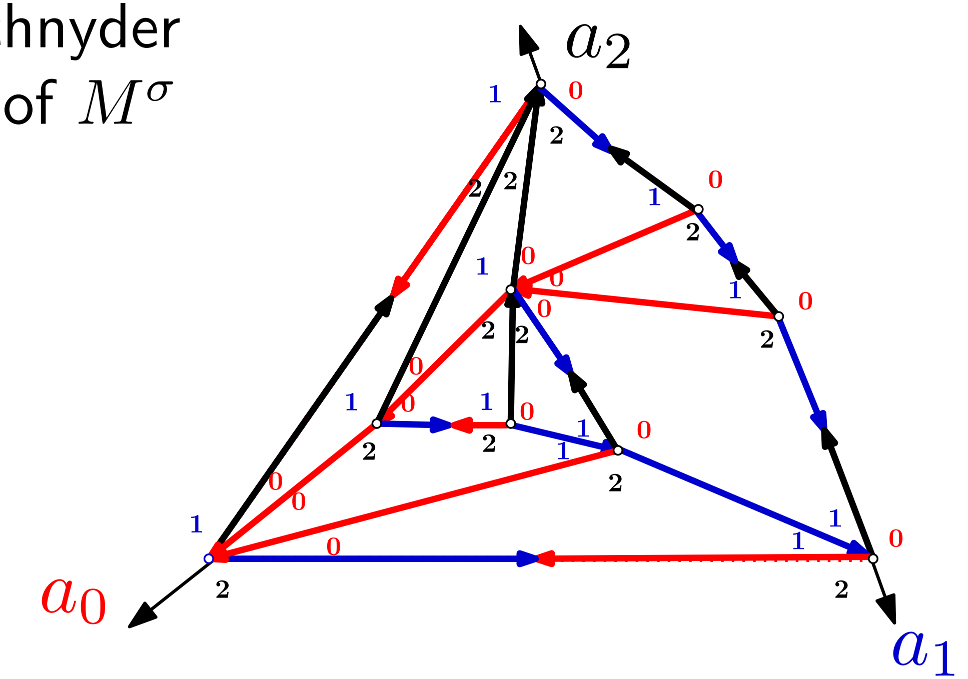
suspension M^σ

Correspondence between Schnyder labelings and Schnyder woods



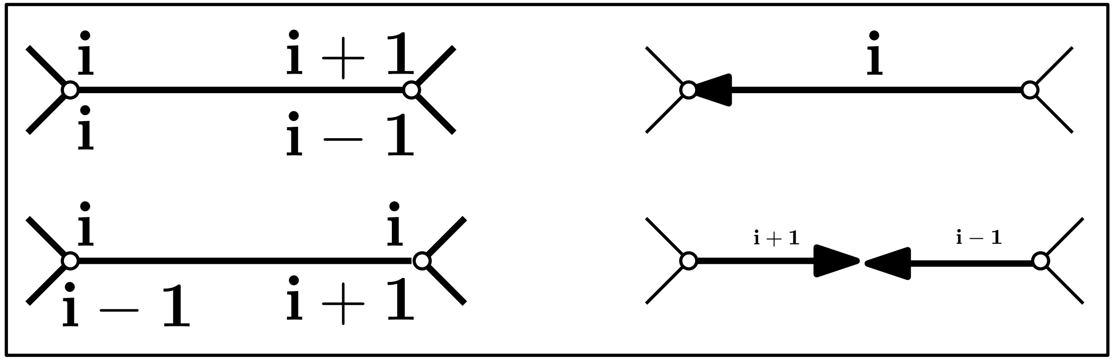
Theorem

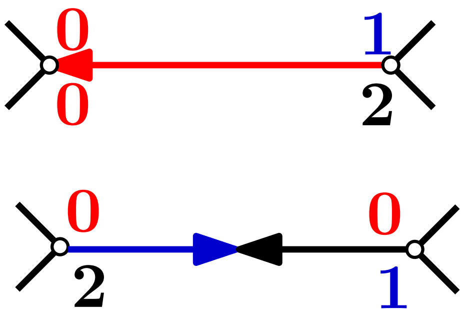
There is a correspondence between the Schnyder labelings of M^σ and the Schnyder woods of M^σ



Schnyder wood+ Schnyder labeling of M^σ

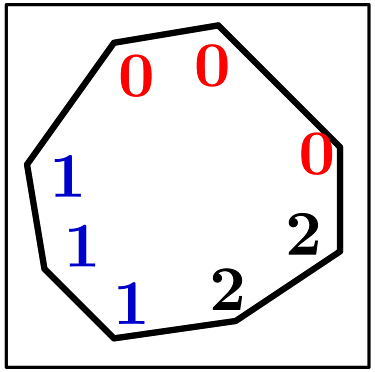
Correspondence between Schnyder labelings and Schnyder woods



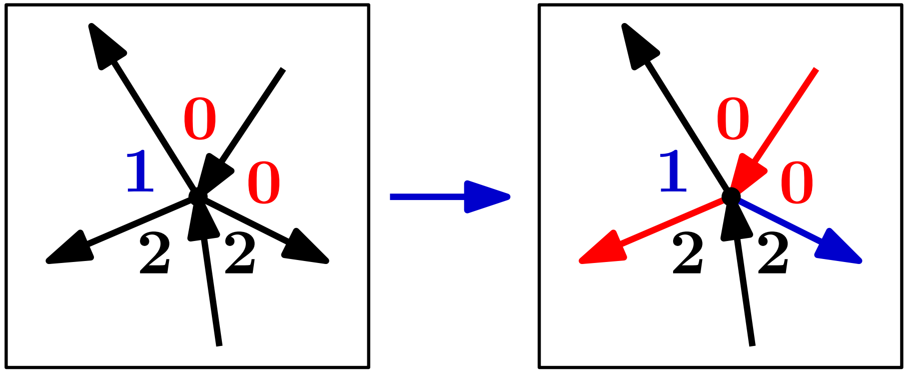


Theorem

There is a correspondence between the Schnyder labelings of M^σ and the Schnyder woods of M^σ

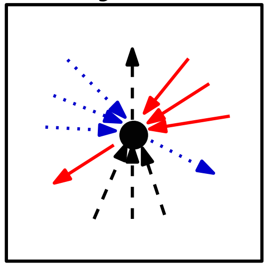


proof: Assume M^σ is endowed with a Schnyder labeling



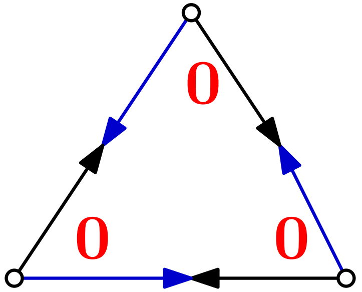
Rule of vertices (A2)

local Schnyder rule (W3)



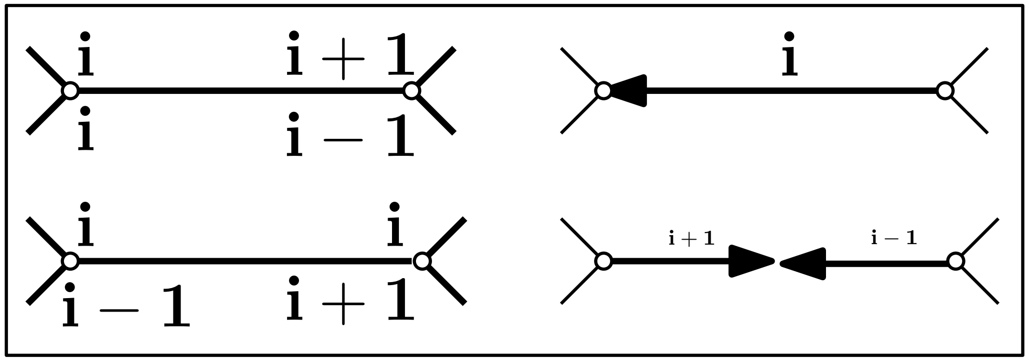
Assume (W4) is violated: there is a cycle in one color

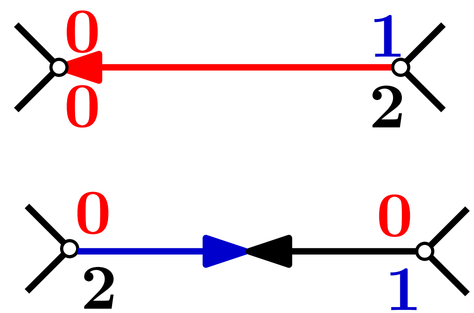
Then the coloring rule of bi-oriented edges implies that all angles have the same color

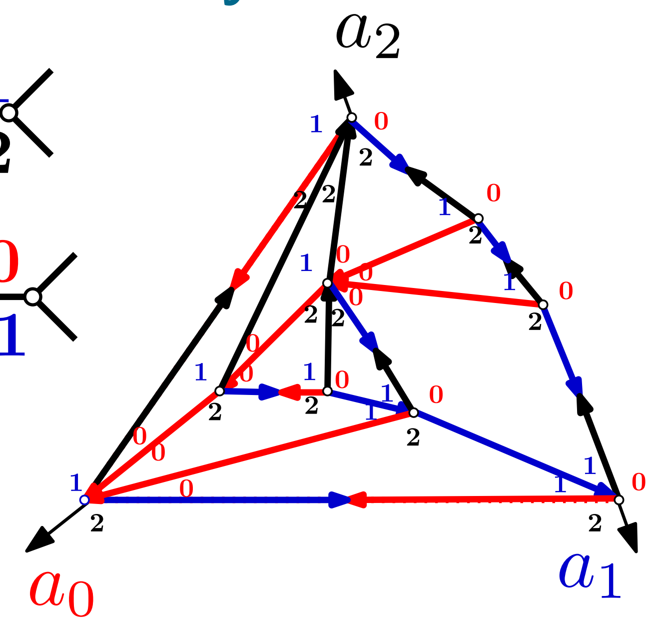


Rule of faces (A3) → **no directed cycles in one color (W4)**

Correspondence between Schnyder labelings and Schnyder woods







Theorem

There is a correspondence between the Schnyder labelings of M^σ and the Schnyder woods of M^σ

proof: Assume M^σ is endowed with a Schnyder wood

use a counting argument (double counts the angles around vertices/faces/edges)

$$d(v) = 3 \qquad d(e) = \begin{cases} 3 & \text{for all (normal) edges} \\ 2 & \text{for the three half-edges} \end{cases}$$

Remark:

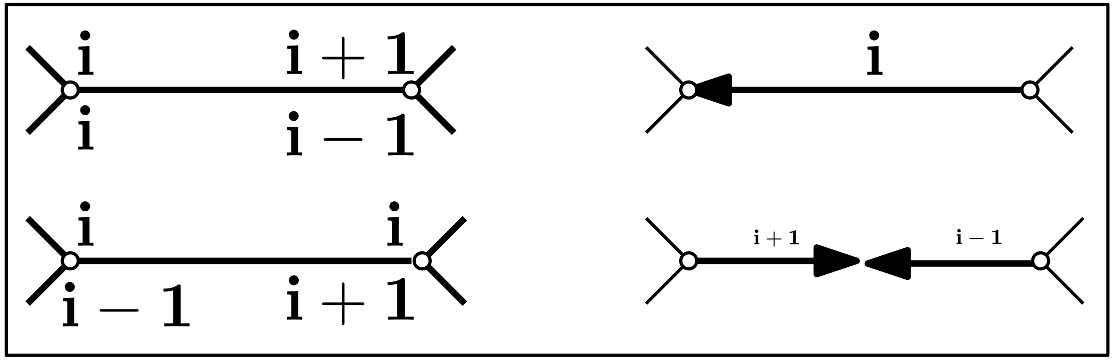
Turning around a face in ccw direction the angle will be i or $i+1$ → The number of changes $d(f)$ is a multiple of 3, and $d(f) > 0$ (otherwise there is a directed cycle of edges in one color)

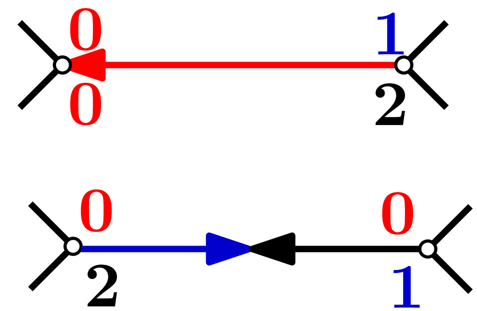
$$\sum_v d(v) + \sum_f d(f) = \sum_e d(e) \quad \rightarrow \quad 3n + \sum_f d(f) = 3|E| + 6$$

Euler formula implies $\sum_f d(f) = 3|F|$ → $d(f) = 3$ for all faces

condition (A3) for faces is true

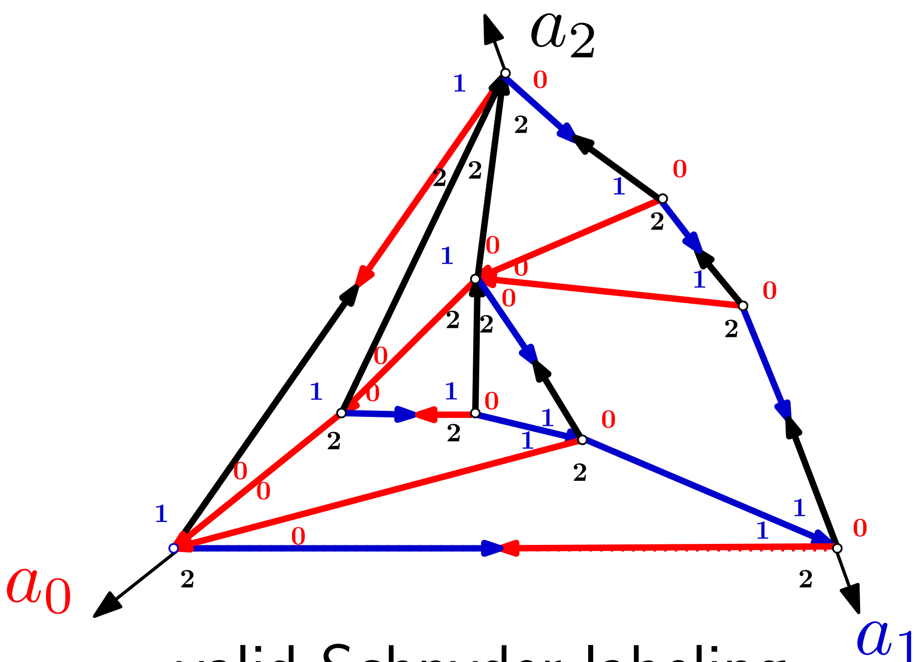
Correspondence between Schnyder labelings and Schnyder woods





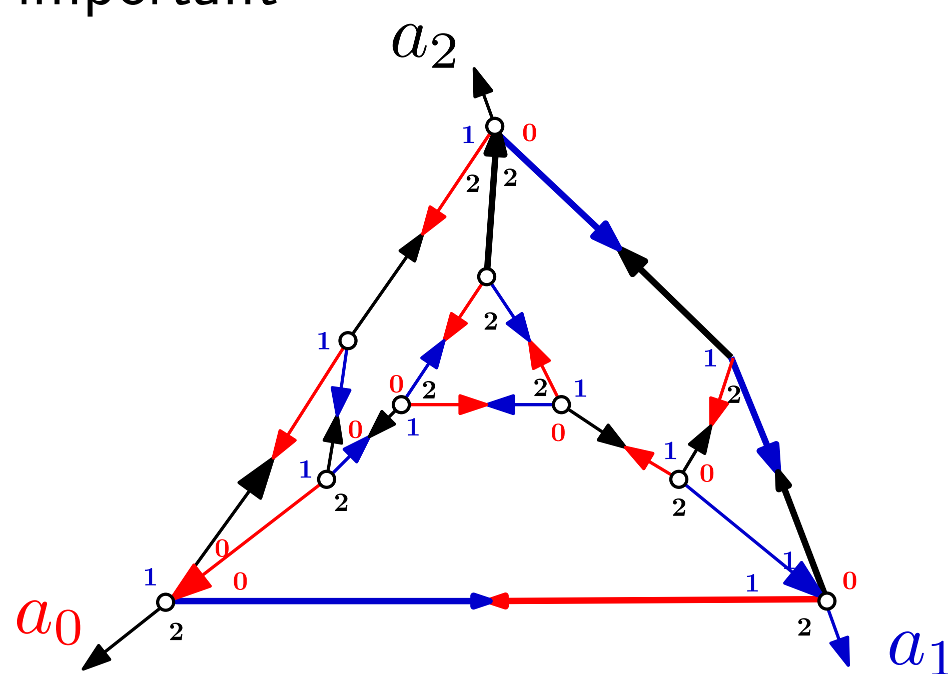
Remark:

The condition (W4) of Schnyder woods is important



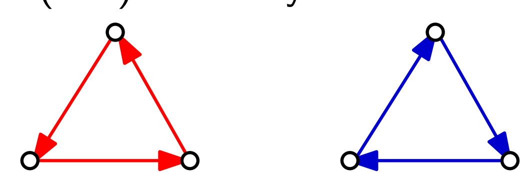
valid Schnyder labeling

conditions (W1)-(W4) of Schnyder woods are satisfied



not valid Schnyder labeling

condition (W4) of Schnyder woods is not satisfied

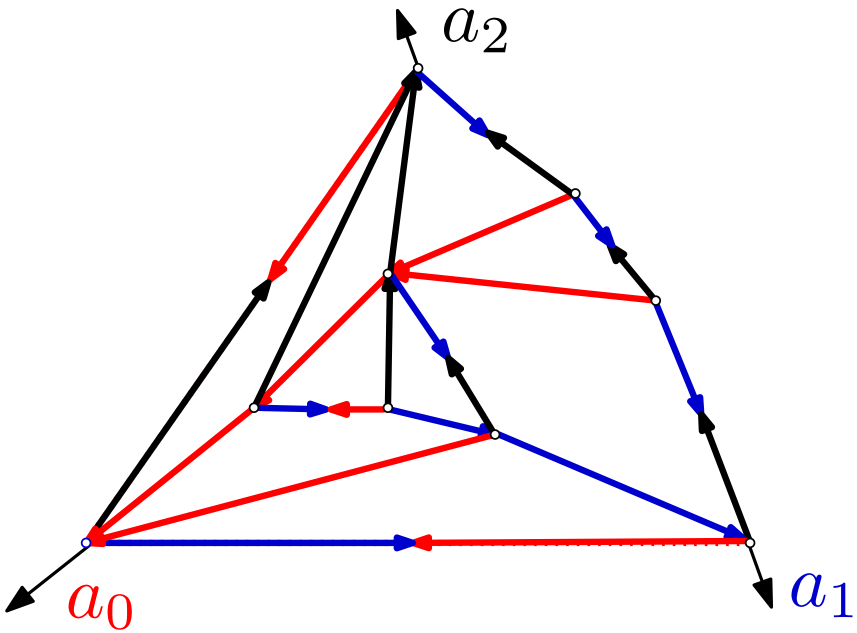


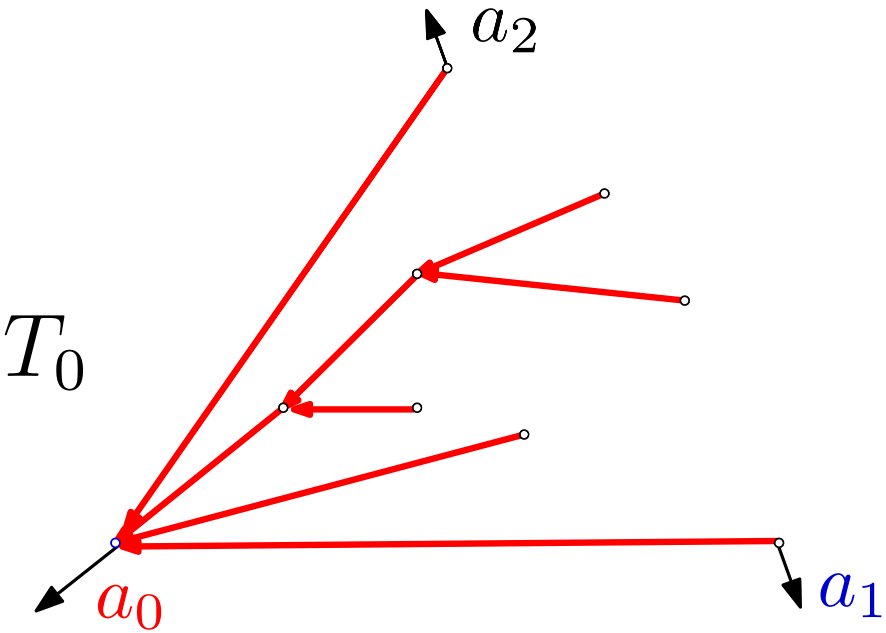
Schnyder woods: spanning property

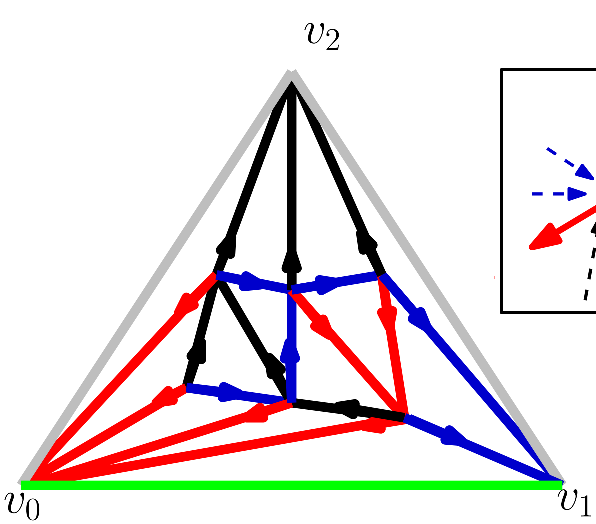
Theorem **[Schnyder '90]**

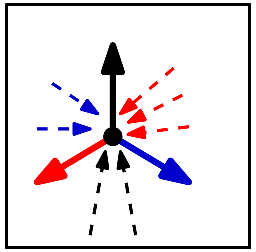
$T_i :=$ digraph defined by directed edges of color i

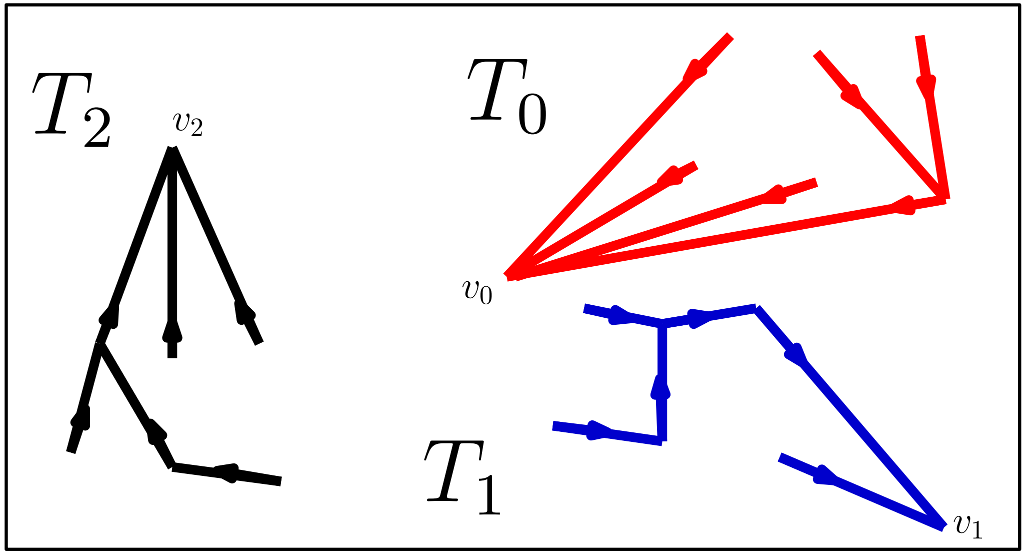
The three sets T_0, T_1, T_2 are spanning trees of the inner vertices of $\mathcal{T}$ (each rooted at vertex v_i)





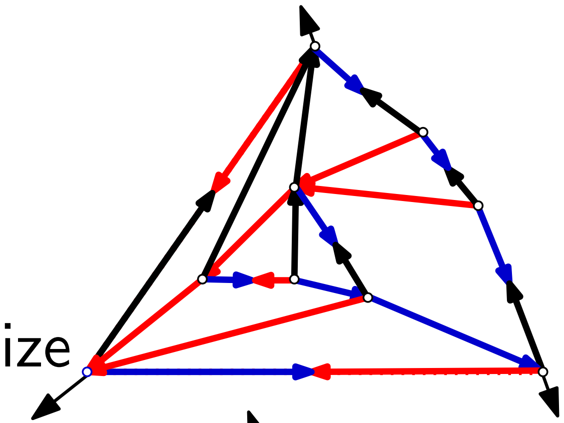






Spanning property for 3-connected maps

$T_i :=$ digraph defined by directed edges of color i



Theorem Let (T_0, T_1, T_2) a Schnyder wood of $\mathcal{M}$.
Then each digraph $D_i := T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ is acyclic

proof:

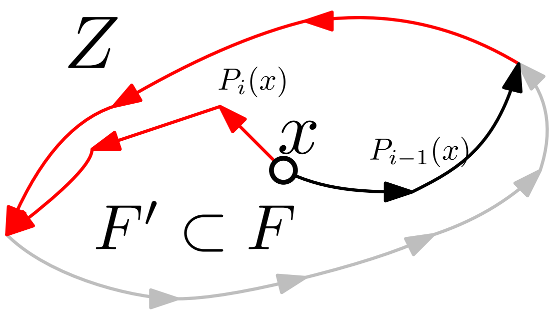
Let Z a directed cycle enclosing a region F of minimal size

Claim 1: F is a single face

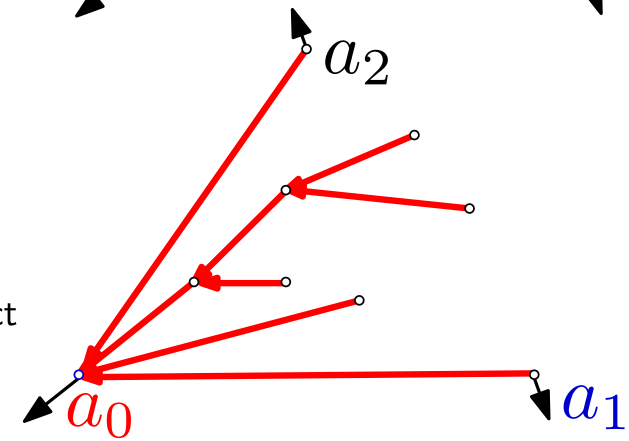
case a: $x \in F$

F' is a smaller than F

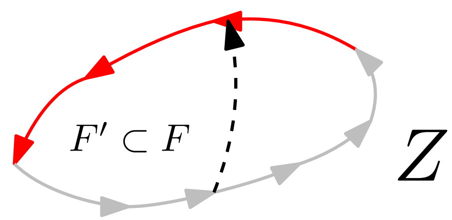
(bounded by a directed cycle)



$P_i(x)$ and $P_{i-1}(x)$ cannot intersect



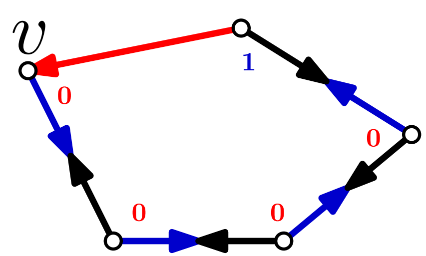
case b: F is empty of vertices
there is an edge inside F

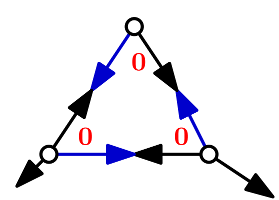


Claim 2: there is no face F whose boundary is a directed cycle

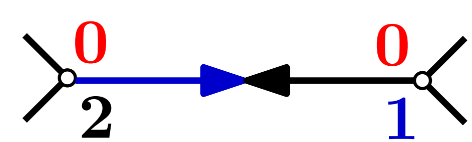
Visit F in ccw order starting from v and propagate colors (first color is i): there is no angle with label $i-1$

The coloring rule for faces is violated





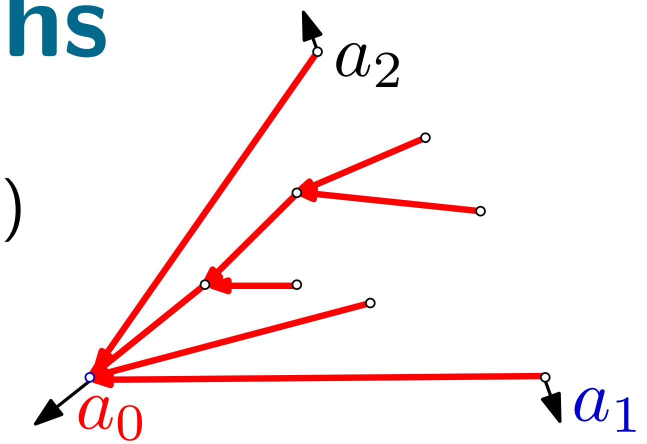
coloring rule for angles



Corollary: Each sets T_i is spanning tree $\mathcal{M}$ (rooted at vertex a_i)

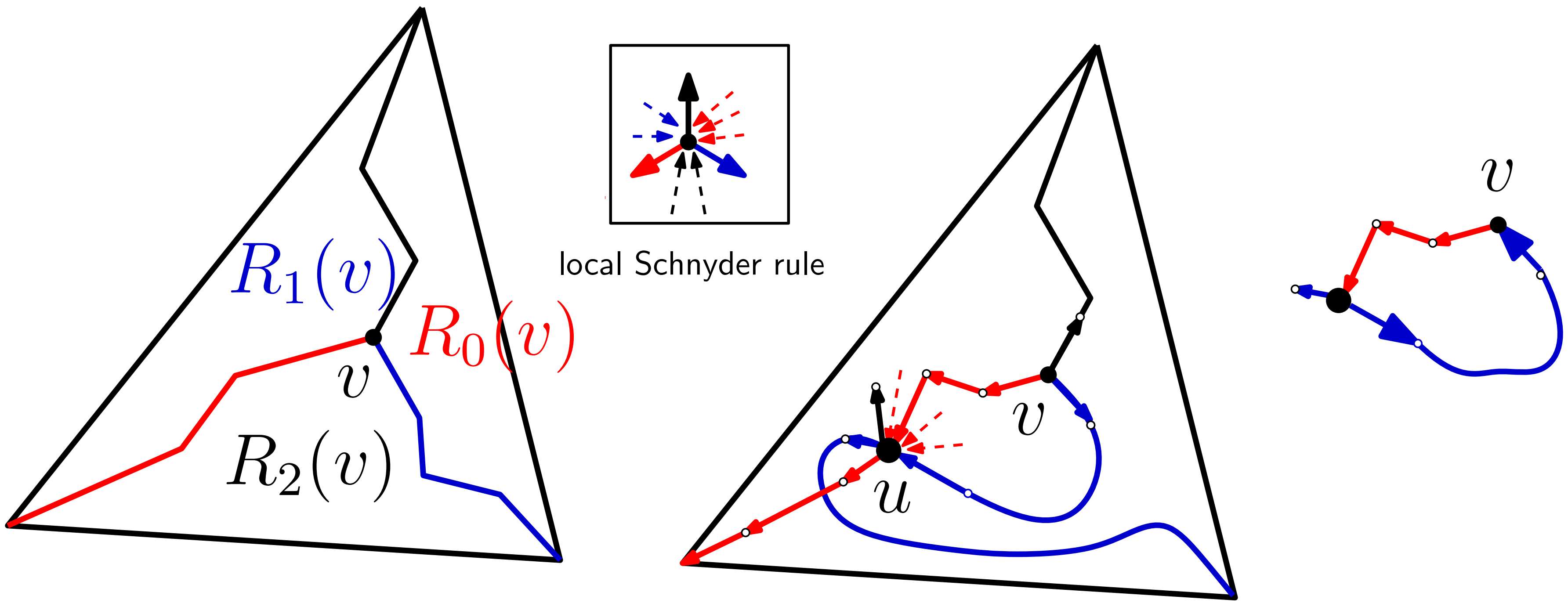
Non crossing paths

Corollary:
Each sets T_i is spanning tree $\mathcal{M}$ (rooted at vertex a_i)



Corollary
For each inner vertex v the three monochromatic paths P_0, P_1, P_2 directed from v toward each vertex a_i are vertex disjoint (except at v) and partition the inner faces into three sets $R_0(v), R_1(v), R_2(v)$

proof: the existence of two paths $P_i(v)$ and $P_{i+1}(v)$ which are crossing would contradicts previous theorem



Planar straight-line drawings

(of planar graphs)

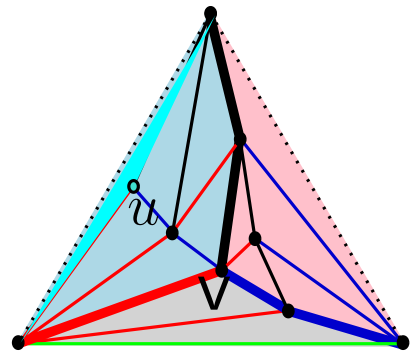
Paths and regions

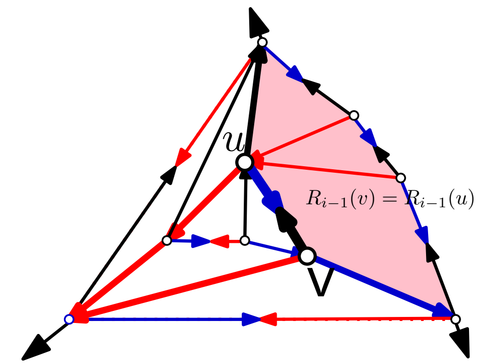
$u \in R_1^{int}(v)$

Lemma Let (T_0, T_1, T_2) a Schnyder wood of $\mathcal{M}$.

If $u \in R_i(v)$ then $R_i(u) \subseteq R_i(v)$

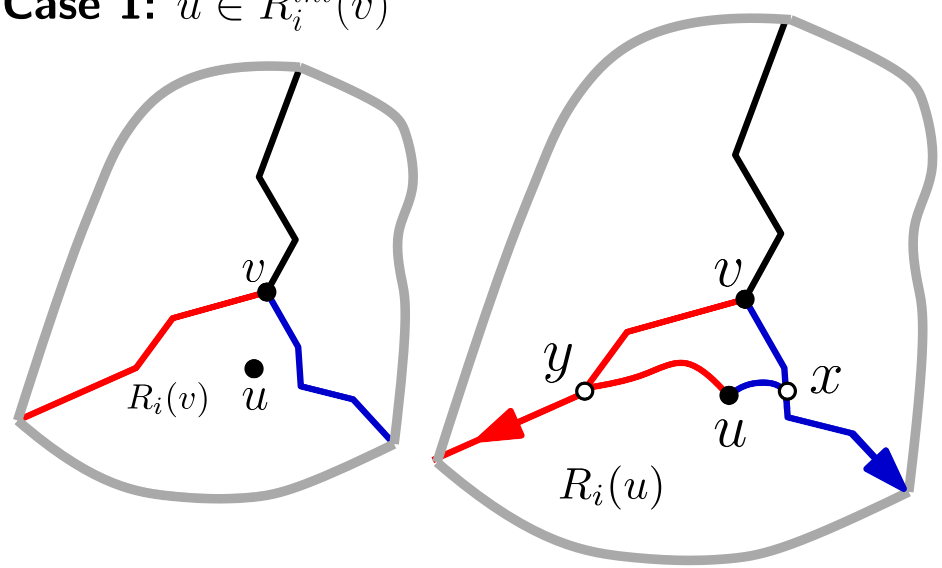
If $u \in R_i^{int}(v)$ then $R_i(u) \subset R_i(v)$





proof:

Case 1: $u \in R_i^{int}(v)$



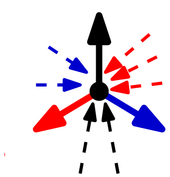
first step: compute the paths $P_{i+1}(u)$ and $P_{i-1}(u)$

They must intersect the boundary of $R_i(v)$ at x and y

Remark: x and y are different from v

and we have $y \in P_{i+1}(u)$ and $x \in P_{i-1}(u)$

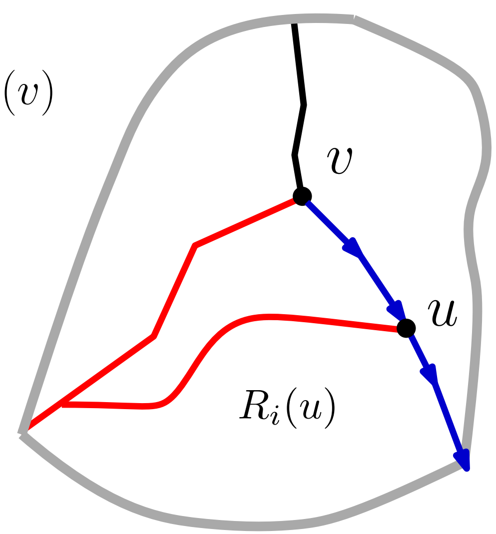
(because of Schnyder rule)



so we have: $R_i(u) \subset R_i(v)$

Case 2a: $u \in P_{i-1}(v)$

$R_i(u) \subset R_i(v)$

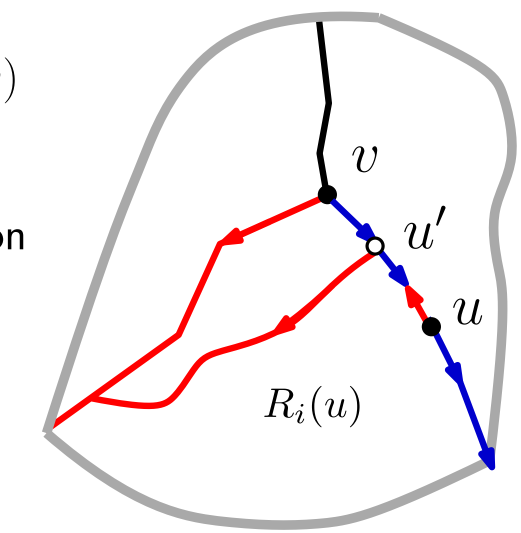


Case 2b: $u \in P_{i-1}(v)$

(u, u') is bi-oriented

Proceed by induction on the path $P_{i-1}(v)$

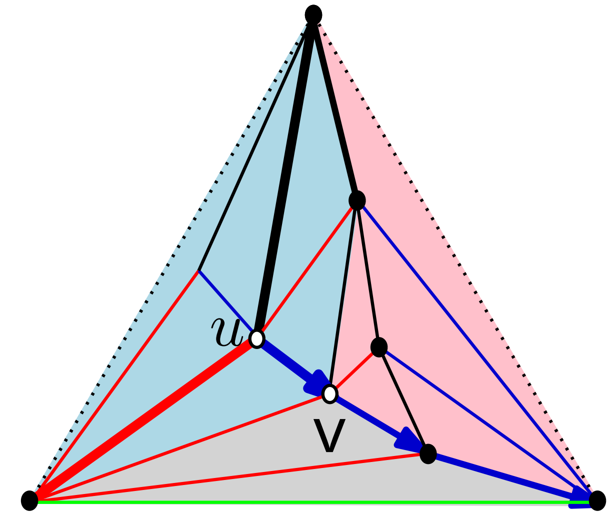
$R_i(u) \subseteq R_i(v)$



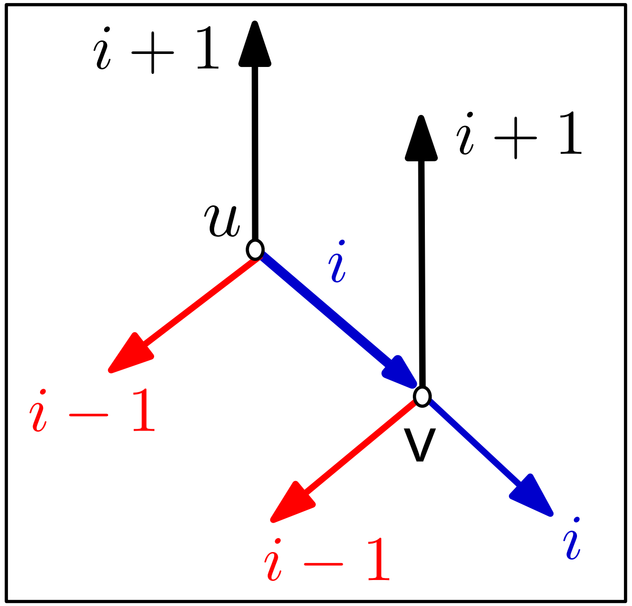
Paths and regions

Remarks: Let (u,v) of color i oriented from u to v

$$v \in P_i(u) \longrightarrow \begin{cases} v \in R_{i+1}(u) \\ v \in R_{i-1}(u) \\ u \in R_i(v) \end{cases}$$

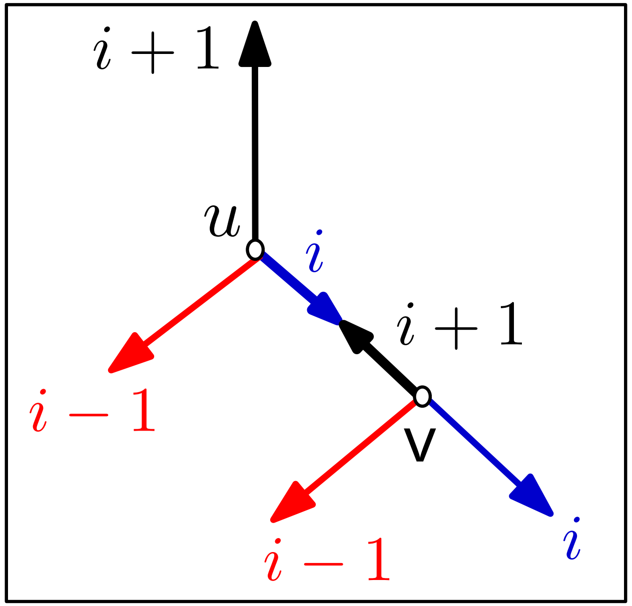


Case 1: (u,v) is unidirectional

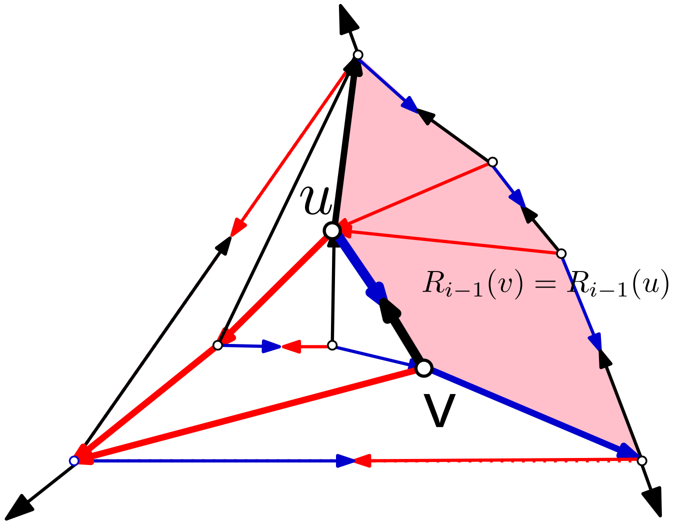


$$R_i(u) \subset R_i(v)$$
$$R_{i+1}(v) \subset R_{i+1}(u)$$
$$R_{i-1}(v) \subset R_{i-1}(u)$$

Case 2: (u,v) is bidirectional

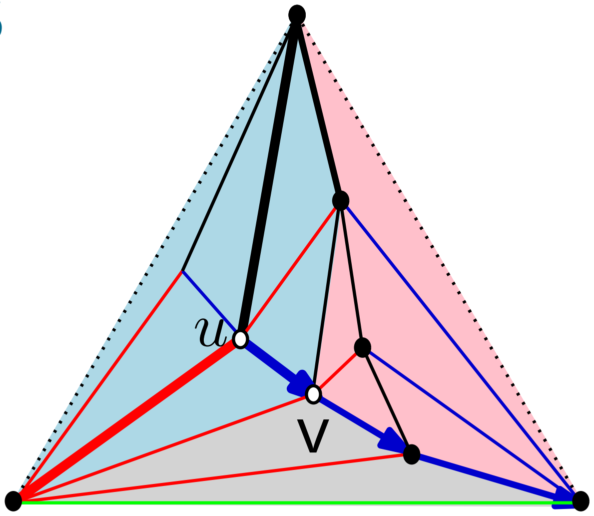


$$R_i(u) \subset R_i(v)$$
$$R_{i-1}(v) \subseteq R_{i-1}(u)$$
$$R_{i+1}(v) \subseteq R_{i+1}(u)$$



Regions and coordinates

Remarks: Let (u, v) of color i oriented from u to v



$$v =: \frac{|R_0(v)|}{|F|-1} x_0 + \frac{|R_1(v)|}{|F|-1} x_1 + \frac{|R_2(v)|}{|F|-1} x_2 =$$
$$= \frac{v_0}{|F|-1} x_0 + \frac{v_1}{|F|-1} x_1 + \frac{v_2}{|F|-1} x_2$$

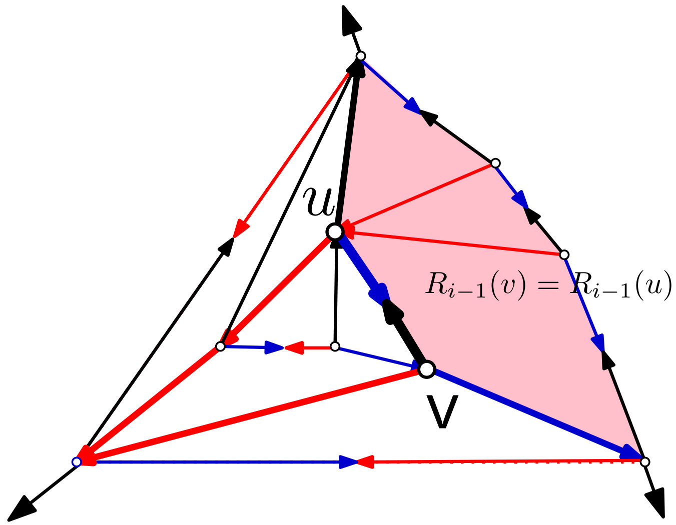
- $R_i(u) \subseteq R_i(v)$ → $|R_i(u)| \leq |R_i(v)|$ → $u_i \leq v_i$

v $(5, 6, 2) := (v_0, v_1, v_2)$

u $(7, 3, 3) := (u_0, u_1, u_2)$

- $v_0 + v_1 + v_2 = f - 1$

- $R_i(u) \subset R_i(v)$, $R_{i+1}(v) \subset R_{i+1}(u)$, $R_{i-1}(v) \subset R_{i-1}(u)$ → $\begin{cases} u_i < v_i \\ u_{i+1} > v_{i+1} \\ u_{i-1} > v_{i-1} \end{cases}$



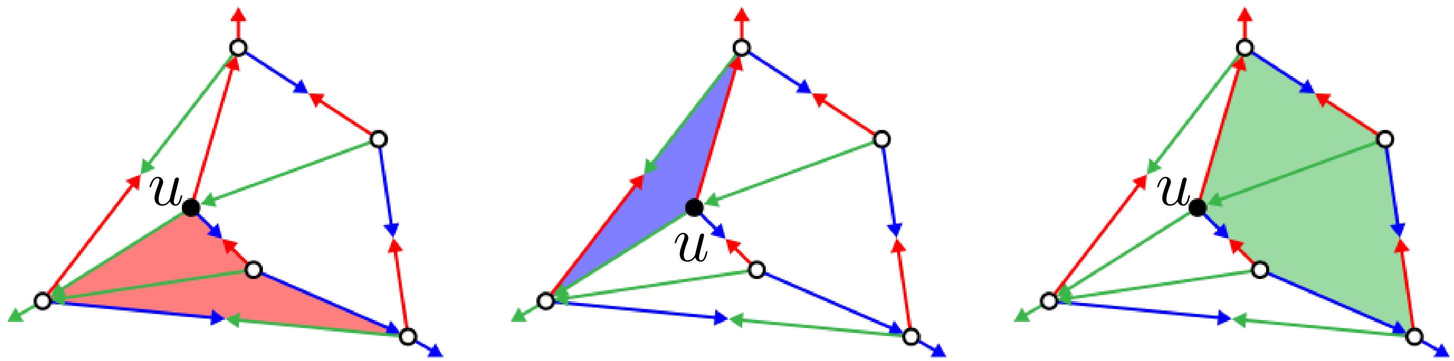
- For every edge (u, v) there are some indices $i, j \in \{0, 1, 2\}$ s.t.

$$u_i < v_i$$
$$u_j > v_j$$

Face counting algorithm

DEF. For a vertex v of M, denote:

- $P_i(v)$ = directed path in T_i to the root v_i,
- $R_i(v)$ = region bounded by the two paths $P_{i-1}(v)$ and $P_{i+1}(v)$,
- $r_i(v)$ = number of faces in region $R_i(v)$.



$r_1(u) = 2$

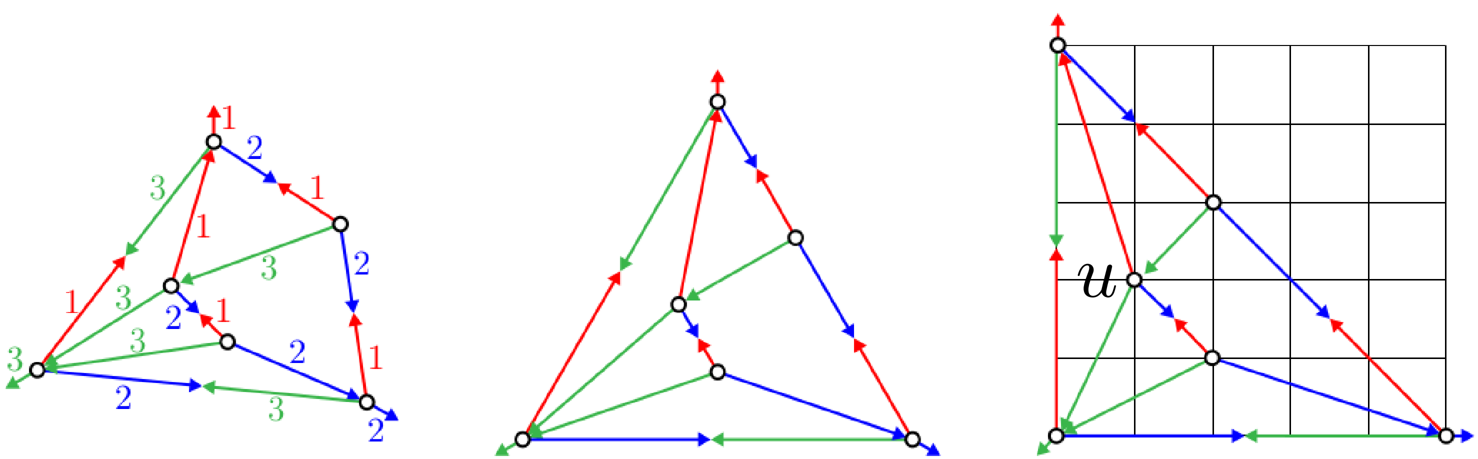
$r_2(u) = 1$

$r_3(u) = 2$

THM. The map

$$\mu : v \longmapsto \frac{1}{f-1}\left(r_1(v)\cdot \boldsymbol{p}_1 + r_2(v)\cdot \boldsymbol{p}_2 + r_3(v)\cdot \boldsymbol{p}_3\right)$$

defines a straightline embedding of M in the plane where all faces are convex.



$u = (1, 2)$

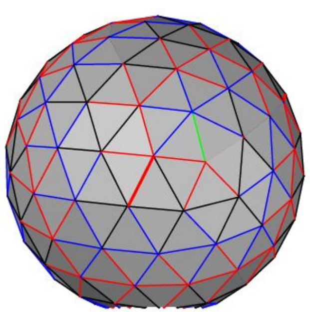
MPRI 2-38-1: Algorithms and combinatorics for geometric graphs

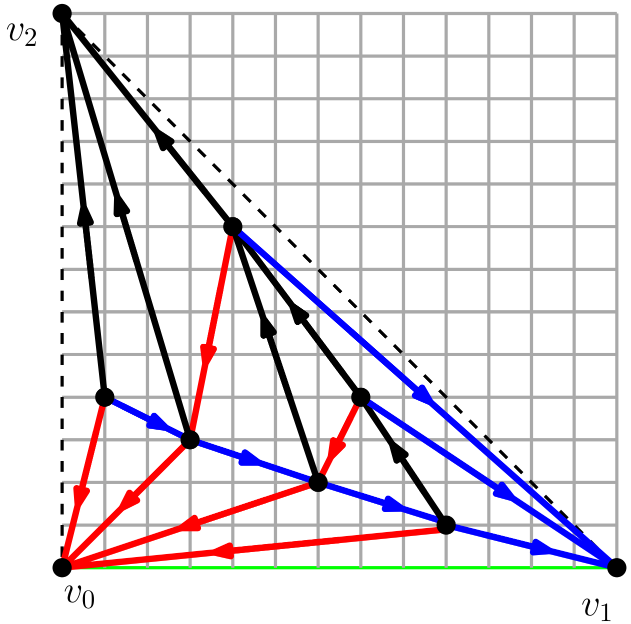
Lecture 6 - part II

Schnyder woods and orthogonal surfaces

october 23, 2024

Luca Castelli Aleardi





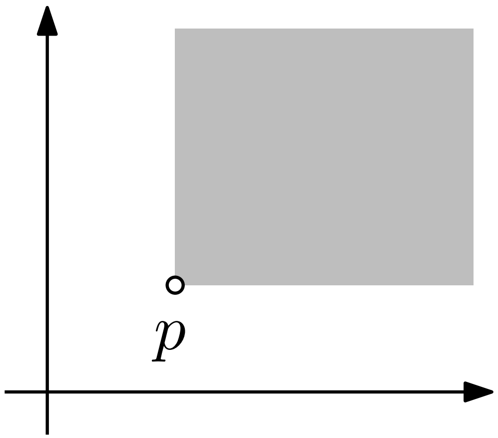
Schnyder woods and orthogonal surfaces

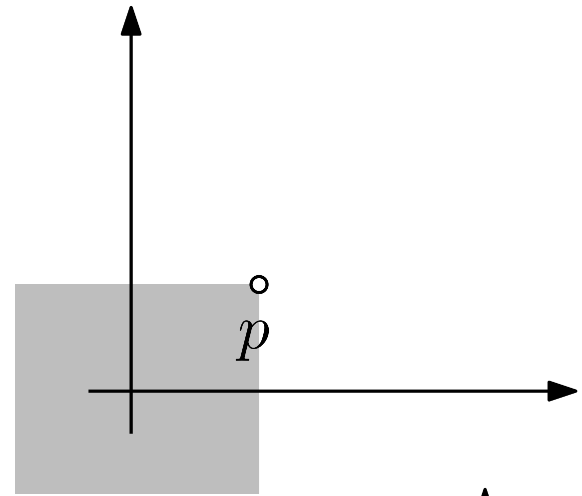
Orthogonal surfaces and elbow geodesics

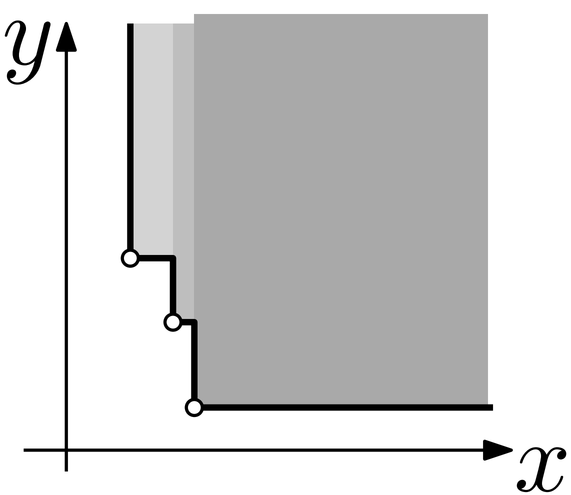
Dominance order $(\mathbf{u}, \mathbf{v} \in \mathbb{Z}^3)$: $\quad \mathbf{u} \leq \mathbf{v} \quad iff \quad u_i \leq v_i, \ \forall i = 0, 1, 2$

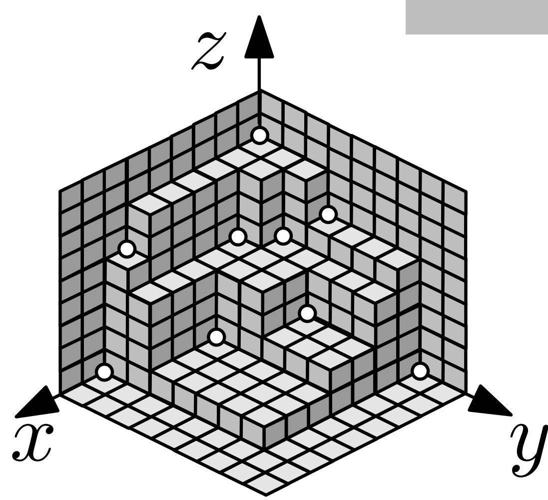
$\triangle_p :=$ cone dominating $p \in \mathbb{R}^3$

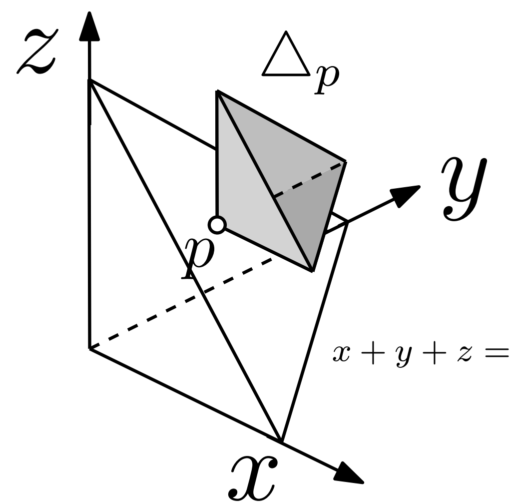
$\nabla_p :=$ cone dominated by $p \in \mathbb{R}^3$











Let $V \subset \mathbb{Z}^3$ be an **antichain**

(elements are pairwise incomparable)

Orthogonal surface $S_V :=$ boundary of $\langle \mathcal{V} \rangle$

$$\langle \mathcal{V} \rangle := \{\alpha \in R^3 | \alpha \geq v, \text{ for some } v \in \mathcal{V}\} = \bigcup_v \triangle_v$$

Orthogonal surfaces and elbow geodesics

Dominance order $(\mathbf{u}, \mathbf{v} \in \mathbb{Z}^3)$

$$\mathbf{u} \leq \mathbf{v} \quad iff \quad u_i \leq v_i, \ \forall i = 0, 1, 2$$

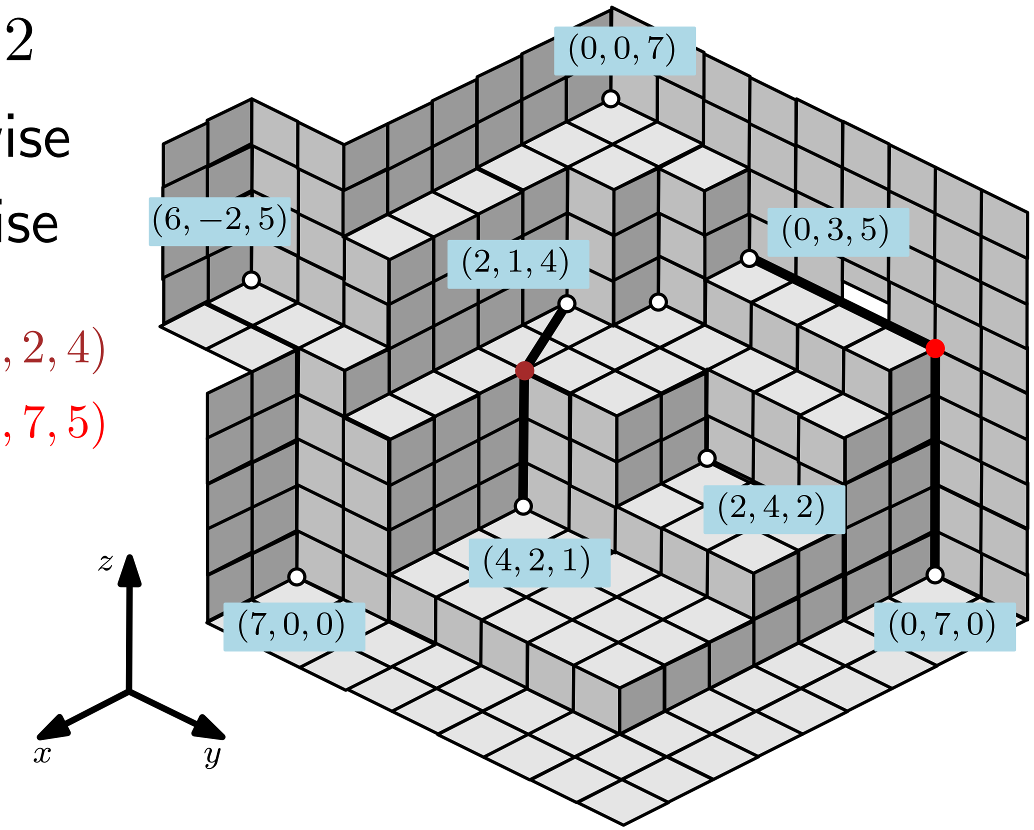
join $\mathbf{u} \vee \mathbf{v} :=$ maximum component-wise

meet $\mathbf{u} \wedge \mathbf{v} :=$ minimum component-wise

$\mathcal{V} = \{\ (0,0,7)\ \ (0,7,0)\ \ (7,0,0)\ \ (2,4,2)\ \ \ldots\ \}$

$(4,2,1) \vee (2,1,4) = (4,2,4)$

$(0,7,0) \vee (0,3,5) = (0,7,5)$



$\langle \mathcal{V} \rangle := \{\alpha \in R^3 | \alpha \geq v,$ for some $v \in \mathcal{V}\}$

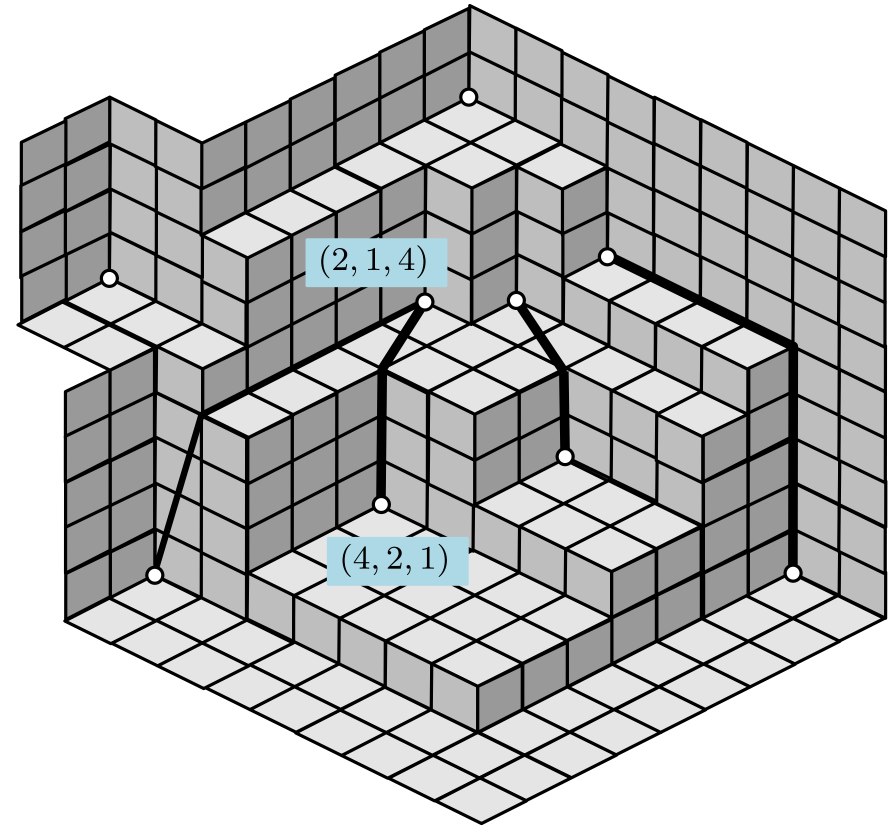
Orthogonal surface $S_V :=$ boundary of $\langle \mathcal{V} \rangle$

Let $V \subset \mathbb{Z}^3$ be an **antichain**

(elements are pairwise incomparable)

Orthogonal surfaces and elbow geodesics

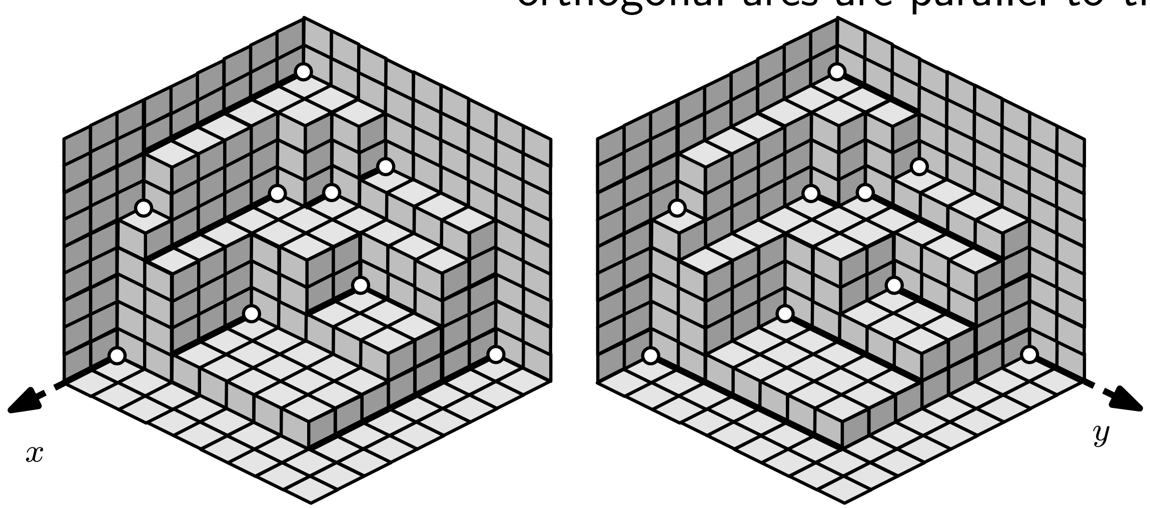
$$(4,2,1) \wedge (2,1,4) = (4,2,4)$$
$$(0,7,0) \wedge (0,3,5) = (0,7,5)$$

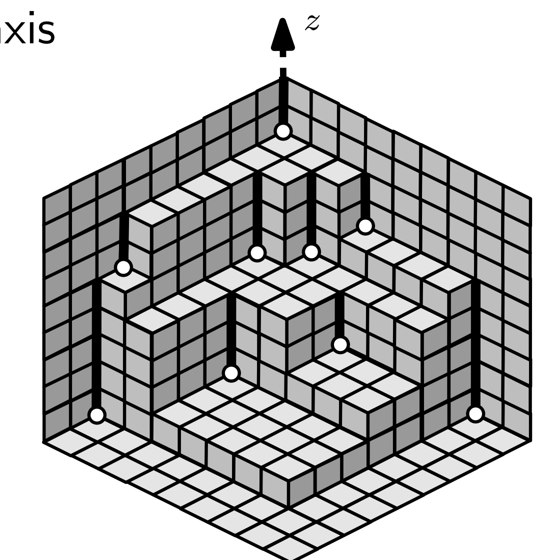


elbow geodesic of u and v:
the union of the two line segments $(u, u \vee v)$ and $(u \vee v, v)$

- every $v \in S_V$ has three orthogonal arcs (parallel to each axis)
- every elbow geodesic contains at least one bounded orthogonal arc

orthogonal arcs are parallel to the 3 axis





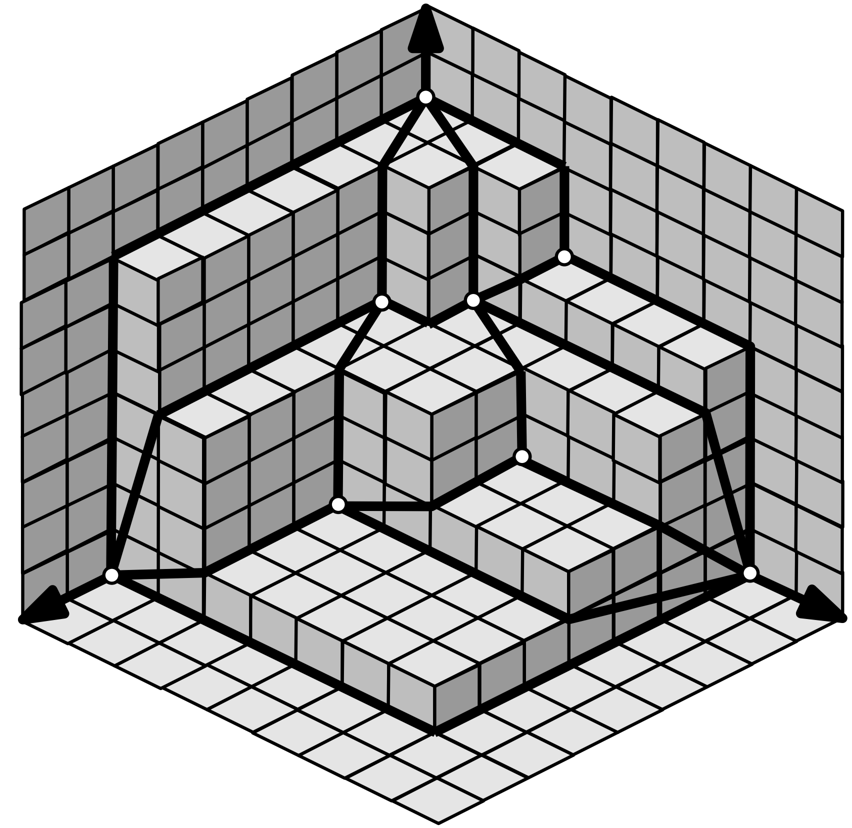
Orthogonal surfaces and elbow geodesics

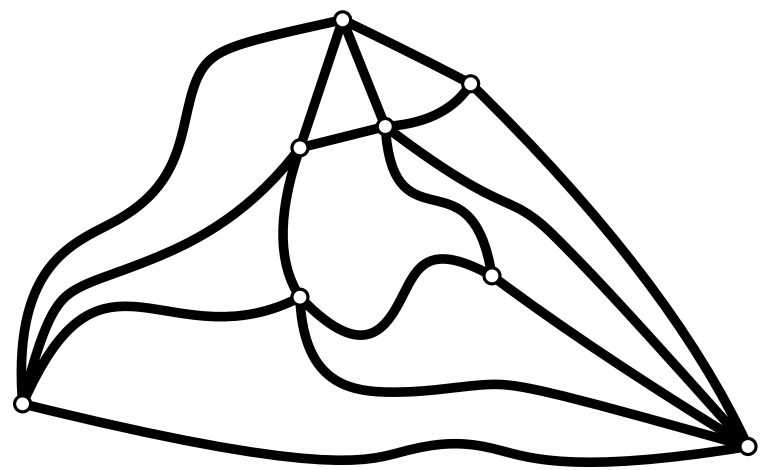
A **geodesic embedding** of a planar map G:
a drawing of G on $S_{\mathcal{V}}$ s.t.

(G1) The vertices of G correspond to the points of $S_{\mathcal{V}}$

(G2) every edge of G is drawn as an elbow geodesic on $S_{\mathcal{V}}$
Every bounded orthogonal arc of $S_{\mathcal{V}}$ is part of an edge of G

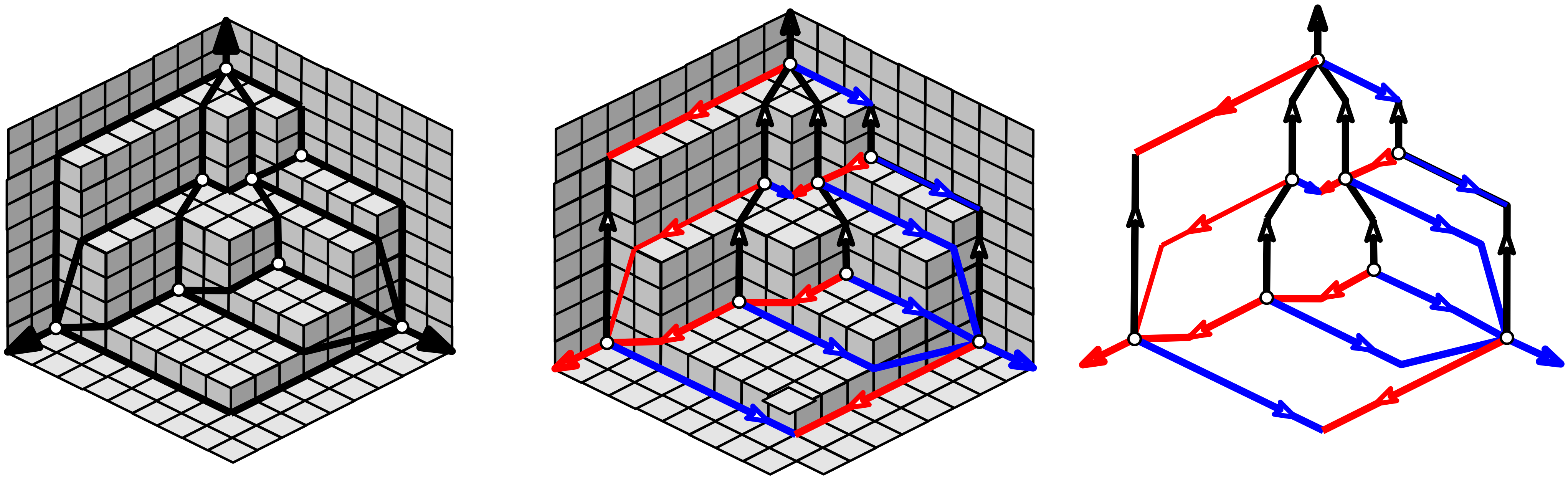
(G3) There are no edge crossings on $S_{\mathcal{V}}$





From geodesic embeddings to Schnyder woods

Thm: Consider a Schnyder wood of a planar map G and the corresponding set of vertex coordinates $\mathcal{V}$ (region vectors). The resulting drawing of G on $S_{\mathcal{V}}$ is a geodesic embedding (no crossings)



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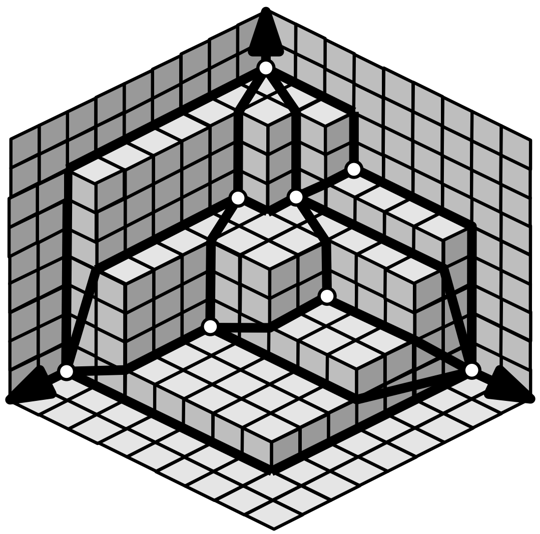
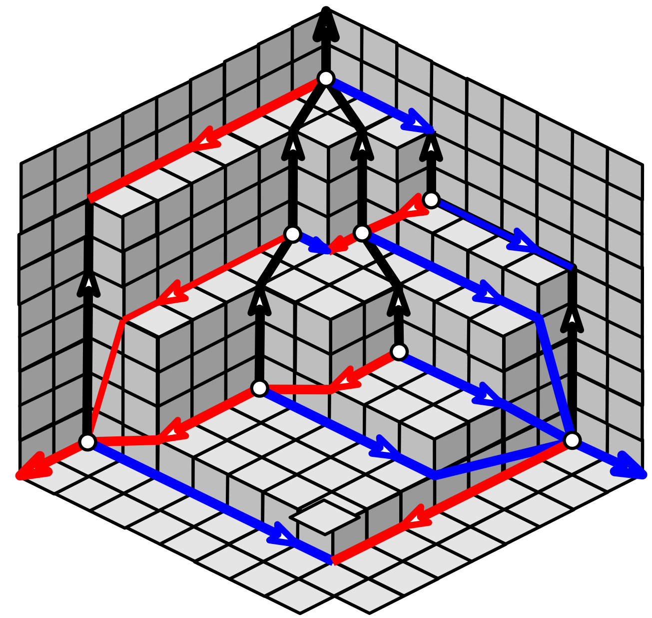
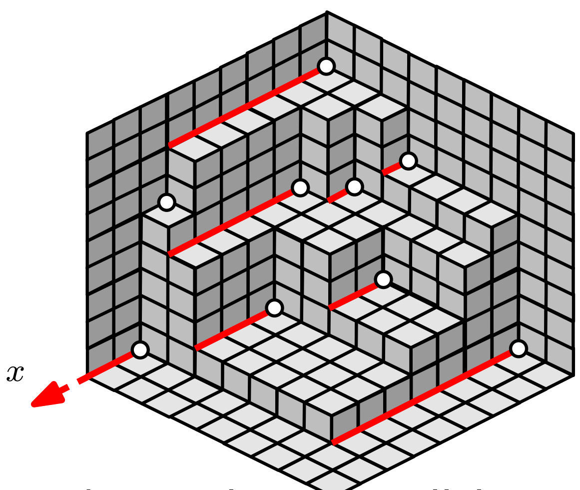
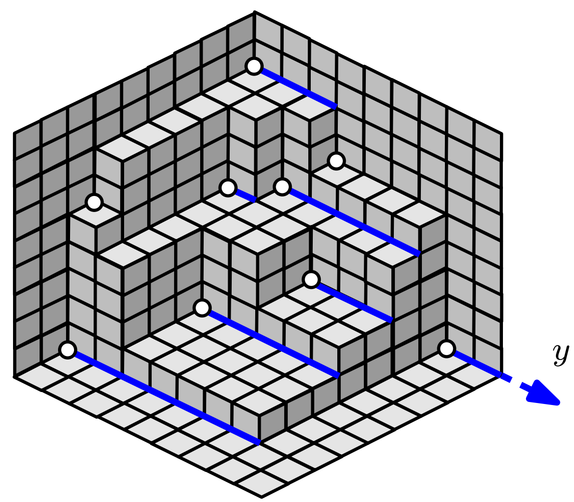


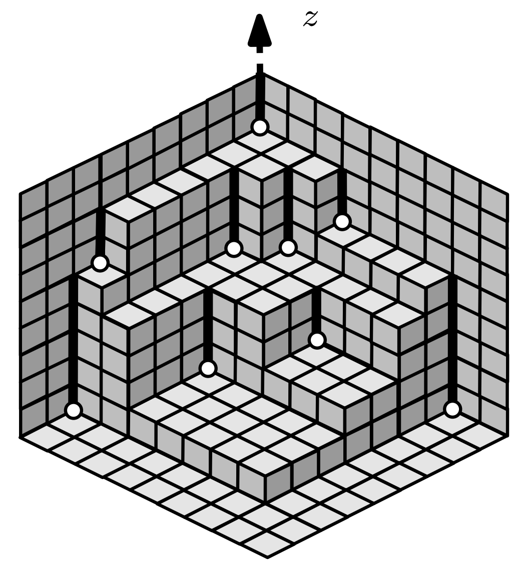




orthogonal arcs parallel to the x-axis are red (color 0)



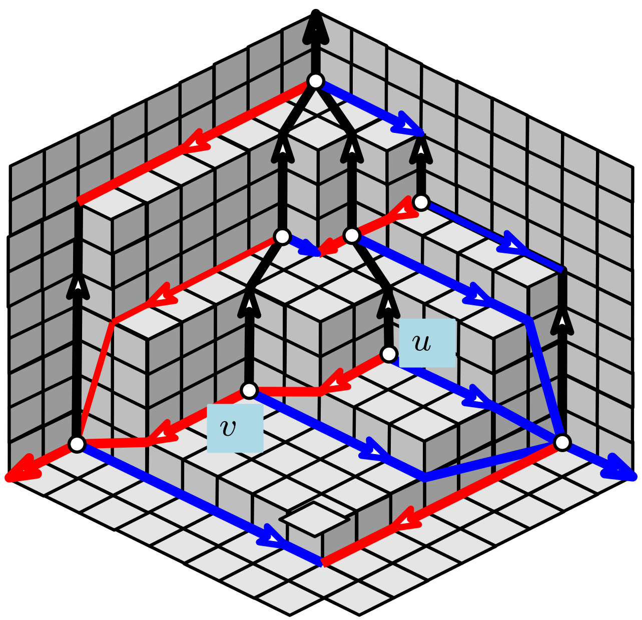
orthogonal arcs parallel to the y-axis are blue (color 1)



orthogonal arcs parallel to the z-axis are black (color 2)

From geodesic embeddings to Schnyder woods

Thm: The edge orientation corresponding to a geodesic embedding is a Schnyder wood

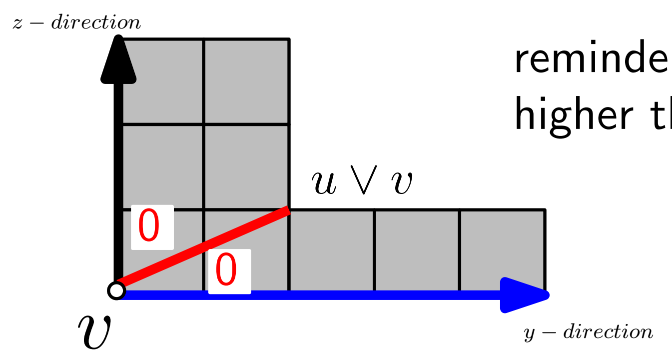


in the example
$u \vee v = (v_0, u_1, u_2)$

Claim 1: The local Schnyder condition (W3) is valid

- Every vertex has 3 outgoing edges (one for each color): the three orthogonal arcs (by construction)
- Let us consider an edge $\{u = (u_0, u_1, u_2), v = (v_0, v_1, v_2)\}$ incident at v in the sector parallel to the vertical yz-plane

The edge $\{u, v\}$ contains the orthogonal arc $(u \vee v, u)$ parallel to the x-direction and lying in the same horizontal plane of u: its color must be red (color 0), and its orientation is outgoing from u.



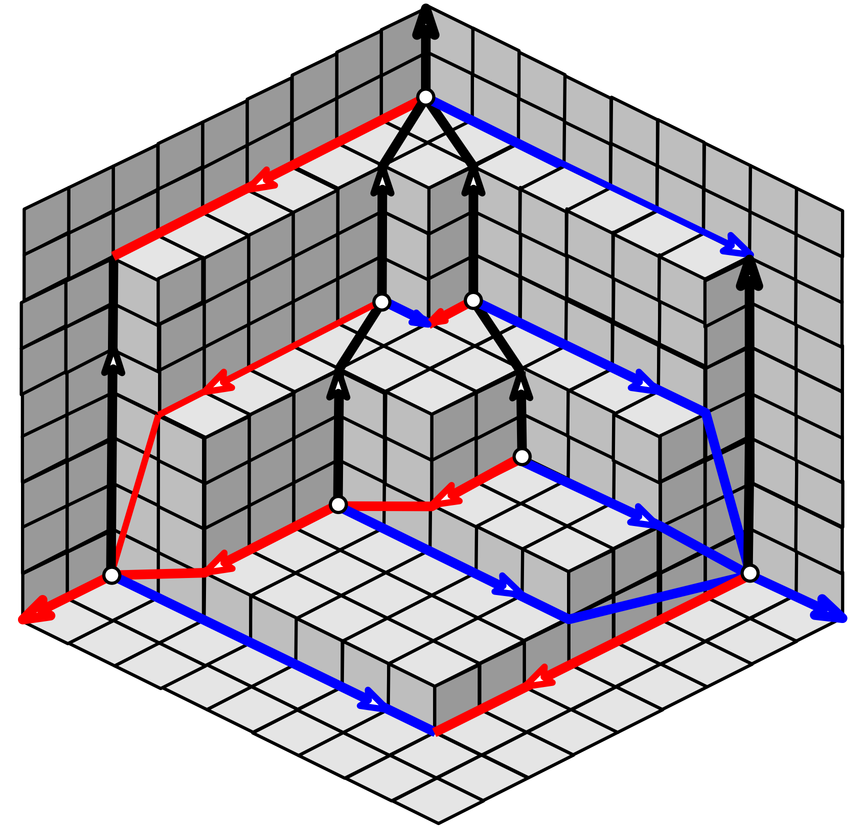
reminder: the join $u \vee v$ is equal or higher than u and v (in every direction)

Claim 2: condition (W4) of the definition is valid

Remark: a path of edges of color i lead to increasing coordinates in i-direction → (W4) no cycles

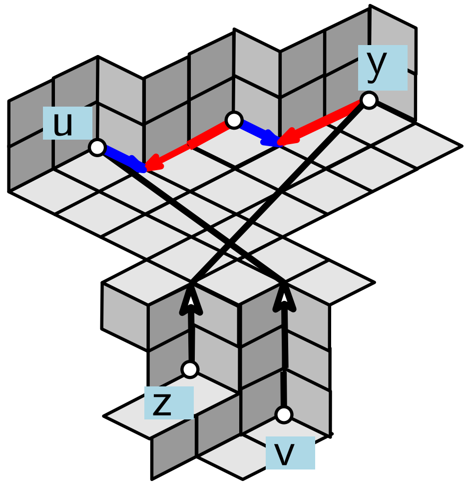
Geodesic embeddings are planar drawings

Thm: Consider a Schnyder wood of a planar map G and the corresponding set of vertex coordinates $\mathcal{V}$ (region vectors). The resulting drawing of G on $S_{\mathcal{V}}$ is a geodesic embedding (no crossings)



proof (assume there are edge crossings)

Fact 1: edge crossing are of the form (as orthogonal arcs cannot cross)

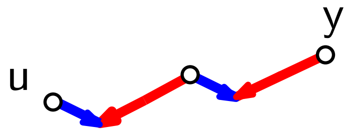


$u_0 > y_0$
$u_1 < y_1$
$z_0 > v_0$

Fact 2: edges (u, v) and (z, y) are of same color, lying on the same plane: $u_2 = y_2$ (in the example)

Fact 3: vertices u and y have the same z-coordinate

thus there is a bi-directed path P^* between u and y



Geodesic embeddings are planar drawings

Thm: Consider a Schnyder wood of a planar map G and the corresponding set of vertex coordinates $\mathcal{V}$ (region vectors). The resulting drawing of G on $S_{\mathcal{V}}$ is a geodesic embedding (no crossings)

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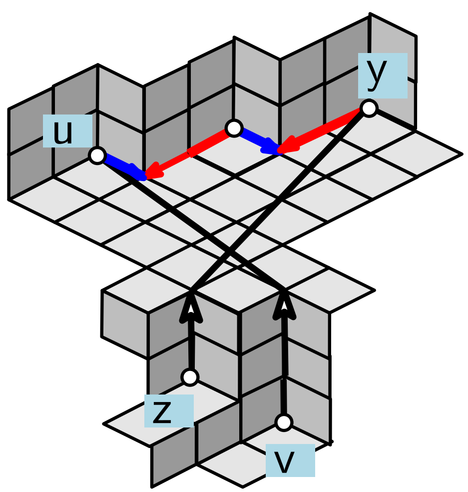
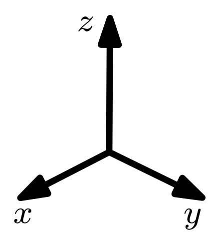
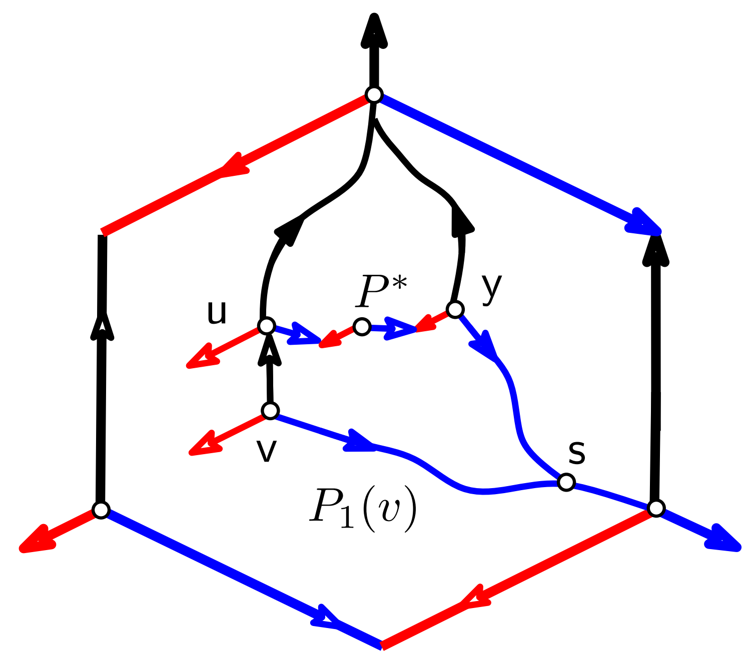


$u_0 > y_0$

$u_1 < y_1$

$z_0 > v_0$

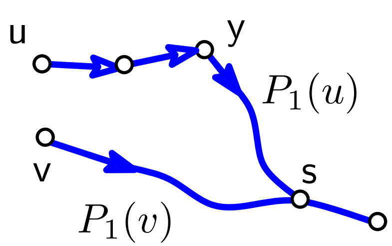


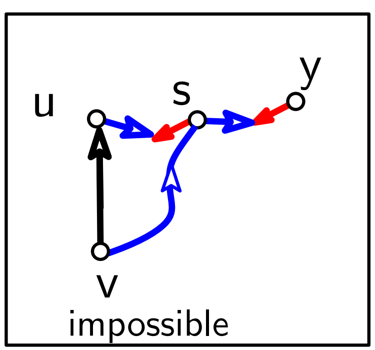


Let $P^* :=$ bi-directed path between u and y

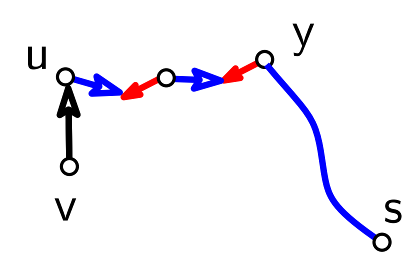
Let $s :=$ first vertex at the crossing of $P_1(u)$ and $P_1(v)$

Claim: s cannot belong to the path P^* ⟶ s belong to $P_1(v)$ and $s \neq y$





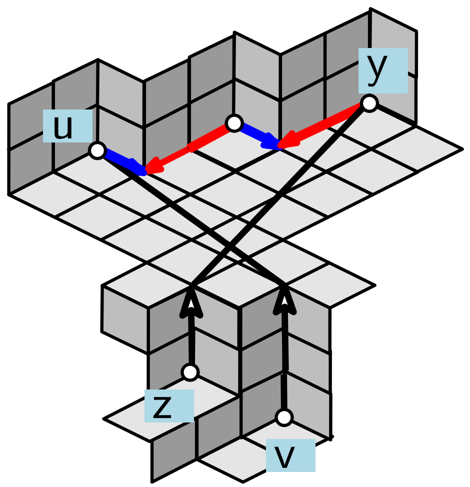
(there is a cycle in $T_2 \cup T_0^{-1} \cup T_1^{-1}$: violates previous theorem)



Geodesic embeddings are planar drawings

Thm: Consider a Schnyder wood of a planar map G and the corresponding set of vertex coordinates $\mathcal{V}$ (region vectors). The resulting drawing of G on $S_{\mathcal{V}}$ is a geodesic embedding (no crossings)

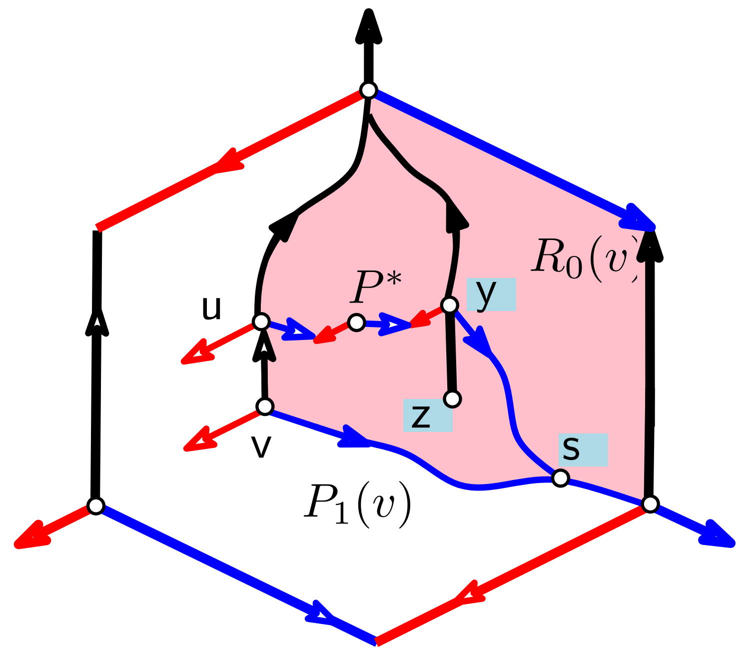
proof (assume there are edge crossings)



$u_0 > y_0$

$u_1 < y_1$

$z_0 > v_0$



Let $s :=$ first vertex at the crossing of $P_1(u)$ and $P_1(v)$

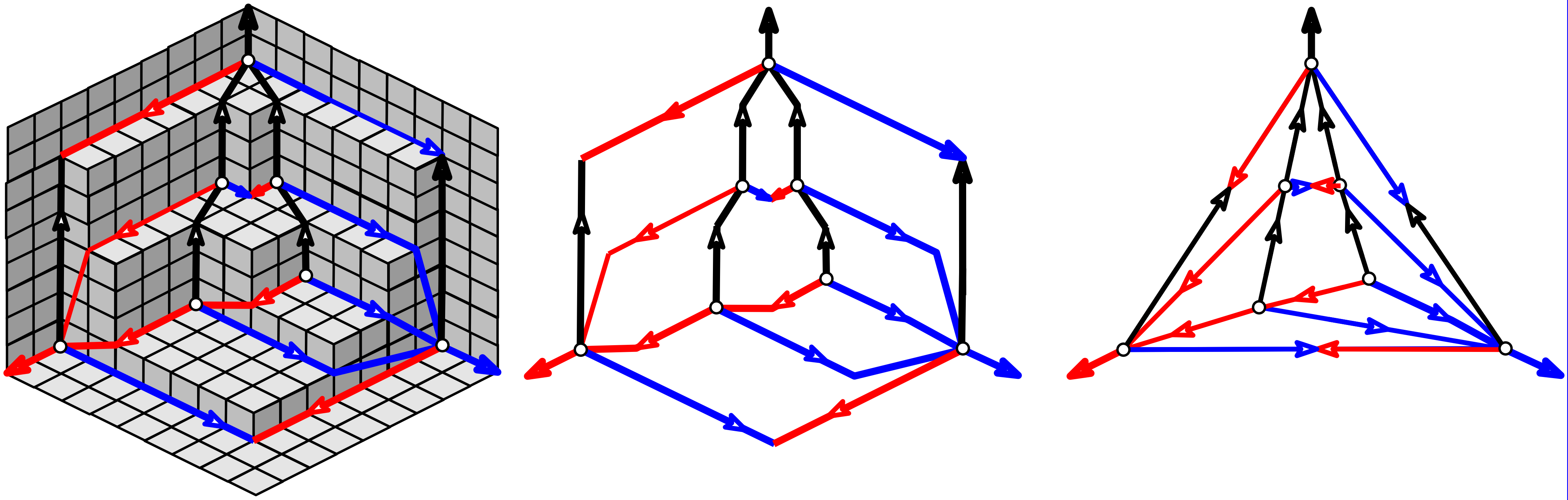
Remark: y is an inner vertex in the (red) region $R_0(v)$

by assumption (z, y) is an edge of G ⟶ (z, y) belong to $R_0(v)$ ⟶ z belong to $R_0(v)$

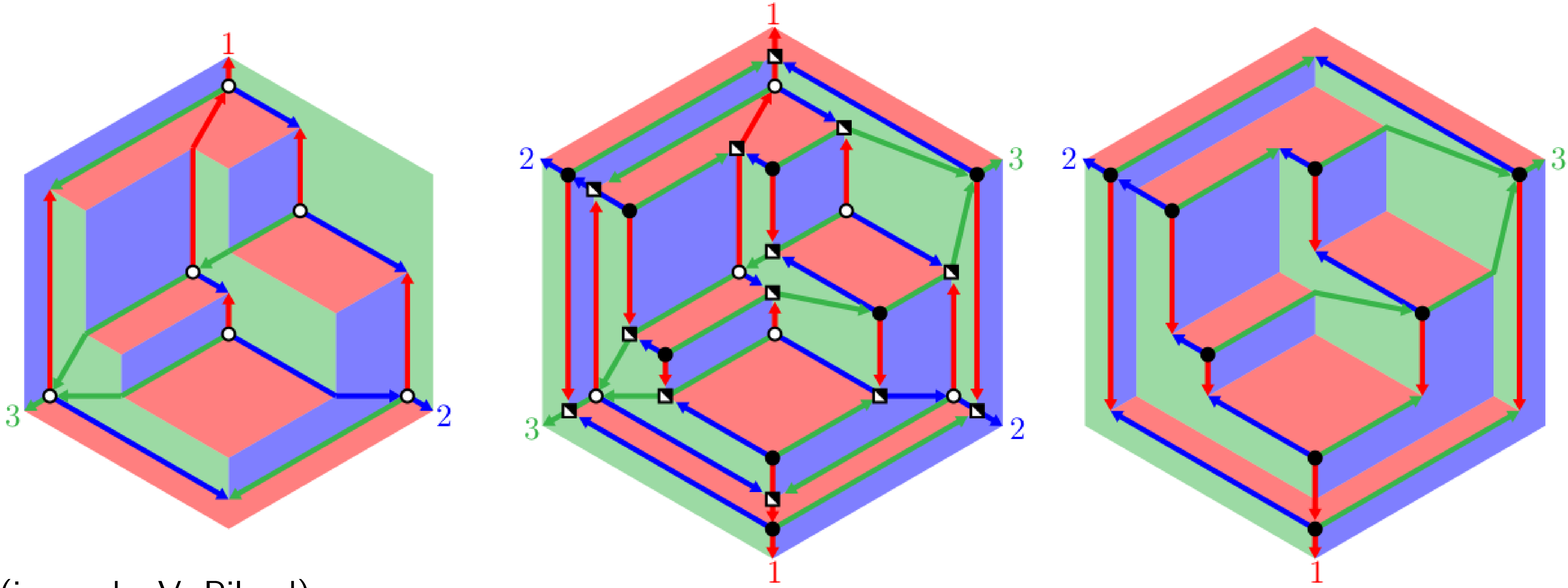
Since (z, y) belongs to $R_0(v)$ we have: $v_0 \geq z_0$ (contradiction)

From geodesic embeddings to straight-line planar drawings

Thm: Given a planar (3-connected) map G, the region counting algorithm leads to a planar straight-line drawing of G (no edge corssings). Moreover, the faces of G are convex.



PRIMAL-DUAL GEODESIC EMBEDDING



(image by V. Pilaud)

THM. Reversing the orientation, the same orthogonal surface admits a geodesic embedding of the map M, of its suspended dual map $M^\star$, and of its primal-dual map $\widetilde{M}$.

Algorithms and combinatorics for geometric graphs

Lecture 6: part III

Efficient algorithms on planar graphs

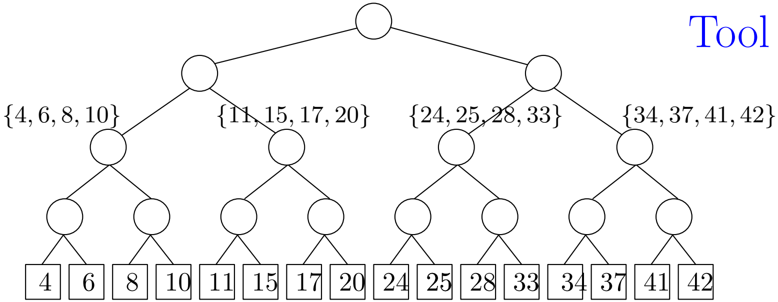
october 23, 2024

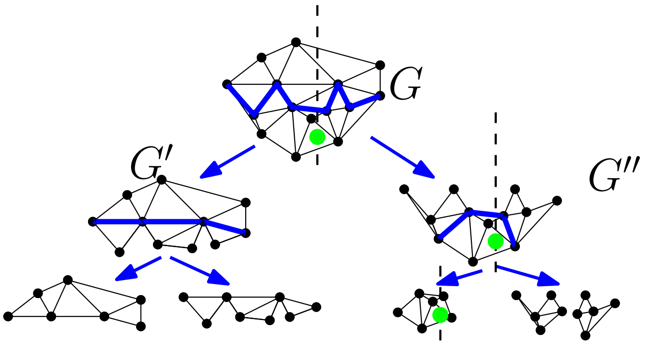
Luca Castelli Aleardi

Graph separators

Divide&Conquer for (planar) graphs: *Small Separators*

Tool for recursive decompositions of graphs





$$T(n) = C \cdot n + t(\frac{n}{2}) + t(\frac{n}{2}) = O(n \log n)$$

Many Algorithmic applications:

Approximation scheme for *Maximum Independent Set*

Graph Encoding: *compression schemes* and *compact representations*

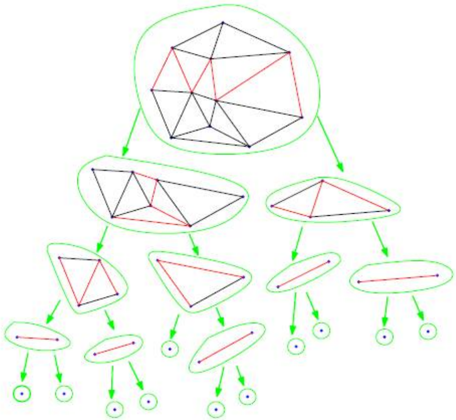
Graph Drawing: *spherical parameterizations*

Point Location (in optimal time)

. . . .

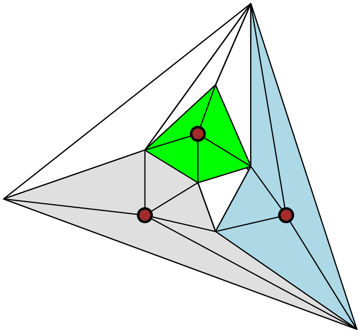
Divide&Conquer for (planar) graphs: *Small Separators*

Encoding planar graphs in $O(n)$ bits



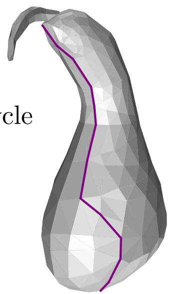
(image by Clément Maria)

Approximation scheme for *Maximum Independent Set*

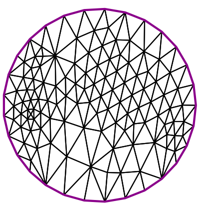


Graph Drawing: *spherical parameterizations*

separating cycle



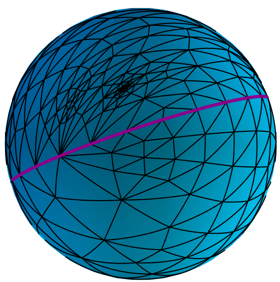
compute 2D layouts of each emisphere



south emisphere

Tutte barycentric layout (in 2D)

project on the sphere

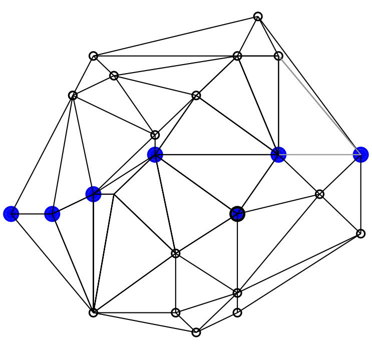


Graph separators: definition

Separators: definitions

Def
Given a graph $G = (V, E)$ with n vertices, an ε-separator is a partition (A, B, S) of the vertices such that:

- (ε-balance) every connected component of $G \setminus S$ has size at most εn
- (separation) there are no edges between A and B
- S is small: $|S| = o(n)$



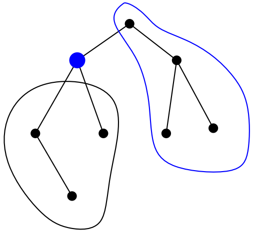
Separators: definitions

Def

Given a weighted graph $G = (V, E)$ *with* n *vertices and total weight* W, *a* separator *is a partition* (A, B, S) *of the vertices such that:*

- (balance) every connected component of $G \setminus S$ has weight at most $\frac{1}{2}W$
- (separation) there are no edges between A and B
- S is small: $|S| = O(\sqrt{n})$

Separators for trees

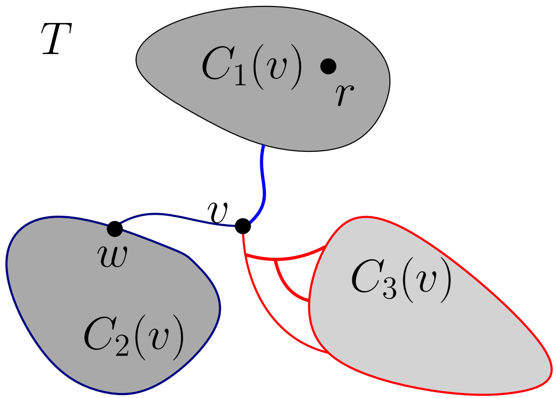


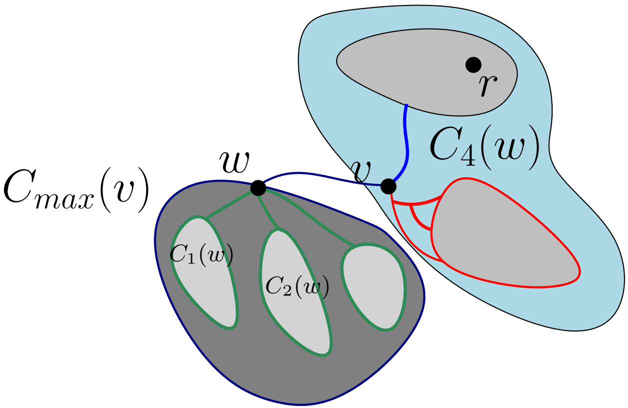
Lemma: A weighted tree T admits a separator consisting of a single vertex (computable in $O(n)$ time)

Proof:

First step: compute for each vertex $v \in T$ the weight of the sub-tree t_v rooted at v (total overall cost: linear time) → we know in $O(1)$ time the weight of each component $C_i(v)$

$$T \setminus v := C_1(v) \cup C_2(v) \cup \dots$$





Case 1: $W(C_i(v)) \leq \frac{1}{2}W \; \forall i$

`return` v

Case 2: $W(C_{max}(v)) > \frac{1}{2}W$

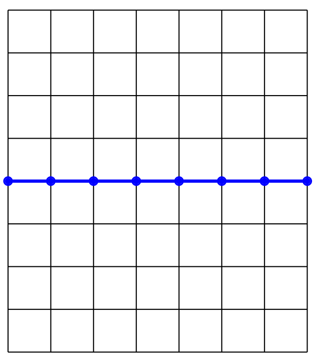
`move to the descendant` $w \in C_{max}(v)$

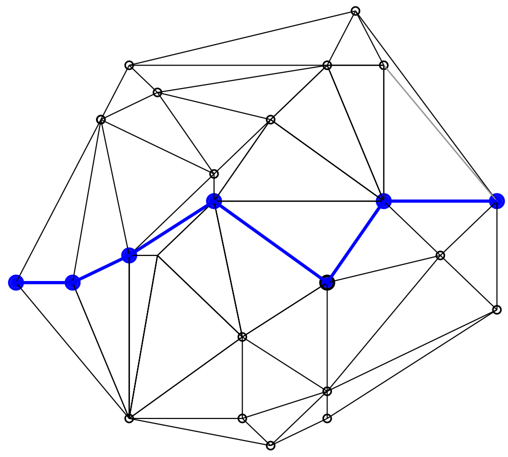
`restart from` w

Correctness: The algorithm visit each vertex at most once (we move from v to its descendant w)
The component $C_i(w)$ containig v is small: $W(C_i(w)) \leq W - W(C_{max}(v) \leq \frac{1}{2}$

Separators: definitions

examples: what about grid graphs? and general graphs? planar graphs?



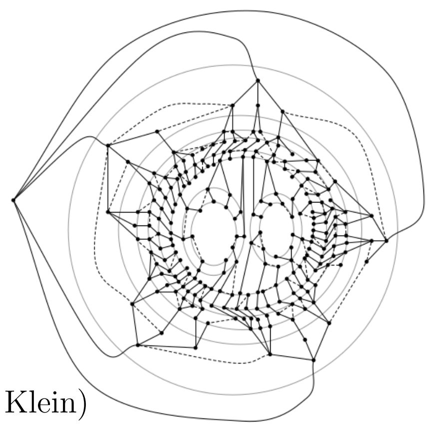


Planar Separators theorems

Thm (Lipton-Tarjan, '79)

Every planar graph with n vertices admits a $\frac{2}{3}$-separator of size at most $4\sqrt{n}$, that can be computed in linear time.

(purely combinatorial proof: perform a BFS traversal)



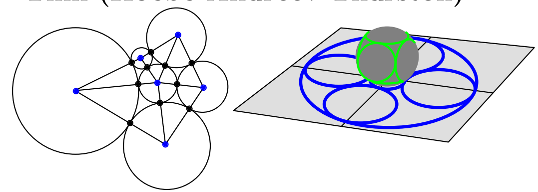
(Image by Klein)

Thm (Spielman and Teng)

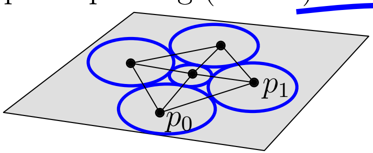
Every planar graph with n vertices admits a $\frac{3}{4}$-separator of size (in expectation) at most $2\sqrt{n}$.

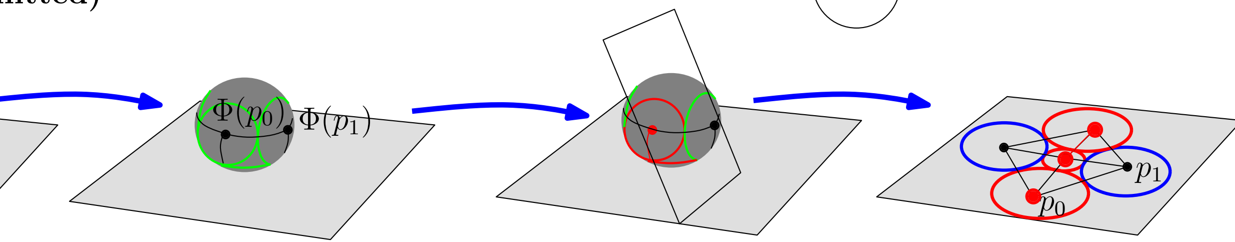
(geometric proof: omitted)

Thm (Koebe-Andreev-Thurston)



sphere packing (Koebe)





stereographic projection + Möbius transformation

Compute intersections with a random hyperplane passing through the origin

Planar Separators for graphs of small radius

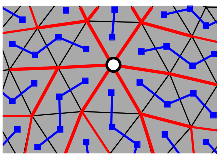
Theorem

Let G be a planar weighted graph with n vertices. Let U be a BFS spanning tree of T of depth at most d, rooted at r. Then we can compute in linear time a separator of size at most $3d+1$.

Proof (assume the graph is triangulated)

Construct a weighted dual graph G^*: each face (a dual vertex) get the weight of a vertex in G

each vertex assigns its weight to a unique incident face

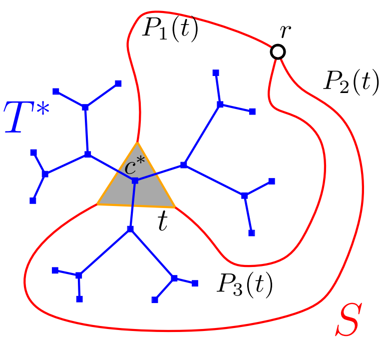


Define the spanning tree $T^* := G^* \setminus U^*$

Apply previous Lemma to T^*, getting a separating vertex c^*
(all component of $T^* \setminus c^*$ are small, of cost at most $\frac{1}{2}$)

computes three shortest paths $P_i(t)$ from t to the root vertex r

$$S := t \cup P_1 \cup P_2 \cup P_3$$



Claim 1: The separator S has at most $3d+1$ vertices

Claim 2: Each component C of $G \setminus S$ has weight at most $\frac{1}{2}$

since each component C^* of $T^* \setminus c^*$ has weight at most $\frac{1}{2}$
and the total (inner) weight of C is at most the weight of C^*

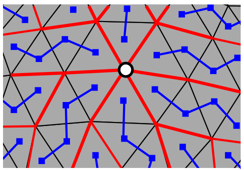
Planar Separators for graphs of small radius

Theorem

Let G be a connected planar graph with n vertices. Then we can compute in linear time a separator of size at most $O(\sqrt{n})$.

Proof: Compute a BFS spanning tree T of G, rooted at r

Claim 1:

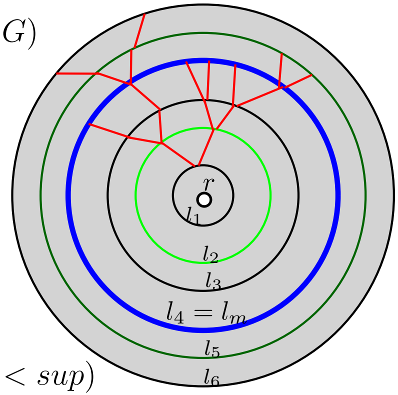


The set of vertices L_i at level l_i are a separator (splitting G)

define $l_m :=$ median level

$$\sum_{i<m} W(L_i) \leq \frac{1}{2}$$

$$\sum_{i>m} W(L_i) \leq \frac{1}{2}$$



define $l_{inf} :=$ largest level l_j $(j < i)$ such that $|L_{l_{inf}}| \leq \sqrt{n}$

define $l_{sup} :=$ smallest level l_j $(j > i)$ such that $|L_{l_{sup}}| \leq \sqrt{n}$

Remark:

the levels l_k between l_{inf} and l_{sup} are large: $|L_k| \geq \sqrt{n}+1$ (for $inf < k < sup$)

Claims:

- number of levels l_k between l_{inf} and l_{sup}: $l_{sup} - l_{inf} \leq \frac{n}{\sqrt{n}+1} < \sqrt{n}$
- The set of vertices $S' := L_{inf} \cup L_{sup}$ is small: $|S'| \leq 2\sqrt{n}$
- The connected components of $G \setminus S'$ which are large (weight larger than $\frac{1}{2}$) are between the levels l_{inf} and l_{sup}

(by definition $l_m :=$ median level)

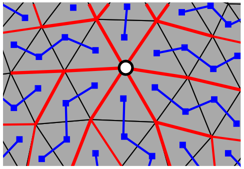
Planar Separators for graphs of small radius

Lemma

Let G be a connected planar graph with n vertices. Then we can compute in linear time a separator of size at most $O(\sqrt{n})$.

Proof: Compute a BFS spanning tree T of G, rooted at r

Claim 1: The set of vertices L_i at level l_i are a separator (splitting G)



define $l_m :=$ median level

$\sum_{i<m} W(L_i) \leq \frac{1}{2}$

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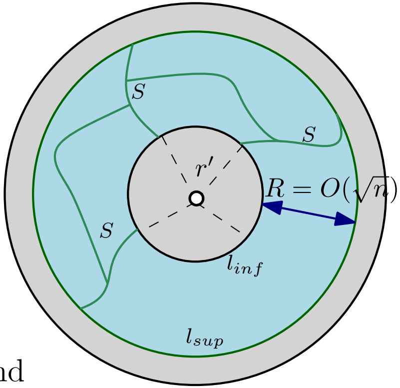


Last step:

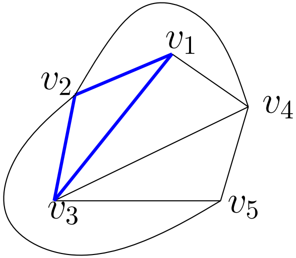
Take the graph G' induced by the vertices strictly between the levels l_{inf} and l_{sup}
G' is not necessarily connected: create a graph G'' by adding a dummy vertex r' and connecting it to vertices in l_{inf}

Apply previous Lemma to graph G': its radius is $O(\sqrt{n})$,
so the separator S has size $O(\sqrt{n})$

`return` $L_{l_{inf}} \cup L_{l_{sup}} \cup S$

Graph separators: algorithmic applications

(classical) Graph representations



adjacency matrix

$$A_G[i,j] = \begin{cases} 1 & v_i \text{ adjacent } v_j \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$O(n^2)$ bits

Adjacency list (and its variants)

neighbors in arbitrary order — d_i, $O(n \log n)$ bits

	d_i				
v_1	3	2	3	4	
v_2	4	1	4	5	3
v_3	4	5	4	1	2
		...			

sorted neighbors — $O(n \log n)$ bits

3	2	3	4	
4	1	3	4	5
4	1	2	4	5
	...			

difference encoding — $O(n \log n)$ bits

3	1	1	1	
4	-1	2	1	1
4	-2	1	2	1
	...			

difference encoding — $O(n \log n)$ bits

d_i	sign	positive differences			
3	1	1	1	1	
4	0	1	2	1	1
4	0	2	1	2	1
...	...	...			

Encoding of planar graphs in $O(n)$ bits

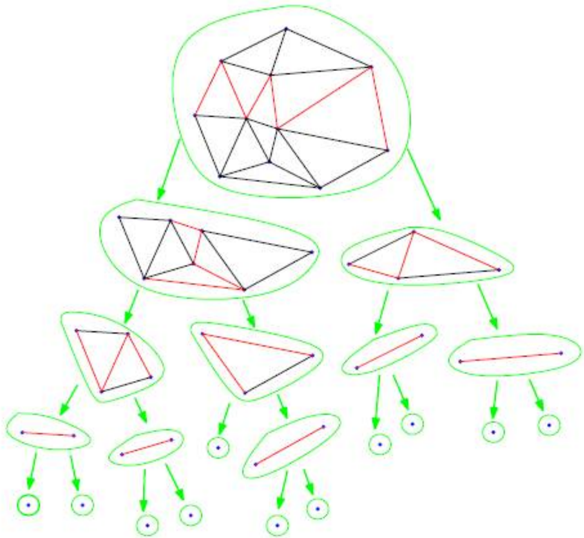
Thm

Any planar graph with n vertices can be encoded with at most $O(n)$ bits.

Solution: use difference encoding of adjacency lists + separators

this time we get $O(n)$ bits

Why does it work? Because vertices which are "close" in the graph get "close indices"



d_i	sign	positive differences			
3	1	1	1	1	
4	0	1	2	1	1
4	0	2	1	2	1
...	...	...			

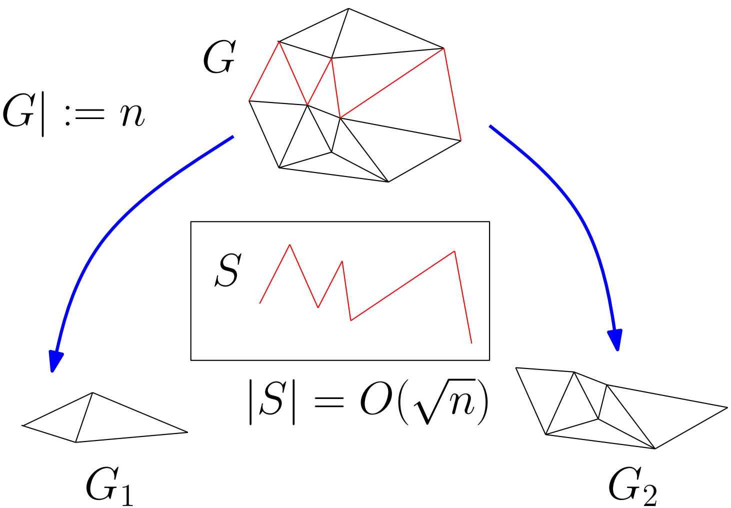
Encoding of planar graphs in $O(n)$ bits

Thm

Any planar graph with n vertices can be encoded with at most $O(n)$ bits.

Proof (overview):

Step 1: compute a recursive decomposition using (edge) separators



$|G| := n$

$|S| = O(\sqrt{n})$

$|G_1| \leq O(\alpha n)$ $|G_2| \leq O(\alpha n)$

Step 2: encode using adjacency lists with difference encoding

encode the edges in S as usual

$$size(S) = O(|S| \log |S|)$$

$$size(S) = O(\sqrt{|G|} \log |G|)$$

encode each piece G_i recursively

$$size(G) = size(S) + size(G_1) + size(G_2)$$

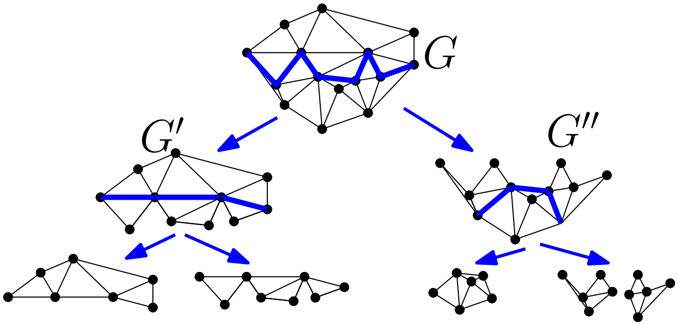
$$size(n) = C \cdot \sqrt{n \log n} + size(\alpha n) + size(\alpha n)$$

$$size(n) = O(n)$$

Recursive graph decompositions and hierarchical representations

Thm (Lipton Tarjan)

Given a planar graph G of size n and weight $W = 1$, and a parameter $0 \leq \varepsilon \leq 1$. Then it is possible to compute a separator $S \subset V$ of size at most $|S| = O(\sqrt{\frac{n}{\varepsilon}})$, such that each connected component of $G \setminus S$ has size at most ε. The computation time is $O(n \log n)$.



Trade-offs

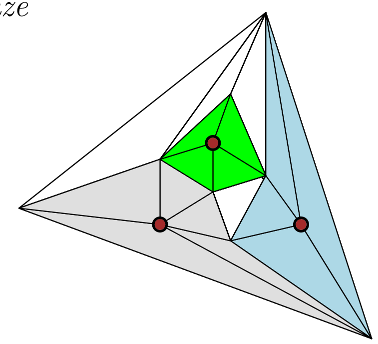
Separator size	Component size
$O(\sqrt{n})$	$O(n)$
$O(\sqrt{\frac{n}{\varepsilon}})$	$O(\varepsilon)$
$O(n^c)$	$O(n^{2-2c})$
$O(n^{\frac{2}{3}})$	$O(n^{\frac{2}{3}})$
$O(n^{\frac{3}{5}})$	$O(n^{\frac{4}{5}})$

Maximum Independent Set

Thm (approx scheme)

LetG be a planar graph on n vertices. Show that you can compute in $O(n \log n)$ time an approximated independent set of vertices I whose size, for large values of n, is closed to the size of a maximum independent set I_{opt}: $\frac{|I|-|I_{opt}|}{|I_{opt}|}$ tends to 0 with increasing n.

Def: *maximum independent set I_{opt}: a set of non adjacent vertices (no edges between pairs of vertices in I_{opt}) of maximal size*



Proof:

Maximum Independent Set

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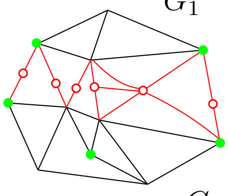


Thm (approx scheme)

Let$G$ be a planar graph on n vertices. Show that you can compute in $O(n \log n)$ time an approximated independent set of vertices I whose size, for large values of n, is closed to the size of a maximum independent set I_{opt}: $\frac{|I|-|I_{opt}|}{|I_{opt}|}$ tends to 0 with increasing n.

Proof: use uniform weights: $w(v_i) = \frac{1}{n}$

Idea: apply previous result with parameter $\varepsilon = \frac{\log\log n}{n}$

sub-components G_i have size $|G_i| \leq \frac{W(G_i)}{\frac{1}{n}} = O(\log\log n)$

the vertex separator S has size at most $|S| = O(\frac{n}{\sqrt{\log\log n}})$

Trick: in each G_i use `brute-force` to compute a maximal independent set (checking all subsets) ⟶ $|I_{opt}| - |I| \leq |S| = O(\frac{n}{\sqrt{\log\log n}})$

for each $G_i$of size n_i it takes: $O(n_i \cdot 2^{n_i})$ ⟶ in overall: $O(\frac{n}{\log\log n}(\log\log n) \cdot 2^{\log\log n}) = O(n \log n)$

Remark: planar graphs are 4-colorable ⟶ $|I_{opt}| \geq \frac{n}{4}$

$$\frac{|I|-|I_{opt}|}{|I_{opt}|} \leq \frac{O(n/\sqrt{\log\log n})}{n/4} = O(\frac{1}{\sqrt{\log\log n}})$$

Computing triangles and cliques in planar graphs

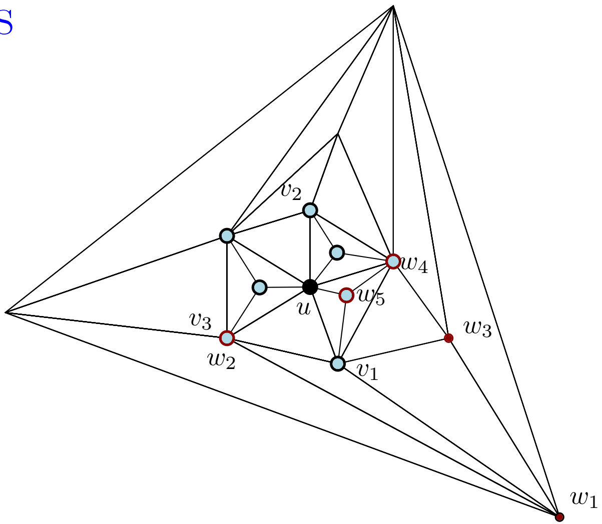
Counting triangles

Thm

Let G be a graph on n vertices and m edges. Then it is possible to count (or list) the triangles of G in $O(nm)$ time.

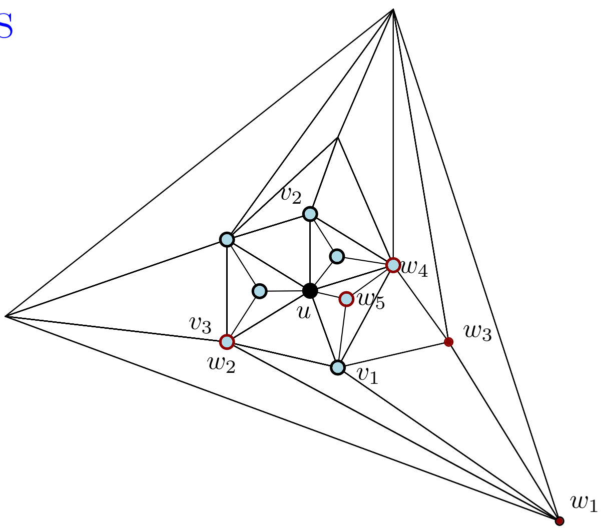
Proof:

procedure CountTriangles($G = (V, E)$)
  $Count := 0$;
  for each vertex $u \in V$
  do {
    mark all vertices which are neighbors of u in G;
    for each marked vertex $v \in V$
    do {
      for each vertex w which is a neighbor of v in G
        do if w is marked **then** $Count := Count + 1$;
      unmark vertex v;
    }
    $G := G \setminus \{u\}$; // vertex removal in $O(d_u)$ time
  }
  return $Count$;



triangle:= cycle of size 3 (complete graph on 3 vertices)

Counting triangles



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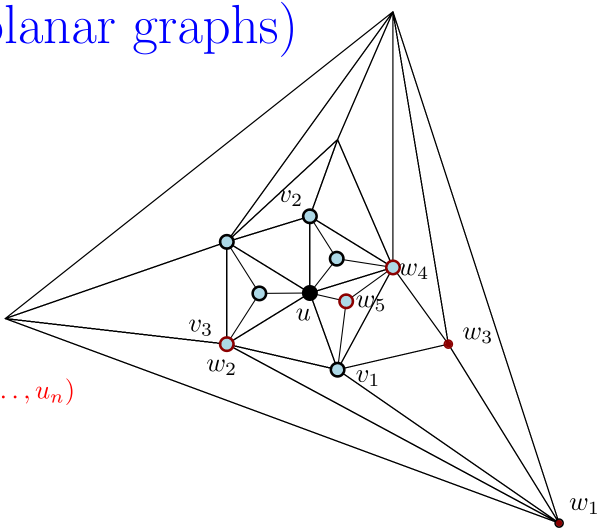
        do if w is marked **then** $Count := Count + 1$;
      unmark vertex v;
    $G := G \setminus \{u\}$; // vertex removal in $O(d_u)$ time
  return $Count$;

- each vertex v is marked at most $deg(v)$ times: each time the inner loop performs at most $deg(v)$ iterations: the cost per vertex is thus at most $(deg(v))^2$
- $\sum_{v \in V} \deg^2(v) \leq (\max_{v \in V} \deg(v)) \cdot (\sum_{v \in V} \deg(v)) \leq (|V| - 1) \sum_{v \in V} \deg(v) = O(|V||E|)$ □

Counting triangles in linear time (in planar graphs)

Thm

Let G be a planar graph on n vertices and m edges. Then it is possible to count (or list) the triangles of G in $O(n)$ time.



Proof:

procedure COUNTTRIANGLES($G = (V, E)$)
  $Count := 0$; order vertices of V according to non-increasing degree as $(u_1, \ldots, u_n)$
  for each vertex $u \in V$ // visit vertices according the computed order
  do
    mark all vertices which are neighbors of u in G;
    for each marked vertex $v \in V$
    do
      for each vertex w which is a neighbor of v in G
        do if w is marked **then** $Count := Count + 1$;
      unmark vertex v;
    $G := G \setminus \{u\}$; // vertex removal in $O(d_u)$ time
  return $Count$;

- for any edge $\{u, v\}$ for a pair of vertices u, v considered in the algorithm, we have $\deg(v) \leq \deg(u)$
- the time complexity becomes $\sum_{(u,v)\in E} \min(d_u, d_v)$

Claim (exercise, homework I)

Show that in a planar graph with n vertices we have:

$$\sum_{(u,v)\in E} \min\{\deg(u), \deg(v)\} \leq 18n$$

□

Counting 4-cliques in linear time (in planar graphs)

Thm

Let G be a planar graph on n vertices and m edges. Then it is possible to count (or list) all 4-cliques of G in $O(n)$ time.

Proof: [case analysis, exercise]

Hint: compute a BFS of G and partition the vertices into $k+1$ sets $\{V_0, V_1, \ldots, V_k\}$

$V_k :=$ vertices at the distance k from the root (seed) vertex

define $E_j :=$ set of edges $e = (u, v)$ s. t. $u \in V_{j-1}$ and $v \in V_j$

(an edge belongs to E_j if it is connecting two vertices on levels V_j and V_{j-1})

Claim 1:

- Consider a 4-clique $Q = \{u, v, w, x\}$ in G.
  Show that the four vertices u, v, w, x cannot all belong to the same level V_j.

Claim 2: consider a 4-clique $Q = \{u, v, w, x\}$ in G, and let j be a positive integer $\leq k$.

- assume $u \in V_{j-1}$ and $v, w, x \in V_j$. Show that for one of the tree vertices v, w, x the only incident edge lying in E_j has u has other extremity.

- assume $u, w, x \in V_{j-1}$ and $x \in V_j$. Show that the edges incident to x lying in E_j are exactly (u, x), (v, x) and (w, x).

- assume $u, v \in V_{j-1}$ and $w, x \in V_j$. Show that one of the vertices w, x has exactly two incident edges lying in E_j (whose other extremities are u and v).

Counting 4-cliques in linear time (in planar graphs)

Thm

Let G be a planar graph on n vertices and m edges. Then it is possible to count (or list) all 4-cliques of G in $O(n)$ time.

Proof: [case analysis, exercise]

Hint: compute a BFS of G and partition the vertices into $k+1$ sets $\{V_0, V_1, \dots, V_k\}$

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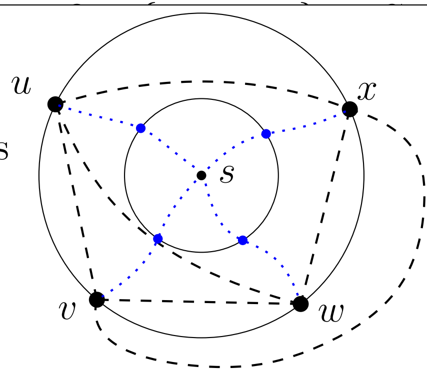
- Consider a 4-clique $Q = \{u, v, w, x\}$ in G.
  Show that the four vertices u, v, w, x cannot all belong to the same level V_j.

Cla

Claim 1:

Contracting the edges we get K_5



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