Algorithms and combinatorics for geometric graphs (Geomgraphs)

Lecture 5

Schnyder woods for 3-connected plane graphs

october 16, 2025

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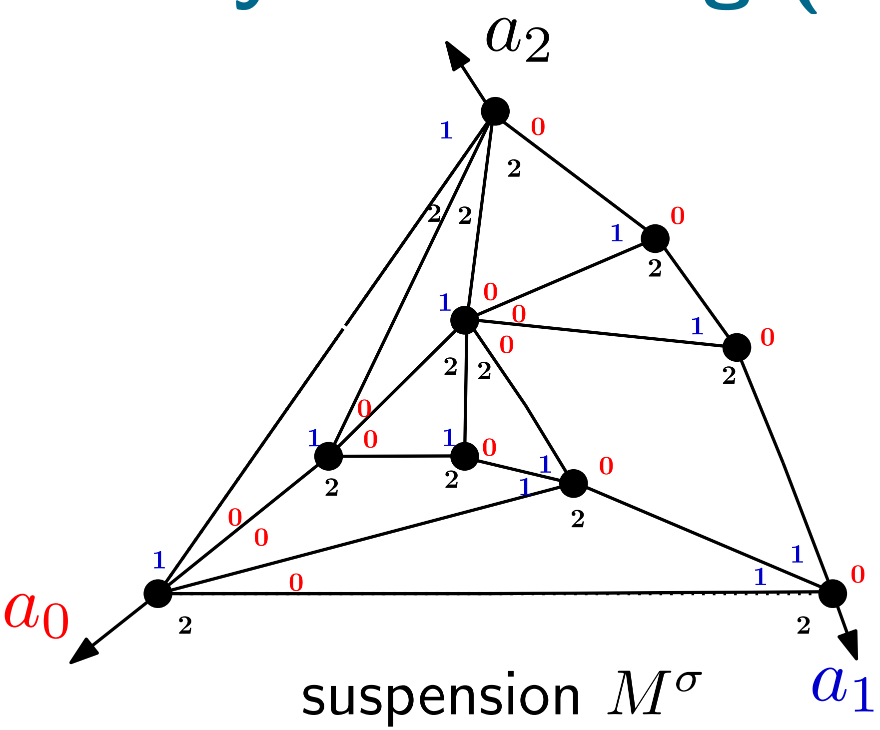
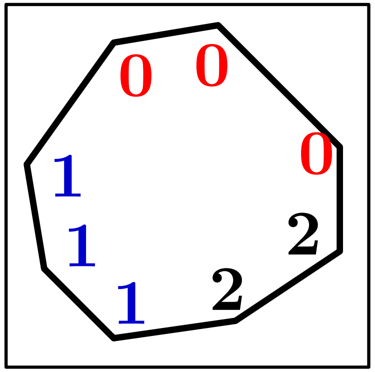
Schnyder woods

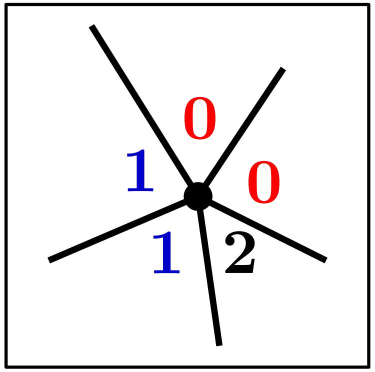
(definitions)

Schnyder labeling (3-connected maps): definition

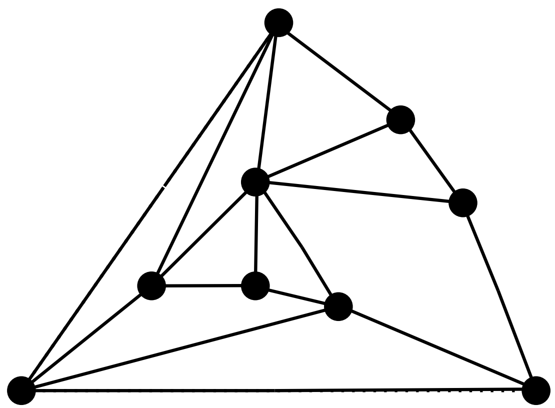


suspension M^σ





3-connected graphs [Felsner]



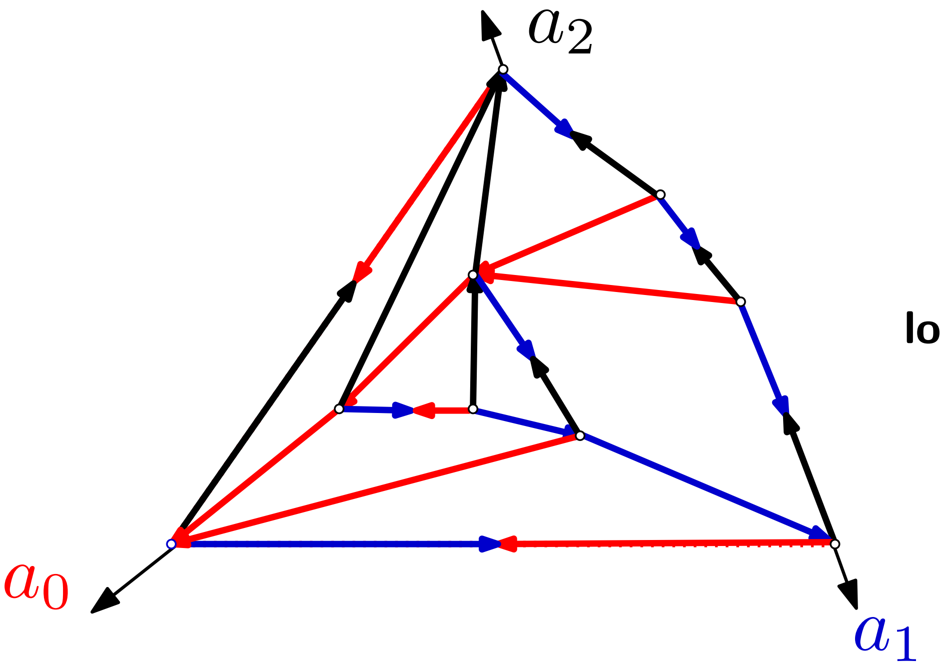
3-connect. map M

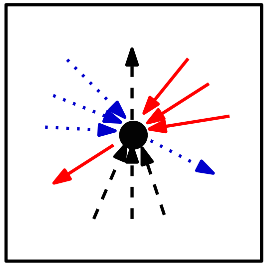
A1) the angles at a_i have labels $i+1$, $i-1$

A2) **rule for vertices:** at each vertex there are non-empty intervals of labels $0, 1$ and 2 (listed counter-clockwise)

A3) **rule for faces:** at each inner faces the angles define three non-empty intervals of labels $0, 1$ and 2 in ccw order. For the outer face the angles are listed clockwise.

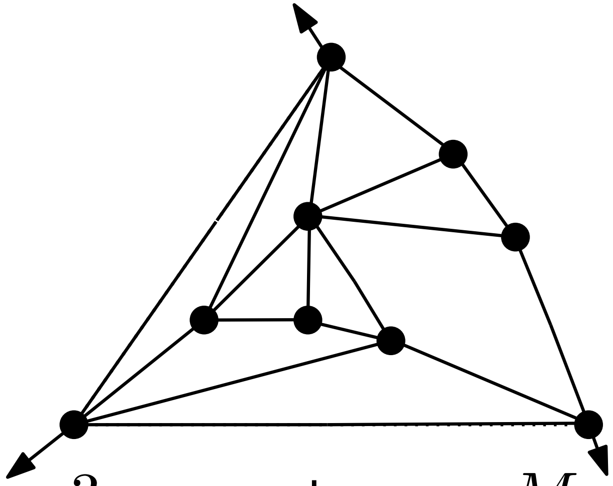
Schnyder woods (3-connected maps): definition





local Schnyder rule

3-connected graphs [Felsner]



3-connect. map M

W1) edges have one or two (opposite) orientations. If an edge 3 is bo-oriented than the two direction have distinct colors

W2) the edges at a_i are outgoing of color i

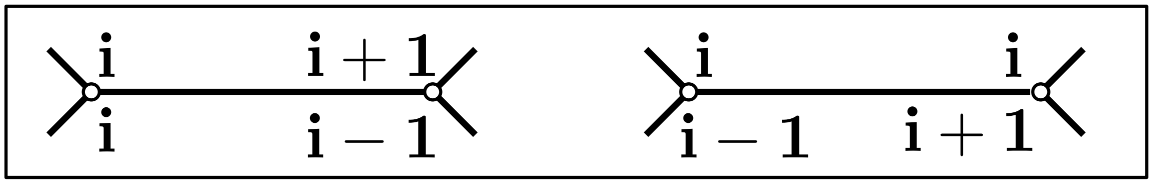
W3) **local rule for vertices:** at each vertex there are three outgoing edges (one in each color) satisfying the local Schnyder rule

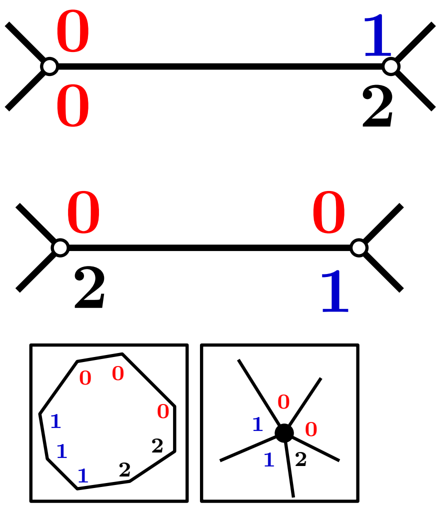
W4) there is no interior face whose boundary is a directed cycle in one color

Schnyder labelings: angles around edges

Lemma

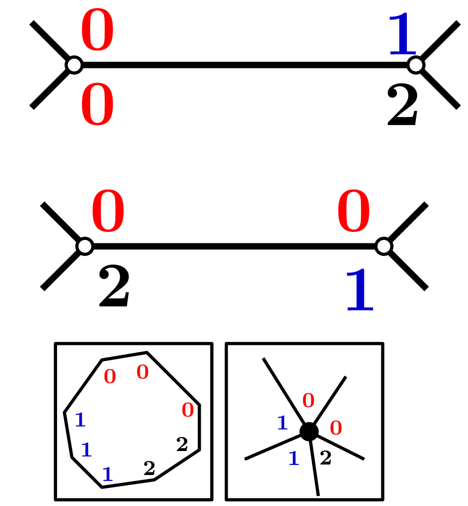
Given a Schnyder labeling of M^{σ}, the angles of each edges have colors $0, 1, 2$ and are of the following 2 types:





proof:

Schnyder labelings: angles around edges



Lemma

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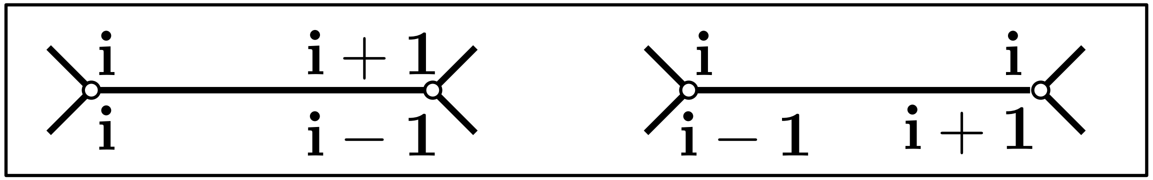
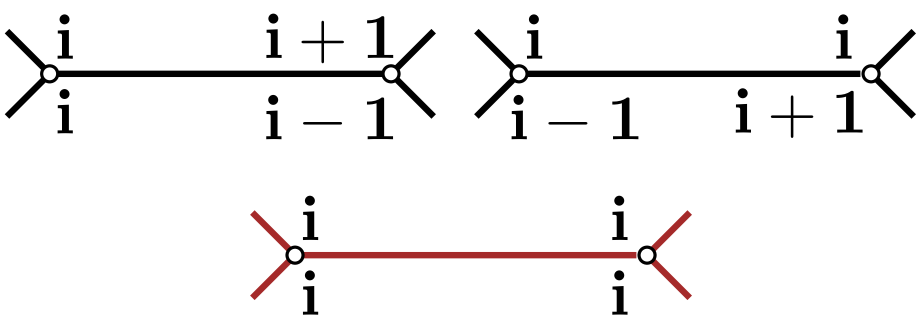
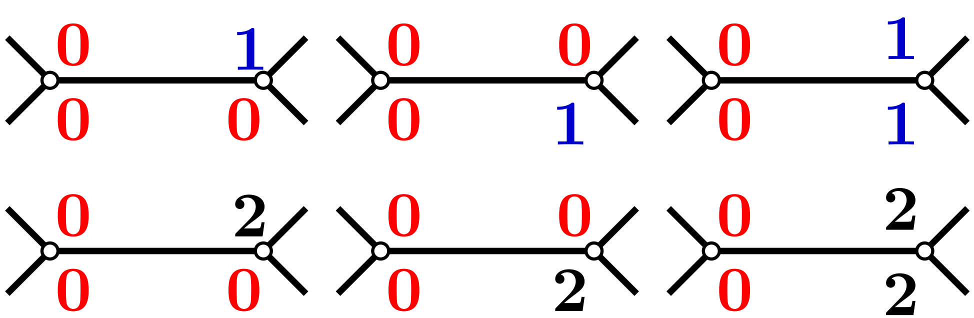


proof:

possibly valid configurations



forbidden configurations



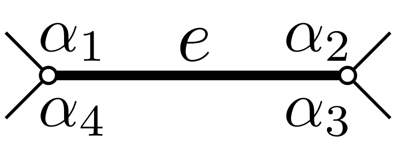
use a counting argument (double counts the angles)

$d(v)$:= number of label changes for the angles around v

$d(f)$:= number of label changes for the angles in face f

$$\sum_v d(v) + \sum_f d(f) = 3n + 3|f| = 3|E| + 6$$

use Euler formula: 3n+3(2+—E—-n)



at vertex a_i there are two label changes

$\epsilon(e)$ = number of label changes at the angles around e

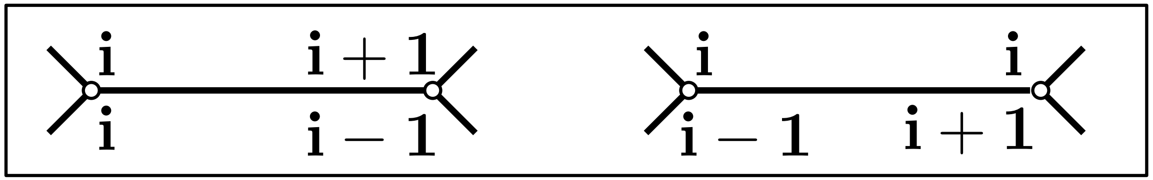
$$\epsilon(e) = \begin{cases} 0 \\ 3 \end{cases}$$

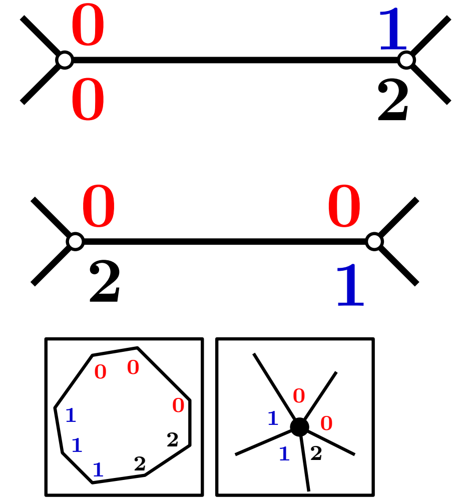
$\longrightarrow$ $\epsilon(e) = 3$ for all (normal) edges

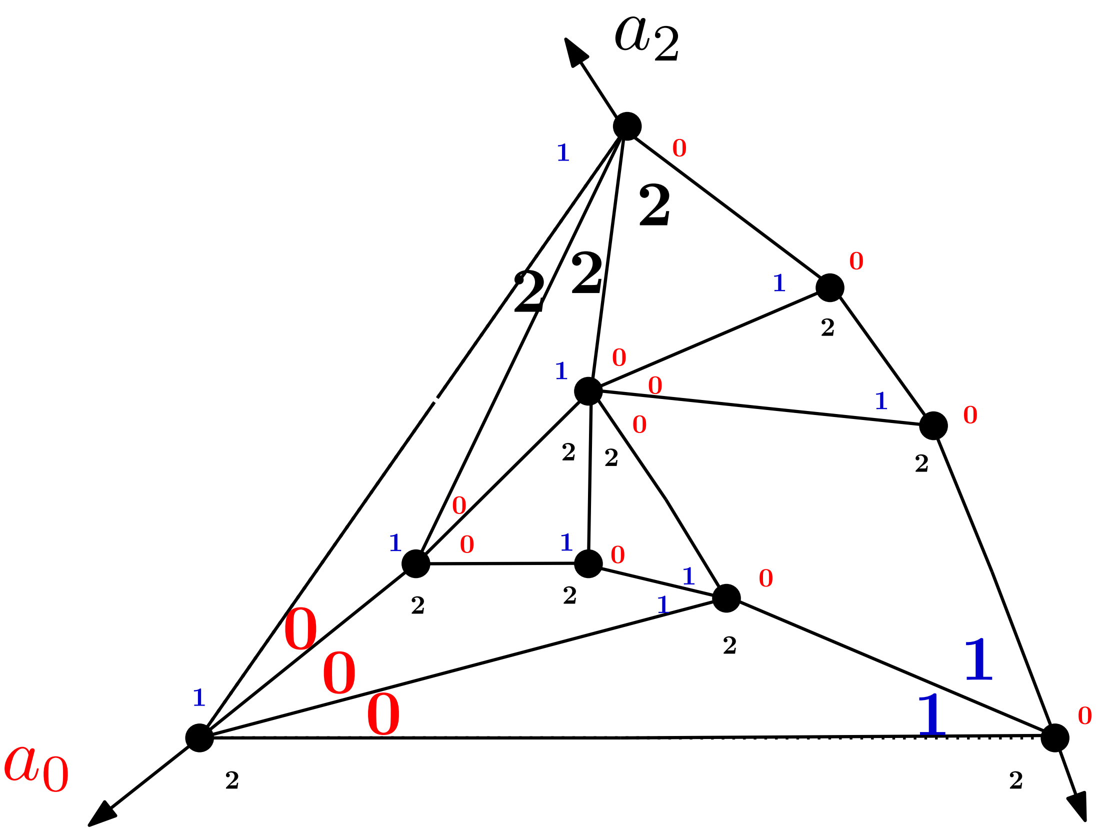
Schnyder labelings: angles at exterior vertices

Corollary

Given a Schnyder labeling of M^σ, all interior angles at a vertex a_i have label i

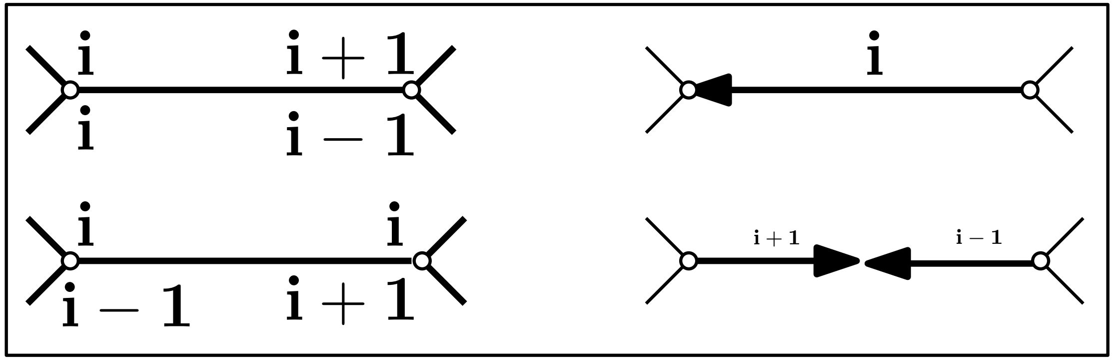


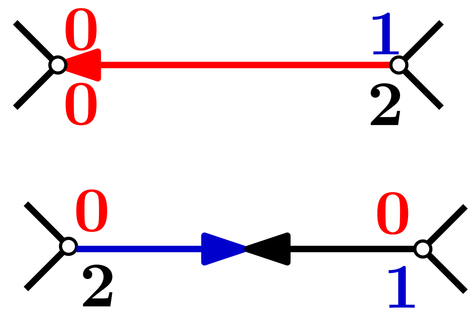




suspension M^σ

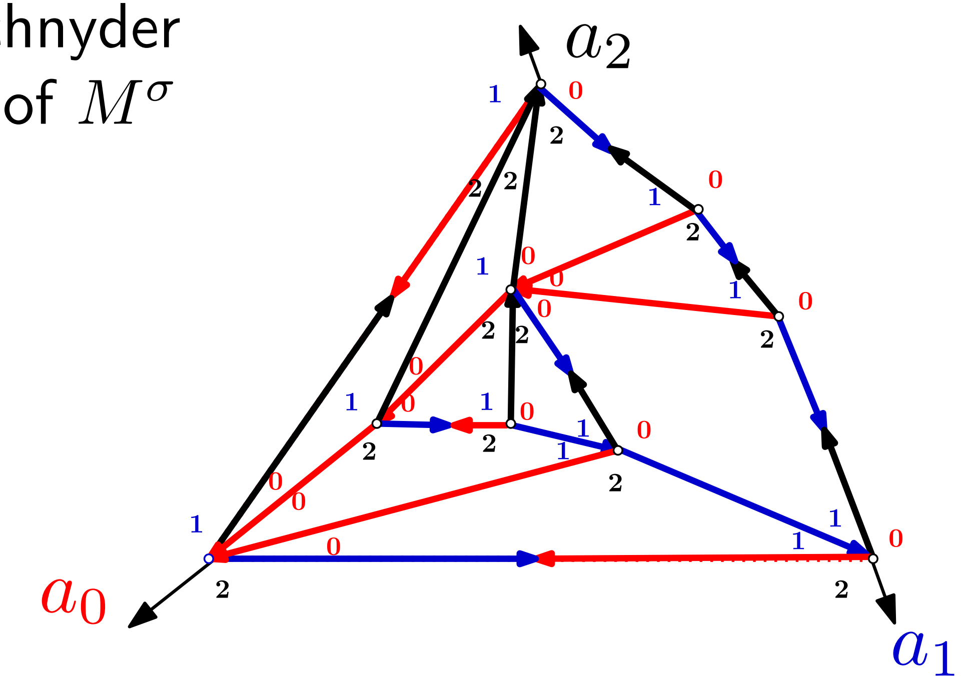
Correspondence between Schnyder labelings and Schnyder woods





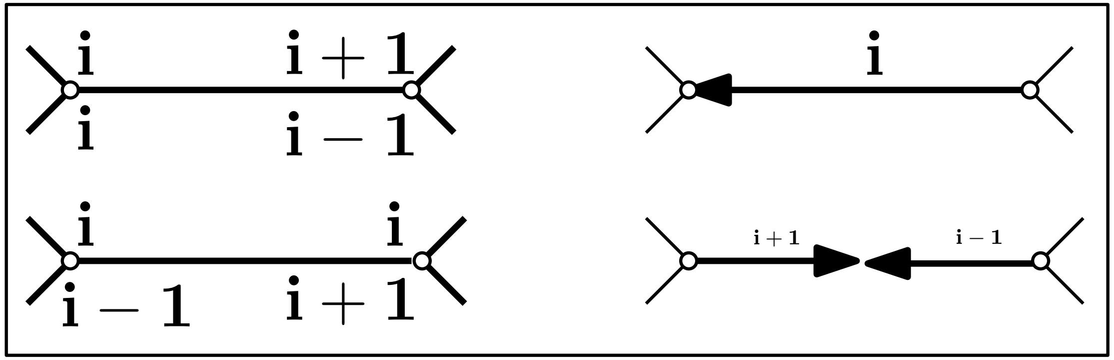
Theorem

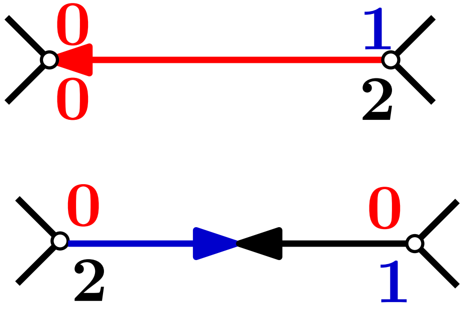
There is a correspondence between the Schnyder labelings of M^σ and the Schnyder woods of M^σ



Schnyder wood+ Schnyder labeling of M^σ

Correspondence between Schnyder labelings and Schnyder woods





Theorem

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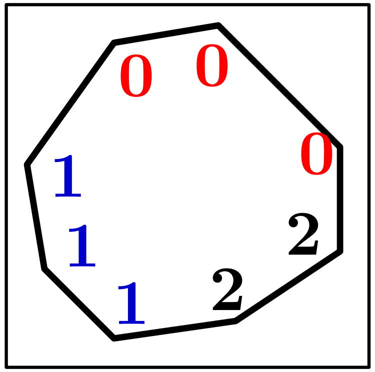
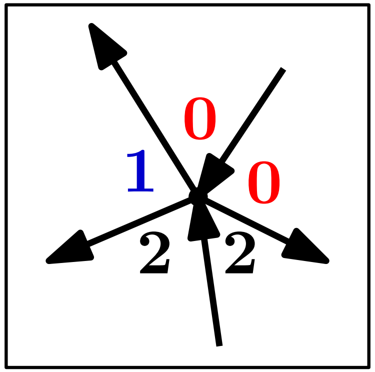
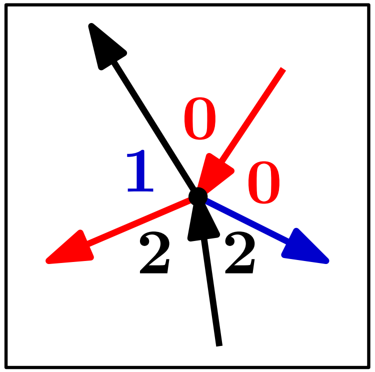


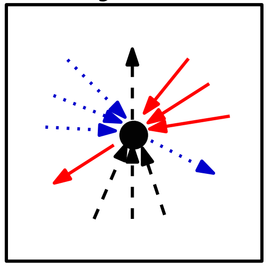
proof: Assume M^σ is endowed with a Schnyder labeling



Rule of vertices (A2)

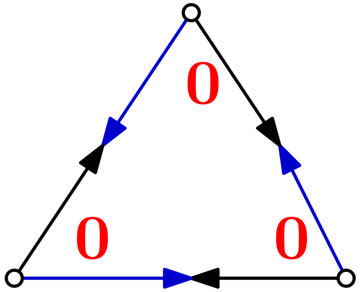


local Schnyder rule (W3)



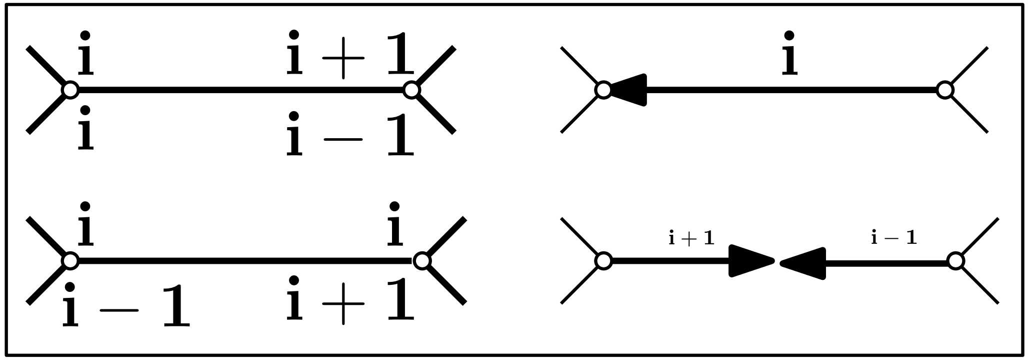
Assume (W4) is violated: there is a cycle in one color

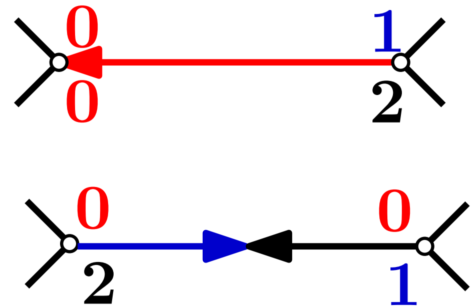
Then the coloring rule of bi-oriented edges implies that all angles have the same color

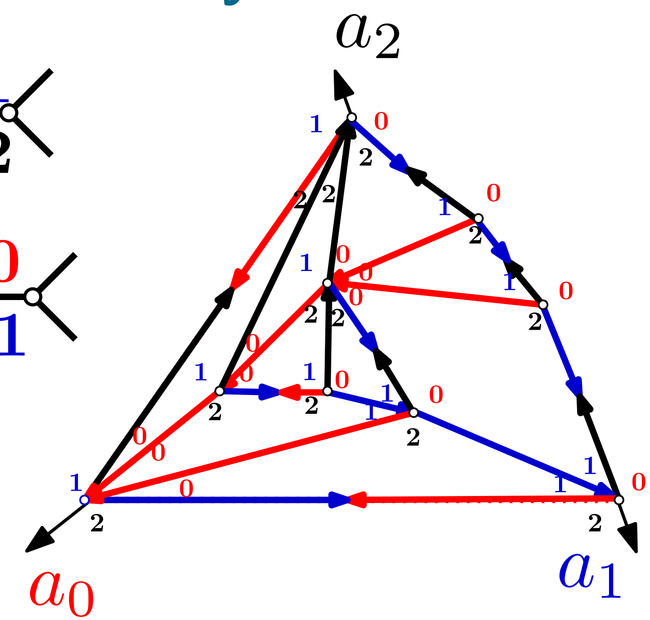


Rule of faces (A3) → **no directed cycles in one color (W4)**

Correspondence between Schnyder labelings and Schnyder woods







Theorem

There is a correspondence between the Schnyder labelings of M^σ and the Schnyder woods of M^σ

proof: Assume M^σ is endowed with a Schnyder wood

use a counting argument (double counts the angles around vertices/faces/edges)

$$d(v) = 3 \qquad d(e) = \begin{cases} 3 & \text{for all (normal) edges} \\ 2 & \text{for the three half-edges} \end{cases}$$

Remark:

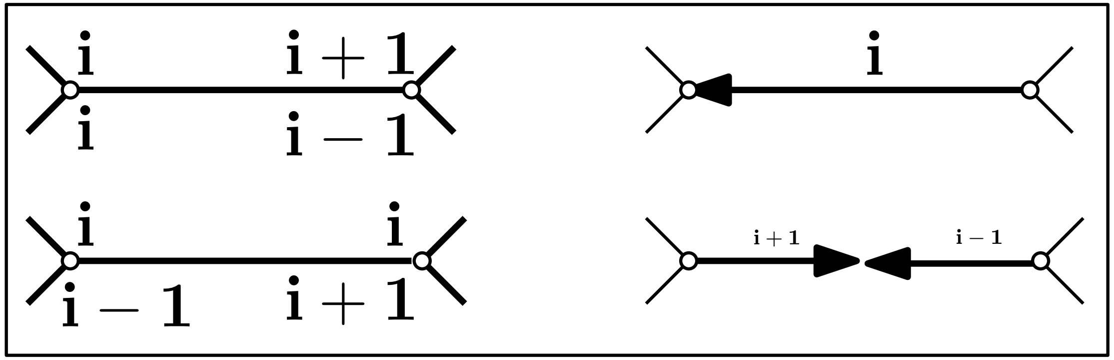
Turning around a face in ccw direction the angle will be i or $i+1$ → The number of changes $d(f)$ is a multiple of 3, and $d(f) > 0$ (otherwise there is a directed cycle of edges in one color)

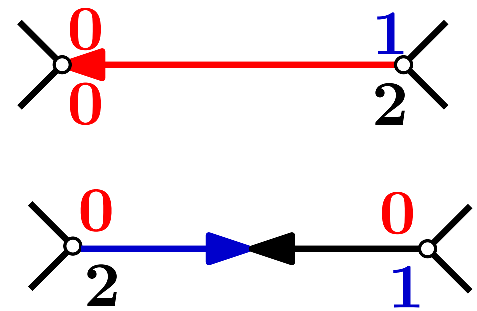
$$\sum_v d(v) + \sum_f d(f) = \sum_e d(e) \quad \rightarrow \quad 3n + \sum_f d(f) = 3|E| + 6$$

Euler formula implies $\sum_f d(f) = 3|F|$ → $d(f) = 3$ for all faces

condition (A3) for faces is true

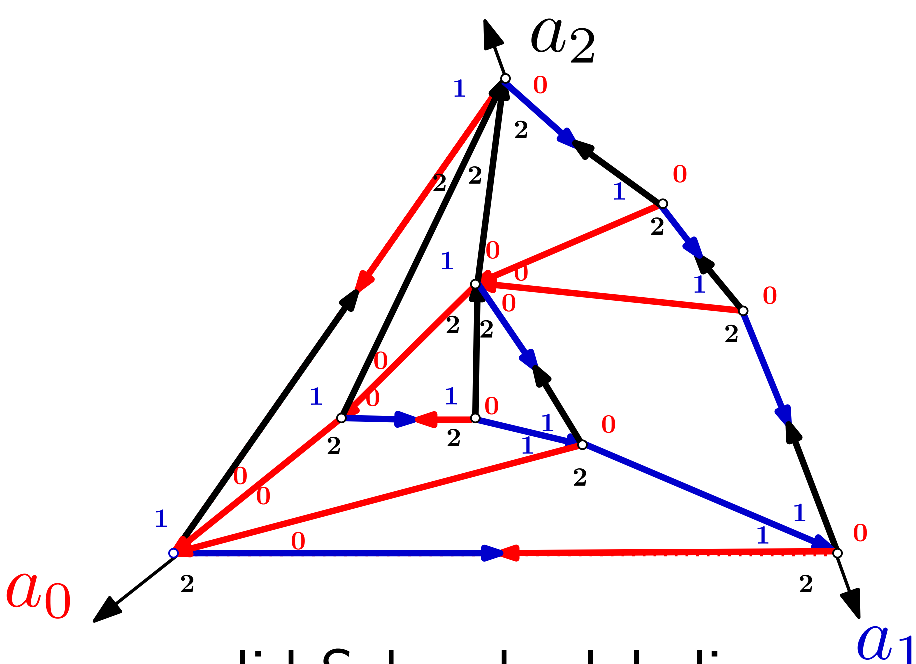
Correspondence between Schnyder labelings and Schnyder woods





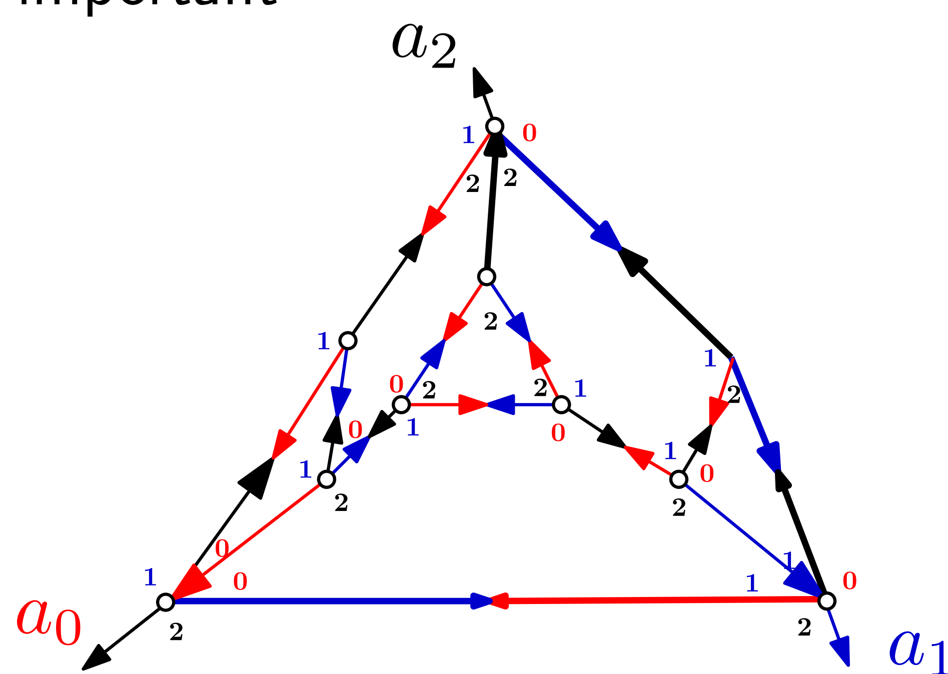
Remark:

The condition (W4) of Schnyder woods is important



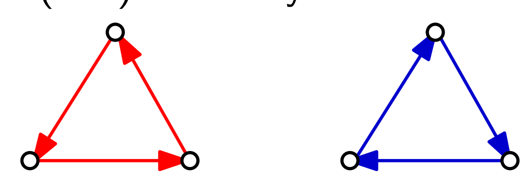
valid Schnyder labeling

conditions (W1)-(W4) of Schnyder woods are satisfied



not valid Schnyder labeling

condition (W4) of Schnyder woods is not satisfied

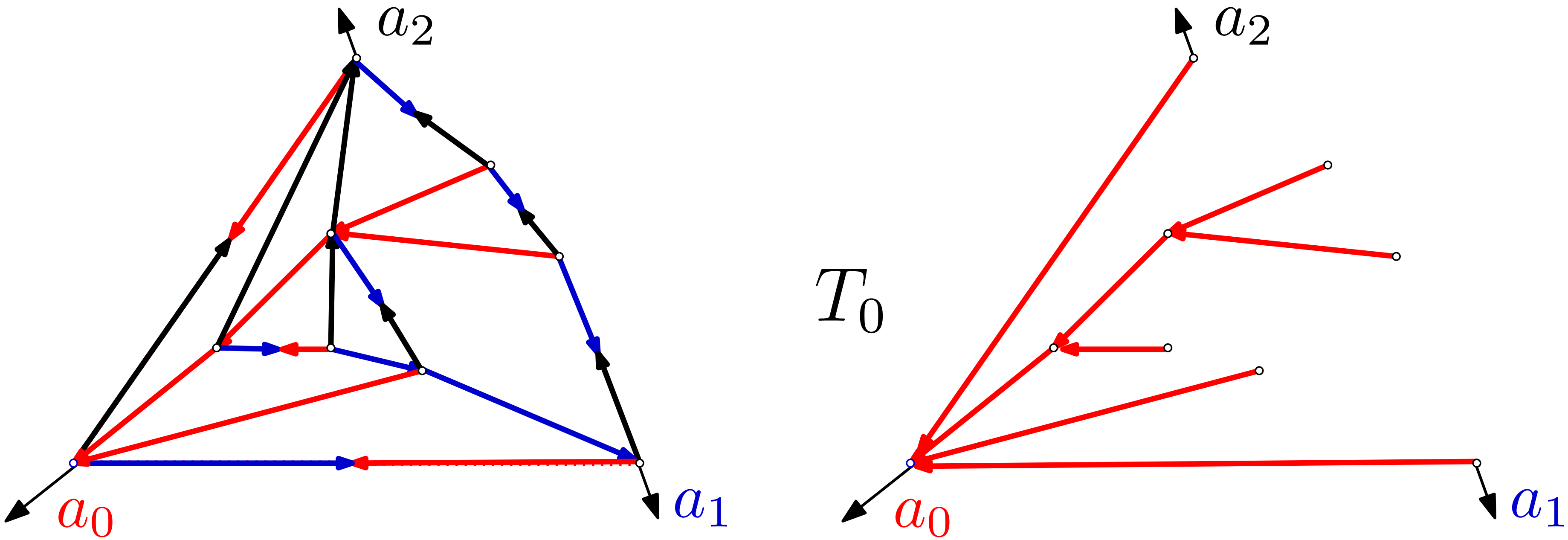


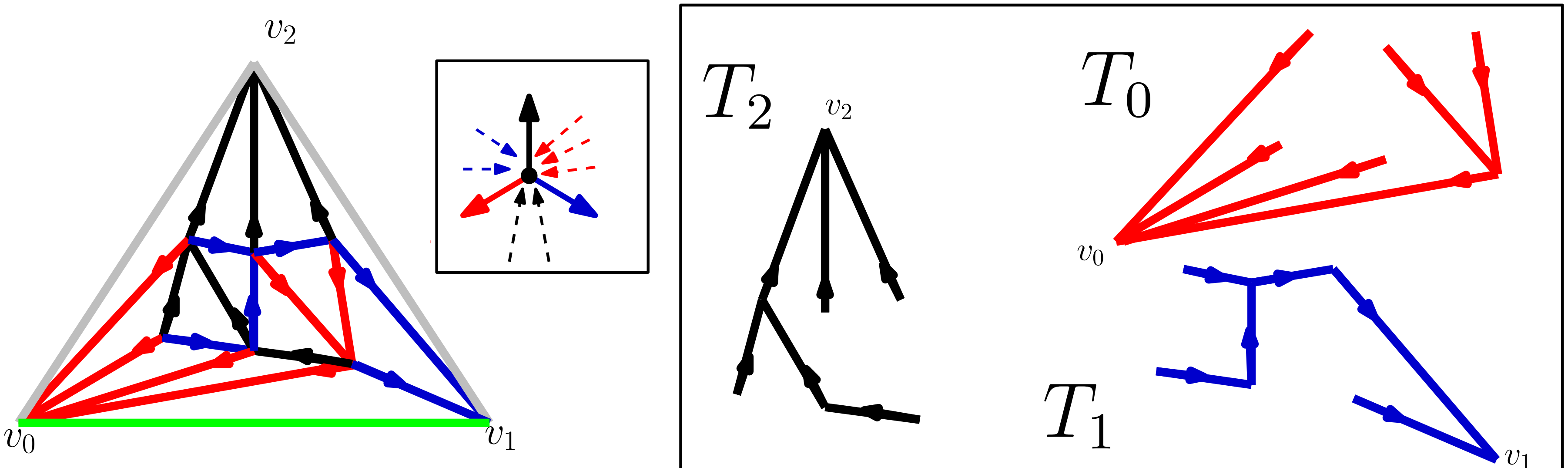
Schnyder woods: spanning property

Theorem **[Schnyder '90]**

$T_i :=$ digraph defined by directed edges of color i

The three sets T_0, T_1, T_2 are spanning trees of the inner vertices of $\mathcal{T}$ (each rooted at vertex v_i)

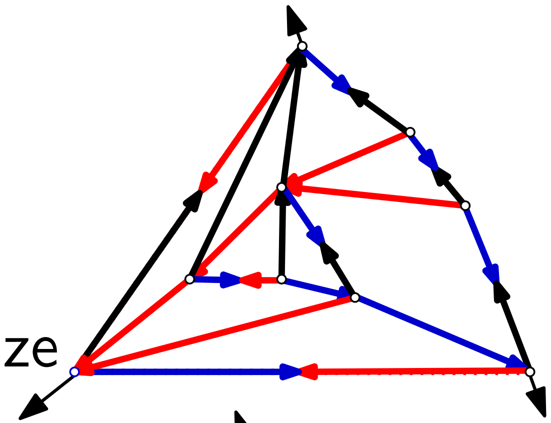




Spanning property for 3-connected maps

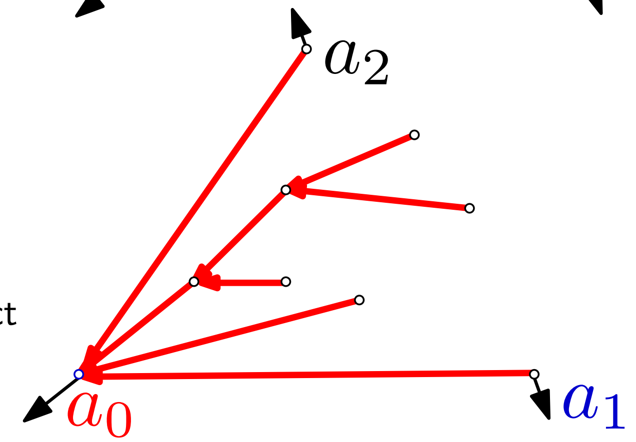
$T_i :=$ digraph defined by directed edges of color i

Theorem Let (T_0, T_1, T_2) a Schnyder wood of $\mathcal{M}$.
Then each digraph $D_i := T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ is acyclic

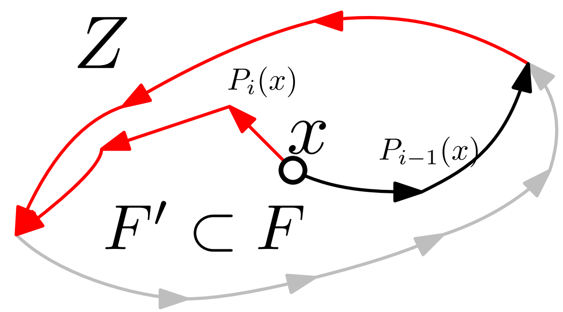


proof:

Let Z a directed cycle enclosing a region F of minimal size



Claim 1: F is a single face



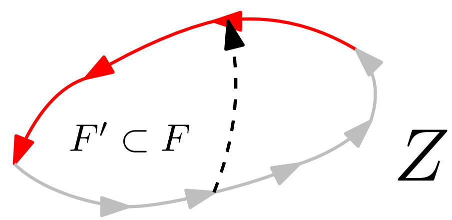
$P_i(x)$ and $P_{i-1}(x)$ cannot intersect

case a: $x \in F$

F' is a smaller than F

(bounded by a directed cycle)

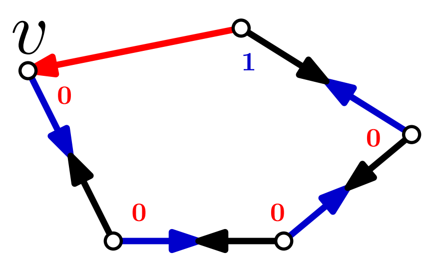
case b: F is empty of vertices
there is an edge inside F

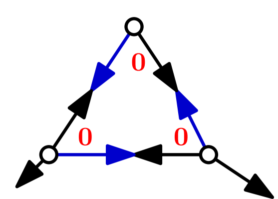


Claim 2: there is no face F whose boundary is a directed cycle

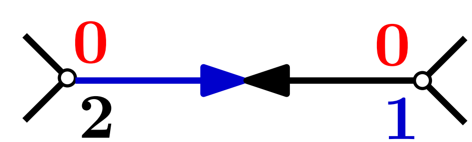
Visit F in ccw order starting from v and propagate colors (first color is i): there is no angle with label $i-1$

The coloring rule for faces is violated





coloring rule for angles

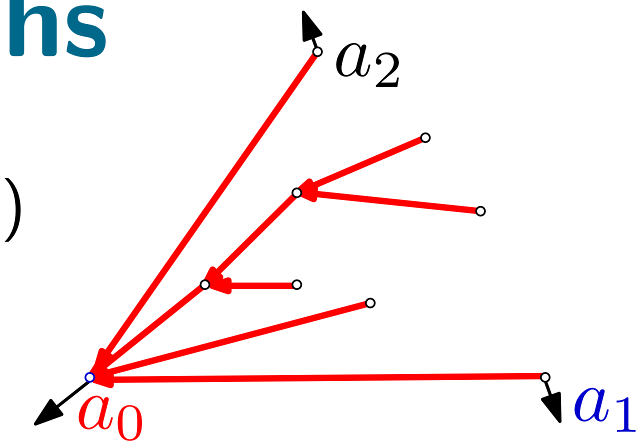


Corollary: Each sets T_i is spanning tree $\mathcal{M}$ (rooted at vertex a_i)

Non crossing paths

Corollary:

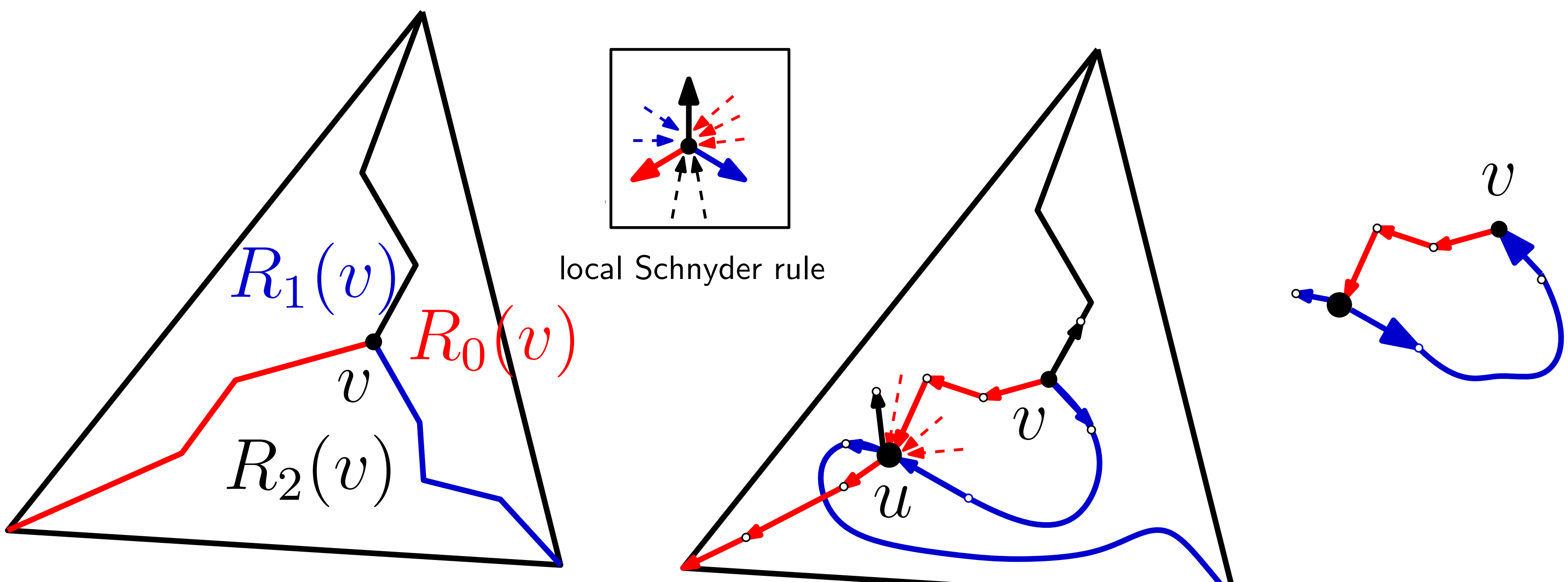
Each sets T_i is spanning tree $\mathcal{M}$ (rooted at vertex a_i)



Corollary

For each inner vertex v the three monochromatic paths P_0, P_1, P_2 directed from v toward each vertex a_i are vertex disjoint (except at v) and partition the inner faces into three sets $R_0(v), R_1(v), R_2(v)$

proof: the existence of two paths $P_i(v)$ and $P_{i+1}(v)$ which are crossing would contradicts previous theorem



Planar straight-line drawings

(of 3-connected planar graphs)

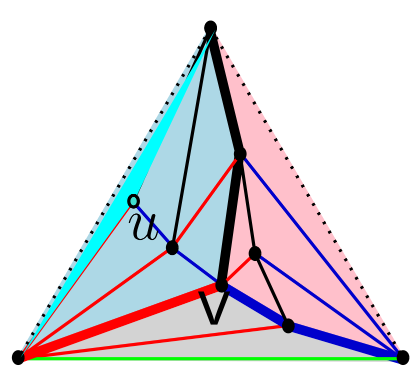
Paths and regions

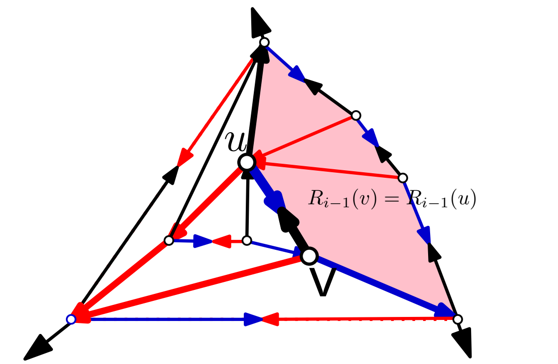
$u \in R_1^{int}(v)$

Lemma Let (T_0, T_1, T_2) a Schnyder wood of $\mathcal{M}$.

If $u \in R_i(v)$ then $R_i(u) \subseteq R_i(v)$

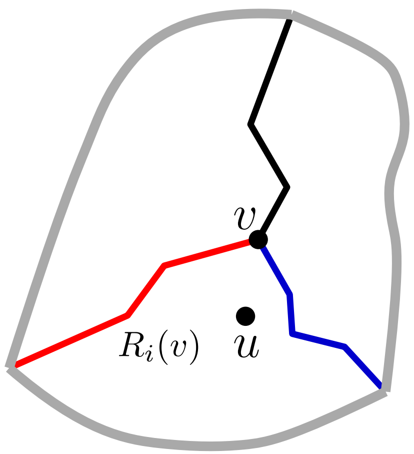
If $u \in R_i^{int}(v)$ then $R_i(u) \subset R_i(v)$

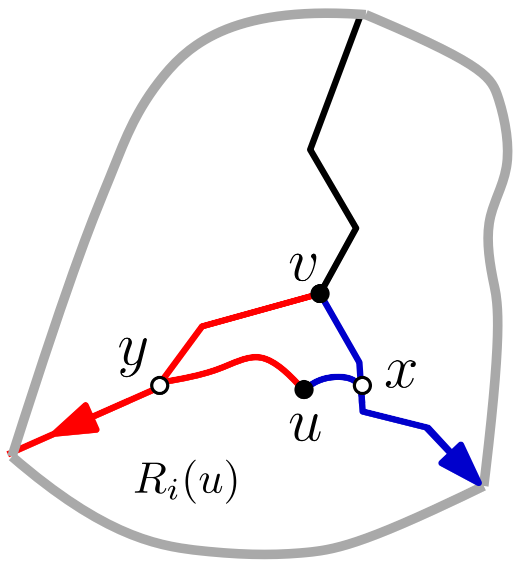




proof:

Case 1: $u \in R_i^{int}(v)$





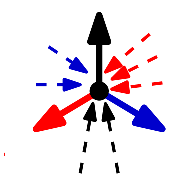
first step: compute the paths $P_{i+1}(u)$ and $P_{i-1}(u)$

They must intersect the boundary of $R_i(v)$ at x and y

Remark: x and y are different from v

and we have $y \in P_{i+1}(u)$ and $x \in P_{i-1}(u)$

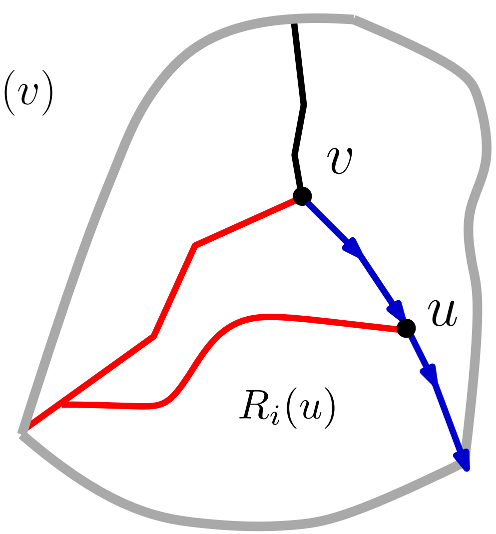
(because of Schnyder rule)



so we have: $R_i(u) \subset R_i(v)$

Case 2a: $u \in P_{i-1}(v)$

$R_i(u) \subset R_i(v)$

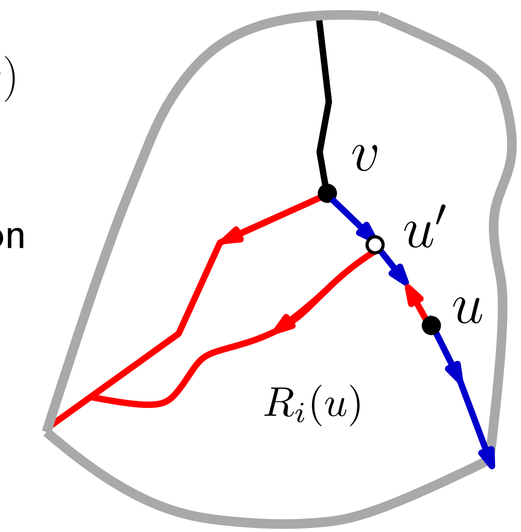


Case 2b: $u \in P_{i-1}(v)$

(u, u') is bi-oriented

Proceed by induction on the path $P_{i-1}(v)$

$R_i(u) \subseteq R_i(v)$

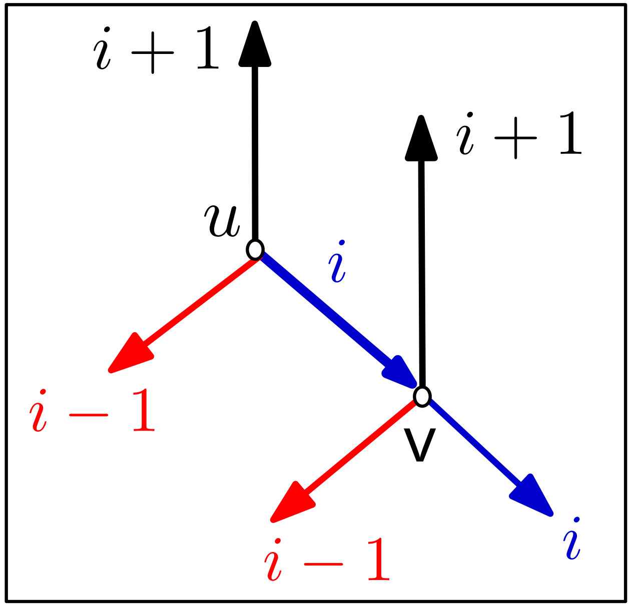


Paths and regions

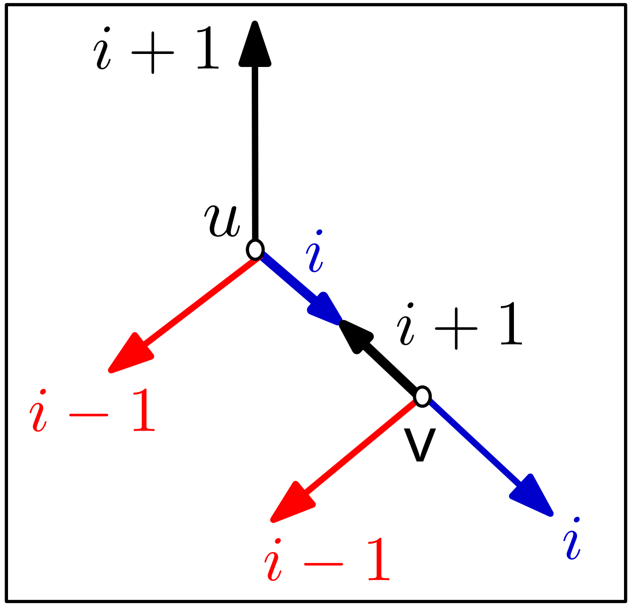
Remarks: Let (u, v) of color i oriented from u to v

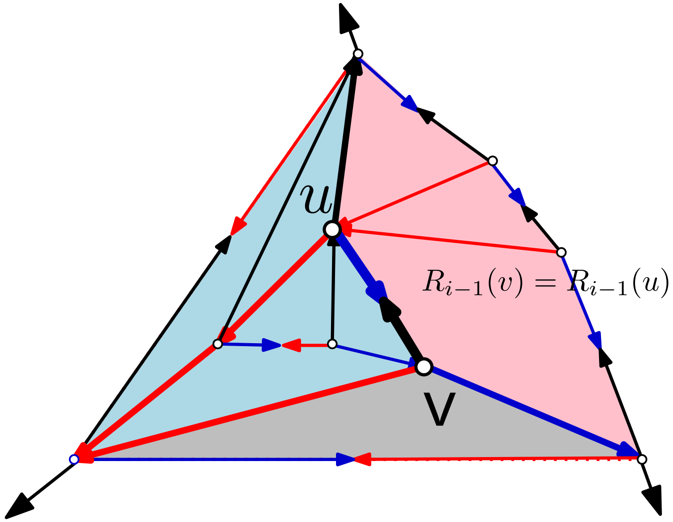
$$v \in P_i(u) \longrightarrow \begin{cases} v \in R_{i+1}(u) \\ v \in R_{i-1}(u) \\ u \in R_i(v) \end{cases}$$

Case 1: (u, v) is unidirectional



Case 2: (u, v) is bidirectional





Case 1:

$$R_i(u) \subset R_i(v)$$
$$R_{i+1}(v) \subset R_{i+1}(u)$$
$$R_{i-1}(v) \subset R_{i-1}(u)$$

Case 2:

$$R_i(u) \subset R_i(v)$$
$$R_{i-1}(v) \subseteq R_{i-1}(u)$$
$$R_{i+1}(v) \subseteq R_{i+1}(u)$$

Regions and coordinates

Remarks: Let (u,v) of color i oriented from u to v

$$v =: \frac{|R_0(v)|}{|F|-1}x_0 + \frac{|R_1(v)|}{|F|-1}x_1 + \frac{|R_2(v)|}{|F|-1}x_2 =$$
$$= \frac{v_0}{|F|-1}x_0 + \frac{v_1}{|F|-1}x_1 + \frac{v_2}{|F|-1}x_2$$

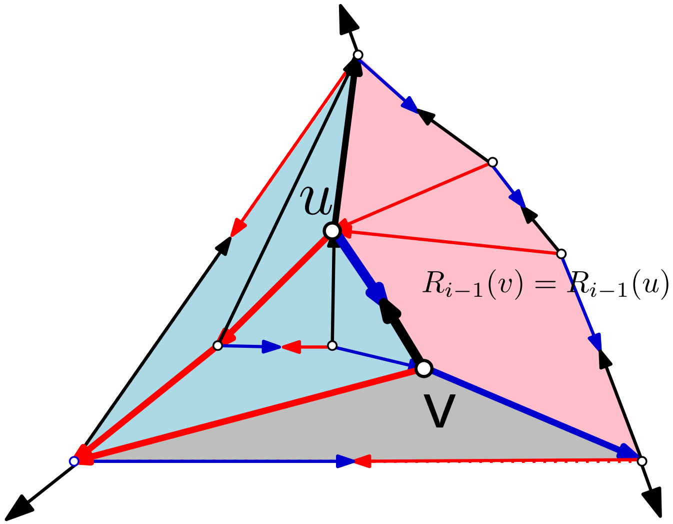
- $R_i(u) \subseteq R_i(v) \rightarrow |R_i(u)| \leq |R_i(v)| \rightarrow \boxed{u_i \leq v_i}$

- $v_0 + v_1 + v_2 = f - 1$

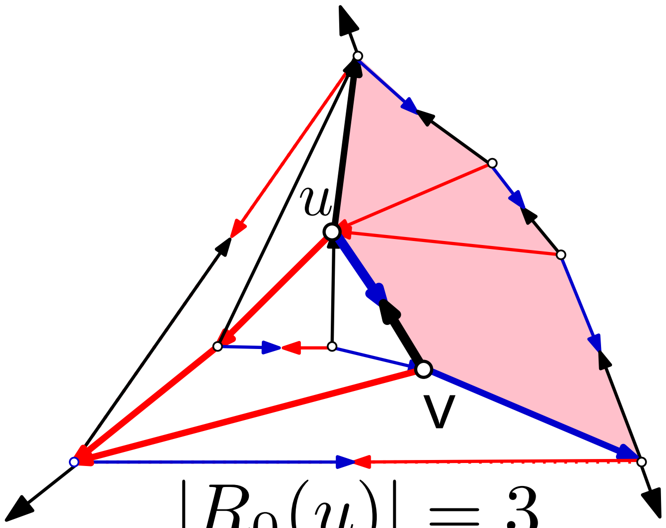
- $R_i(u) \subset R_i(v)$, $R_{i+1}(v) \subset R_{i+1}(u)$, $R_{i-1}(v) \subset R_{i-1}(u)$ $\rightarrow$ $\begin{cases} u_i < v_i \\ u_{i+1} > v_{i+1} \\ u_{i-1} > v_{i-1} \end{cases}$

- For every edge (u,v) there are some indices $i,j \in \{0,1,2\}$ s.t.

$$\boxed{\begin{array}{c} u_i < v_i \\ u_j > v_j \end{array}}$$

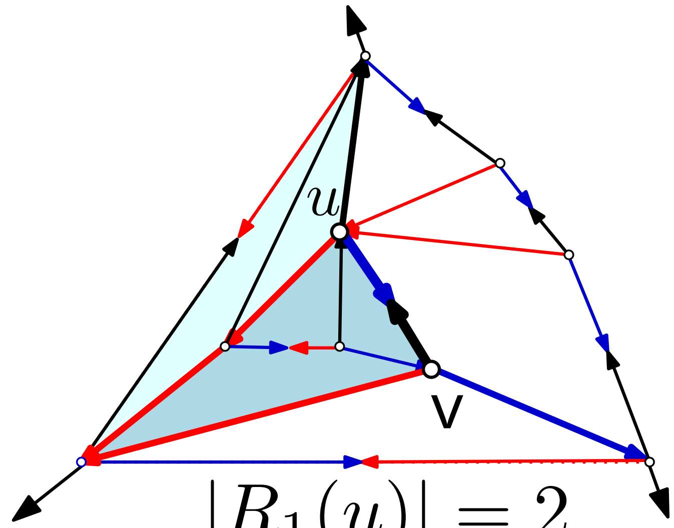


Face counting algorithm



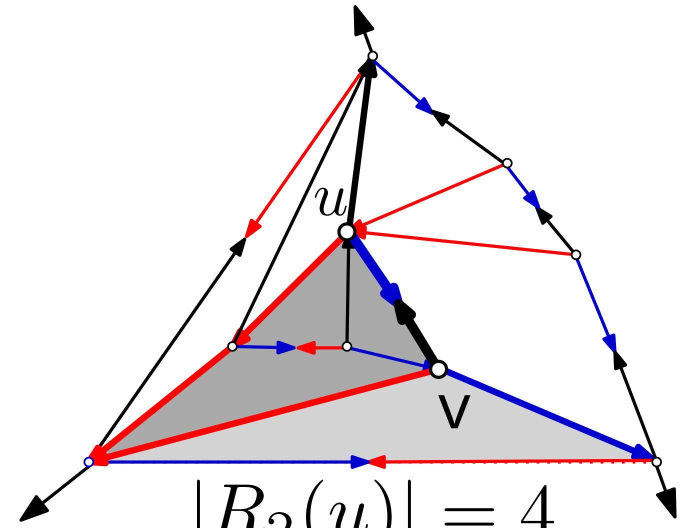
$|R_0(u)| = 3$

$|R_0(v)| = 3$



$|R_1(u)| = 2$

$|R_1(v)| = 5$



$|R_2(u)| = 4$

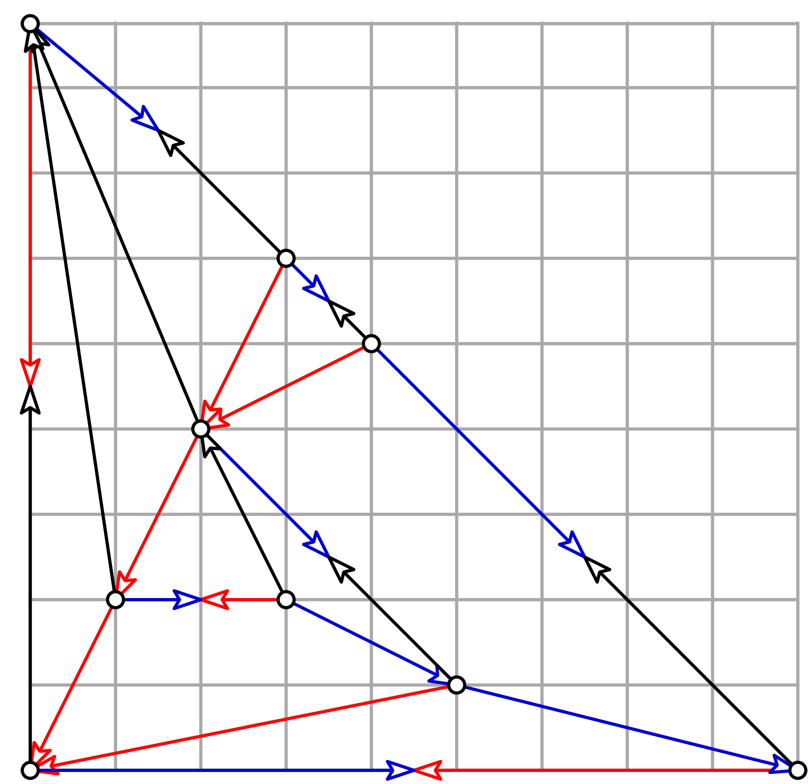
$|R_2(v)| = 1$

Theorem

Given a 3-connected planar graph $\mathcal{G}$ having $|F|$ vertices, the map above defines a straight-line crossing free planar drawing of G (where all faces are convex).

$$v = \frac{|R_0(v)|}{|F|-1} x_0 + \frac{|R_1(v)|}{|F|-1} x_1 + \frac{|R_2(v)|}{|F|-1} x_2$$

where $|R_i(v)|$ is the number of triangles in $R_i(v)$

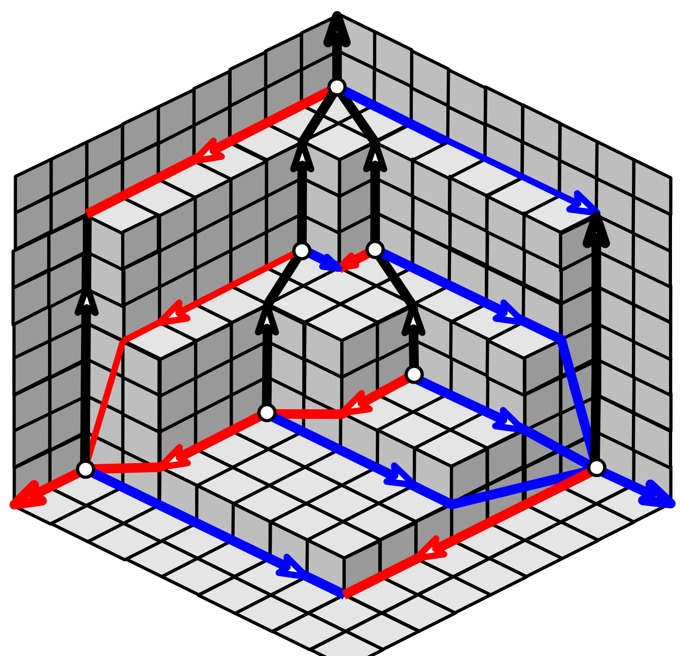


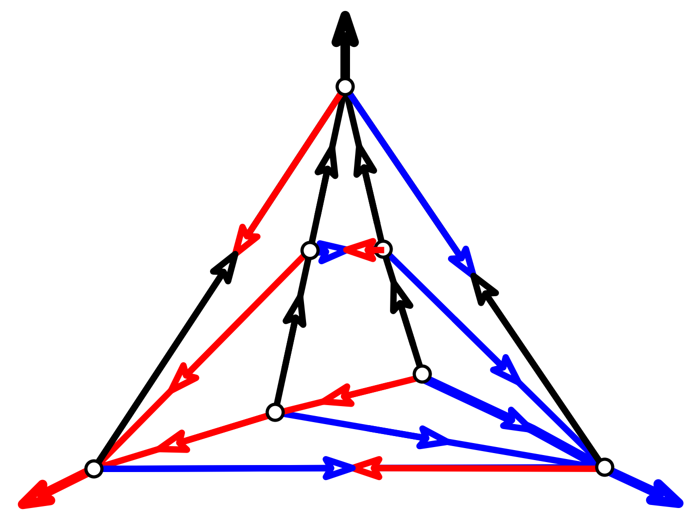
Algorithms and combinatorics for geometric graphs (Geomgraphs)

Lecture 5 - part II

Schnyder woods and orthogonal surfaces

Luca Castelli Aleardi





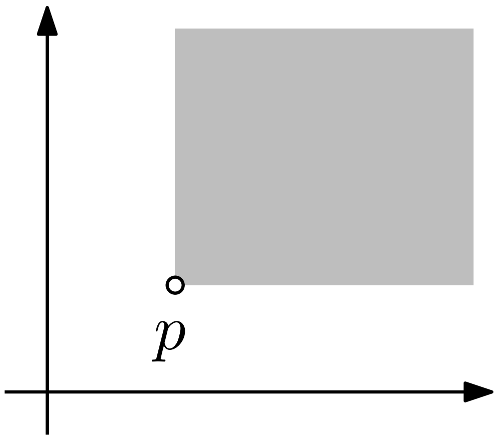
Schnyder woods and orthogonal surfaces

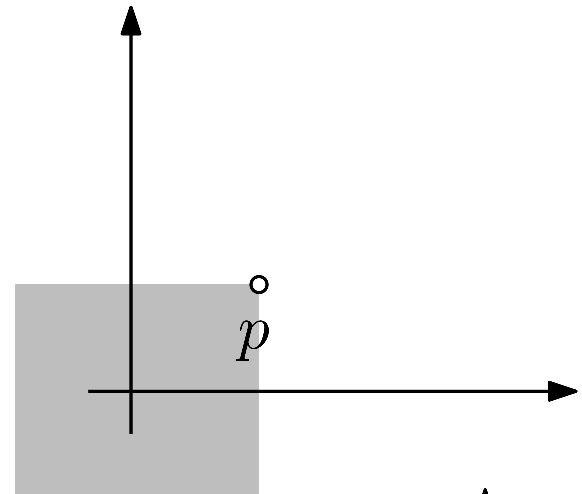
Orthogonal surfaces and elbow geodesics

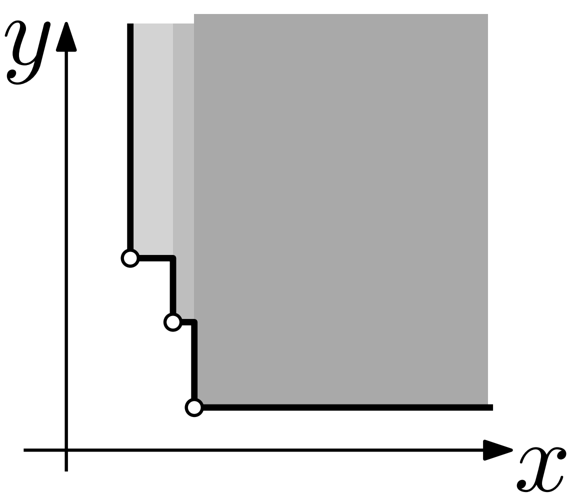
Dominance order $(\mathbf{u}, \mathbf{v} \in \mathbb{Z}^3)$: $\quad \mathbf{u} \leq \mathbf{v} \quad iff \quad u_i \leq v_i, \; \forall i = 0, 1, 2$

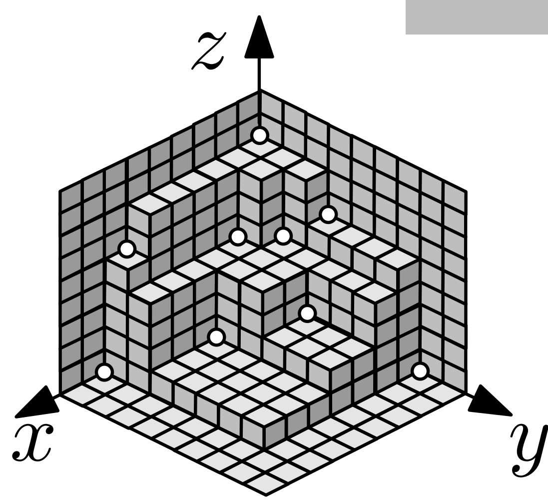
$\triangle_p :=$ cone dominating $p \in \mathbb{R}^3$

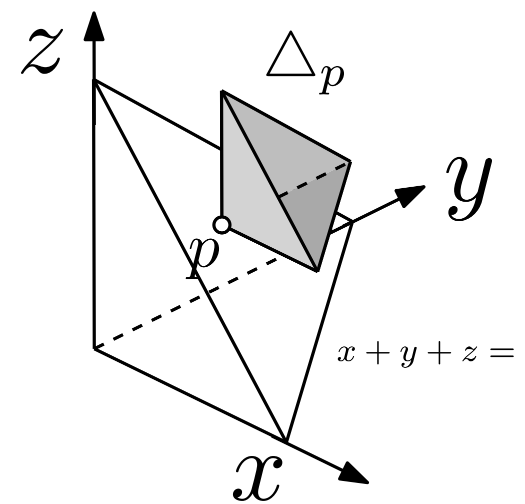
$\nabla_p :=$ cone dominated by $p \in \mathbb{R}^3$











Let $V \subset \mathbb{Z}^3$ be an **antichain**

(elements are pairwise incomparable)

Orthogonal surface $S_V :=$ boundary of $\langle \mathcal{V} \rangle$

$\langle \mathcal{V} \rangle := \{\alpha \in R^3 | \alpha \geq v, \text{ for some } v \in \mathcal{V}\} = \bigcup_v \triangle_v$

Orthogonal surfaces and elbow geodesics

Dominance order $(\mathbf{u}, \mathbf{v} \in \mathbb{Z}^3)$

$$\mathbf{u} \leq \mathbf{v} \quad iff \quad u_i \leq v_i, \ \forall i = 0, 1, 2$$

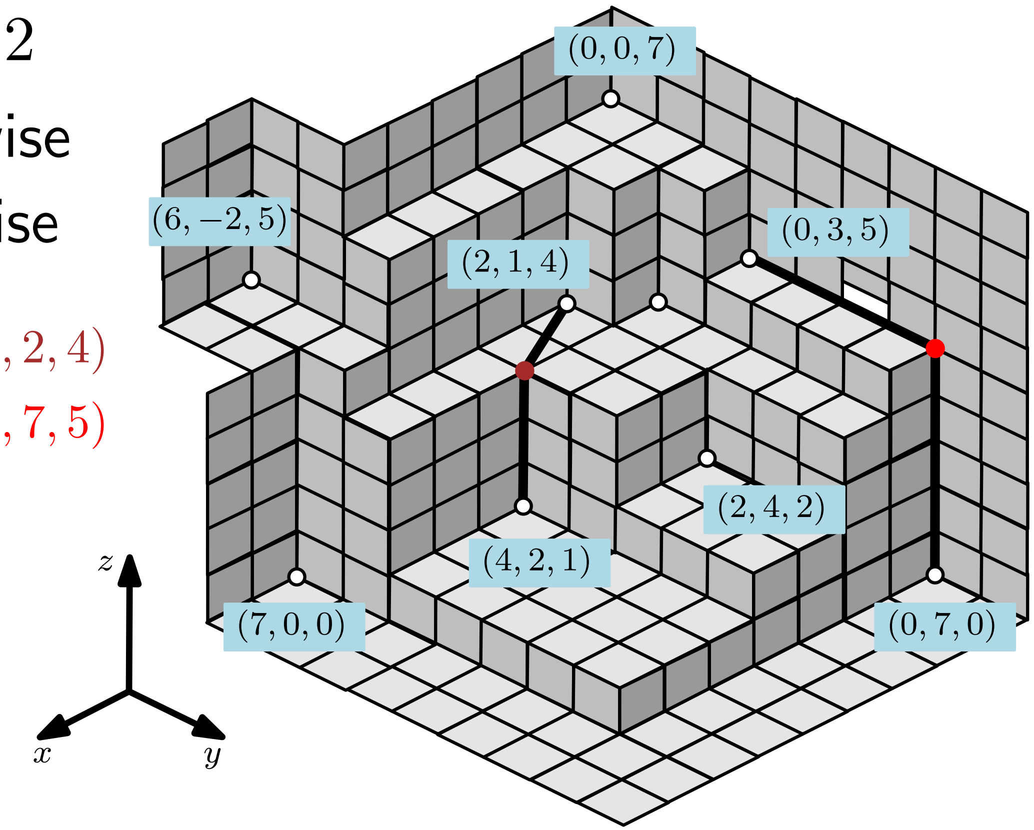
join $\mathbf{u} \vee \mathbf{v} :=$ maximum component-wise

meet $\mathbf{u} \wedge \mathbf{v} :=$ minimum component-wise

$\mathcal{V} = \{ (0,0,7) \ (0,7,0) \ (7,0,0) \ (2,4,2) \ \dots \ \}$

$(4,2,1) \vee (2,1,4) = (4,2,4)$

$(0,7,0) \vee (0,3,5) = (0,7,5)$



$\langle \mathcal{V} \rangle := \{\alpha \in R^3 | \alpha \geq v, \ \text{for some } v \in \mathcal{V}\}$

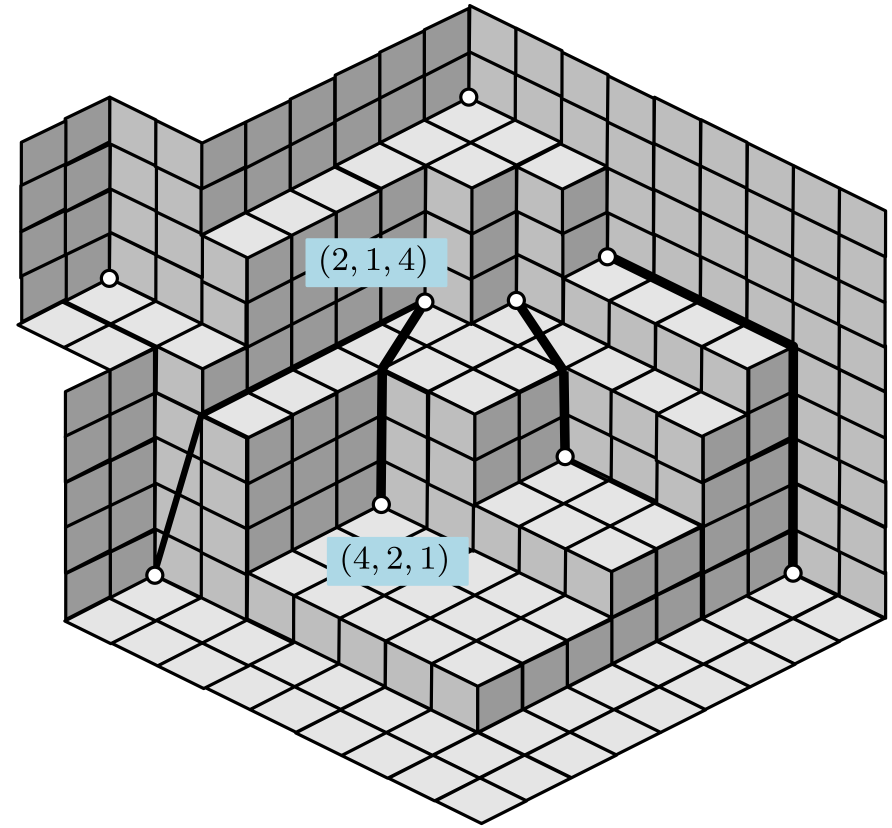
Orthogonal surface $S_V :=$ boundary of $\langle \mathcal{V} \rangle$

Let $V \subset \mathbb{Z}^3$ be an **antichain**

(elements are pairwise incomparable)

Orthogonal surfaces and elbow geodesics

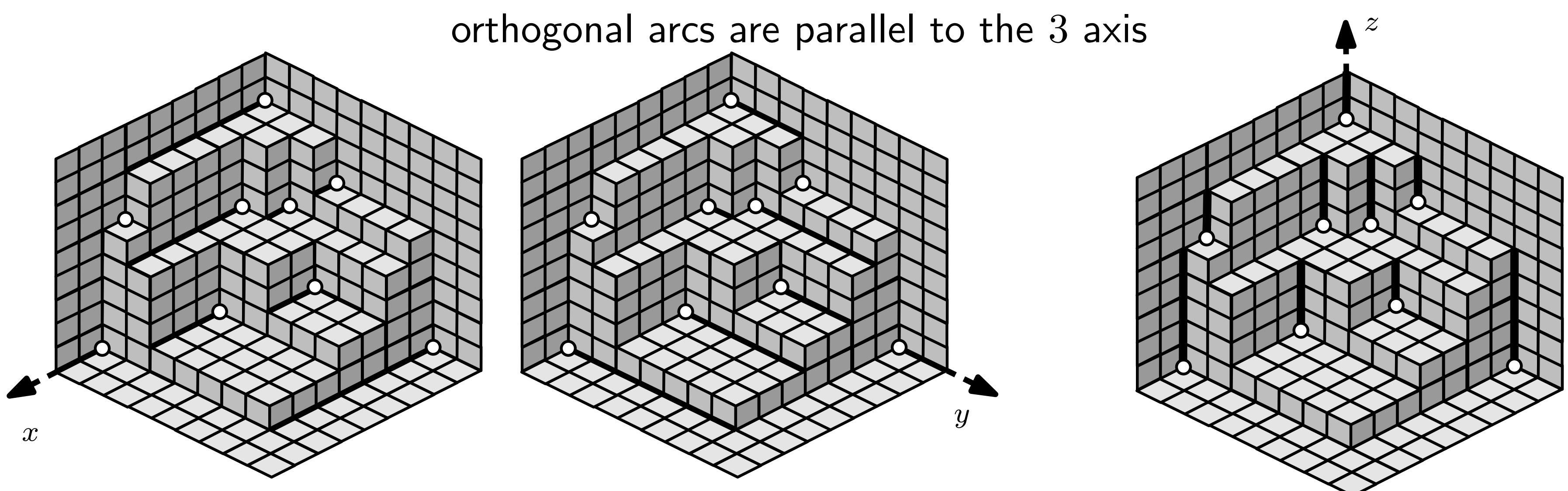
$$(4,2,1) \wedge (2,1,4) = (4,2,4)$$
$$(0,7,0) \wedge (0,3,5) = (0,7,5)$$



elbow geodesic of u and v:
the union of the two line segments $(u, u \vee v)$ and $(u \vee v, v)$

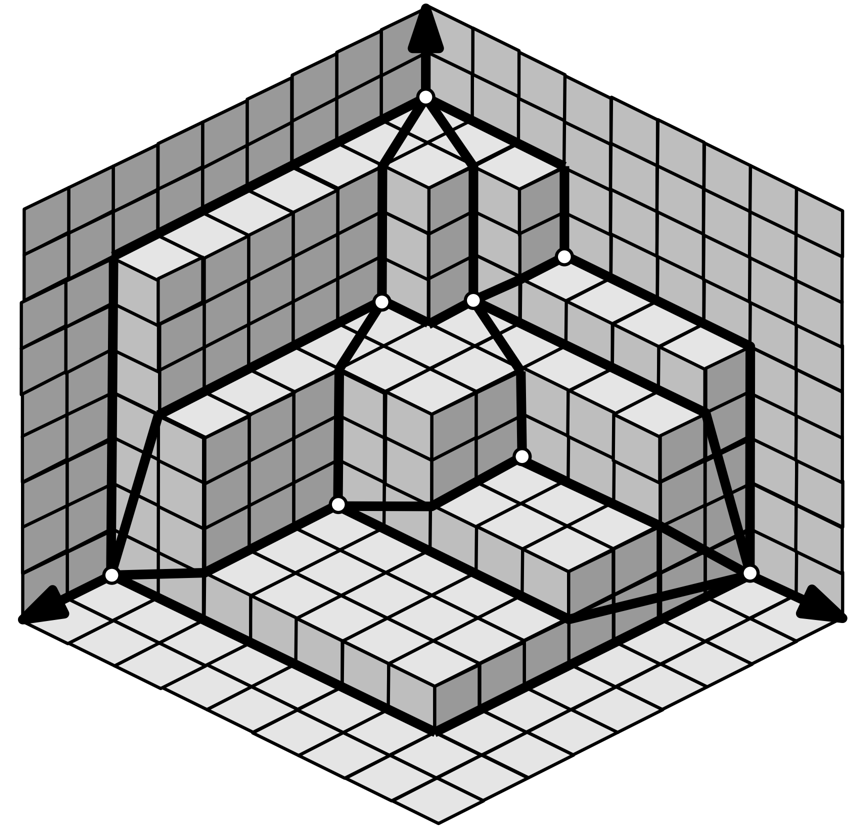
- every $v \in S_V$ has three orthogonal arcs (parallel to each axis)
- every elbow geodesic contains at least one bounded orthogonal arc

orthogonal arcs are parallel to the 3 axis



Orthogonal surfaces and elbow geodesics

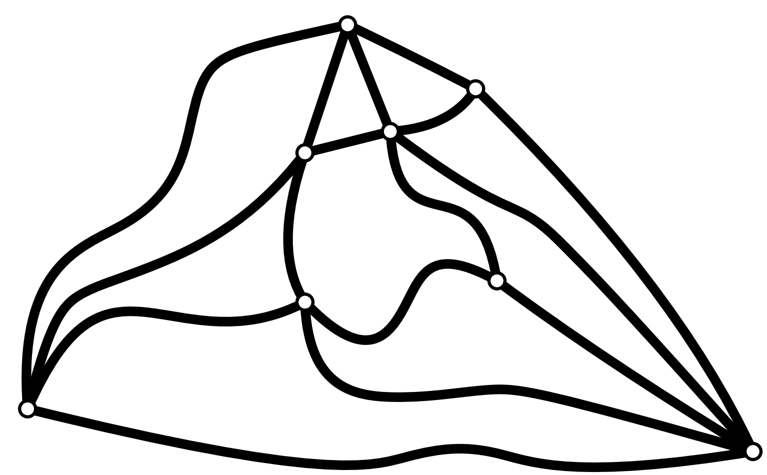
A **geodesic embedding** of a planar map G:
a drawing of G on $S_{\mathcal{V}}$ s.t.



(G1) The vertices of G correspond to the points of $S_{\mathcal{V}}$

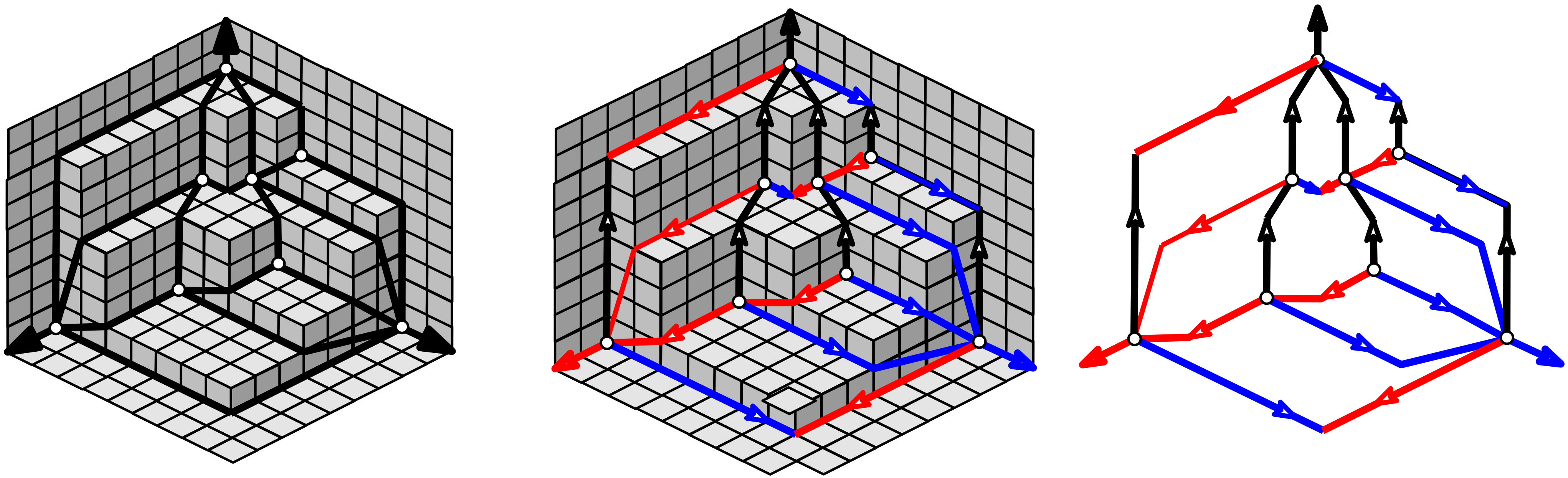
(G2) every edge of G is drawn as an elbow geodesic on $S_{\mathcal{V}}$
Every bounded orthogonal arc of $S_{\mathcal{V}}$ is part of an edge of G

(G3) There are no edge crossings on $S_{\mathcal{V}}$



From geodesic embeddings to Schnyder woods

Thm: Consider a Schnyder wood of a planar map G and the corresponding set of vertex coordinates $\mathcal{V}$ (region vectors). The resulting drawing of G on $S_{\mathcal{V}}$ is a geodesic embedding (no crossings)



From geodesic embeddings to Schnyder woods

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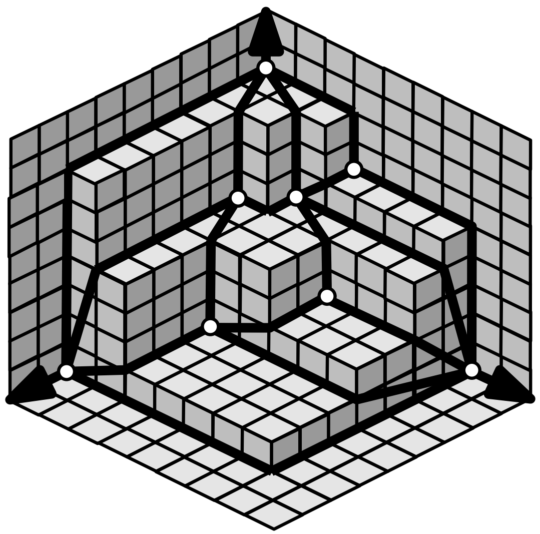
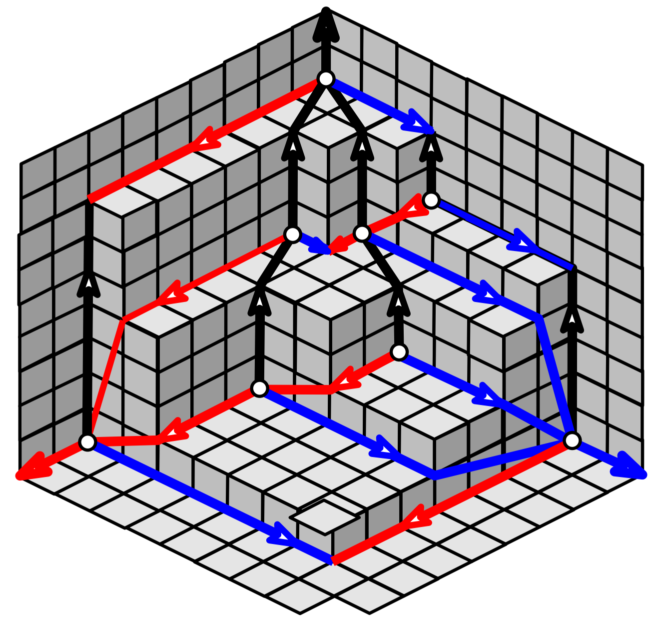
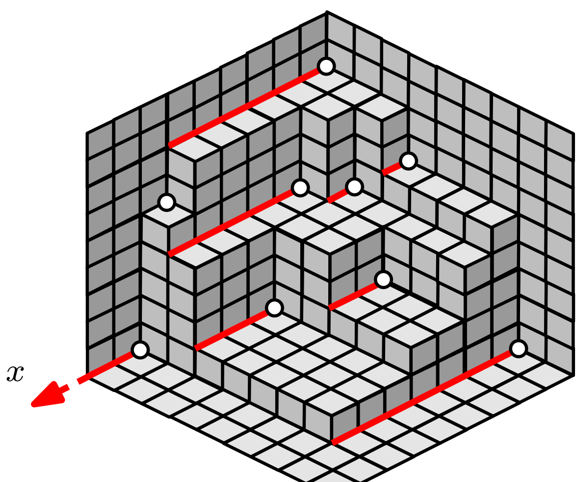
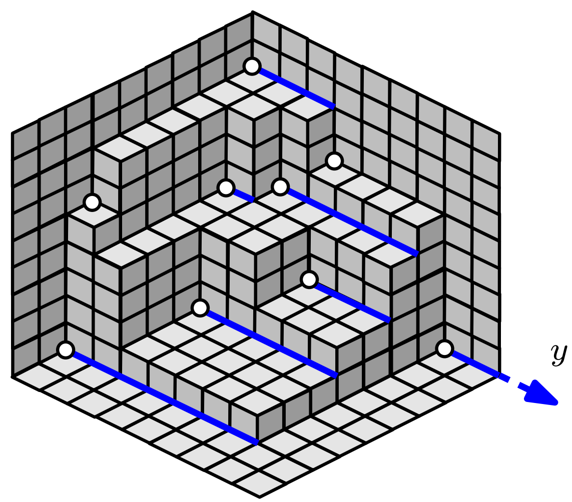


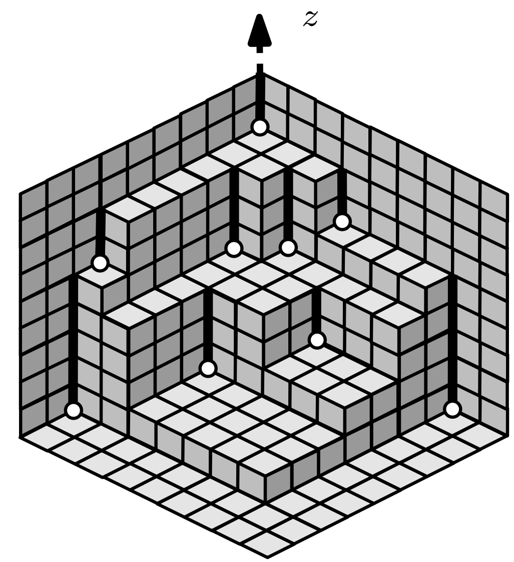




orthogonal arcs parallel to the x-axis are red (color 0)



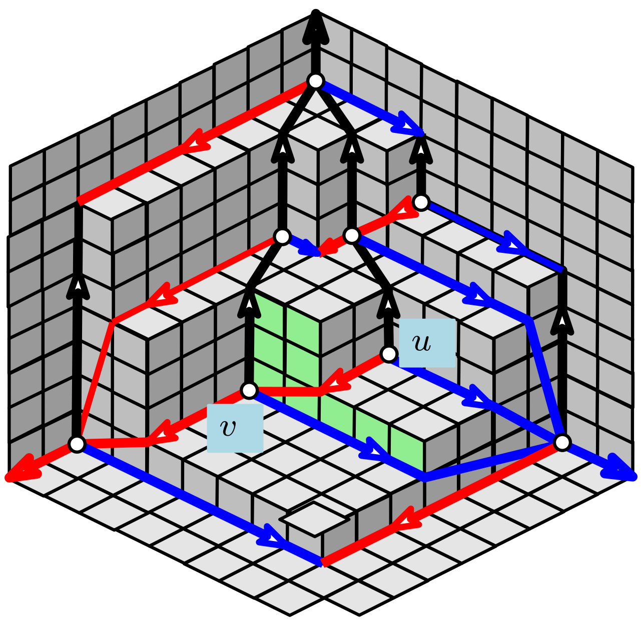
orthogonal arcs parallel to the y-axis are blue (color 1)



orthogonal arcs parallel to the z-axis are black (color 2)

From geodesic embeddings to Schnyder woods

Thm: The edge orientation corresponding to a geodesic embedding is a Schnyder wood

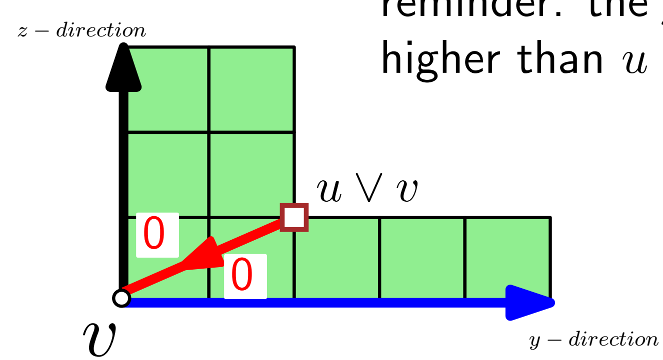


in the example
$u \vee v = (v_0, u_1, u_2)$

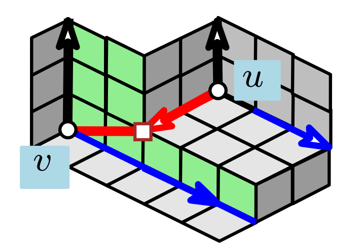
Claim 1: The local Schnyder condition (W3) is valid

- Every vertex has 3 outgoing edges (one for each color): the three orthogonal arcs (by construction)
- Let us consider an edge $\{u = (u_0, u_1, u_2), v = (v_0, v_1, v_2)\}$ incident at v in the sector parallel to the vertical yz-plane

The edge $\{u, v\}$ contains the orthogonal arc $(u \vee v, u)$ parallel to the x-direction and lying in the same horizontal plane of u: its color must be red (color 0), and its orientation is outgoing from u.



reminder: the join $u \vee v$ is equal or higher than u and v (in every direction)

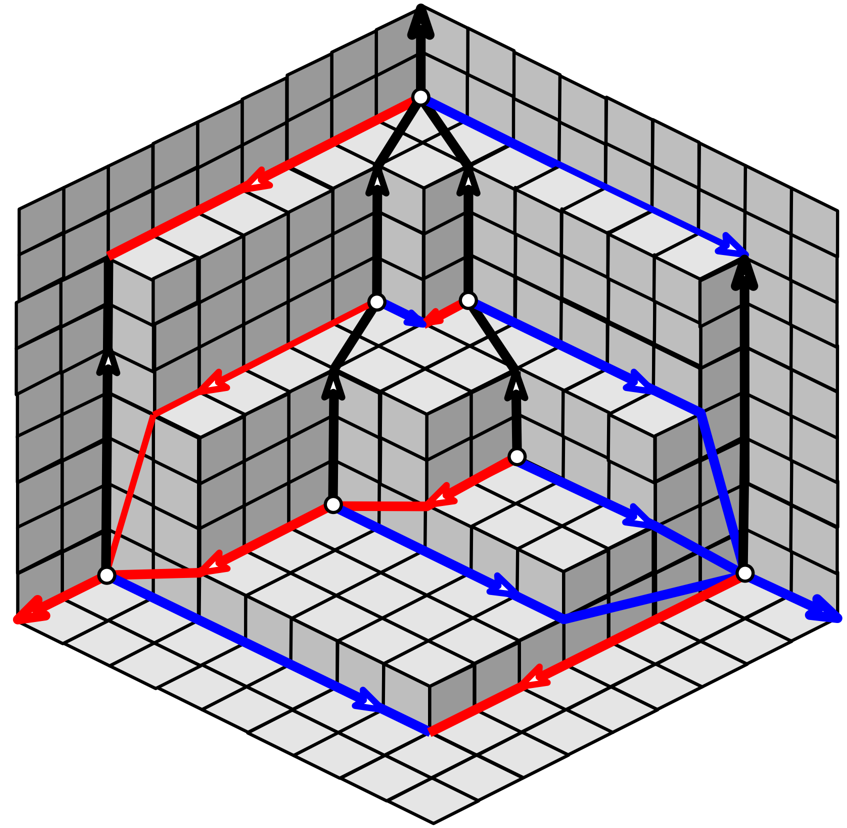


Claim 2: condition (W4) of the definition is valid

Remark: a path of edges of color i lead to increasing coordinates in i-direction → (W4) no cycles

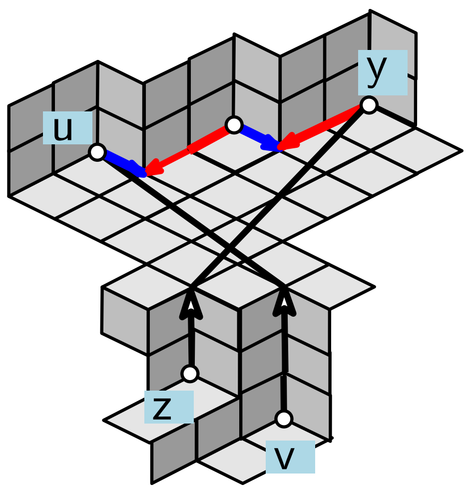
Geodesic embeddings are planar drawings

Thm: Consider a Schnyder wood of a planar map G and the corresponding set of vertex coordinates $\mathcal{V}$ (region vectors). The resulting drawing of G on $S_{\mathcal{V}}$ is a geodesic embedding (no crossings)



proof (assume there are edge crossings)

Fact 1: edge crossing are of the form
(as orthogonal arcs cannot cross)



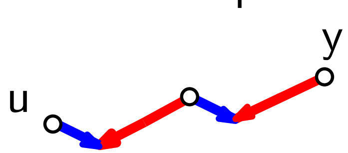
$u_0 > y_0$

$u_1 < y_1$

$z_0 > v_0$

Fact 2: edges (u, v) and (z, y) are of same color, lying on the same plane: $u_2 = y_2$ (in the example)

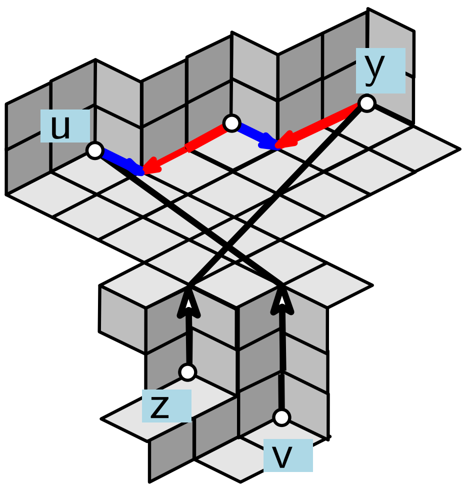
Fact 3: vertices u and y have the same z-coordinate
thus there is a bi-directed path P^* between u and y



Geodesic embeddings are planar drawings

Thm: Consider a Schnyder wood of a planar map G and the corresponding set of vertex coordinates $\mathcal{V}$ (region vectors). The resulting drawing of G on $S_{\mathcal{V}}$ is a geodesic embedding (no crossings)

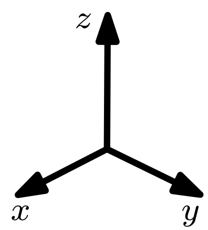
proof (assume there are edge crossings)

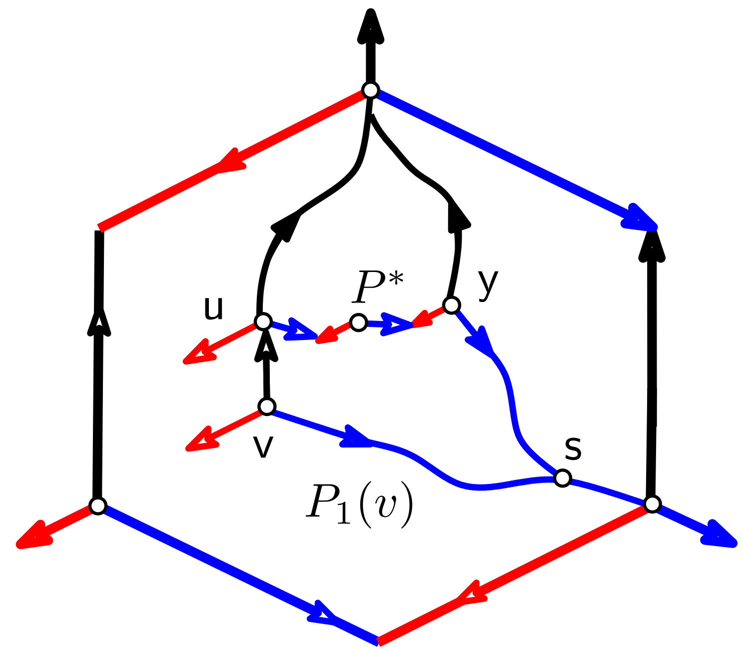


$u_0 > y_0$

$u_1 < y_1$

$z_0 > v_0$

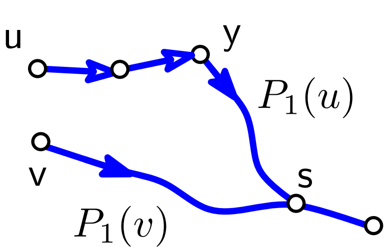


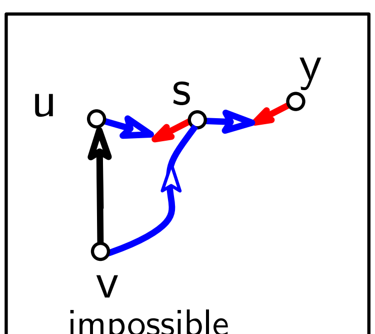


Let $P^* :=$ bi-directed path between u and y

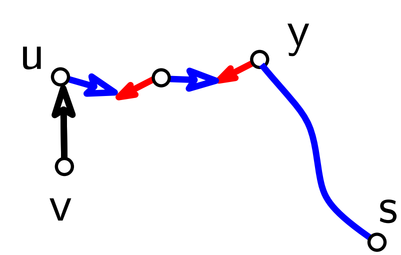
Let $s :=$ first vertex at the crossing of $P_1(u)$ and $P_1(v)$

Claim: s cannot belong to the path P^* ⟶ s belong to $P_1(v)$ and $s \neq y$





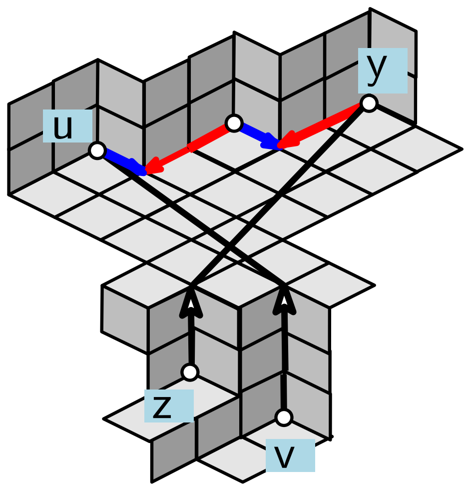
(there is a cycle in $T_2 \cup T_0^{-1} \cup T_1^{-1}$: violates previous theorem)



Geodesic embeddings are planar drawings

Thm: Consider a Schnyder wood of a planar map G and the corresponding set of vertex coordinates $\mathcal{V}$ (region vectors). The resulting drawing of G on $S_{\mathcal{V}}$ is a geodesic embedding (no crossings)

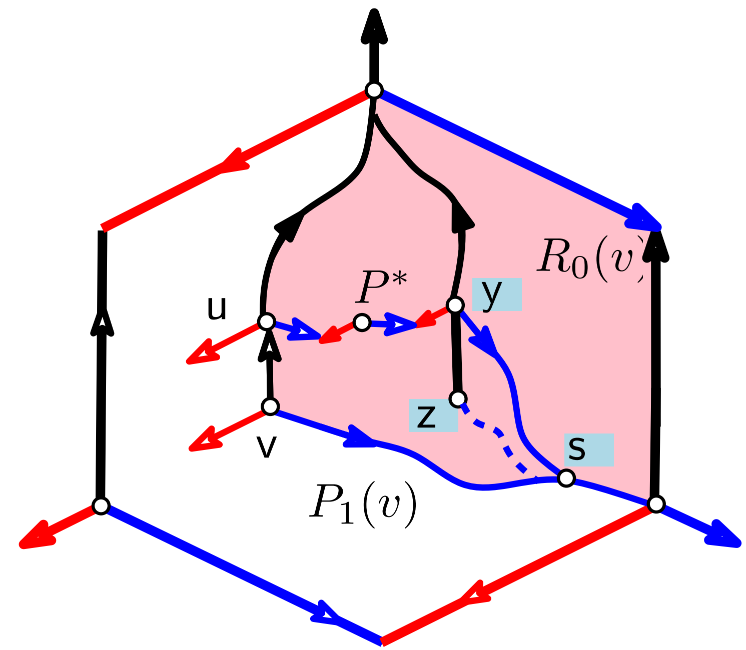
proof (assume there are edge crossings)



$u_0 > y_0$

$u_1 < y_1$

$z_0 > v_0$



Let $s :=$ first vertex at the crossing of $P_1(u)$ and $P_1(v)$

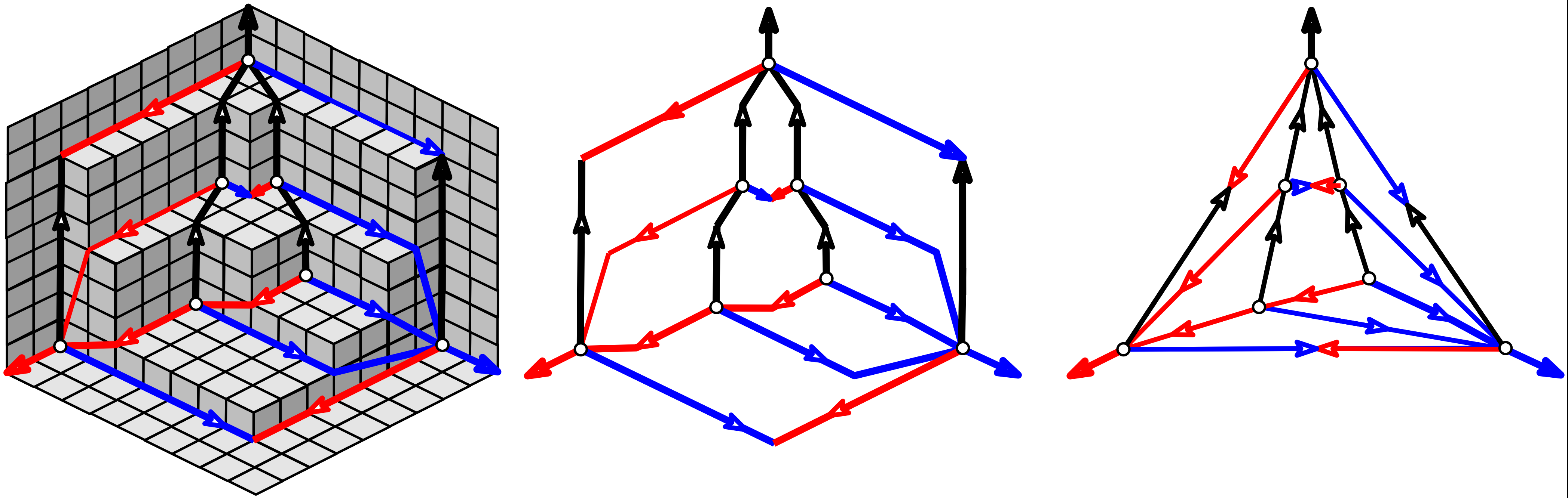
Remark: y is an inner vertex in the (red) region $R_0(v)$
(since there is red path from y to u)

by assumption (z, y) is an edge of G ⟶ (z, y) belong to $R_0(v)$ ⟶ z belong to $R_0(v)$

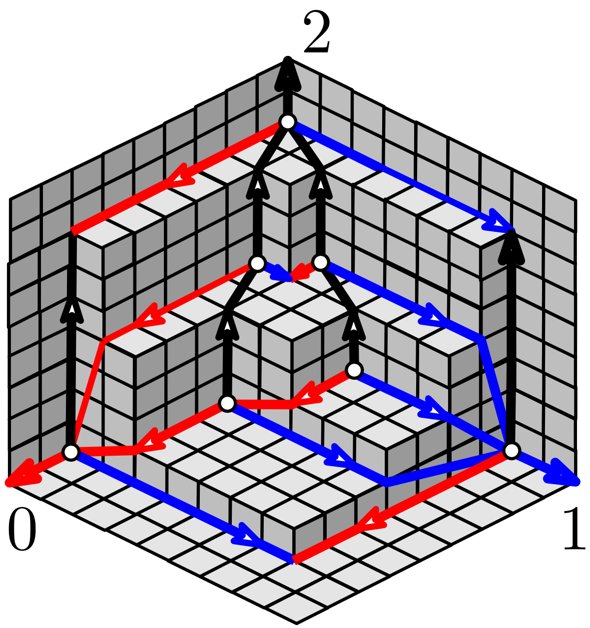
Since (z, y) belongs to $R_0(v)$ we have: $R_0(z) \subset R_0(v)$, implying $v_0 \geq z_0$
(contradiction)

From geodesic embeddings to straight-line planar drawings

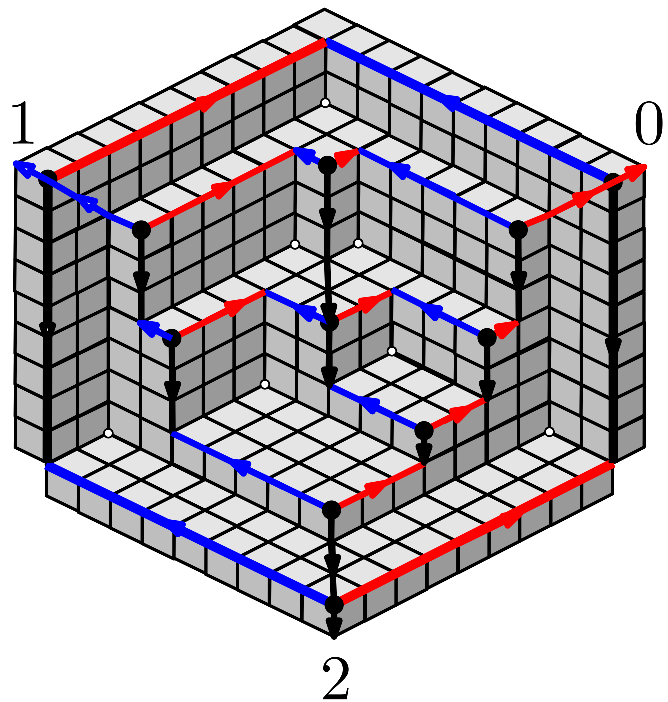
Thm: Given a planar (3-connected) map G, the region counting algorithm leads to a planar straight-line drawing of G (no edge corssings). Moreover, the faces of G are convex.



Primal/dual geodesic embeddings



Schnyder wood of the primal graph



Schnyder wood of the dual graph