Schnyder woods and applications

October 13, 2021

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Some facts about planar graphs

(“As I have known them”)
Some facts about planar graphs

Thm (Schnyder, Trotter, Felsner)

$G$ planar if and only if $\text{dim}(G) \leq 3$

Thm (Kuratowski, excluded minors)

$G$ planar if and only if $G$ contains neither $K_5$ nor $K_{3,3}$ as minors

Thm (Koebe-Andreev-Thurston)

Every planar graph with $n$ vertices is isomorphic to the intersection graph of $n$ disks in the plane.

Thm (Tutte)

\[
E(\rho) := \sum_{(i,j) \in E} |x(v_i) - x(v_j)|^2 = \sum_{(i,j) \in E} (x_i - x_j)^2 + (y_i - y_j)^2
\]

\[
x(v_i) = \sum_{j \in \mathcal{N}(i)} \frac{1}{\deg(v_i)} x(v_j)
\]
Straight-line planar drawings of planar graphs

**Thm (Schnyder 1990)**

- Face counting via Schnyder woods

**Thm (De Fraysseix, Pack Pollack 1989)**

- Shift algorithm via Canonical orderings
- Tutte barycentric embedding
- FPP algorithm

**Spring embedder (Eades, 1984)**
- Force-directed paradigm
- Easy to implement
- Pretty slow: $O(n^2)$ or $O(n \log n)$ time per iteration
  
  $F_a(v) = c_1 \cdot \sum_{(u,v) \in E} \frac{\log(dist(u,v))}{c_2}$

  $F_r(v) = c_3 \cdot \sum_{u \in V} \frac{1}{\sqrt{dist(u,v)}}$

**FPP algorithm**

- Linear time algorithms
- $O(n) \times O(n)$ grid drawings
- Not trivial to implement
- Extremely fast: they can process millions of vertices per second

**Tutte barycentric embedding**

- Minimize the spring energy
- Solve large sparse linear systems
- Easy to implement
- Not very fast: they can process approximately $10^4$ vertices per second
Straight-line planar drawings of planar graphs

**Thm (Schnyder 1990)**

- Face counting via Schnyder woods

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- FPP algorithm
  - Shift algorithm via Canonical orderings

**Linear time algorithms**

- \(O(n) \times O(n)\) grid drawings
- Not trivial to implement: they can process millions of vertices per second

**Tutte barycentric embedding**

- Solve large sparse linear systems
  - Easy to implement
- Not very fast: they can process \(\approx 10^4\) vertices per second

**Timing performances**

- Schnyder drawing or FPP algorithm:
  - Less than 1 second
  - (Java, 2.66GHz Intel i7 CPU)

- Chinese dragon (655k vert.)

- Solve sparse linear systems with the conjugate gradient solver of MTJ (Java) library
  - Numeric precision 10^{-6}

\[
E(\rho) := \sum_{(i,j) \in E} |x(v_i) - x(v_j)|^2 = \sum_{(i,j) \in E} (x_i - x_j)^2 + (y_i - y_j)^2
\]

- Minimize the spring energy

- ISP layout
- PC layout

- (Numeric precision 10^{-6})

- Solve sparse linear systems with the conjugate gradient solver of MTJ (Java) library
  - Numeric precision 10^{-6}

- Easy to implement
- Not very fast: they can process \(\approx 10^4\) vertices per second
**Thm (Koebe-Andreev-Thurston)**

Every planar graph with $n$ vertices is isomorphic to the intersection graph of $n$ disks in the plane.

**Voronoi cell:**

$$C(s_i) = \{x / d(s_i, x) \leq d(s_j, x) \forall i \neq j\}$$

**Delaunay Triangulation:**

$s_i$ is a neighbour $s_j$ if $C(s_i) \cap C(s_j) \neq \emptyset$

**General Position:**

- No 3 points collinear
- No 4 points co-circular

**Alternative def:** There is an edge $(s_i, s_j)$ if there is an empty circle supporting $s_i$ and $s_j$.

$\Rightarrow$: each face is supported by an empty circle.
Using triangles to measure distances

Thm (de Fraysseix, Ossona de Mendez, Rosenstiehl, ’94)

Every planar triangulation is TD-Delaunay realizable

Chew, ’89

TD-Delaunay: triangular distance Delaunay triangulations

- Distance triangulaire :
  \[ d(u,v) = \text{taille du plus petit triangle équilatérale à base horizontale centré en } u \text{ contenant } v. \]

- Rq : \( d(u,v) \neq d(v,u) \) en général
Schnyder woods and canonical orderings: overview of applications

(graph drawing, graph encoding, succinct representations, compact data structures, exhaustive graph enumeration, bijective counting, greedy drawings, spanners, contact representations, planarity testing, untangling of planar graphs, Steinitz representations of polyhedra, ... )
Some (classical) applications

(Chuang, Garg, He, Kao, Lu, Icalp’98)
(He, Kao, Lu, 1999)

Graph encoding \((4n \text{ nits})\)

\[
\begin{align*}
\overline{T}_0 &= ([]) ([]) ([]) ([]) ([]) ([]) ([]) ([]) ([]) ([]) ([]) \\
\overline{T}_2 &= 00000101010100110111
\end{align*}
\]

\[c_n = \frac{2(4n+1)!}{(3n+2)!(n+1)!}\]

\(\Rightarrow\) optimal encoding \(\approx 3.24\) bits/vertex

Thm (Schnyder '90)

Planar straight-line grid drawing (on a \(O(n \times n)\) grid)
More ("recent") applications

Schnyder woods, TD-Delaunay graphs, orthogonal surfaces and Half-Θ₆-graphs

Every planar triangulation admits a greedy drawing (Dhandapani, Soda08)

(conjectured by Papadimitriou and Ratajczak for 3-connected planar graphs)
Schnyder woods
(definitions)
A Schnyder wood of a (rooted) planar triangulation is partition of all inner edges into three sets $T_0$, $T_1$ and $T_2$ such that

i) edge are colored and oriented in such a way that each inner nodes has exactly one outgoing edge of each color

ii) colors and orientations around each inner node must respect the local Schnyder condition
Schnyder woods: equivalent formulation

[Schnyder labeling]

[3-orientation]
Schnyder labeling (3-connected maps): definition

3-connected graphs [Felsner]

3-connected map $M$

A1) the angles at $a_i$ have labels $i + 1, i - 1$

A2) rule for vertices: at each vertex there are non-empty intervals of labels 0, 1 and 2 (listed counter-clockwise)

A3) rule for faces: at each inner faces the angles define three non-empty intervals of labels 0, 1 and 2 in ccw order. For the outer face the angles are listed clockwise.
Schnyder woods (3-connected maps): definition

W1) edges have one or two (opposite) orientations. If an edge is bo-oriented than the two direction have distinct colors
W2) the edges at $a_i$ are outgoing of color $i$
W3) **local rule for vertices:** at each vertex there are three outgoing edges (one in each color) satisfying the local Schnyder rule
W4) there is no interior face whose boundary is a directed cycle in one color

3-connected graphs [Felsner]
Lemma

Given a Schnyder labeling of $M^\sigma$, the angles of each edges have colors 0, 1, 2 and are of the following 2 types:

\[
\begin{align*}
&\begin{array}{ccc}
\times & \times & \times \\
\sigma & \sigma & \sigma \\
\end{array} \\
&\begin{array}{ccc}
\times & \times & \times \\
\sigma & \sigma & \sigma \\
\end{array}
\]

proof:
Lemma

Given a Schnyder labeling of $M^\sigma$, the angles of each edges have colors 0, 1, 2 and are of the following 2 types:

<p>| | | |</p>
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proof:

possibly valid configurations

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forbidden configurations

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</table>

use a counting argument (double counts the angles)

$$\sum d(v) + \sum d(f) = 3n + 3|f| = 3|E| + 6$$

where

- $d(v)$: number of label changes for the angles around $v$
- $d(f)$: number of label changes for the angles in face $f$

Let $\alpha_i$ be a vertex

$\alpha_1 e \alpha_2$

$\alpha_4 e \alpha_3$

at vertex $\alpha_i$ there are two label changes

$\epsilon(e) = 3$ for all (normal) edges

$$\epsilon(e) = \begin{cases} 
0 & \text{if } e \text{ is an edge} \\
3 & \text{otherwise}
\end{cases}$$
Schnyder labelings: angles at exterior vertices

Corollary
Given a Schnyder labeling of $M^\sigma$, all interior angles at a vertex $a_i$ have label $i$.

\[
\begin{align*}
&\text{angle at vertex } a_i \\
&\quad\begin{array}{c}
\begin{array}{c}
\text{angle at vertex } a_i \\
\text{angle at vertex } a_i \\
\text{angle at vertex } a_i \\
\end{array}
\end{array}
\end{align*}
\]
Theorem
There is a correspondence between the Schnyder labelings of $M^\sigma$ and the Schnyder woods of $M^\sigma$. 

Correspondence between Schnyder labelings and Schnyder woods

Schnyder wood + Schnyder labeling of $M^\sigma$
Theorem
There is a correspondence between the Schnyder labelings of \( M^\sigma \) and the Schnyder woods of \( M^\sigma \)

proof: Assume \( M^\sigma \) is endowed with a Schnyder labeling

Assume (W4) is violated: there is a cycle in one color

Then the coloring rule of bi-oriented edges implies that all angles have the same color
Correspondence between Schnyder labelings and Schnyder woods

**Theorem**

There is a correspondence between the Schnyder labelings of $M^\sigma$ and the Schnyder woods of $M^\sigma$

**proof:** Assume $M^\sigma$ is endowed with a Schnyder wood

use a counting argument (double counts the angles around vertices/faces/edges)

\[
d(v) = 3 \quad d(e) = \begin{cases} 
3 & \text{for all (normal) edges} \\
2 & \text{for the three half-edges}
\end{cases}
\]

**Remark:**

Turning around a face in ccw direction

The number of changes $d(f)$ is a multiple of 3, and $d(f) > 0$

the angle will be $i$ or $i + 1$

(otherwise there is a directed cycle of edges in one color)

\[
\sum_v d(v) + \sum_f d(f) = \sum_e d(e) \quad 3n + \sum_f d(f) = 3|E| + 6
\]

Euler formula implies $\sum_f d(f) = 3|F| + 6$

$d(f) = 3$ for all faces
Correspondence between Schnyder labelings and Schnyder woods

Remark:

The condition (W4) of Schnyder woods is important

valid Schnyder labeling
conditions (W1)-(W4) of Schnyder woods are satisfied

not valid Schnyder labeling
condition (W4) of Schnyder woods is not satisfied
Theorem [Schnyder '90] \( T_i := \) digraph defined by directed edges of color \( i \)

The three sets \( T_0, T_1, T_2 \) are spanning trees of the inner vertices of \( T \) (each rooted at vertex \( v_i \))
**Spanning property for triangulations**

**Theorem [Schnyder '90]**

The three sets $T_0$, $T_1$, $T_2$ are spanning trees of the inner vertices of $\mathcal{T}$ (each rooted at vertex $v_i$)

**proof (use a counting argument)**

**Claim 1:** $T_i$ does not contain cycles

(assume there are monochromatic cycles, by contradiction)

**Case 1:** $C$:=non oriented monochromatic cycle of size $k$

there is a vertex $u$ violating Schnyder rule

**Case 2:** $C$:=monochromatic cycle of size $k$ (cw or ccw) oriented

Schnyder local rule implies:

(count edges in the triangulation bounded by the cycle)

$e_i = 3n_i + k$

3 outgoing edges for inner vertices

1 outgoing edges for boundary vertices

**Triangulations with a boundary**

$f_i = 2n_i + k - 2$

$e_i = 3n_i + (k - 3)$

$k := \#\text{boundary edges} = \#\text{boundary vertices}$

$n_i := \#\text{inner vertices}$

$\text{Schnyder local rule implies:}$

3 outgoing edges for inner vertices

1 outgoing edges for boundary vertices
Spanning property for triangulations

**Theorem** [Schnyder '90]
The three sets $T_0$, $T_1$, $T_2$ are spanning trees of the inner vertices of $T$ (each rooted at vertex $v_i$)

**proof** (use a counting argument)

**Claim 2:** $T_i$ is connected
(by contradiction, assume there are several disjoint components)

Let $G$ be a connected component not containing $v_i$

$G$ is connected and without cycles
then $G$ is a tree: $|G|$ vertices and $|G| - 1$ edges
all vertices of $G$ are inner vertices (distinct from $v_0$, $v_1$ and $v_2$)

there is a vertex $u \in G$ violating Schnyder rule:
no outgoing edge of color $i$
Spanning property for 3-connected maps

$T_i := \text{digraph defined by directed edges of color } i$

**Theorem** Let $(T_0, T_1, T_2)$ a Schnyder wood of $\mathcal{M}$. Then each digraph $D_i := T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ is acyclic

**proof:**

Let $Z$ a directed cycle enclosing a region $F$ of minimal size

**Claim 1:** $F$ is a single face

**case a:** $x \in F$

$F'$ is a smaller directed cycle

**case b:** $F$ is empty of vertices

there is an edge inside $F$

**Claim 2:** there is no face $F$ whose boundary is a directed cycle

Visit $F$ in ccw order starting from $v$ and propagate colors (first color is $i$): there is no angle with label $i-1$

The coloring rule for faces is violated

**Corollary:** Each sets $T_i$ is spanning tree $\mathcal{M}$ (rooted at vertex $a_i$)
Corollary:
Each sets $T_i$ is spanning tree $\mathcal{M}$ (rooted at vertex $a_i$)

Corollary
For each inner vertex $v$ the three monochromatic paths $P_0, P_1, P_2$ directed from $v$ toward each vertex $a_i$ are vertex disjoint (except at $v$) and partition the inner faces into three sets $R_0(v), R_1(v), R_2(v)$

proof: the existence of two paths $P_i(v)$ and $P_{i+1}(v)$ which are crossing would contradicts previous theorem
**Consequences**

**Efficient graph data structure for planar graphs**
There exist a (simple) data structure of size $O(n \log n)$ bits supporting constant time adjacency test between vertices

**Adjacency matrix**

\[
A_G[i, j] = \begin{cases} 
1 & v_i \text{ adjacent } v_j \\ 
0 & \text{otherwise} 
\end{cases}
\]

space: $O(n^2)$ bits
adjacency: $O(1)$ time

**Menger theorem for planar triangulations**

Schnyder woods allows us to compute in linear time, for any pair of vertices $(u, v)$, 3 vertex disjoint paths between $u$ and $v$

**Adjacency lists**

space: $O(n \log n)$ bits
adjacency: $O(n)$ time

**Thm (Menger)**
If $G$ is $k$-connected, then for each pair \( u, v \) there exist $k$ disjoint path from $u$ to $v$
Efficient graph data structure for planar graphs
There exist a (simple) data structure of size $O(n \log n)$ bits supporting constant time adjacency test between vertices

Truncated adjacency lists: store only 3 successors

Menger theorem for planar triangulations
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Case 1:

Case 2:
Efficient graph data structure for planar graphs
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Menger theorem for planar triangulations
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Case 1:

Case 2:
Number and structure of Schnyder woods

**Counting Schnyder woods:** (there are graphs admitting an exponential number)

- **[Bonichon '05]**
  \[\# \text{Schnyder woods of triangulations of size } n: \approx 16^n\]
  (all Schnyder woods over all distinct triangulations of size \(n\))

- **[Felsner Zickfeld '08]**
  \[2.37^n \leq \max_{T \in \mathcal{T}_n} |SW(T)| \leq 3.56^n\]
  (count of Schnyder woods of a fixed triangulation)

\(\mathcal{T}_n := \text{class of planar triangulations of size } n\)

\(SW(T) := \text{set of all Schnyder woods of the triangulation } T\)

---

reversal of oriented triangles
**Thm:** The set $S(T)$ of all distinct Schnyder woods of a triangulation $T$ is a distributive lattice

The min is the unique $S \in S(T)$ with no clockwise circuit.

Flip: \[ \begin{array}{c} \vdots \quad \vdots \quad \vdots \quad \vdots \\ \text{min} \\ \text{max} \end{array} \] to \[ \begin{array}{c} \vdots \quad \vdots \quad \vdots \quad \vdots \\ \text{min} \\ \text{max} \end{array} \]

[Ossona de Mendez'94], [Felsner'03]
The traversal starts from the root face

**Theorem**
Every planar triangulation admits a Schnyder wood, which can be computed in linear time.
The traversal starts from the root face

**Theorem**
Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

perform a vertex conquest at each step
Theorem

Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

The traversal starts from the root face

perform a vertex conquest at each step

The diagram illustrates the construction process, starting from the root face $v_0$ and ending at $v_1$, with intermediate steps emphasizing the vertex conquests at each level $G_k$.
The traversal starts from the root face

Theorem
Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

perform a vertex conquest at each step

\[ G_k \]

\[ G_{k-1} \]
The traversal starts from the root face

**Theorem**
Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

perform a vertex conquest at each step

$G_k \
\downarrow \
G_{k-1}$
The traversal starts from the root face

**Theorem**
Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

**Invariant:**
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**Theorem**

Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

**Invariant:**
Schnyder woods: existence (algorithm I)

[incremental vertex shelling, Brehm’s thesis]

The traversal starts from the root face

**Theorem**

Every planar triangulation admits a Schnyder wood, which can be computed in linear time.

**Invariant:**
Planar straight-line drawings
(of planar graphs)
Planar straight-line drawings

[Wagner’36]
[Fary’48]
Planar straight-line drawings

existence of straight-line drawing

Classical algorithms:

spring-embedding

incremental (Shift-algorithm)

face-counting principle

[Wagner’36]
[Fary’48]
[Stein’51]
[De Fraysseix, Pach, Pollack 89]
[Tutte’63]
[Schnyder’90]
Planar straight-line drawings

⇒

[Wagner’36]
[Fary’48]
Face counting algorithm
(Schnyder algorithm, 1990)
**Face counting algorithm**

Geometric interpretation

\[ v = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 \]

where \( \alpha_i \) is the normalized area

\[
\mathbf{v} \rightarrow (5, 6, 2) := (v_0, v_1, v_2)
\]

\[
\mathbf{u} \rightarrow (7, 3, 3) := (u_0, u_1, u_2)
\]

\[
v = \frac{|R_0(v)|}{|F|-1} x_0 + \frac{|R_1(v)|}{|F|-1} x_1 + \frac{|R_2(v)|}{|F|-1} x_2 + |R_i(v)| \text{ is the number of triangles in } R_i(v)
\]

**Theorem**

For a 3-connected planar map \( \mathcal{M} \) having \( f \) vertices, there is drawing on a grid of size \((f - 1) \times (f - 1)\)

**Theorem (Schnyder, Soda ’90)**

For a triangulation \( \mathcal{T} \) having \( n \) vertices, we can draw it on a grid of size \((2n - 5) \times (2n - 5)\), by setting \( x_0 = (2n - 5, 0) \), \( x_1 = (0, 0) \) and \( x_2 = (0, 2n - 5) \).
Face counting algorithm: example

\( \mathcal{T} \) endowed with a Schnyder wood

Input: \( \mathcal{T} \)

\[
\begin{align*}
    a & \rightarrow (0, 0) \\
    b & \rightarrow (0, 1) \\
    i & \rightarrow (1, 0) \\
    c & \rightarrow \left( \frac{9}{13}, \frac{1}{13} \right) \\
    d & \rightarrow \left( \frac{5}{13}, \frac{6}{13} \right) \\
    e & \rightarrow \left( \frac{7}{13}, \frac{4}{13} \right) \\
    f & \rightarrow \left( \frac{3}{13}, \frac{3}{13} \right) \\
    g & \rightarrow \left( \frac{4}{13}, \frac{8}{13} \right) \\
    h & \rightarrow \left( \frac{1}{13}, \frac{4}{13} \right)
\end{align*}
\]
Face counting algorithm: example

Input: $\mathcal{T}$

$\mathcal{T}$ endowed with a Schnyder wood

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$\rightarrow (13,0,0)$</td>
</tr>
<tr>
<td>b</td>
<td>$\rightarrow (0,13,0)$</td>
</tr>
<tr>
<td>c</td>
<td>$\rightarrow (9,3,1)$</td>
</tr>
<tr>
<td>d</td>
<td>$\rightarrow (5,6,2)$</td>
</tr>
<tr>
<td>e</td>
<td>$\rightarrow (2,7,4)$</td>
</tr>
<tr>
<td>f</td>
<td>$\rightarrow (7,3,3)$</td>
</tr>
<tr>
<td>g</td>
<td>$\rightarrow (1,4,8)$</td>
</tr>
<tr>
<td>h</td>
<td>$\rightarrow (8,1,4)$</td>
</tr>
<tr>
<td>i</td>
<td>$\rightarrow (0,0,13)$</td>
</tr>
</tbody>
</table>
Input: $\mathcal{T}$

$\mathcal{T}$ endowed with a Schnyder wood

$x + y + z = 2n - 5$

$\begin{align*}
a & \rightarrow (13, 0, 0) \\
b & \rightarrow (0, 13, 0) \\
c & \rightarrow (9, 3, 1) \\
d & \rightarrow (5, 6, 2) \\
e & \rightarrow (2, 7, 4) \\
f & \rightarrow (7, 3, 3) \\
g & \rightarrow (1, 4, 8) \\
h & \rightarrow (8, 1, 4) \\
i & \rightarrow (0, 0, 13)
\end{align*}$
Lemma Let $(T_0, T_1, T_2)$ a Schnyder wood of $\mathcal{M}$.
If $u \in R_i(v)$ then $R_i(u) \subseteq R_i(v)$
If $u \in R_i^{int}(v)$ then $R_i(u) \subset R_i(v)$

proof:

Case 1: $u \in R_i^{int}(v)$

first step: compute the paths $P_{i+1}(u)$ and $P_{i-1}(u)$
They must intersect the boundary of $R_i(v)$ at $x$ and $y$
Remark: $x$ and $y$ are different from $v$
and we have $y \in P_{i+1}(u)$ and $x \in P_{i-1}(u)$
(because of Schnyder rule)
so we have: $R_i(u) \subset R_i(v)$

Case 2a: $u \in P_{i-1}(v)$

$R_i(u) \subset R_i(v)$

Case 2b: $u \in P_{i-1}(v)$

(u, u') is bi-oriented
Proceed by induction on the path $P_{i-1}(v)$
$R_i(u) \subseteq R_i(v)$
Remarks: Let \((u, v)\) of color \(i\) oriented from \(u\) to \(v\)

\[ v \in P_i(u) \]

\[ \begin{cases} 
  v \in R_{i+1}(u) \\
  v \in R_{i-1}(u) \\
  u \in R_i(v) 
\end{cases} \]

**Case 1:** \((u, v)\) is unidirectional

\[ R_i(u) \subset R_i(v) \]
\[ R_{i+1}(v) \subset R_{i+1}(u) \]
\[ R_{i-1}(v) \subset R_{i-1}(u) \]

**Case 2:** \((u, v)\) is bidirectional

\[ R_i(u) \subset R_i(v) \]
\[ R_{i-1}(v) \subseteq R_{i-1}(u) \]
\[ R_{i+1}(v) \subseteq R_{i+1}(u) \]
**Regions and coordinates**

**Remarks:** Let \((u, v)\) of color \(i\) oriented from \(u\) to \(v\)

\[
v =: \frac{|R_0(v)|}{|F|-1} x_0 + \frac{|R_1(v)|}{|F|-1} x_1 + \frac{|R_2(v)|}{|F|-1} x_2 = \frac{v_0}{|F|-1} x_0 + \frac{v_1}{|F|-1} x_1 + \frac{v_2}{|F|-1} x_2
\]

- \(R_i(u) \subseteq R_i(v)\), \(|R_i(u)| \leq |R_i(v)|\), \(u_i \leq v_i\)
  
  - \(v_0 + v_1 + v_2 = f - 1\)
  
  - \(R_i(u) \subset R_i(v)\)
  
  - \(R_{i+1}(v) \subset R_{i+1}(u)\)
  
  - \(R_{i-1}(v) \subset R_{i-1}(u)\)

- For every edge \((u, v)\) there are some indices \(i, j \in \{0, 1, 2\}\) s.t.
  
  \[
  u_i < v_i \\
  u_j > v_j
  \]
**Regions and coordinates**

**Remarks:** Let \((u, v)\) of color \(i\) oriented from \(u\) to \(v\)

\[ v = \frac{|R_0(v)|}{|F|-1} x_0 + \frac{|R_1(v)|}{|F|-1} x_1 + \frac{|R_2(v)|}{|F|-1} x_2 = \]
\[= \frac{v_0}{|F|-1} x_0 + \frac{v_1}{|F|-1} x_1 + \frac{v_2}{|F|-1} x_2 \]

- \(R_i(u) \subseteq R_i(v) \quad |R_i(u)| \leq |R_i(v)| \quad \Rightarrow \quad u_i \leq v_i\)

- \(v_0 + v_1 + v_2 = f - 1\)

**Remark:**

is \(u_i < v_i\) the \(u\) lies in the white sector

the outgoing edges \((v, w)\) lie in the gray sectors
Barycentric representation of a planar graph
(validity of the Schnyder drawing)
Definition: A barycentric representation of a graph $G$ is defined by a mapping $f(v) \rightarrow (v_0, v_1, v_2) \in \mathbb{R}^3$ satisfying:

- $v_0 + v_1 + v_2 = 1$, for each vertex $v$
- for each edge $(x, y) \in E$ and each vertex $z \notin \{x, y\}$ there is an index $k \in \{0, 1, 2\}$ such that $x_k < z_k$ and $y_k < z_k$.

Remark: The Schnyder drawing of a planar triangulation $\mathcal{T}$ is a barycentric mapping.

proof: it follows from previous inclusion properties of regions $R_i$.
**Barycentric representation of a planar graph**

**Definition:** A barycentric representation of a graph \( G \) is defined by a mapping \( f(v) \rightarrow (v_0, v_1, v_2) \in \mathbb{R}^3 \) satisfying:

- \( v_0 + v_1 + v_2 = 1 \), for each vertex \( v \)
- for each edge \((x, y) \in E\) and each vertex \( z \notin \{x, y\}\) there is an index \( k \in \{0, 1, 2\} \) such that

Example:

\[
\begin{align*}
(0, 0, 1) & \rightarrow f(x), f(y) \\
(1, 0, 0) & \rightarrow f(z) \\
(1, 0, 0) & \rightarrow f(z) \\
(1, 0, 0) & \rightarrow f(z)
\end{align*}
\]

**Remark:** The Schnyder drawing of a planar triangulation \( T \) is a barycentric mapping.

**Proof:** it follows from previous inclusion properties of regions \( R_i \).
Barycentric representation of a planar graph

Theorem
A barycentric representation defines a planar straight-line drawing of \( G \), in the plane spanned by \((1,0,0),(0,1,0)\) and \((0,0,1)\).

proof:

Claim 1: for each edge \((x,y) \in E\) and each vertex \(z \notin \{x,y\}\), \(f(z)\) cannot lie on \((f(x),f(y))\).

\[
f(z) = tf(x) + (1-t)f(y)\ , \text{ for } t \in [0,1] \\
z_k = tx_k + (1-t)y_k < t z_k + (1-t)z_k = z_k \\
\text{for } k \in \{0,1,2\}
\]
**Theorem**

A barycentric representation defines a planar straight-line drawing of $G$, in the plane spanned by $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

**Proof:**

**Claim 2:** given two edges $(x,y)$, $(u,v)$ of $G$ they cannot cross.

By definition there are four indices $i, j, k, l \in \{0, 1, 2\}$

\[
\begin{align*}
  u_i, v_i &< x_i & x_k, y_k &< u_k \\
  u_j, v_j &< y_j & x_l, y_l &< v_l
\end{align*}
\]

**Fact:** $i \neq k$

If $i = k$ we would have

\[
\begin{align*}
  u_k &< x_k \\
  v_k &< x_k
\end{align*}
\]

contradicting

\[
x_k, y_k < u_k
\]

\[
i \neq k, l \text{ and } j \neq k, l
\]

In the example above we have $i = j = 2$

There exists a separating line $l$ parallel to one of the sides of the outer triangle, that separates $(u,v)$ and $(x,y)$.

The line $l$ parallel to $[(1,0,0), (0,1,0)]$ separates $(u,v)$ and $(x,y)$. 

In the example above we have $i = j = 2$.
**Problem:** how to efficiently compute $|R_i(v)|$ (for all $v \in V$)?

**Remark:** the number of faces $|R_i(v)|$ can be retrieved from: the number of inner vertices and the number of vertices on the path $P_{i+1}(v)$ and $P_{i-1}(v)$
Problem: how to efficiently compute $|R_i(v)|$ (for all $v \in V$)?

Remark: the number of faces $|R_i(v)|$ can be retrieved from: the number of inner vertices and the number of vertices on the path $P_{i+1}(v)$ and $P_{i-1}(v)$

$$R_i(v) = 4$$ (inner faces)

$$\partial R_i(v) := (P_{i+1}(v) + P_{i-1}(v)) - 1 = 4$$ (outer vertices)

$$\sum_{w \in P_{i+1}} |t_w| + \sum_u |t_u| = 1$$ (inner vertices)
**Practical performances**

average timings (over 100 executions)

Timing performances (pure **Java**, on a core i7-5600 U, 2.60GHz, 1GB Ram): Schnyder woods can process $\approx 1.43M - 1.92M$ vertices/seconds

Two Schnyder drawings of a sphere graph
Practical performances

<table>
<thead>
<tr>
<th>Tutte</th>
<th>Schnyder</th>
<th>FPP layout</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Tutte" /></td>
<td><img src="image2" alt="Schnyder" /></td>
<td><img src="image3" alt="FPP layout" /></td>
</tr>
</tbody>
</table>

**Timings**

- Fish model
  - FFP layout
  - Dragon
  - Eros

- Random
  - Tutte embedding
  - Dragon
  - Eros
Schnyder woods and orthogonal surfaces

(next slides are courtesy of Vincent Pilaud)
GEODESIC MAPS ON ORTHOGONAL SURFACES
DEF. dominance order in $\mathbb{R}^3 = u \leq v \iff u_i \leq v_i$ for all $i \in [3]$ (componentwise).

DEF. cone dominating $y \in \mathbb{R}^3$
$\Delta_y = \{ z \in \mathbb{R}^3 \mid y \leq z \}$

cone dominated by $y \in \mathbb{R}^3$
$\nabla_y = \{ x \in \mathbb{R}^3 \mid x \leq y \}$

(= upper ideal of $y$)  (= lower ideal of $y$)
DEF. dominance order in $\mathbb{R}^3 = u \leq v \iff u_i \leq v_i$ for all $i \in [3]$ (componentwise).

DEF. cone dominating $y \in \mathbb{R}^3$

$\Delta_y = \{ z \in \mathbb{R}^3 \mid y \leq z \}$

cone dominated by $y \in \mathbb{R}^3$

$\nabla_y = \{ x \in \mathbb{R}^3 \mid x \leq y \}$

DEF. $\langle V \rangle = \{ z \in \mathbb{R}^3 \mid v \leq z \text{ for some } v \in V \} = \bigcup_{v \in V} \Delta_v$.

orthogonal surface $S_V = \text{boundary of } \langle V \rangle$ (assume now that $V = \text{antichain}$)
DEF. On an orthogonal surface $S_V$, define

- **elbow geodesic** = union of the segments from $u, v \in V$ to $u \lor v = [\max(u_i, v_i)]_{i \in [n]}$,
- **coordinate arcs** = (not always bounded) segments from $v \in V$ in an axis direction.
DEF. geodesic embedding of a map $M$ on a surface $S_V = \text{drawing of } M$ on $S_V$ st:

(G1) there is a bijection between the points of $V$ and the vertices of $M$,

(G2) every edge of $M$ is an elbow geodesic in $S_V$ and every bounded coordinate arc is part of an edge of $M$,

(G3) the drawing is crossing-free.
THM. If $V$ is an axial antichain, then a geodesic embedding of a map $M$ on $S_V$ induces a Schnyder wood on $M$.

proof idea:

- label the angles according to the color of the flat region containing it,
- orient and color the edges according to the three axis. An elbow geodesic can get one or two colors depending on whether it contains one or two bounded coordinate arcs.
**THM.** If \( V \) is an axial antichain, then a geodesic embedding of a map \( M \) on \( S_V \) induces a Schnyder wood on \( M \).

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**THM.** If $V$ is an axial antichain, then a geodesic embedding of a map $M$ on $S_V$ induces a Schnyder wood on $M$.

**THM.** Given a Schnyder wood $W$ on a planar map $M$, the region vectors of the vertices of $M$ with respect to $W$ form an axial antichain $V$ inducing a geodesic embedding of $M$ on $S_V$. 
**THM.** The projection of the geodesic embedding onto the plane \( v_1 + v_2 + v_3 = f - 1 \) gives a planar drawing of \( M \) whose edges are bended segments. Replacing them by straight segments preserves the non-crossing-freeness.

**Proof idea:** when straightening the geodesic embedding, the elbow geodesic joining \( u \) and \( v \) is controlled by \( \nabla_{u\wedge v} \).