Algorithms and combinatorics for geometric graphs (**Geomgraphs**)

Lecture 1

Preliminaries on planar graphs

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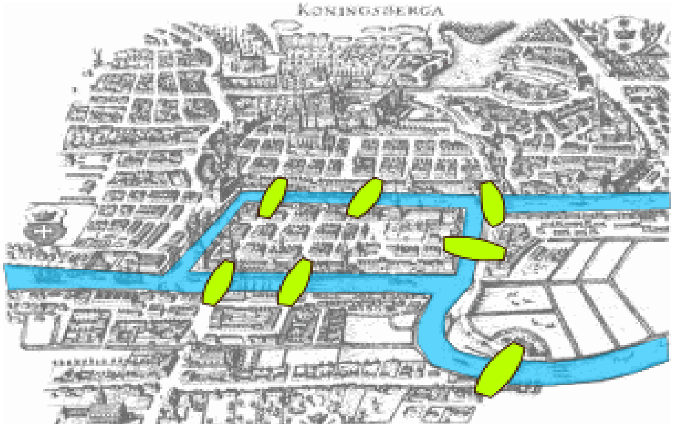
Part 0

Introduction and historical background

A short digression on (planar) graphs and their applications

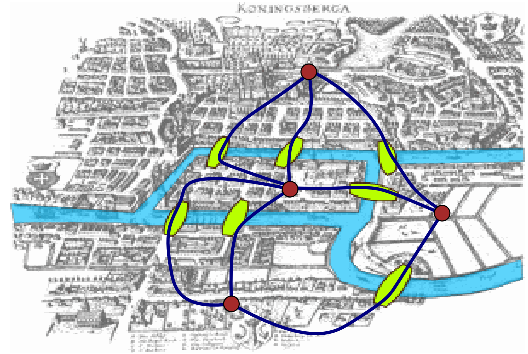
Origins of Graph Theory (back to Euler)

Solutio problematis ad geometriam situs pertinentis
(1735, presented to the St. Petersburg Academy)



(Königsberg, Prussia)

(from Wikipedia)



Eulerian path: it visits every edge exactly once

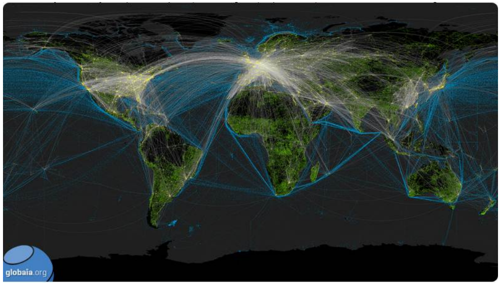
Theorem (Euler 1735, Hierholzer 1873)
A graph G contains an *Eulerian walk (path)* if and only if G is connected and the number of vertices of odd degree is 0 or 2.

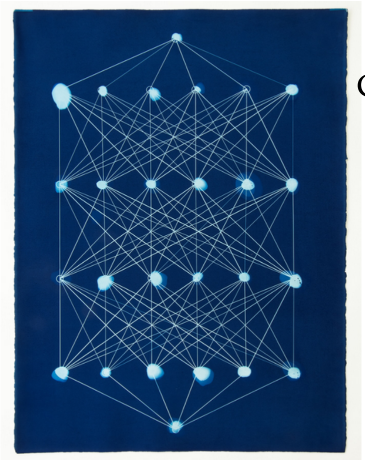
Theorem
A connected graph contains an *Eulerian circuit* if and only if there are no vertices of odd degree.

Nowdays graphs (networks) are ubiquitous

Social networks

Global transportation system



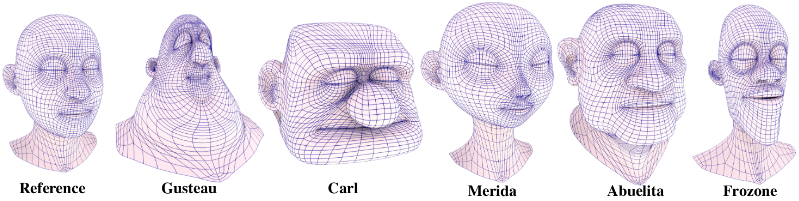


Neural Networks II (2024)

Clemens von Wedemeyer

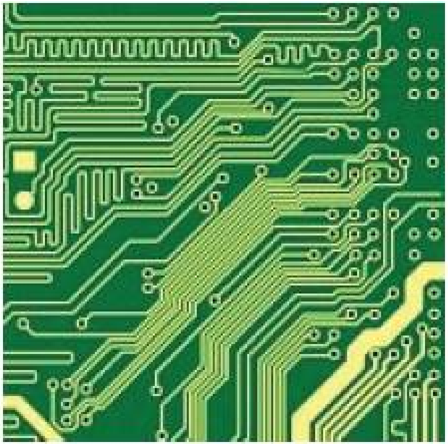
3D Geometric modeling

Pixar characters (image by De Goes et al.)

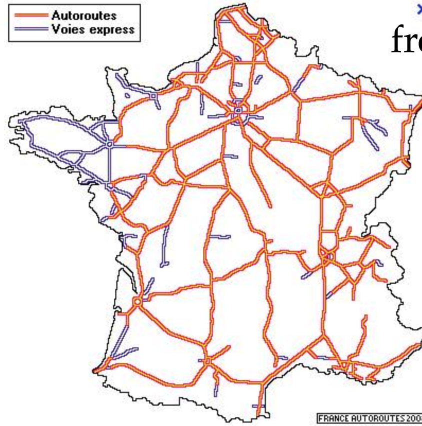


Planar graphs are nice (and important)

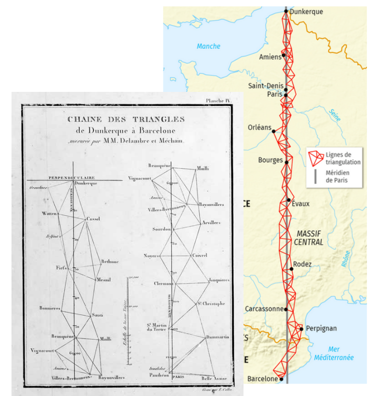
Design of integrated circuits (VLSI)



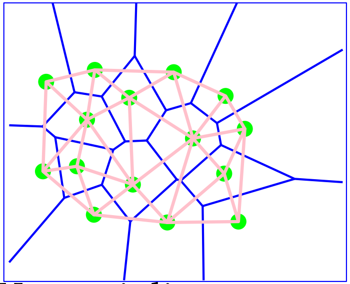
french roads network



triangulations were already used in 18th century: approximation of the meridian
(Delambre et Méchain, 1792)

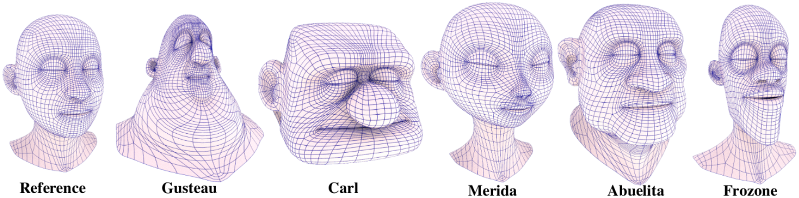


Planar graphs in computational geometry and geometric modeling
(Delaunay triangulations, Voronoi diagrams, 3D meshes, ...)

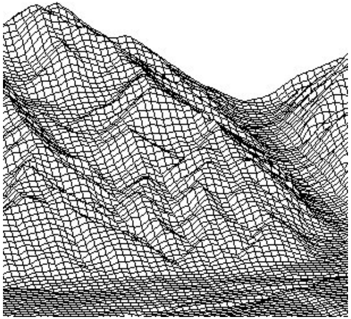


Voronoi diagram
Delaunay triangulation

3D Geometric modeling
Pixar characters (image by De Goes et al.)



GIS Technology



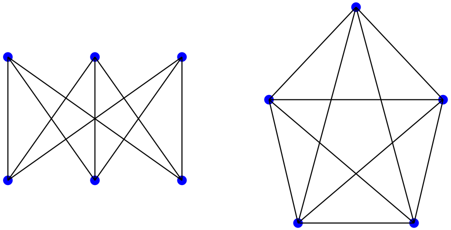
Terrain modelling

3D paper sculpture (*DT Workshop*)

Major results (on planar graphs) in graph theory

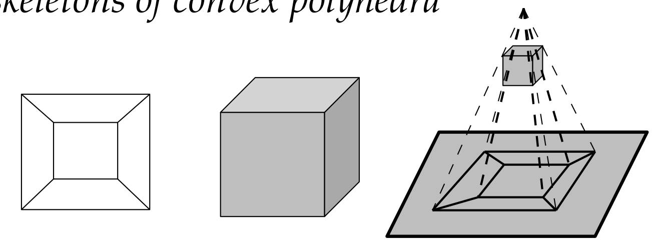
Kuratowski theorem (1930)
(cfr Wagner's theorem, 1937)

- G contains neither K_5 nor $K_{3,3}$ as minors

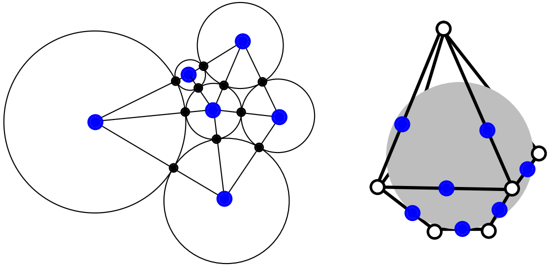


Thm (Steinitz, 1916)

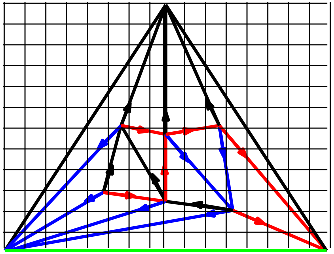
3-connected planar graphs are skeletons of convex polyhedra



Thm (Koebe-Andreev-Thurston)
Every planar graph with n vertices is isomorphic to the intersection graph of n disks in the plane.



Thm (Schnyder '90)



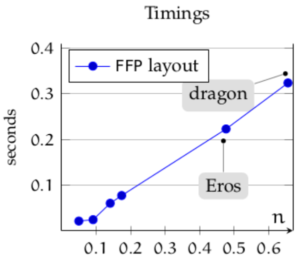
(dimension of partial orders)

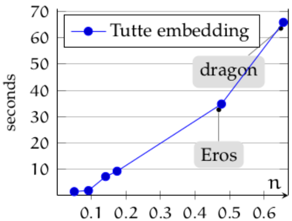
- G planar iff $dim(G) \leq 3$

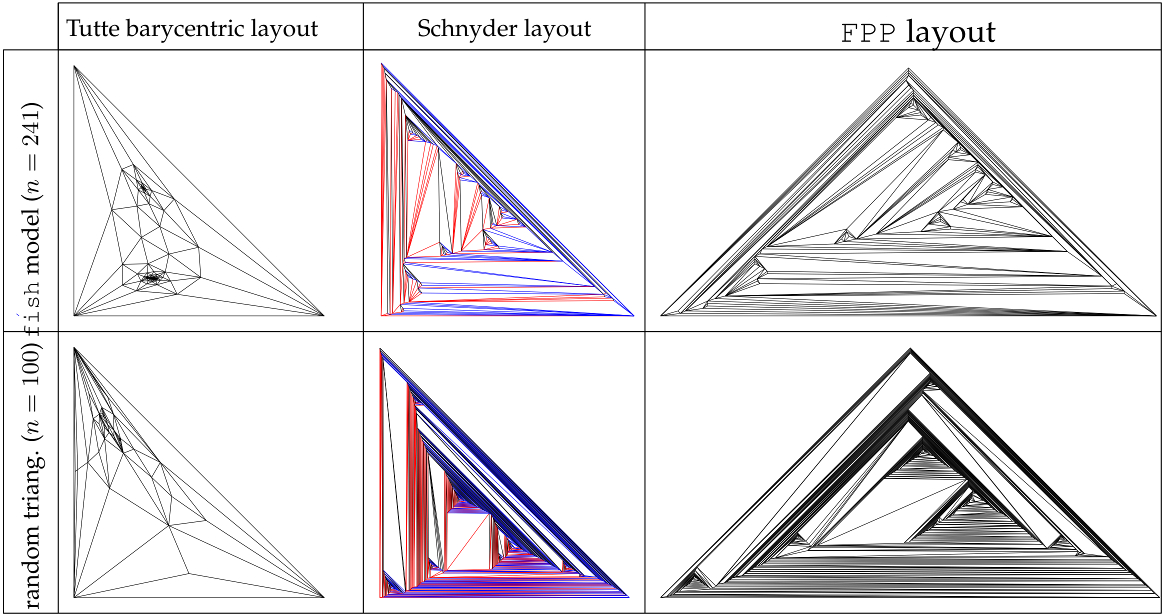
Efficient algorithms on planar graphs

Graph drawing

Planarity testing







Minimum spanning tree

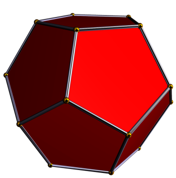
Planar Separators

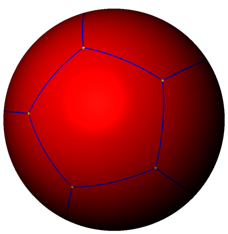
Minimum cut

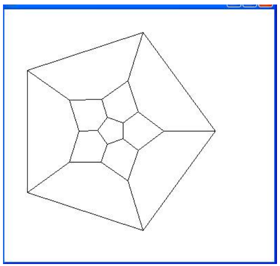
Part I

What is a planar graph?

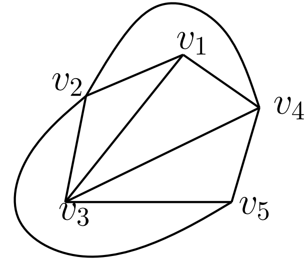
(some terminology: embedded graphs, topological and combinatorial maps)





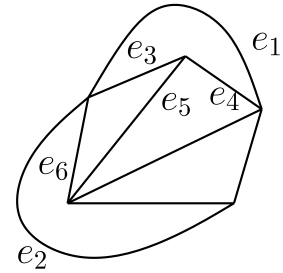


Graphs



A *graph* $G = (V, E)$ is a pair of:

- a set of *vertices* $V = (v_1, \ldots, v_n)$
- a collection of $E = (e_1, \ldots, e_m)$ elements of the cartesian product $V \times V = \{(u, v) \mid u \in V, v \in V\}$ (called *edges*).



circuit: a closed walk without repeated vertices

$$A_G[i,j] = \begin{cases} 1 & v_i \text{ adjacent } v_j \\ 0 & \text{otherwise} \end{cases}$$

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

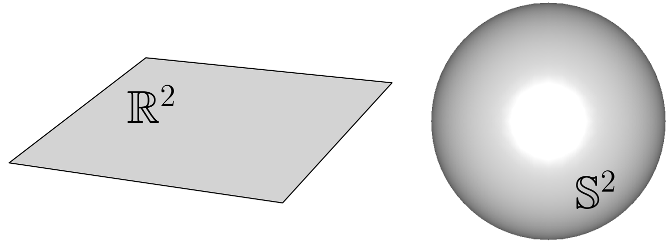
$$\begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{matrix}$$

$$D_G = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \cdots & \cdots & & & & & & & \\ \cdots & & & & & & & & \\ & & & & & & & & \end{bmatrix} \begin{matrix} v_1 \\ v_2 \\ \\ \\ \\ \end{matrix}$$

Planar drawings: some basic notion of topology

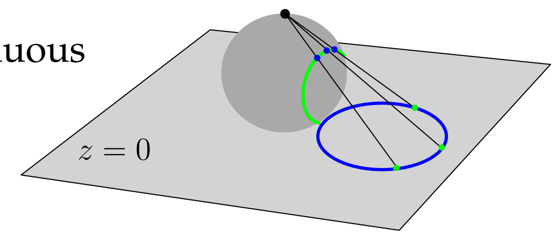
topological space: a set X with a collection of *open sets* (subsets of X) satisfying:

X itself and the empty set are open
the union of open sets is open
any finite intersection of open sets is open



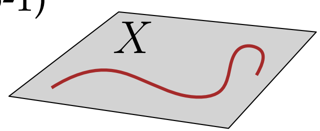
$f : X \to Y$ is *continuous*: the inverse image of an open set of Y is open

$f : X \to Y$ is *homeorphism*: f, f^{-1} are bijective and continuous



$\Pi : \mathbb{S}^2 \setminus N \longrightarrow \mathbb{R}^2$

a *path* is a continuous map $p : [0,1] \to X$
(the path is simple if p is 1-to-1)



Remark:
we consider topological spaces which are *Haussdorff* (any two distinct points have disjoint neighborhoods)

Planar drawings of planar graphs

an *embedding* of G into R^2 is a 1-to-1 continuous map satisfying:

(i) graph vertices are represented as points ;

(ii) edges are represented as paths (curves);

(iii) the images of vertices are distinct points

(iv) the images of edges simple (no self-intersections at the interior)

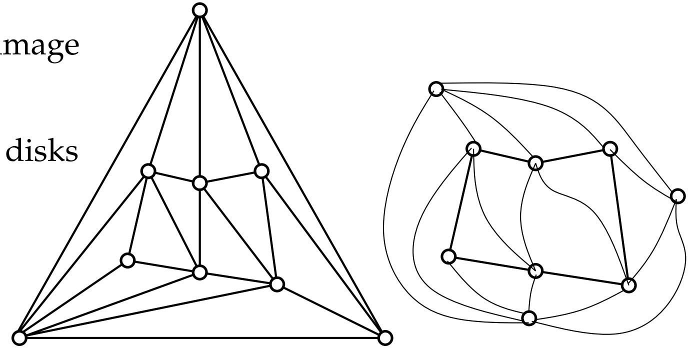
(v) the interior of the images of edges are disjoint (no crossings)

(vI) edges cannot pass trough a vertex (except at its extremities)

faces of a graph embedding: connected component of the image of the vertices/edges of G

cellular embedding: the faces are homeomorphic to open disks

planar graph: a graph admitting an embedding in the plane



two cellular embeddings defining the same planar graph

plane graph: a planar graph + a cellular embedding

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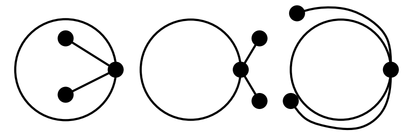
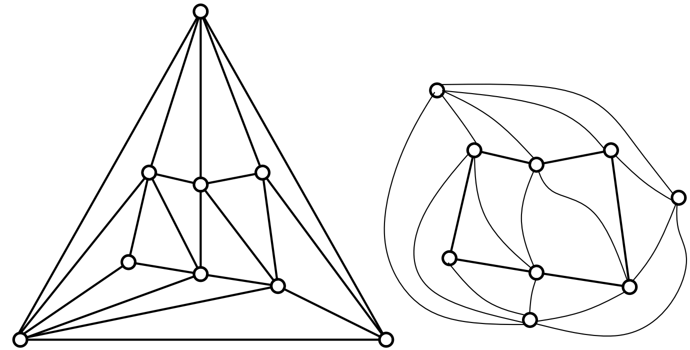


cellular embeddings of a graph defining the same (planar) map

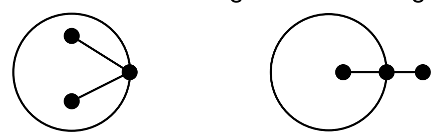


two cellular embeddings defining the same planar map

plane graph: a planar graph + a cellular embedding

(topological) map: cellular embedding up to homeomorphism (equivalence class)

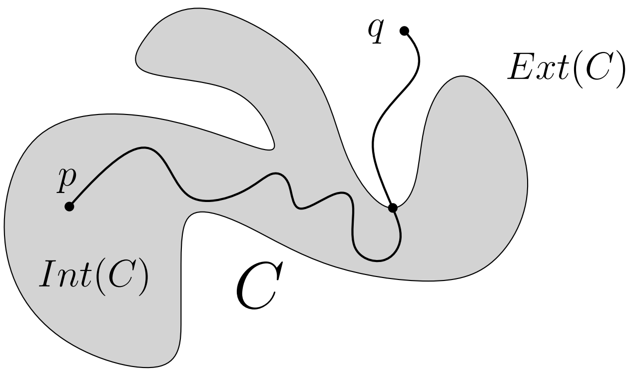
two different embeddings of the same graph



The Jordan curve theorem

Theorem

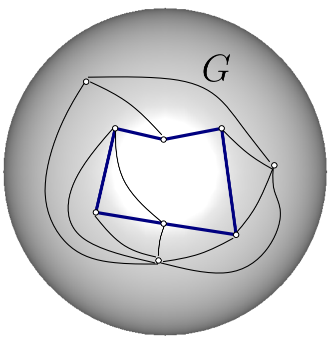
Any simple closed curve C in the plane partitions $\mathbb{R}^2$ into two disjoint arcwise-connected open sets.



($Ext(C)$ and $Int(C)$ are closed sets)

$$Ext(C) \cap Int(C) = C$$

Remark:
Any arc joining a point p in the (open) interior to a point q in the (open) exterior must meet C at least once.



Jordan curve Theorem (reformulation)

Let G a graph embedded on $\mathbb{S}^2$. Then G disconnets $\mathbb{S}^2$ if and only if it contains a circuit

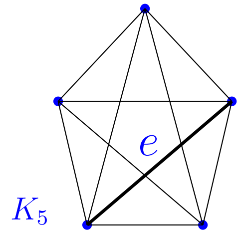
The Jordan curve theorem: application

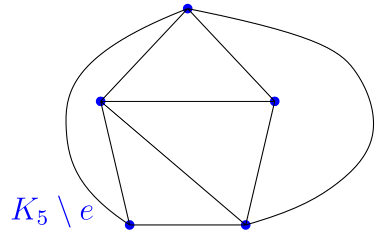
Theorem

Any simple closed curve C in the plane partitions $\mathbb{R}^2$ into two disjoint arcwise-connected open sets.

Theorem

The graph K_5 is not planar





Proof (topological)

(by contradiction) Let G be a planar embedding of K_5

K_5 is complete ⟶ it contains $C := \{v_1, v_2, v_3, v_1\}$ (simple cycle)

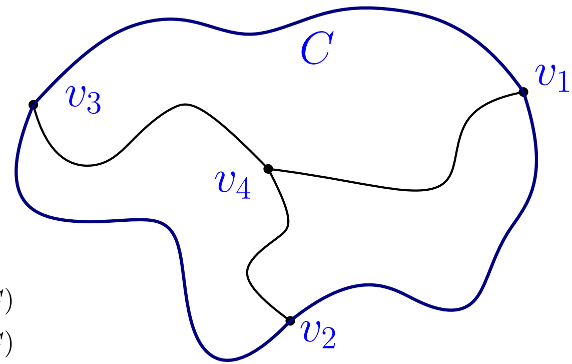
G planar ⟶ $f(C)$ simple closed curve (separating the plane)

w.l.o.g. assume $v_4 \in Int(C)$ ⟶ (G planar,no edge crossings)

$(v_4, v_1) \in Int(C)$

$(v_4, v_2) \in Int(C)$

$(v_4, v_3) \in Int(C)$



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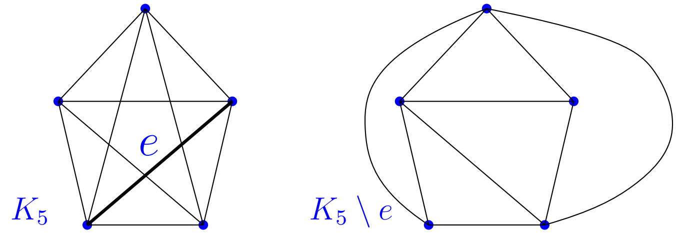

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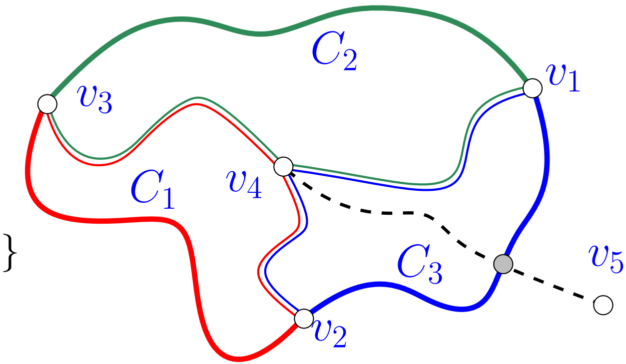


Consider 3 cycles

$$C_1 := \{v_2, v_3, v_4, v_2\}$$
$$C_2 := \{v_3, v_1, v_4, v_3\} \longrightarrow v_i \in ext(C_i)\ (i \in \{1, 2, 3\})$$
$$C_3 := \{v_1, v_2, v_4, v_1\}$$

K_5 is complete → it contains $(v_5, v_1), (v_5, v_2), (v_5, v_3)$ → $v_5 \in ext(C_i)\ (i \in \{1, 2, 3\})$
(Jordan curve theorem)

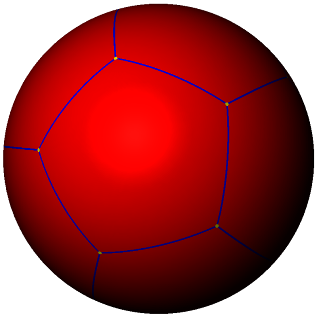
$v_5 \in ext(C)$ → (v_5, v_4) meets C somewhere (edge crossing, contradicting the planarity of G)
(Jordan curve theorem)

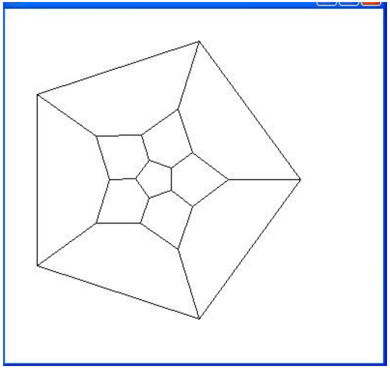
□

Planar graphs and graphs embeddable on the sphere are the same

Theorem

A graph G is embeddable on the sphere $\mathbb{S}^2$ if and only if it is embeddable on the plane





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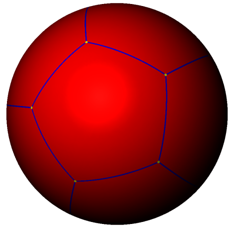
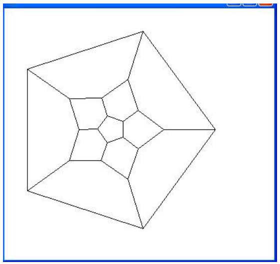
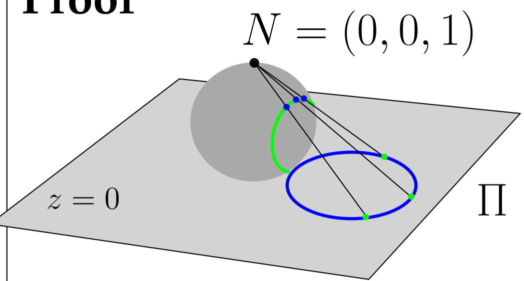




Proof



$$\Pi : \mathbb{S}^2 \setminus N \longrightarrow \mathbb{R}^2$$

Stereographic projection $\Pi : \mathbb{S}^2 \setminus N \longrightarrow \mathbb{R}^2$
(homemorphism: Π *and its inverse are bijective and continuous)*

$$\Pi^{-1}(x,y) \quad \begin{pmatrix} 2x/\chi \\ 2y/\chi \\ 1-2/\chi \end{pmatrix}$$

$$\chi := x^2 + y^2 + 1$$

Remark

To get a planar embedding of a graph G, just take a point N in the interior of a face of G on $\mathbb{S}^2$, and project on $\mathbb{R}^2$

$$\Pi(x,y,z) \quad \begin{pmatrix} \frac{x}{1-z} \\ \frac{y}{1-z} \end{pmatrix}$$

Combinatorial maps: representations and data structures

Cellularly embedded planar graphs as *combinatorial maps*

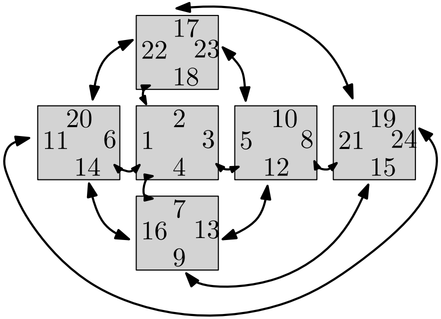
Let G a cellular graph embedding

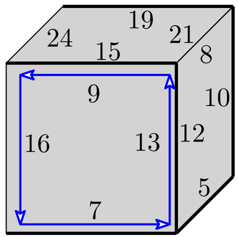
The *combinatorial map* associated to G is the set of closed walks, obtained walking around the boundary of each face (in our example in cw direction)

2 permutations on the set H of the $2m$ *darts*

(i) α involution without fixed point;

(ii) ϕ gives the cyclic ordering of the darts (edges) around each face





$$\phi = (1,2,3,4)(17,23,18,22)(5,10,8,12)\dots$$

$$\alpha = (2,18)(3,5)(4,7)(12,13)(9,15)\dots$$

2 permutations on the set H of the $2m$ *darts*

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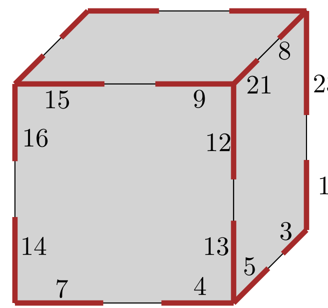

$$\sigma = (1,20,18)(4,5,13)(3,12,7)\dots$$

$$\alpha = (2,18)(3,5)(4,7)(12,13)(9,15)\dots$$

(*) $\alpha\sigma\phi = Id$; The two representations are dual to each other $3 \xrightarrow[\phi]{} 4 \xrightarrow[\sigma]{} 5 \xrightarrow[\alpha]{} 3$

(*) the action of the group generated by σ, α et ϕ is transitive on H.

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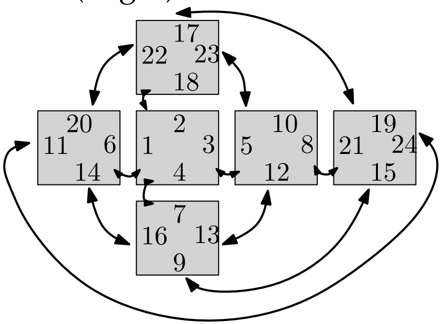

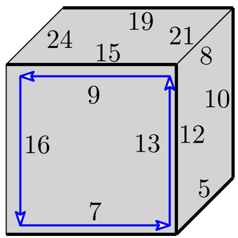

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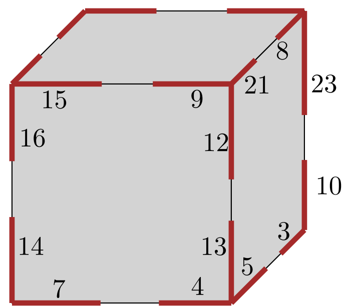
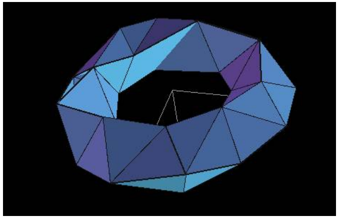
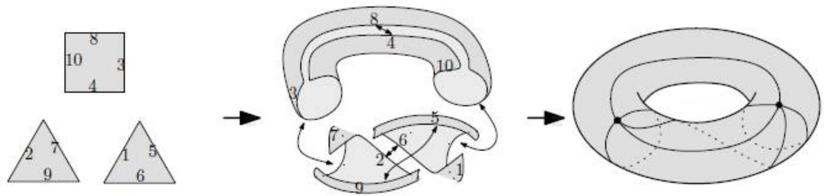




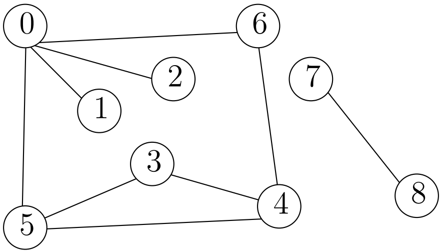


Graph: adjacency lists representation

easy to implement

quite compact

not efficient for traversal



for each face (of degree d), store:

- d references to adjacent vertices

for each vertex, store:

- 1 reference to its coordinates

```
class Point{
  double x;
  double y;
}
```

geometric information

```
class Vertex{
 Point p;
 List<Vertex> neigh
}
```

combinatorial information

Memory cost

$$\sum_i deg(v_i) = 2 \times e$$

Size (number of references)

Queries/Operations

List all vertices

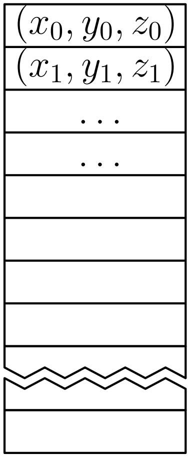
Test adjacency between u and v

Find the 3 neighboring faces of f

List the neighbors of vertex v

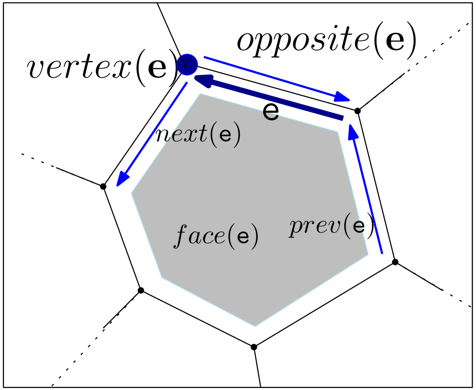
v_0	6 5 1 2
v_1	0
v_2	0
v_3	4 5
	5 6 3 6
	0 4 3
	4 0
	8
	7

vertex locations



Half-edge data structure: polygonal (orientable) meshes

2 half-edges per edge




```
class Halfedge{
  Halfedge  prev, next, opposite;
  Vertex v;
  Face f;
}
class Vertex{
  Halfedge  e;
  Point p;
}
class Face{
  Halfedge  e;
}
```

combinatorial information

```
class Point{
  double x;
  double y;
}
```

geometric information

Size (number of references)

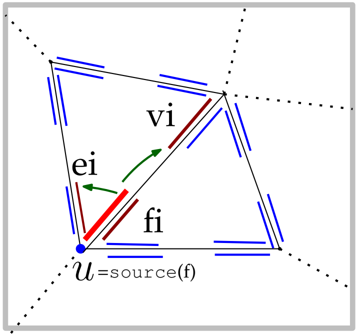
$$f + 5 \times h + n \approx 2n + 5 \times (2e) + n$$

```
public int degree() {
    Halfedge<X> e,p;
    if(this.halfedge==null) return 0;

    e=halfedge; p=halfedge.next;
    int cont=1;
    while(p!=e) {
        cont++;
        p=p.next;
    }
    return cont;
}
```

Flag representation

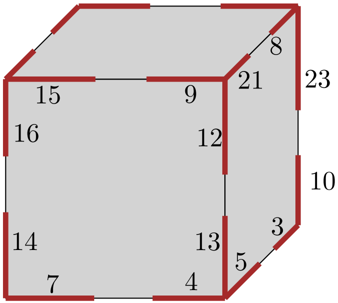
4 flags per edge




```
class Flag{
  Flag ei, fi, vi;
  Vertex u;
}

 class Vertex{
  Flag  f;
  Point p;
}
```

combinatorial information



$$0 \leq f \leq 4 * e - 1$$

$$0 \leq v \leq n - 1$$

navigation around vertices

```
vertexDegree(Flag f) {
  int j=0;
  Flag g=f;
  do {
    ++j;
   g=g.ei().fi();
  } while (g!=f);
  return j;
}
```

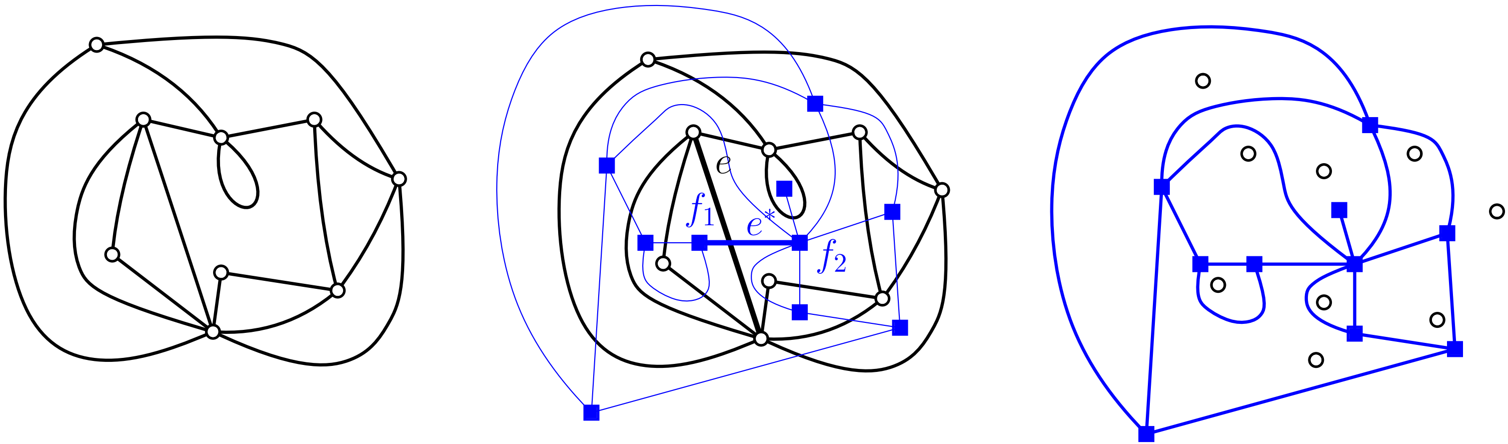
navigation around faces

```
faceDegree(Flag f) {
  int j=0;
  Flag g=f;
  do {
    ++j;
   g=g.ei().vi();
  } while (g!=f);
  return j;
}
```

Duality

Definition

Given a cellular graph embedding G on the sphere, its *dual graph* G^* is a graph embedding for which: we put a (dual) vertex f^* in the inteior of a face $f \in G$; and create a dual edge e^* crossing an edge $e \in G$



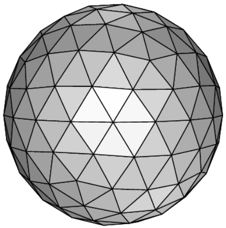
Remarks:

- The dual of a plane graph is connected (**exercise**)
- A dual graph embedding is also cellular
- The combinatorial map of the dual graph is uniquely defined
- $(G^*)^* \cong G$

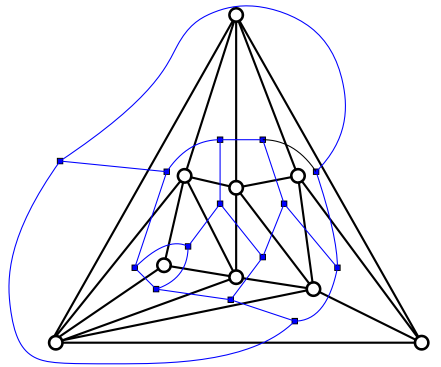
Duality

Definition

Given a cellular graph embedding G on the sphere, its *dual graph* G^* is a graph embedding for which: we put a (dual) vertex f^* in the inteior of a face $f \in G$; and create a dual edge e^* crossing an dedge $e \in G$



(genus 0) triangle mesh



planar triangulation

(simple connected plane graph, with all faces of degree 3)

Remark:

- A simple connected plane graph is a planar triangulation if and only if its dual is a cubic graph

Duality

Exercice:

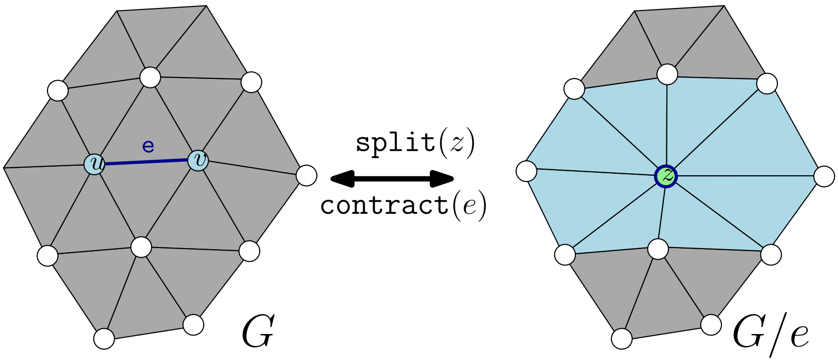
Given a plane graph G with m edges show that:

$$\sum_{f \in F} \texttt{degree}(f) = 2m$$

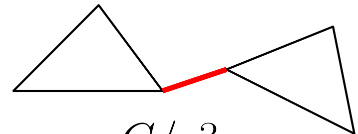
$$\sum_{v \in V} \texttt{degree}(v) = 2m$$

Duality: edge contractions and deletions

edge contraction and *vertex split*

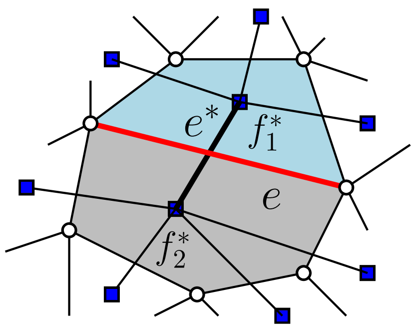


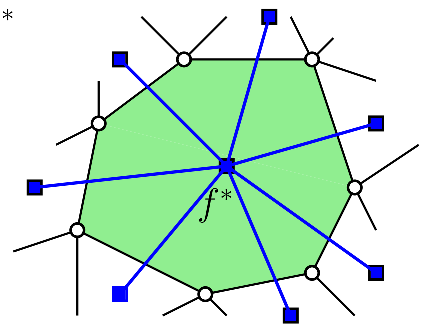
Remark:
Edge contractions and edge deletions preserve some properties



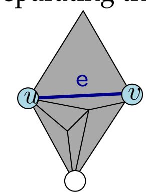
Property
Let G be a connected cellularly embedded graph, and e and non cut edge.
Then

$$(G \setminus e)^* \cong G^*/e^*$$





Remark:
What happens if e belongs to a separating triangle?



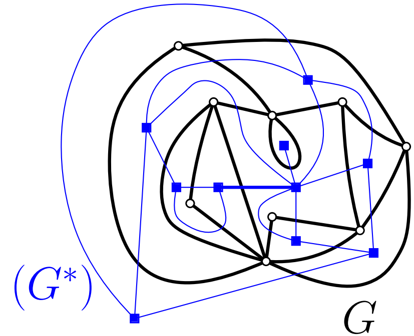
Duality

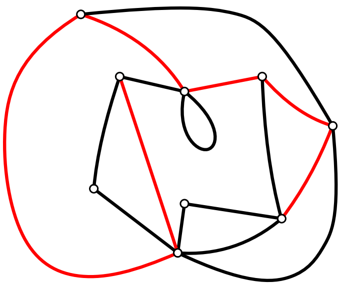
Lemma

Let us consider a graph embedding $G = (V, E)$ and its dual $G^* = (F^*, E^*)$, and a subset of edges $E' \subset E$. Then we have

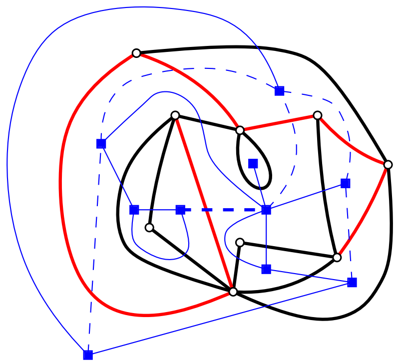
- (V, E') is acyclic if and only if $(F^*, (E \setminus E')^*)$ is connected

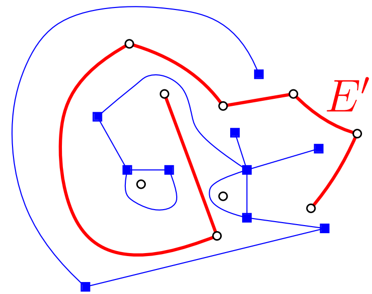
Corollary:
(V, E') is a spanning tree if and only if $(F^*, (E \setminus E')^*)$ is a spanning tree.





(V, E') is acyclic





$(F^*, (E \setminus E')^*)$ is connected

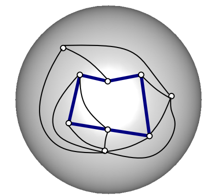
Remove the (dual) blue edges which are crossing the (red) edges in E'

Duality

Lemma

Let us consider a graph embedding $G = (V, E)$ and its dual $G^*(F^*, E^*)$, and a subset of edges $E' \subset E$. Then we have

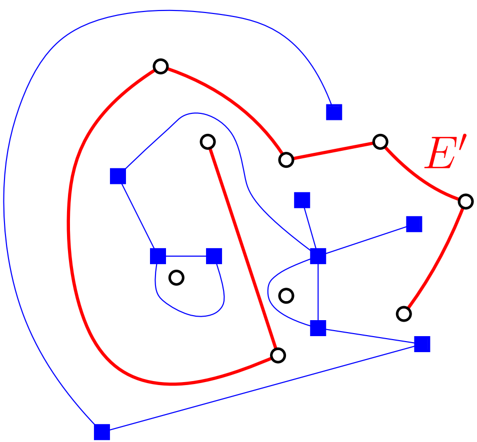
- (V, E') is acyclic if and only if $(F^*, (E \setminus E')^*)$ is connected.



(Jordan curve theorem)

Proof

(V, E') is acyclic $\Leftrightarrow$ $(\mathbb{S}^2 \setminus E')$ is connected



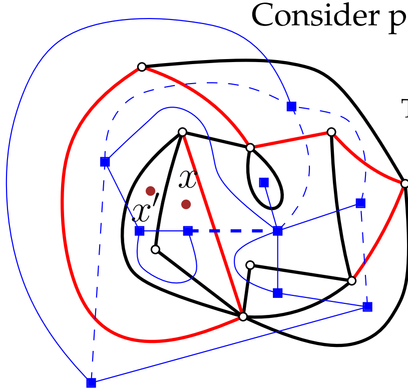
Claim: $(\mathbb{S}^2 \setminus E')$ is connected $\Leftrightarrow$ $(F^*, (E \setminus E')^*)$ is connected

Consider points $x \in f$ and $x' \in f'$ (in the embedding G)

There is a path $P_{xx'}$ { avoiding the edges in E' and; $P_{xx'}$ is lying only in f and f' }

$\Leftrightarrow$ f^* and f'^* are adjacent by some edge not in E'

$\Leftrightarrow$ f^* and f'^* are adjacent in $(F^*, (E \setminus E')^*)$



Part II

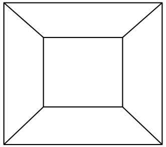
Euler formula and its consequences

Euler-Poincaré characteristic: topological invariant

$$\chi := n - e + f$$

One of the (11) world's most beautiful equations

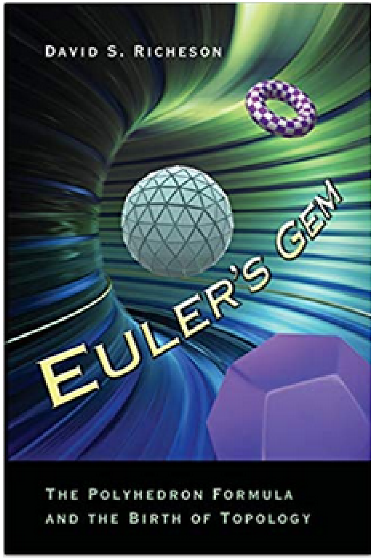
(according to `livescience.com`)



planar map

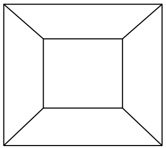
$$n - e + f = 2$$

Euler's relation

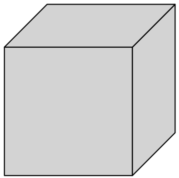


Euler-Poincaré characteristic: topological invariant

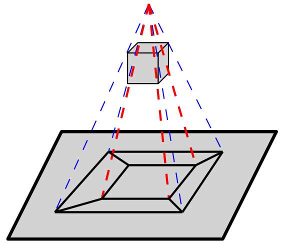
$$\chi := n - e + f$$



planar map



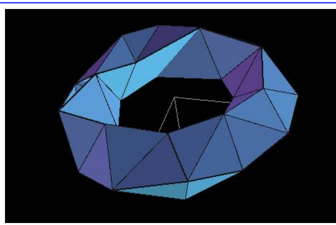
(convex) polyhedron



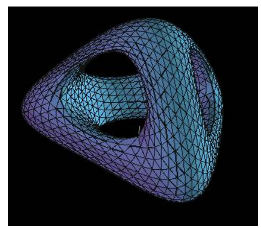
$$n - e + f = 2$$

Euler's relation

$\chi = 0$

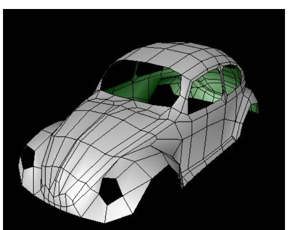


$\chi = -4$



$n = 1660$
$e = 4992$
$f = 3328$
$g = 3$

$$n - e + f = 2 - 2g$$



$n = 364$
$e = 675$
$f = 302$
$b = 11$
$g = 0$

$$n - e + f = 2 - b$$

Euler's relation: first proof

Theorem (Euler's relation)
Given a connected plane graph G we have:

$$v(G) - e(G) + f(G) = 2$$

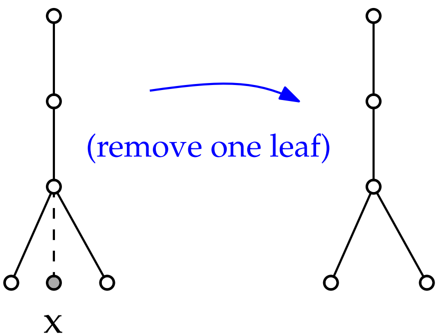
Let us first prove a preliminary result

Lemma

If G is a tree then we have: $e(G) = v(G) - 1$

proof: (induction on the nodes)

$$v(G \setminus x) = v(G) - 1$$
$$e(G \setminus x) = e(G) - 1$$



base case of the induction

$$v(G) = 1$$
$$e(G) = 0$$

Claim

Any tree contains at least one leaf

(exercise)

Euler's relation: first proof

Theorem (Euler's relation)
Given a connected plane graph G we have:

$$v(G) - e(G) + f(G) = 2$$

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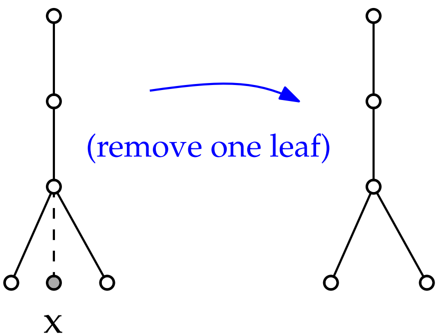


base case of the induction

$$v(G) = 1$$
$$e(G) = 0$$

Claim

Any tree contains at least one leaf

Claim 2
If a graph G all vertices have degree at least 2, then G contains a cycle.

(exercise)

Euler's relation: first proof

Theorem (Euler's relation)
Given a connected plane graph G we have:

$$v(G) - e(G) + f(G) = 2$$

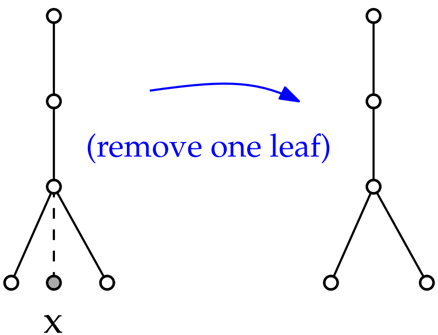
Let us first prove a preliminary result

Lemma

If G is a tree then we have: $e(G) = v(G) - 1$

proof: (induction on the nodes)

$$v(G \setminus x) = v(G) - 1$$
$$e(G \setminus x) = e(G) - 1$$



base case of the induction

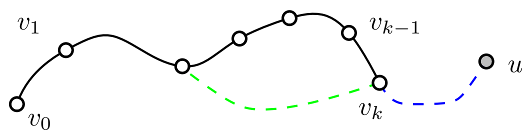
$$v(G) = 1$$
$$e(G) = 0$$

Claim

Any tree contains at least one leaf

Claim 2
If a graph G all vertices have degree at least 2, then G contains a cycle.

(solution) assume G is simple
(otherwise the statement is trivial)



let $P := v_0, v_1 \dots v_{k-1} v_k$ (path of maximal length in G)

$degree(v_k) \geq 2$
- $u \notin P$ — $P \cup \{u\}$ is longer
- $u \in P$ — $u = v_i \ (i \leq k-2)$ it defines a cycle

Euler's relation: first proof

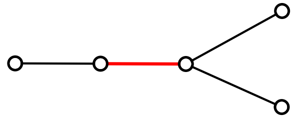
Theorem (Euler's relation)

Given a connected plane graph G we have:

$$v(G) - e(G) + f(G) = 2$$

proof: (induction on the edges)

(base case) $f(G) = 1$



G is a tree (use previous Lemma)

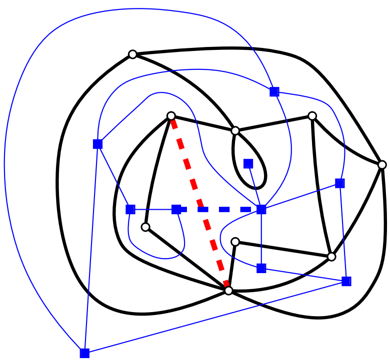
$$e(G) = v(G) - 1$$

(general case) $f(G) \geq 2$

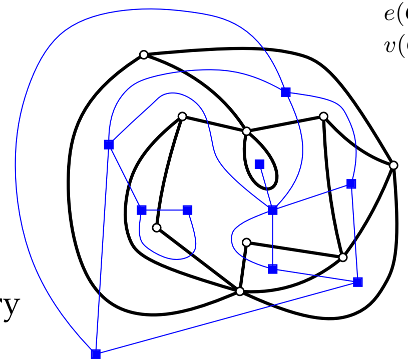
There is a non cut edge e → $G \setminus e$ is connected, $f(G \setminus e) = f(G) - 1$ → (induction hypothesis) $v(G \setminus e) - e(G \setminus e) + f(G \setminus e) = 2$

$f(G \setminus e) = f(G) - 1$
$e(G \setminus e) = e(G) - 1$
$v(G \setminus e) = v(G)$

$$v(G) - e(G) + f(G) = 2$$

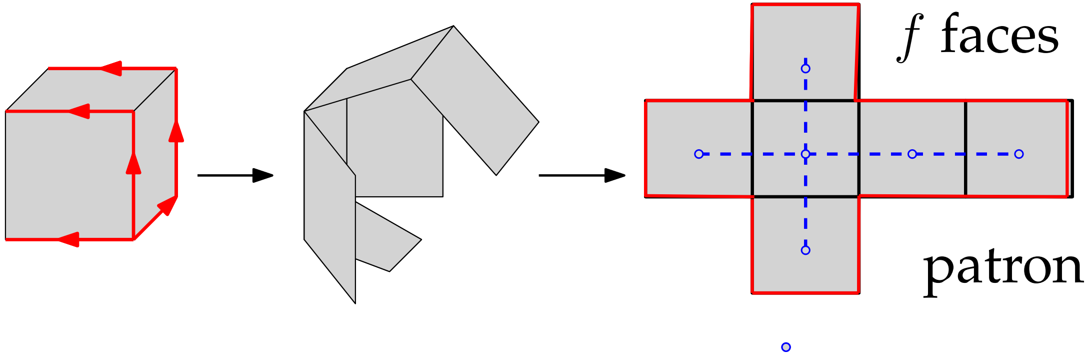


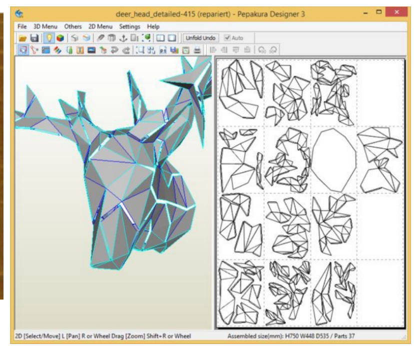
remove an arbitrary non cut edge e



Euler's relation: second proof (via the dual)

$$n - e + f = 2$$



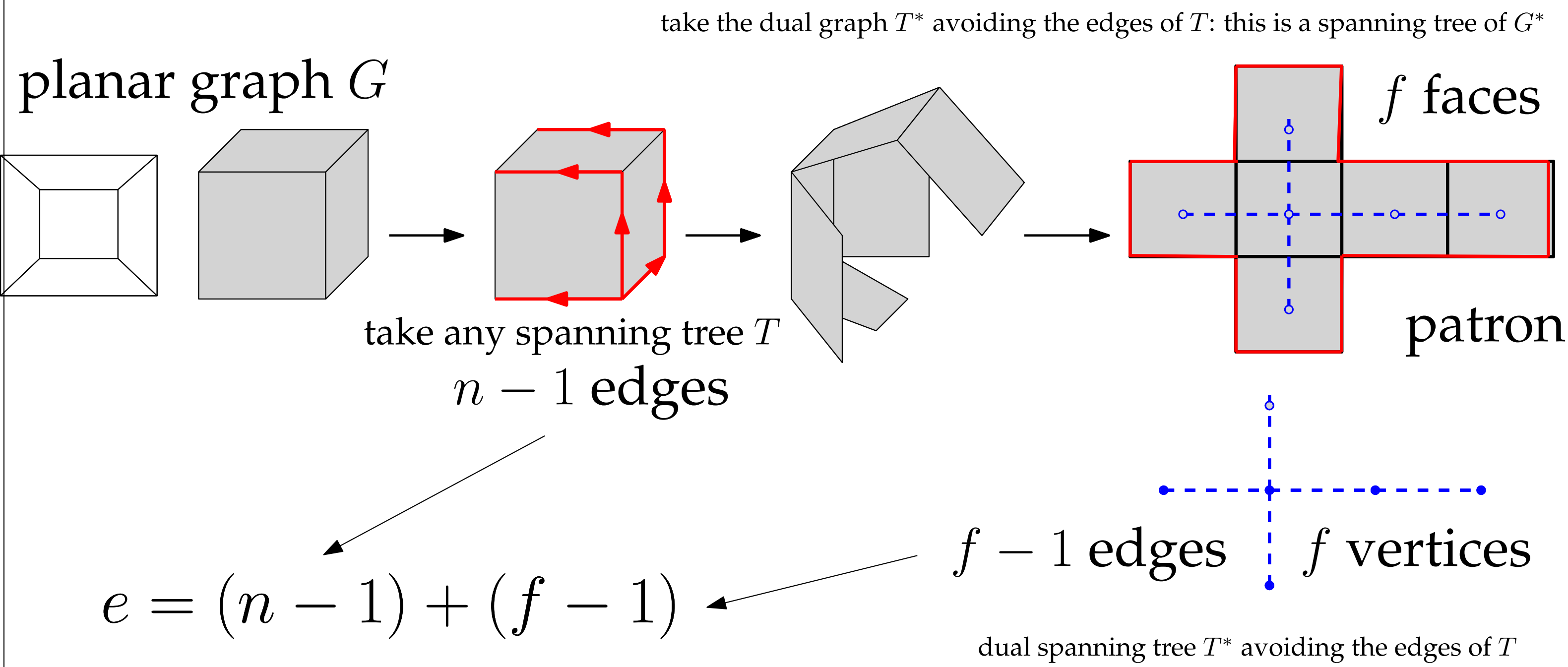


Pepakura software for unfolding polyhedral surfaces

Euler's relation for polyhedral surfaces

Overview of the proof

$$n - e + f = 2$$



$$e = (n - 1) + (f - 1)$$

Euler's relation: consequences

Corollary: linear dependence between edges, vertices and faces

$f \leq 2n - 4$

$$e \leq 3n - 6$$

proof (double counting argument)

$$f = f_1 + f_2 + f_3 + \dots$$
$$n = n_1 + n_2 + n_3 + \dots$$

all faces have degree at least 3 ($\mathcal{G}$ simple simple), then we get

$$f = f_3 + f_4 + \dots$$

every edge appears twice

$$2e = 3 \cdot f_3 + 4 \cdot f_4 + \dots$$

then we get

$$2e - 3f \geq 0$$

Euler's relation: consequences

Corollary: linear dependence between edges, vertices and faces

$f \leq 2n - 4$

$$e \leq 3n - 6$$

given $2e - 3f \geq 0$

by applying Euler formula, we obtain

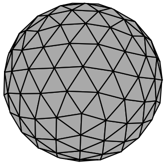
$$3n - 6 = 3(e - f + 2) - 6 = 3e - 3f$$

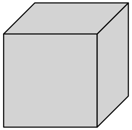
$$3n - 6 = e + (2e - 3f) \geq e$$

$$3n - 6 \geq e$$

Euler's relation: consequences

assume $\mathcal{G}$ is a *simple* planar graph: no multiple edges, no loops





furthermore, assume all **faces have degree at least** 3, then we get

$$f = f_3 + f_4 + \dots$$

every edge appears twice

$$2e = 3 \cdot f_3 + 4 \cdot f_4 + \dots$$

then we get

$$2e - 3f \geq 0$$

by applying Euler formula, we obtain

$$3n - 6 = 3(e - f + 2) = 3e - 3f \geq 0$$

$$e \leq 3n - 6$$

furthermore, assume there are **no cycles of length** 3, then we get

$$f = f_4 + f_5 + \dots$$

every edge appears twice

$$2e = 4 \cdot f_4 + 5 \cdot f_5 + \dots$$

then we get

$$2e - 4f \geq 0$$

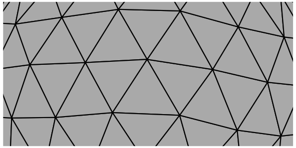
use again Euler formula

$$3n - 6 = 3(e - f + 2) = 3e - 3f \geq 0$$

$$e \leq 2n - 4$$

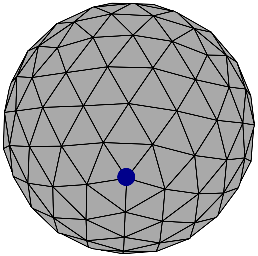
Euler's relation for polyhedral surfaces

can we construct a regular (genus 0) mesh, where every vertex has degree 6?



Euler's relation for polyhedral surfaces

we just showed $2e - 3f \geq 0$



proof (double counting argument)

Assume all the vertices have degrees ≥ 6:

the total number of vertices is: $n = n_6 + n_7 + n_8 + \ldots$

using a double counting of edges: $2e = 6 \cdot n_6 + 7 \cdot n_7 + 8 \cdot n_8 + \ldots$

$$2e - 6 \cdot n \geq 0$$

$$\left.\begin{array}{l} 2e - 6 \cdot n \geq 0 \\ 2e - 3f \geq 0 \end{array}\right\} \longrightarrow 6(e - n - f) = (2e - 6n) + 2(2e - 3f) \geq 0$$

$$e - n - f \geq 0 \longrightarrow e \geq n + f$$

condtradicting Euler formula: $e = n + f - 2$

Euler's relation and Kuratowski theorem (easy direction)

theorem (Kuratowski 1930)

G is planar iff it contains no subdivision of K_5 nor $K_{3,3}$

theorem (Wagner, 1937)

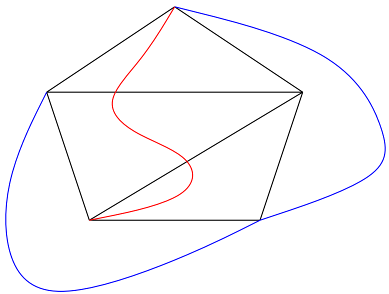
G is planar iff it does not contain K_5 nor $K_{3,3}$ as minors

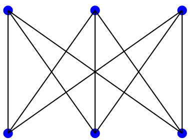
---

Lemma

The graphs K_5 and $K_{3,3}$ are not planar

Exercise: give a combinatorial proof





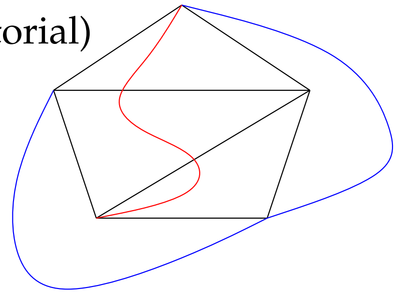
$K_{3,3}$ bipartite:

Euler's relation and Kuratowski theorem (easy direction)

Lemma

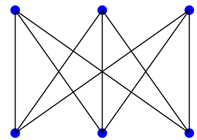
The graphs K_5 and $K_{3,3}$ are not planar

Proof: (combinatorial)



$$e \leq 3n - 6 = 9$$

but we have $e(K_5) = \binom{5}{2} = 10$



$K_{3,3}$ bipartite:

no cycle of length 3: faces have degree ≥ 4

$$4f(G) \leq \sum_{f\in F} deg(f) = 2e(G) = 18$$

so the number of faces is $f(G) \leq 4$

$$2 = v(G) - e(G) + f(G) \leq 6 - 9 + 4 = 1$$

Euler's relation and Kuratowski theorem (easy direction)

theorem (Kuratowski 1930)

G is planar iff it contains no subdivision of K_5 nor $K_{3,3}$

theorem (Wagner, 1937)

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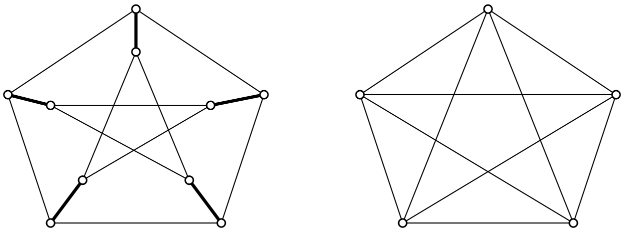
---

A graph G' is a *minor* of a graph G it can be obtained from G with a sequence of vertex/edge deletions and edge contractions

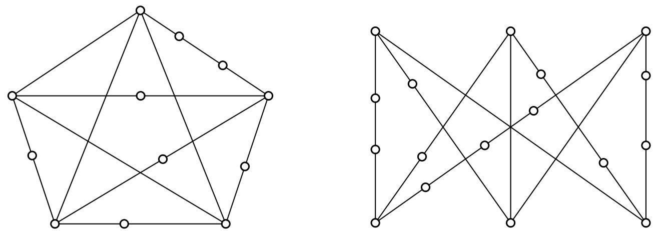


K_5 is a minor of the Petersen graph

Remark

Minors of planar graphs are planar

A graph G' is a *subdivision* of a graph G if it can be obtained from G with a sequence of edge subdivisions



Subdivisions of K_5 and $K_{3,3}$

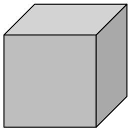
Remark

A graph G is planar if and only if every subdivision of G is planar

Part III

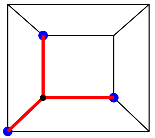
3-connectedness and planar graphs

3-connectedness



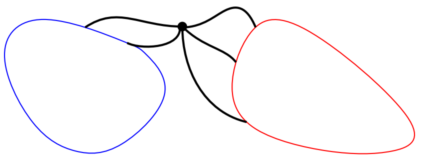
Defini

G is 3-connected if { *is connected and* / *the removal of one or two vertices does not disconnect G*

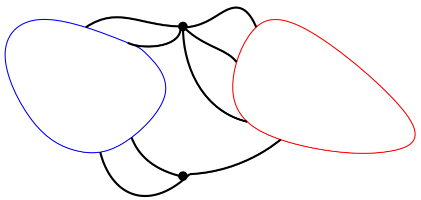


at least 3 vertices are required to disconnect the graph

cut-vertex



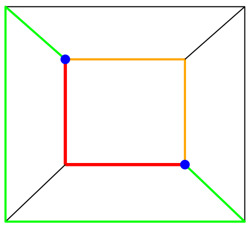
cut-pair



Menger Theorem

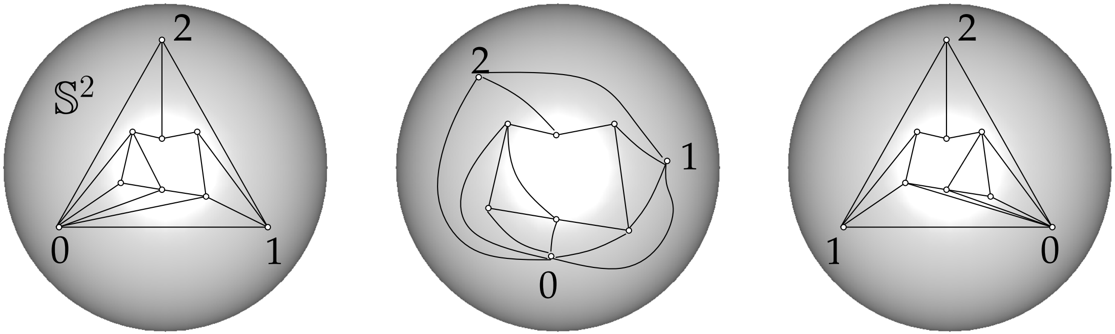
if G is 3-connected then for every pair of vertices u and v there exist 3 vertex disjoint paths (intersecting only at u and v)

(see Lecture 5, for a simple proof in the triangulated planar case)



3-connected planar graphs: Whitney theorem

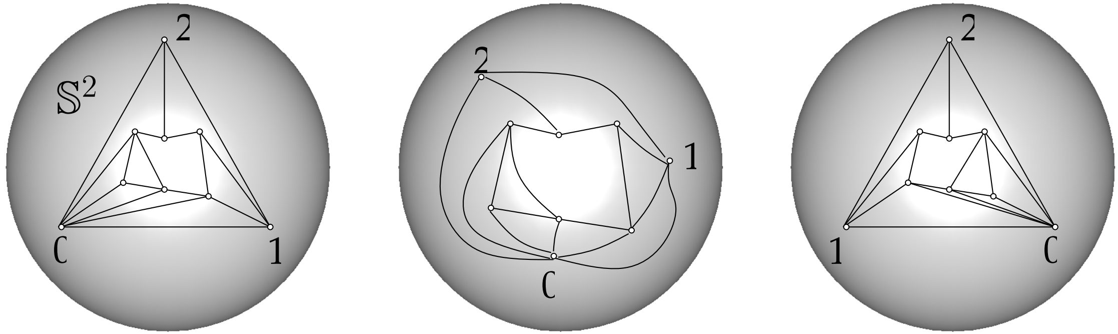
Thm (Whitney, 1933)
3-connected planar graphs admit an unique embedding (up to homeomorphism and inversion of the sphere $\mathbb{S}^2$).



Remark: why 3-connectedness is important?

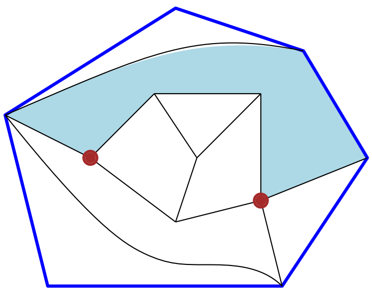
3-connected planar graphs: Whitney theorem

Thm (Whitney, 1933)
3-connected planar graphs admit an unique embedding (up to homeomorphism and inversion of the sphere $\mathbb{S}^2$).

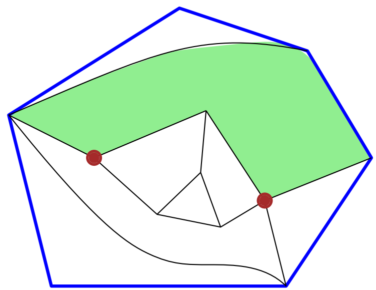


$f_i :=$ number of faces of degree i

$f_3 = 4$
$f_4 = 2$
$f_5 = 1$
$f_6 = 0$
$f_7 = 1$



$|F| = 9$



$f_3 = 4$
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$f_7 = 0$

two different (non equivalent) embeddings of the same graph

Bridges: some terminology

G a connected graph
C a cycle

Bridges:= subgraphs induced by the edges of $E(G) \setminus E(C)$

Remarks

bridges can only intersect at the vertices of C

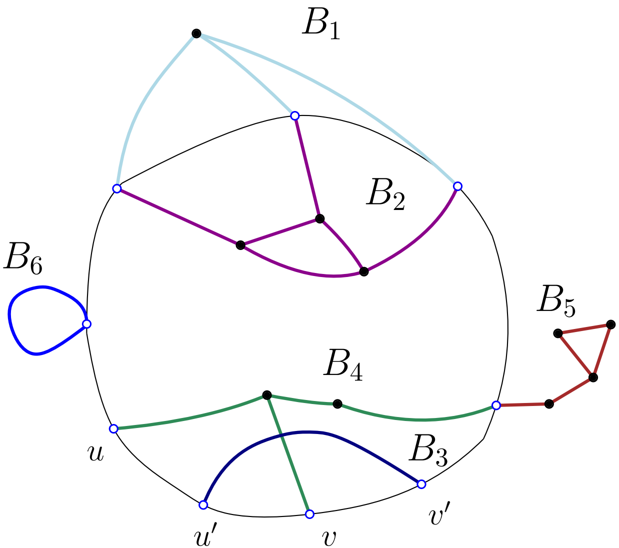
trivial bridges do not have inner vertices: loops, chords

forn any two vertices of a bridge there exists one path internally disjoint from C

if G is non-separable then there are two vertices of attachment

k-bridge is a bridge with k vertices of attachment

equivalent bridges: same point of attachment (B_1 and B_2, which are 3-bridges)



B_2 and B_4 are said to *avoid each other*
B_3 and B_4 are said to be *overlapping*

Two bridges B and B' are *skew* if there exist 4 vertices of attachment u, v (on B) and u', v' on B' which are listed consecutively on C: u, u', v, v'

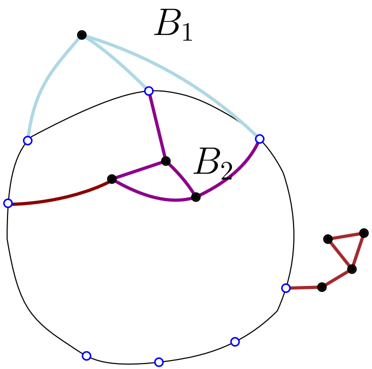
B_3 and B_4 are skew
B_1 and B_2 are not skew

Bridges of cycles: properties

Lemma 1

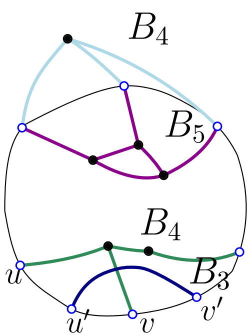
Given a cycle in a graph G the overlapping bridges are either skew or else equivalent 3-bridges.

proof:



Case 1 either B or B' is a chord (2-bridge)
they must be skew (as B_3 and B_4)

Case 2 both B andB' have at least 3 points of attachment
B and B' are not equivalent
they must be skew (as B_1 and B_2)



Case 3a B and B' are equivalent
both B and B' are 3-bridges (as B_4 and B_5)

Case 3b B and B' are equivalent
both B and B' are k-bridges ($k \geq 4$)
they must be skew

Bridges (in planar graphs): properties

Lemma 2

Given a cycle C in a plane graph G the inner (outer) bridges avoid each other.

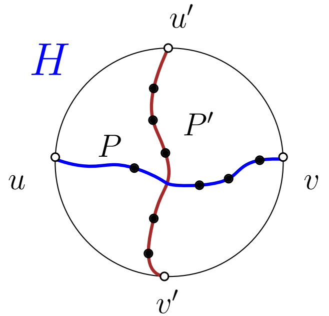
(they are not overlapping)

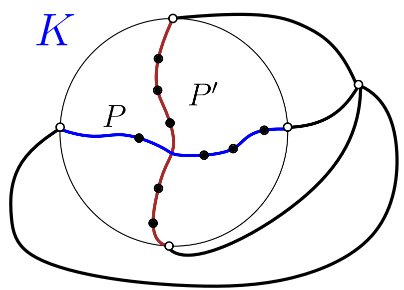
proof:

Case 1 B and B' are skew → there are u, u', v, v' consecutive on C

take the two disjoint paths P and P' (included in B and B')

The graph $H = P \cup C \cup P'$ is planar (subgraph of G)





Define a graph K adding a vertex on the outer face (and connecting it to vertices on C)

The graph K should be planar by construction

But K is also a subdivision of K_5 (non planar)

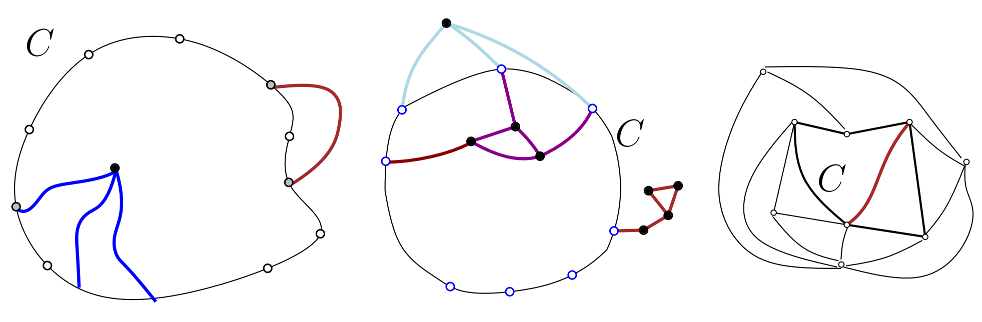
(contradiction)

Case 2 B and B' are equivalent 3-bridges

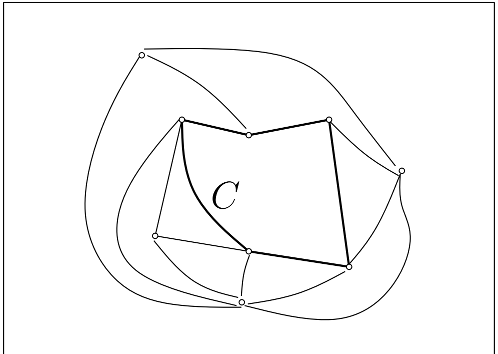
exercise

3-connected planar graphs: Whitney theorem

Thm (Tutte, 1963)
A cycle in a 3-connected planar graph is a facial cycle (bounding a face) if and only if is non-separating.



cycles which are not *non-separating*



non-separating cycle

Def:
a cycle is *non-separating* if it has no chords and at most one non trivial bridge

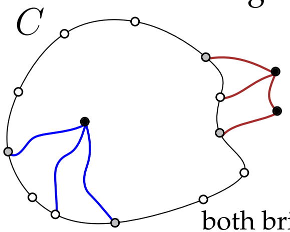
3-connected planar graphs: Whitney theorem

Thm (Tutte, 1963)

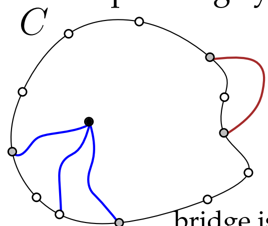
A cycle in a 3-connected planar graph is a facial cycle (bounding a face) if and only if is non-separating.

proof:

Assume C is not a facial cycle → There is at least one inner and one outer bridge (not loops) → C cannot be a non-separating cycle



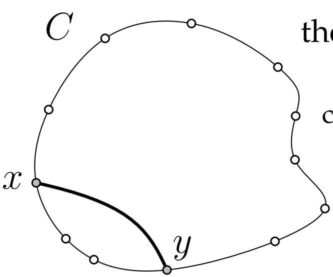
both bridges are non trivial



bridge is a chord

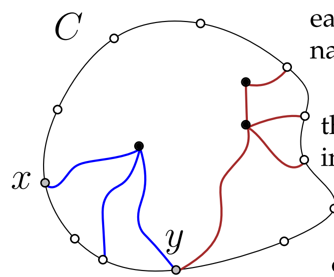
Let C a facial cycle (assume it is bounding the outer face)

All bridges are in the interior of C and avoid each other (previous Lemma)



the chord defines a 2-cut $\{x, y\}$

contradicting 3-connectedness



there are at least two briges (avoiding each other: their segments are internally disjoint)

the pair $\{x, y\}$ is a vertex cut of the inner vertices of the bridges

contradicting 3-connectedness

Algorithms and combinatorics for geometric graphs (**Geomgraphs**)

Lecture 1, part II

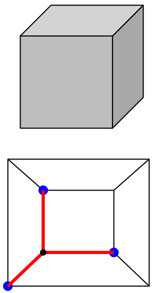
Graph Drawing: embedding algorithm

september 18, 2025

Luca Castelli Aleardi

Computing a planar embedding

3-connectedness



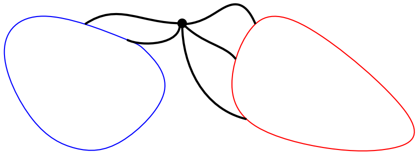
at least 3 vertices are required to disconnect the graph

Defini

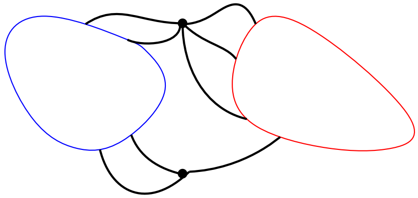
G is 3-connected if

- *is connected and*
- *the removal of one or two vertices does not disconnect G*

cut-vertex



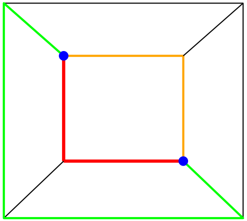
cut-pair



Menger Theorem

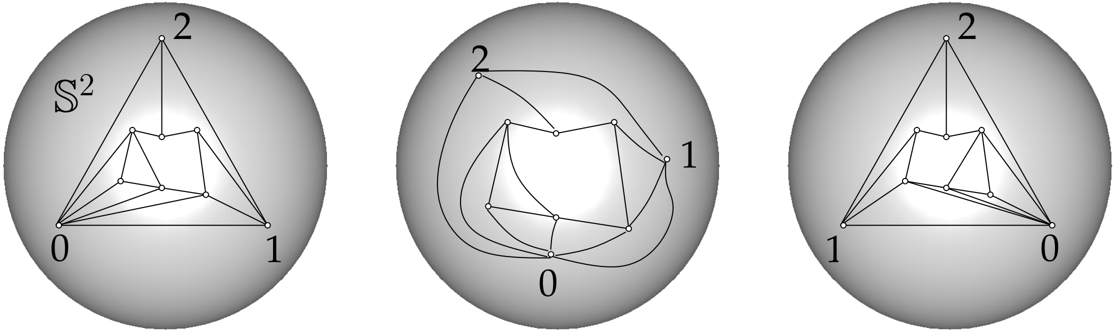
if G is 3-connected then for every pair of vertices u and v there exist 3 vertex disjoint paths (intersecting only at u and v)

(see Lecture on Schnyder woods, for a simple proof in the triangulated planar case)



3-connected planar graphs: Whitney theorem

Thm (Whitney, 1933)
3-connected planar graphs admit an unique embedding (up to homeomorphism and inversion of the sphere $\mathbb{S}^2$).



Remark: why 3-connectedness is important?

3-connected planar graphs: Whitney theorem

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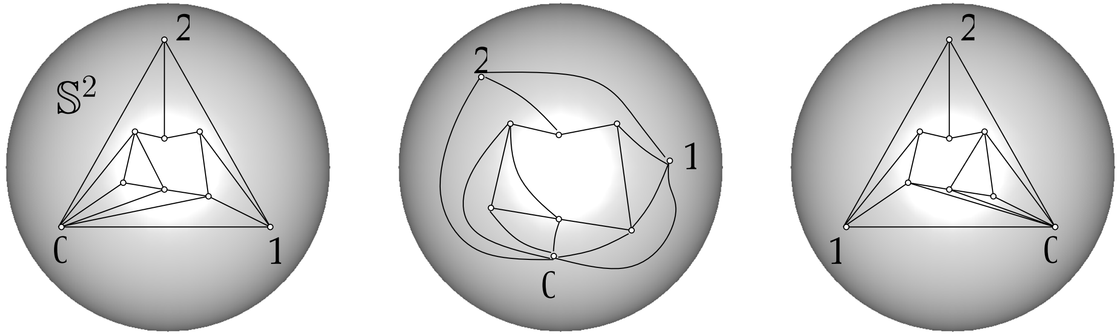
3-connected planar graphs admit an unique embedding (up to homeomorphism and inversion of the sphere $\mathbb{S}^2$).



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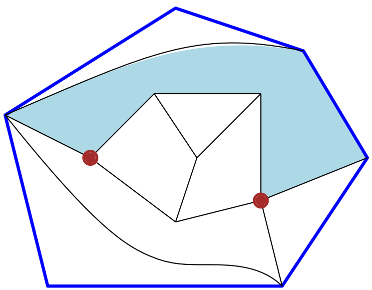

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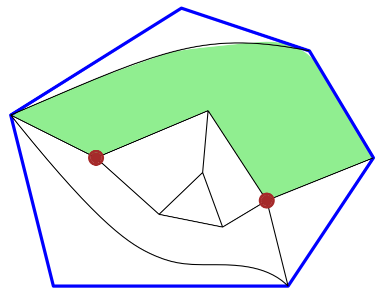

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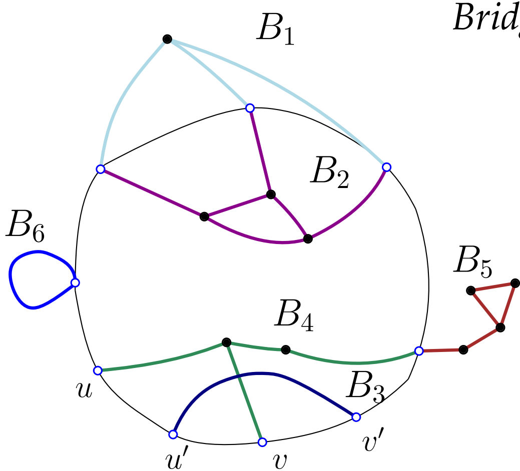
two different (non equivalent) embeddings of the same graph

Bridges

G a connected graph

C a cycle



Bridges:= subgraphs induced by the edges of $E(G) \setminus E(C)$

Remarks

bridges can only intersect at the vertices of C

trivial bridges do not have inner vertices: loops, chords

from any two vertices of a bridge there exists one path internally disjoint from C

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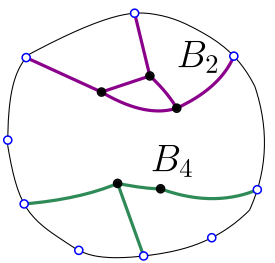
equivalent bridges: same point of attachment (B_1 and B_2, which are 3-bridges)



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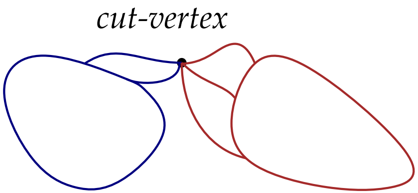
Two bridges B and B' are *skew* if there exist 4 vertices of attachment u, v (on B) and u', v' on B' which are listed consecutively on C: u, u', v, v'

B_3 and B_4 are skew

B_1 and B_2 are not skew

Block decomposition

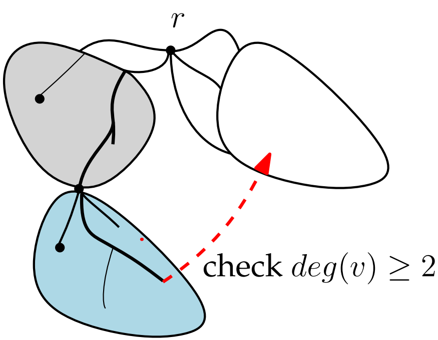


Definition
A *block* is a maximal sub-graph (with respect to inclusion) that has no cut vertex

Remark
A graph G is planar if and only if all its blocks are planar.

Lemma
Given a graph G all its blocks can be computed in linear time.

proof: Compute a DFS tree from an arbitrary vertex r



case 1: $v = r$

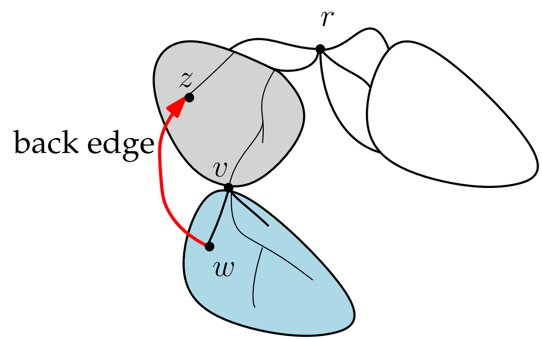
For each vertex v compute:

depth(v)

lowpoint(v)

(process vertices in post-order)

lowpoint(v) := smallest depth of the extremity of a back (red) edge (w, z) (where w is a descendant of v)



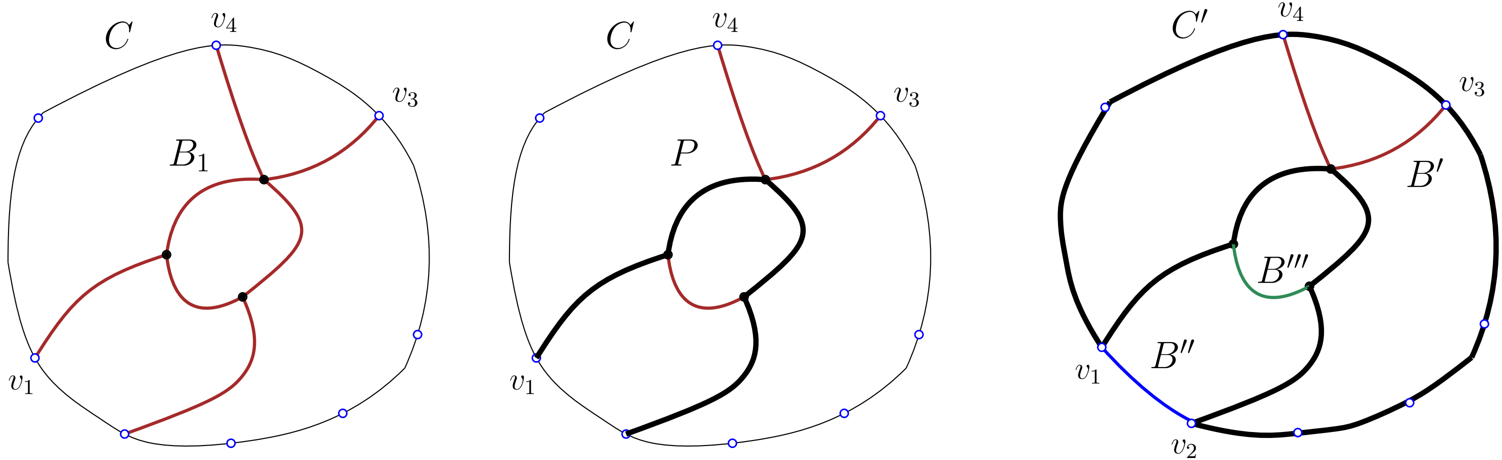
case 2: $v \neq r$

Bridges

Lemma
Given a 2-connected graph G we can in linear time either compute a circuit of G having at least two bridges, or certify that G is planar.

proof: Compute an arbitrary cycle C

Assume there is a single bridge B_1 (otherwise we are already done)



assuming $P \neq B_1$
otherwise $G = C \cup P$ is planar

computhe a path $P :=$ in B_1 from v_1 and v_2

Bridges

Lemma

Let C a circuit of G. The graph G is planar if and only if:

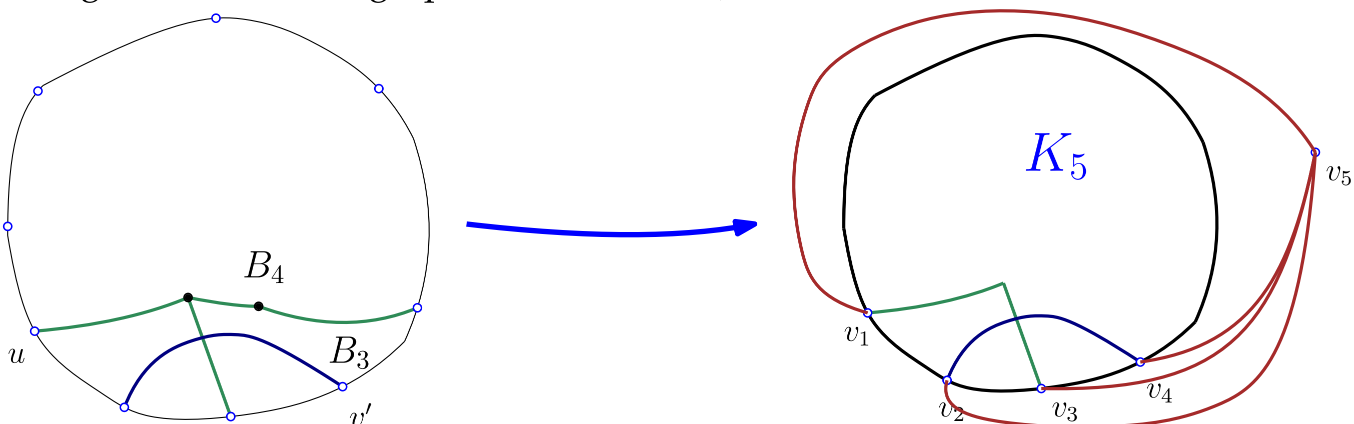
The conflict graph of the bridges of C is bipartite (bridges are either outside or inside)

For every bridge B (with respect to C), the graph $H = B \cup C$ is planar

proof:

One direction: assume G is planar

Two bridges B and B' drawn both inside (or outside) cannot be overlapping (no edge in the conflict graph between them)



The original graph would be non planar

Bridges

Lemma

Let C a circuit of G. The graph G is planar if and only if:

The conflict graph of the bridges of C is bipartite

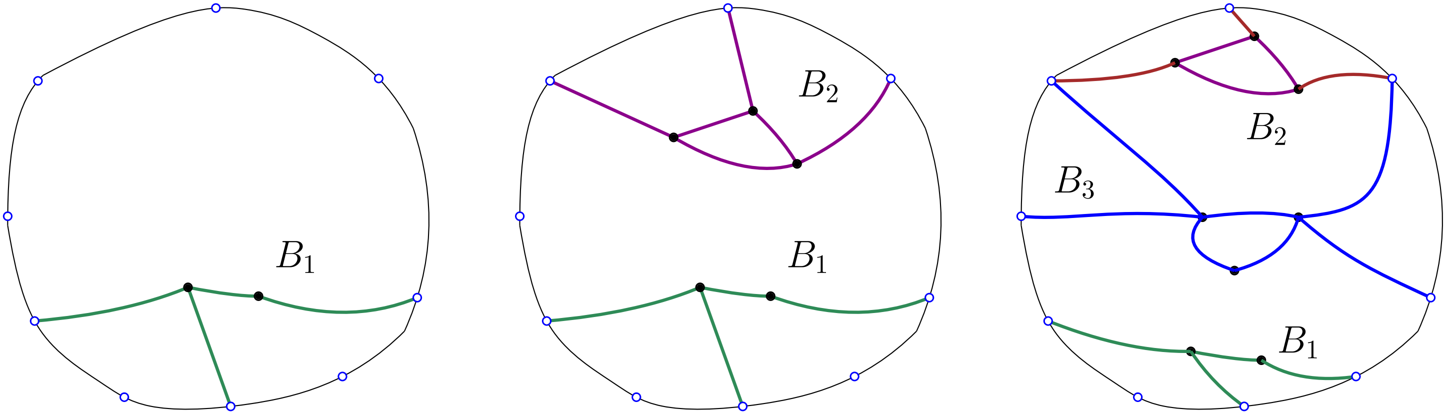
For every bridge B (with respect to C), the graph $H = B \cup C$ is planar

proof:

Other direction (we want to embed the graph, without crossings)

Solution: since (inner) bridges are without conflict, we can add all (inner) bridges iteratively one by one

B_3 is not overlapping with B_1 and B_2



Embedding algorithm

`Embed(G, C)`

Compute the bridges of G with respect to C

Compute the conflict graph of $B_1, B_2, B_3, \ldots$

`if` *the conflict graph is not bipartite,* `return` *non-planar*

`For each` *bridge B of G (not a path):*

`let` $G' := C \cup B$

`let` $C' := \texttt{extract}(G', C)$ (apply previous Lemma)

`embed`(G', C') (recursive call)

`if` G' *is non-planar,* `return` *non-planar*

`return` *planar*

Embedding algorithm $O(n^3)$

```
Embed(G, C)
```

Compute the bridges of G with respect to C $O(n)$

Compute the conflict graph of $B_1, B_2, B_3, \ldots$ $O(n^2)$

`if` *the conflict graph is not bipartite,* `return` *non-planar* $O(n)$

`For each` *bridge B of G (not a path):*

`let` $G' := C \cup B$

`let` $C' := \mathtt{extract}(G', C)$ (apply previous Lemma) $O(n')$

`embed`(G', C') (recursive call) $O(n)$ recursive calls

`if` G' *is non-planar,* `return` *non-planar*

`return` *planar*

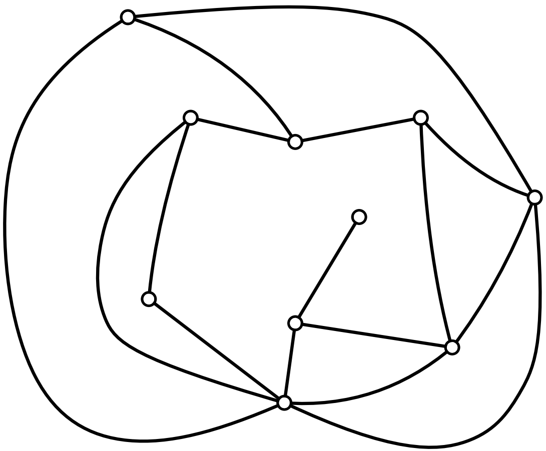
Triangulating a planar graph

Lemma

Let G be a simple plane graph (celullarly embedded). Then it is possible to triangulate G in linear time obtaining a simple triangulation T (super graph of G).

proof:

Any idea?



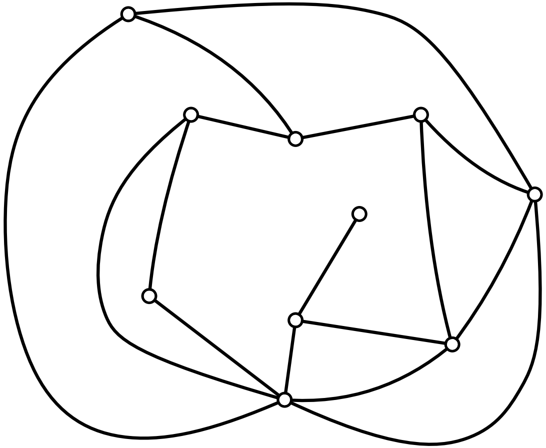
Triangulating a planar graph

Lemma

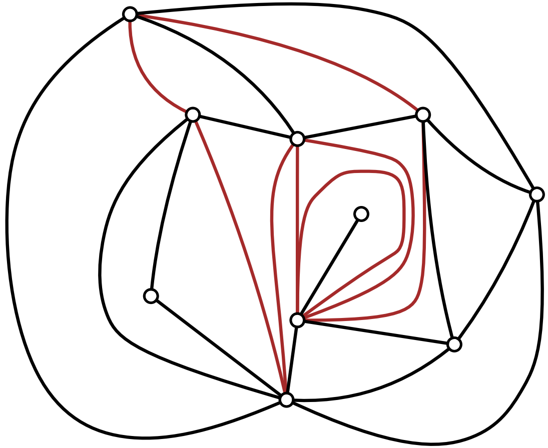
Let G be a simple plane graph (celullarly embedded). Then it is possible to triangulate G in linear time obtaining a simple triangulation T (super graph of G).

proof:

Solution: triangulate faces



Problem: eliminate loops and multiple edges



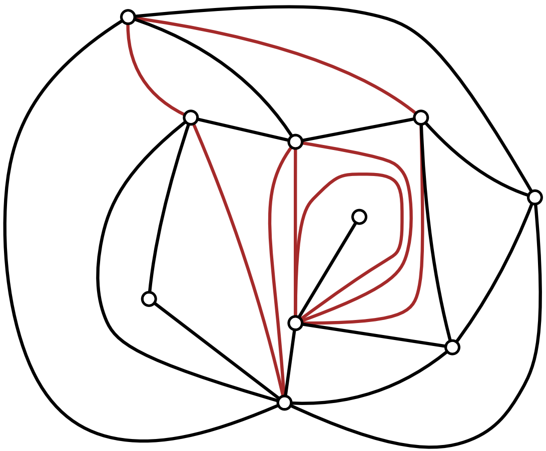
Triangulating a planar graph

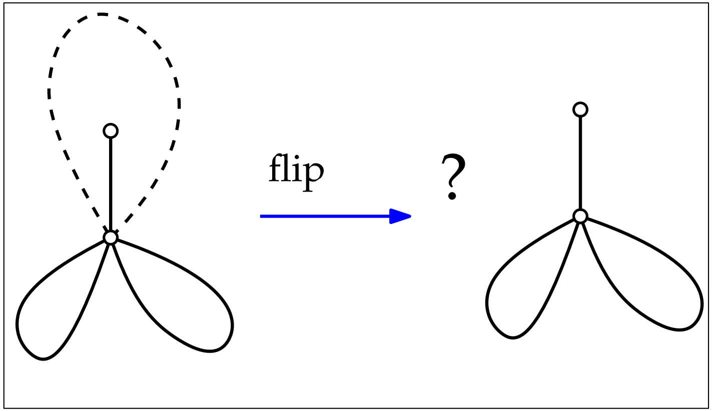
Lemma

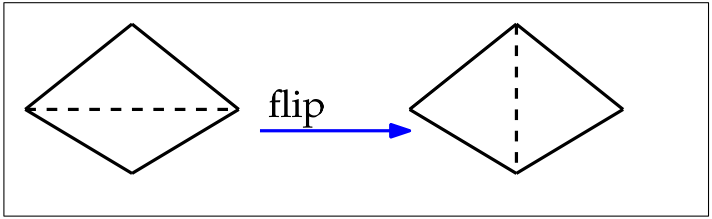
Let G be a simple plane graph (celullarly embedded). Then it is possible to triangulate G in linear time obtaining a simple triangulation T (super graph of G).

proof:

Idea: eliminate loops and multiple edges (via edge flipping)







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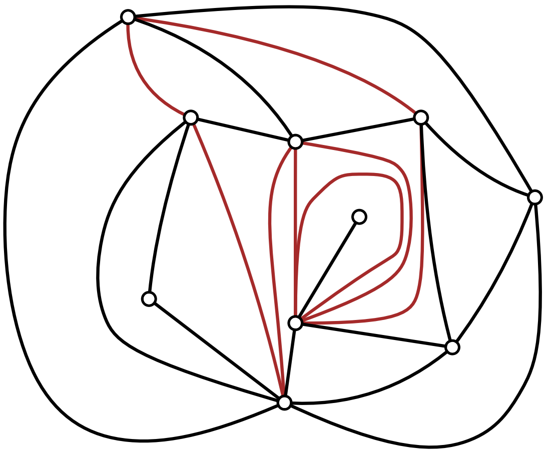

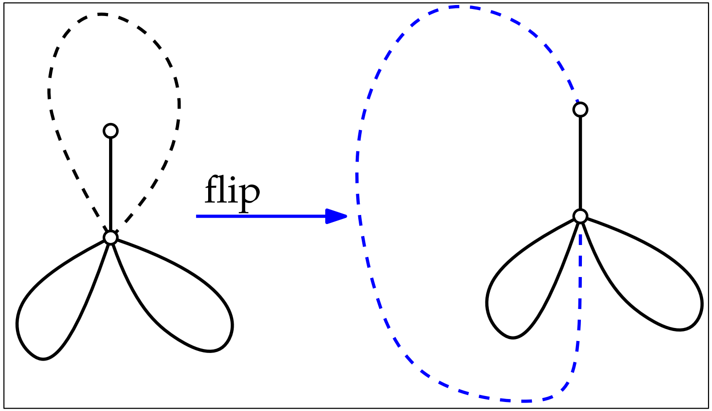

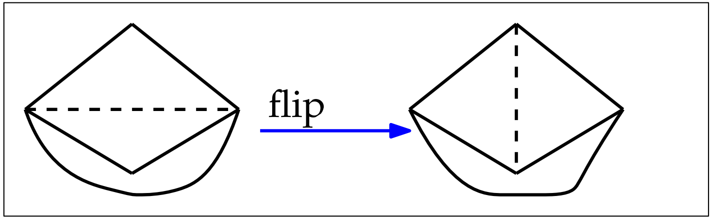

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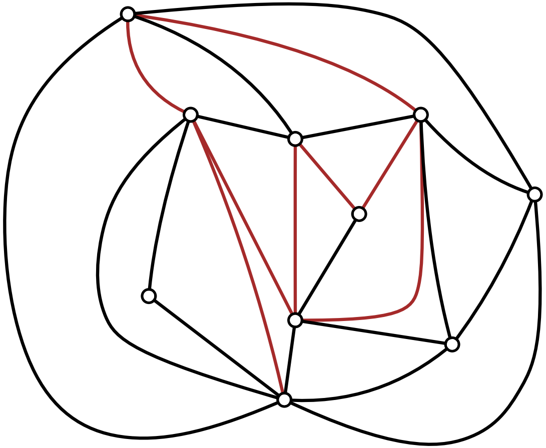
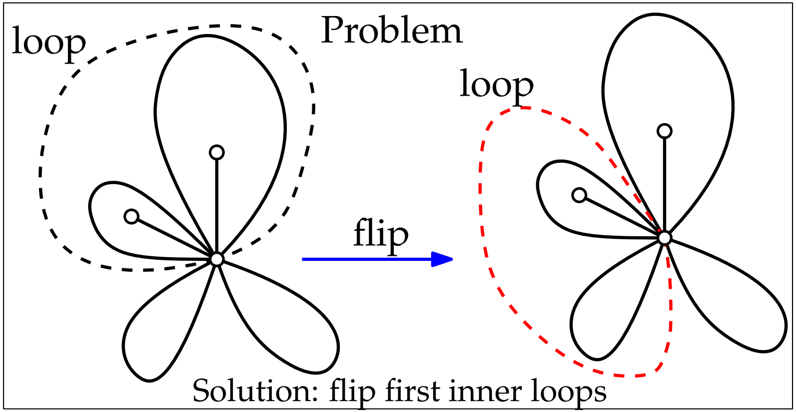
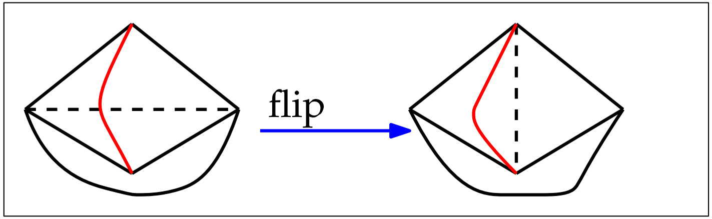






Algorithms and combinatorics for geometric graphs (**Geomgraphs**)

Lecture 1, part II

Graph Drawing: Tutte barycentric method

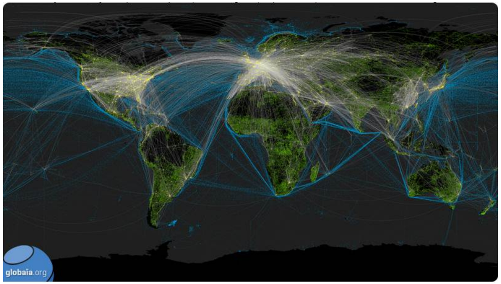
september 18, 2025

Luca Castelli Aleardi

Graph drawing: introduction and applications

Graph drawing and data visualization

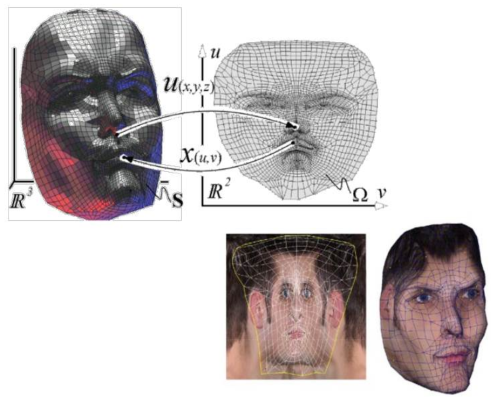
Global transportation system



Social networks

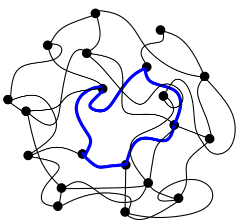
Roads, railways, ...

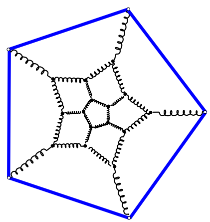
Parameterization problem
(known in Geometry Processing)



Bennis et al., 1991
Maillot et al., 1993

Compute a crossing-free drawing of planar graphs





Graph drawing: motivation

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Challenge: what kind of graph does A_G represent?

Graph drawing: motivation

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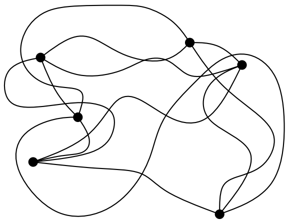

adjacency matrix

$$A_G[i,j] = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Graph drawing: motivation

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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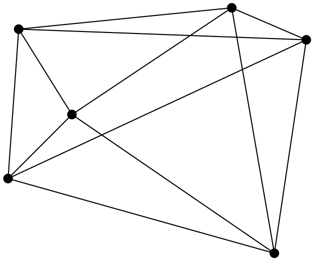

Graph drawing: motivation

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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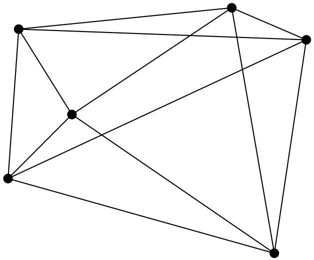

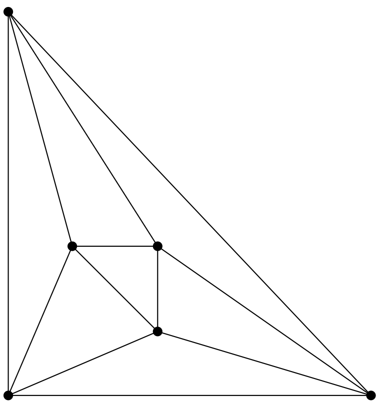

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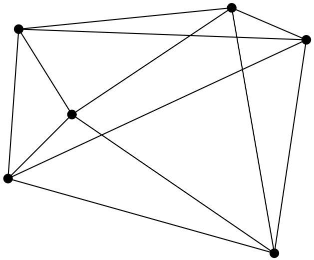

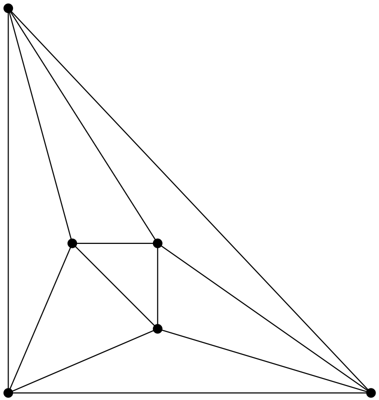

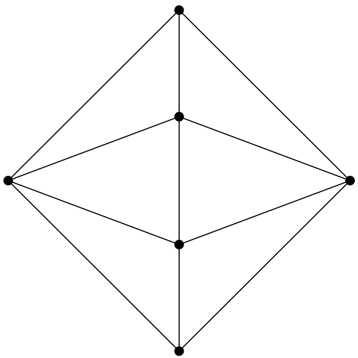

Graph drawing: motivation

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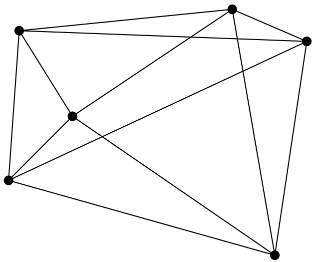

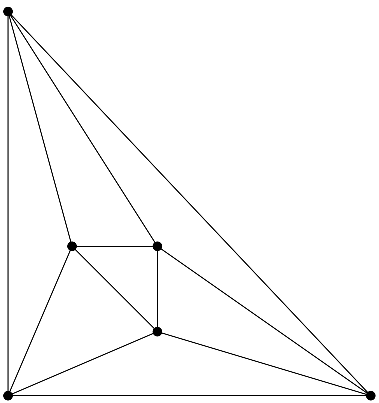

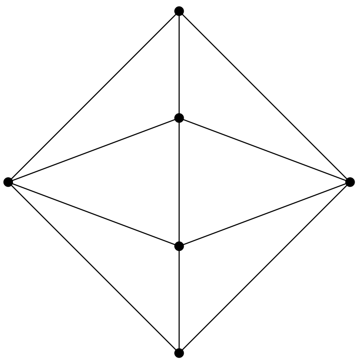

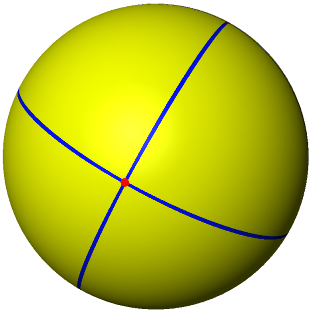

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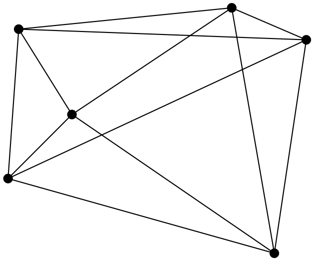

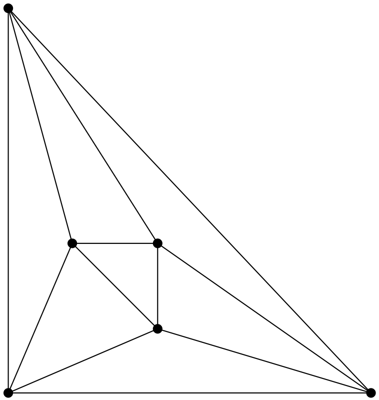

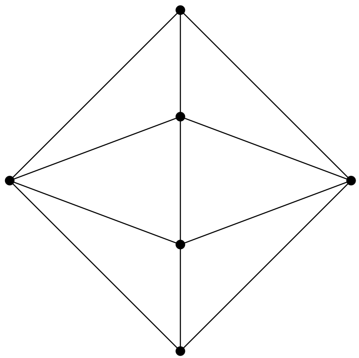

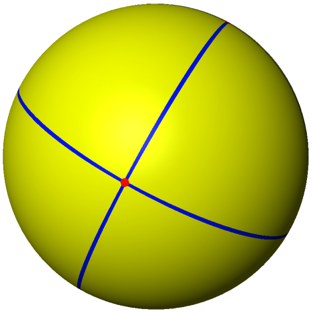

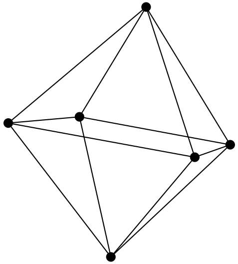

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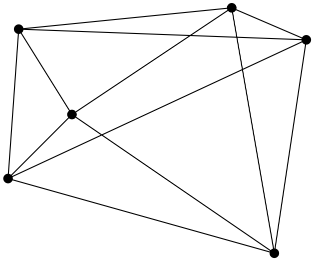
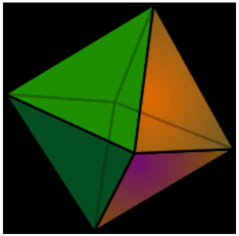
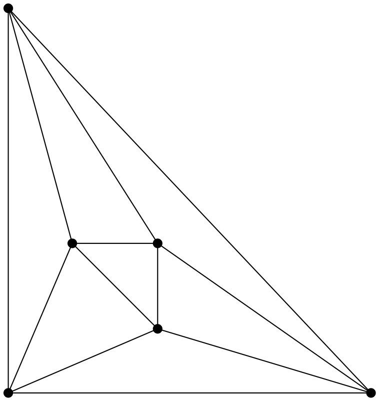
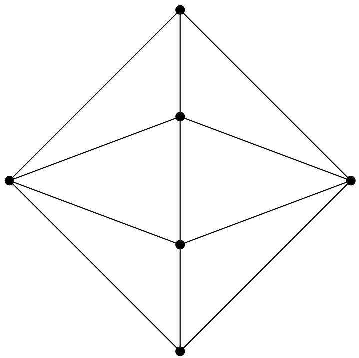
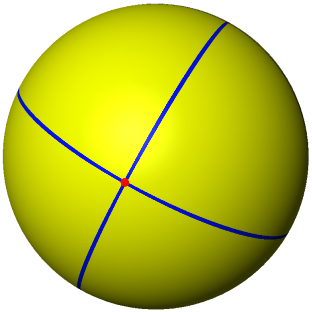
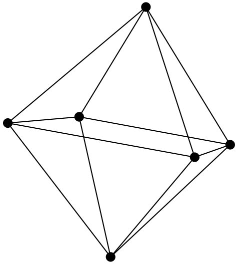








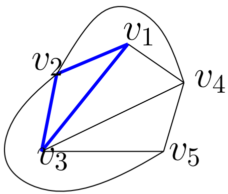




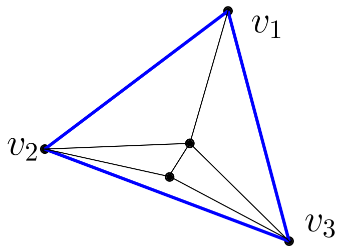
Major results in Graph Drawing (for planar graphs)

Fáry theorem (1947) **(exercise)**

- Every (simple) planar graph admits a straight line planar embedding (no edge crossings)



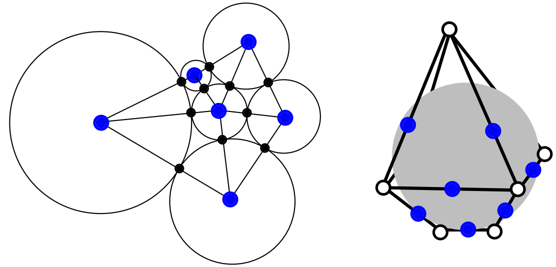
planar drawing



straight-line planar drawing

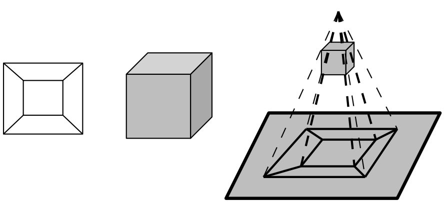
Thm (Koebe-Andreev-Thurston) **(not covered)**

Every planar graph with n vertices is isomorphic to the intersection graph of n disks in the plane.



Thm (Steinitz, 1916)

3-connected planar graphs are the 1-skeletons of convex polyhedra



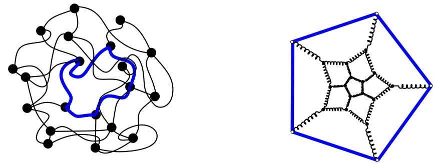
(Lecture 1)

Thm (Tutte barycentric method, 1963)

Every 3-connected planar graph G admits a barycentric representation ρ in R^2.

$$\rho(v_i) = \sum_{j \in N(i)} w_{ij}\rho(v_j)$$

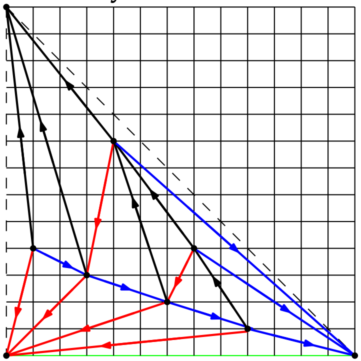
($\sum_j w_{ij} = 1$ and $w_{ij} > 0$



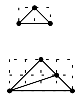
Graph drawing paradigms

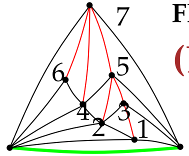
(Lecture 5)

Thm (Schnyder 1990)



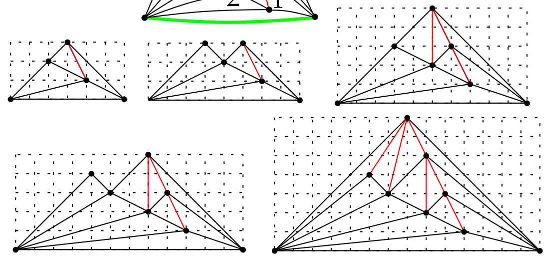
Thm (De Fraysseix, Pack Pollack 1989)





FPP algorithm

(Lecture 2)



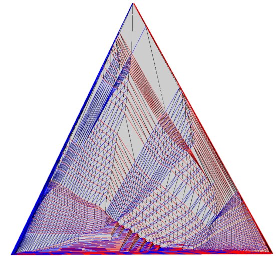
shift algorithm via Canonical orderings

linear time algorithms

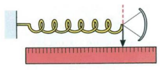
$O(n) \times O(n)$ grid drawings

not trivial to implement

extremey fast: they can process millions of vertices per second



Spring embedder (Eades, 1984)
(Fruchterman and Reingold, 1991)



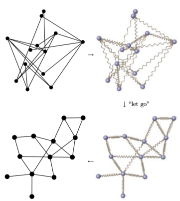
force-directed paradigm

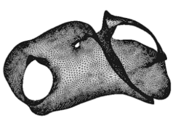
easy to implement

pretty slow: $O(n^2)$ or $O(n \log n)$ time per iteration

$$\mathbf{F}_a(v) = c_1 \cdot \sum_{(u,v)\in E} \log(dist(u,v)/c_2)$$

$$\mathbf{F}_r(v) = c_3 \cdot \sum_{u\in V} \frac{1}{\sqrt{dist(u,v)}}$$





images from Kaufman Wagner (Springer, 2001)

[Tutte'63] Tutte barycentric embedding

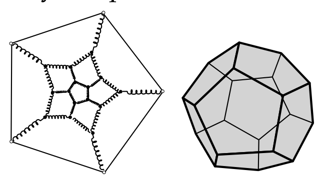
minimize the spring energy

$$E(\rho) := \sum_{(i,j)\in E} |\mathbf{x}(v_i) - \mathbf{x}(v_j)|^2 = \sum_{(i,j)\in E} (x_i - x_j)^2 + (y_i - y_j)^2$$

solve large sparse linear systems $\mathbf{x}(v_i) = \sum_{j\in N(i)} \frac{1}{deg(v_i)} \mathbf{x}(v_j)$

easy to implement

not very fast: they can process $\approx 10^4$ vertices per second

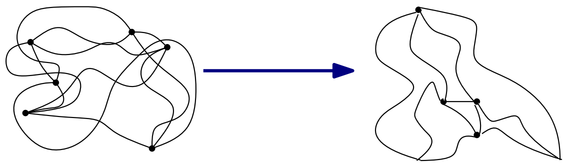


Straight-line planar drawings of planar graphs

Problem definition (Planarity testing, Embedding a planar graph)

Input: a planar graph

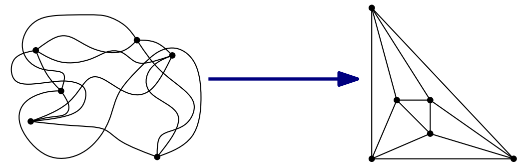
Output: the planar map (cellulaly embedded graph)



Problem definition (drawing in the plane)

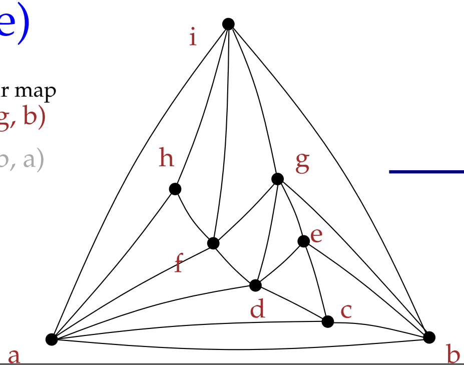
Input: a planar graph (or planar map)

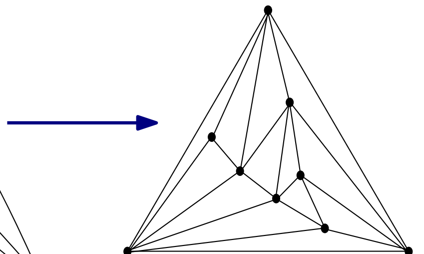
Output: a straight-line planar drawing (crossing-free)



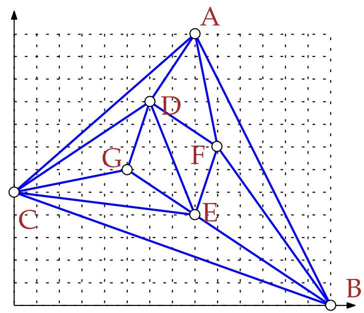
Input of the problem: planar map

(a, b, c)	(d, e, g)	(i, g, b)
(a, c, d)	(e, b, g)	(i, b, a)
(d, c, e)	(a, f, h)	
(c, b, e)	(a, h, i)	
(a, d, f)	(i, h, f)	
(f, d, g)	(i, f, g)	





straight-line crossing-free drawing



straight-line grid drawing

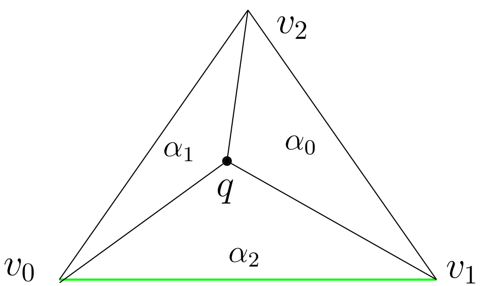
Tutte's barycentric method

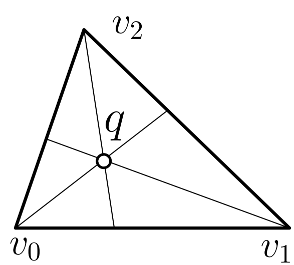
Preliminaries: barycentric coordinates

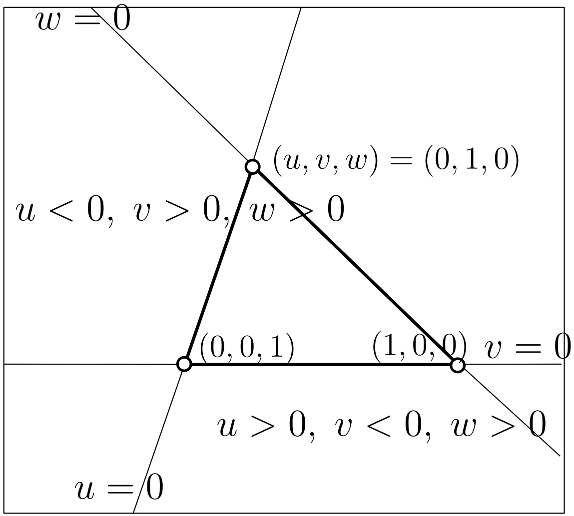
$$q = \boxed{\sum_i^n \alpha_i v_i \text{ (avec } \sum_i \alpha_i = 1)}$$

coefficients $(\alpha_1, \ldots, \alpha_n)$ are called *barycentric coordinates* of q (relative to $v_1, \ldots, v_n$)

Geometric interpretation of barycentric coordinates



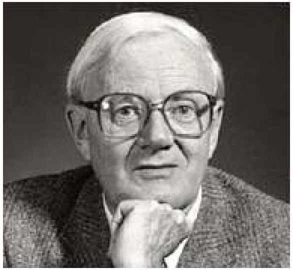




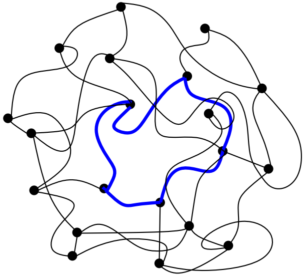
$$q = \alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2$$

$$q = \frac{area(v, v_1, v_2) v_0 + area(v_0, v, v_2) v_1 + area(v_0, v_1, v) v_2}{area(v_0, v_1, v_2)}$$

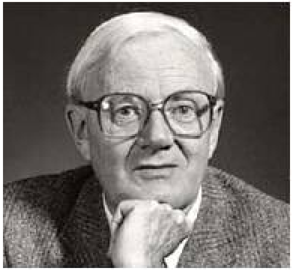
Tutte's theorem



Thm (Tutte barycentric method, 1963)
Every 3-connected planar graph G admits a convex representation ρ in R^2.



Tutte's theorem

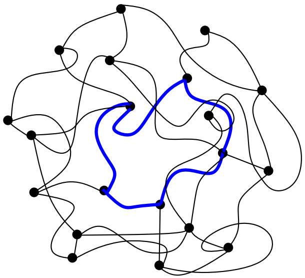


Thm (Tutte barycentric method, 1963)

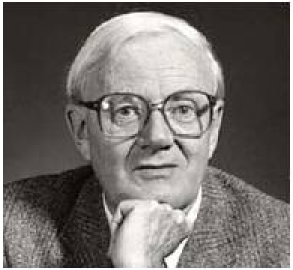
Every 3-connected planar graph G admits a convex representation ρ in R^2.

$$\rho : (V_G) \longrightarrow R^2$$

ρ is *convex* the images of the faces of G are convex polygons



Tutte's theorem



Thm (Tutte barycentric method, 1963)

Every 3-connected planar graph G admits a convex representation ρ in R^2.

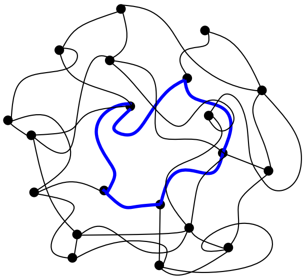
$$\rho : (V_G) \longrightarrow R^2$$

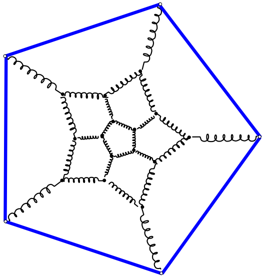
ρ is *barycentric* the images of interior vertices are barycenters of their neighbors

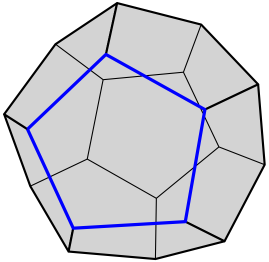
$$\rho(v_i) = \sum_{j \in N(i)} w_{ij} \rho(v_j)$$

where w_{ij} satisfy $\sum_j w_{ij} = 1$, and $w_{ij} > 0$

according to Tutte: $w_{ij} = \frac{1}{deg(v_i)}$







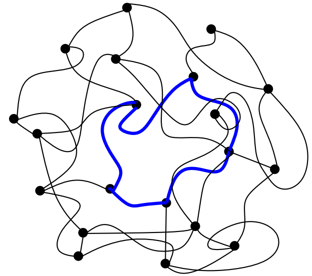
Tutte's theorem: main steps

- chose a cycle F (the outer face of G) in the right way

a cycle such that $G \setminus F$ is connected

(deletion of vertices and edges)

Tutte's theorem: main steps



- chose a cycle F (the outer face of G) in the right way

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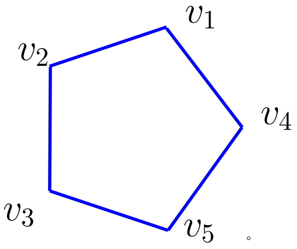


- choose a convex polygon P of size $k = |F|$

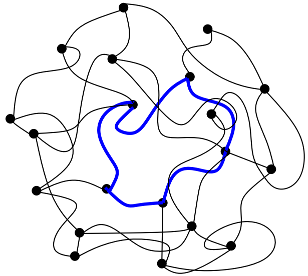
  such that $\rho(F) = P$

Tutte's theorem: main steps

- chose a cycle F (the outer face of G) in the right way

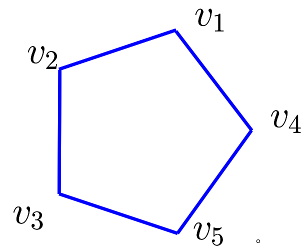
a cycle such that $G \setminus F$ is connected

(deletion of vertices and edges)



- choose a convex polygon P of size $k = |F|$

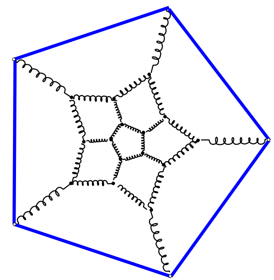
such that $\rho(F) = P$



- solve equations for images of inner vertices $\rho(v_i)$:

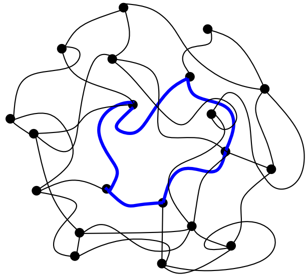
$$\rho(v_i) = \sum_{j \in N(i)} w_{ij} \rho(v_j)$$

$$\rho(v_i) - \sum_{j \in N(i)} w_{ij} \rho(v_j) = 0$$



according to Tutte: $w_{ij} = \frac{1}{deg(v_i)}$

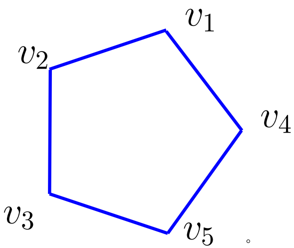
Tutte's theorem: main steps



- chose a cycle F (the outer face of G) in the right way

  a cycle such that $G \setminus F$ is connected

  (deletion of vertices and edges)



- choose a convex polygon P of size $k = |F|$

  such that $\rho(F) = P$

- solve two linear systems:

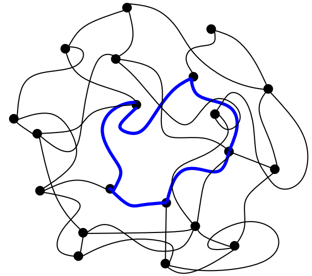
$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases} \longleftrightarrow \begin{cases} \rho_x(v_i) - \sum_{j \in N(i)} w_{ij} \rho_x(v_j) = 0 \\ \rho_y(v_i) - \sum_{j \in N(i)} w_{ij} \rho_y(v_j) = 0 \end{cases}$$

Tutte's theorem: main steps

- chose a cycle F (the outer face of G) in the right way

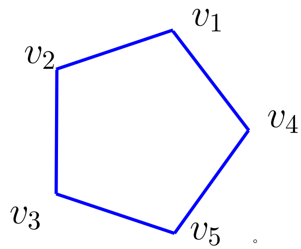
  a cycle such that $G \setminus F$ is connected

  (deletion of vertices and edges)



- choose a convex polygon P of size $k = |F|$

  such that $\rho(F) = P$



- solve a linear system:

Validity of Tutte's theorem: main results

- show that the linear system admit a (unique) solution:

$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases}$$

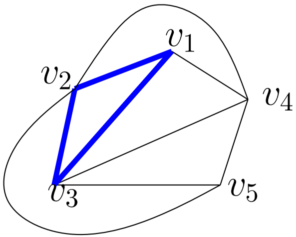
matrix $(I - W)$ is inversible

- a barycentric drawing is planar: no edge crossing

- a 3-connected planar graph G has a non-separating cycle

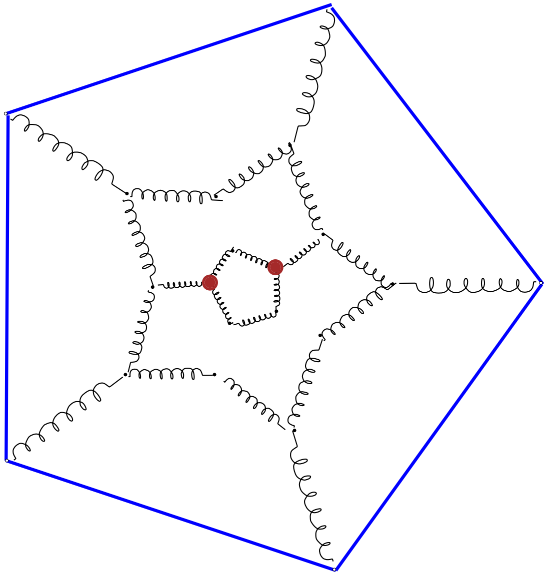
Claim (existence of no-separating cycles)

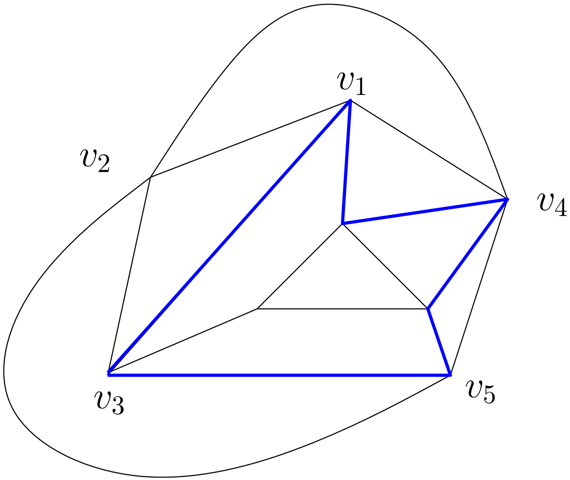
In a 3-connected planar graph peripheral cycles are exactly the faces (of the embedding)



Validity of Tutte's theorem: main results

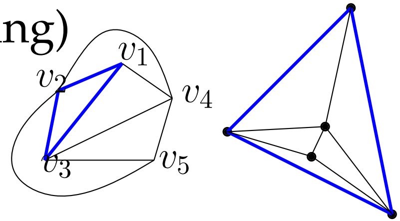
why 3-connectedness and peripheral cycles are important:





Advantages of Tutte's drawing

- the drawing is guaranteed to be planar (no edge crossing)

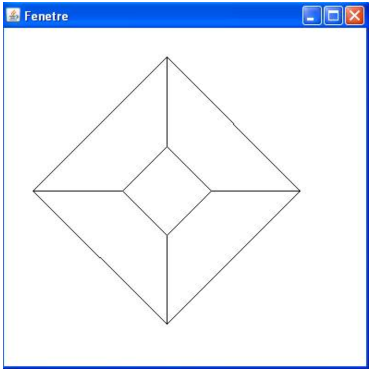


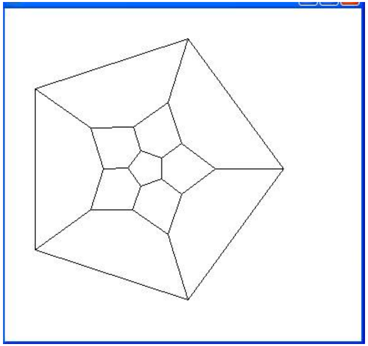
- no need of the map structure
  graph structure + a peripheral cycle

- very easy to implement: no need of sophisticated data structure or preprocessing

linear systems to solves

- nice drawings
  (detection of symmetries)





Drawbacks of Tutte's drawing

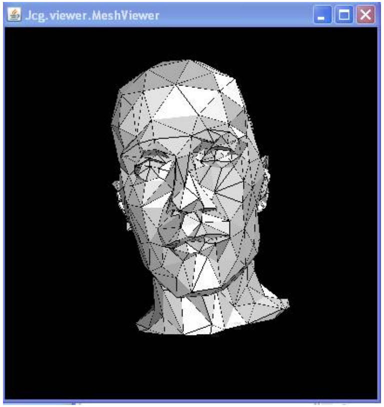
- requires to solve linear systems of equations (of size n)

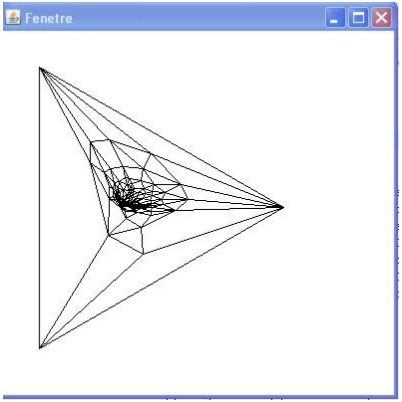
$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases}$$

complexity $O(n^3)$
or $O(n^{3/2})$ with methods more involved

- exponential size of the resulting vertex coordinates (with respect to n)

- drawings are not always "nice"





Tutte's spring embedder: iterative version

- choose an outer face F, and a convex polygon P
- put exterior vertices $v \in F$ on the polygon
- repeat (until convergence)

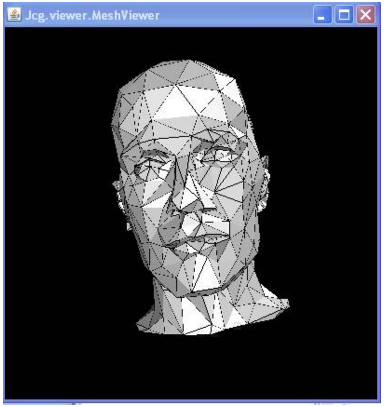
for each inner vertex $v \in V_i$ compute

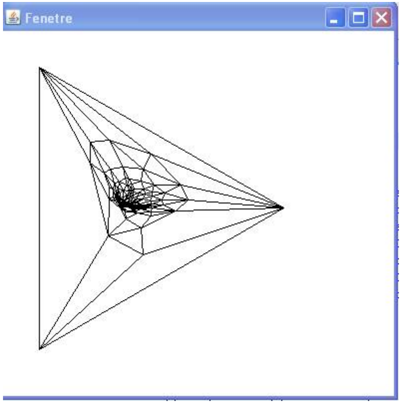
$$x_v = \frac{1}{deg(v)} \sum_{(u,v)\in E} x_u$$

$$y_v = \frac{1}{deg(v)} \sum_{(u,v)\in E} y_u$$

V_i inner vertices

(u, v) edge connecting v and u





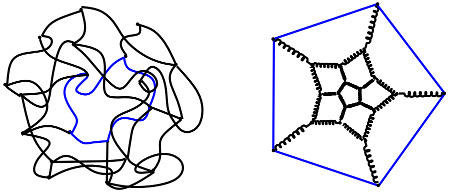
Tutte's spring embedder: several interpretations

- choose an outer face F, and a convex polygon P
- put exterior vertices $v \in F$ on the polygon
- repeat (until convergence)

for each inner vertex $v \in V_i$ compute

$$x_v = \frac{1}{deg(v)} \sum_{(u,v)\in E} x_u$$

$$y_v = \frac{1}{deg(v)} \sum_{(u,v)\in E} y_u$$



Resolution of linear systems

$$\rho(v_i) = \frac{1}{d_i} \sum_{j \in N(i)} \rho(v_j)$$

$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases}$$

Force directed method, with total force:

$$\mathbf{F}(v) = F_a(v) + F_r(v) = \sum_{(u,v)\in E} (\mathbf{p}_u - \mathbf{p}_v)$$

Energy minimization

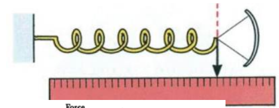
$$E(\rho) := \sum_{(i,j)\in E} |\rho(v_i) - \rho(v_j)|^2 = \sum_{(i,j)\in E} (x_i - x_j)^2 + (y_i - y_j)^2$$

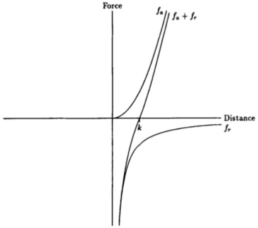
find ρ minimizing

$$\begin{cases} E(\rho) \\ \text{subject to } \rho(v_k) = p_k = (x_k, y_k) \text{ (for exterior vertices } v_k) \end{cases}$$

Related drawing paradigm: force-directed algorithms

Spring electrical model (Fruchterman and Reingold, 1991) **(not covered)**





$$F_a(u) = \sum_{(u,v)\in E} \frac{\|\mathbf{x}(u) - \mathbf{x}(v)\|}{K} (\mathbf{x}(v) - \mathbf{x}(v))$$

$$F_r(u) = \sum_{v\in V, v\neq u} \frac{-CK^2(\mathbf{x}(v) - \mathbf{x}(u))}{\|\mathbf{x}(u) - \mathbf{x}(v)\|^2}$$

```
area:= W * L; {W and L are the width and length of the frame}
G := (V, E); {the vertices are assigned random initial positions}
k := sqrt(area/|V|);
function f_a(x) := begin return x^2/k end;
function f_r(x) := begin return k^2/x end;
for i := 1 to iterations do begin
    {calculate repulsive forces}
    for v in V do begin
        {each vertex has two vectors: .pos and .disp
        v.disp := 0;
        for u in V do
            if (u ≠ v) then begin
            {δ is the difference vector between the positions of the two vertices}
                δ := v.pos − u.pos;
                v.disp := v.disp + (δ/|δ|) * f_r(|δ|)
            end
    end
    {calculate attractive forces}
    for e in E do begin
        {each edges is an ordered pair of vertices .vand.u}
        δ := e.v.pos − e.u.pos;
        e.v.disp := e.v.disp − (δ/|δ|) * f_a(|δ|);
        e.u.disp := e.u.disp + (δ/|δ|) * f_a(|δ|)
    end

outside frame}
    for v in V do begin
        v.pos := v.pos + (v.disp/|v.disp|)
        v.pos.x := min(W/2, max(−W/2, v.pos.x));
        v.pos.y := min(L/2, max(−L/2, v.pos.y))
    end


end
```

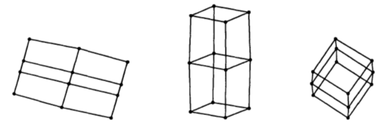


Figure 63. Three-dimensional layouts of twin cubes

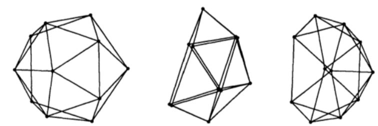
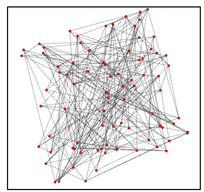
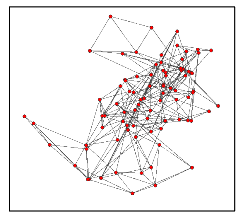
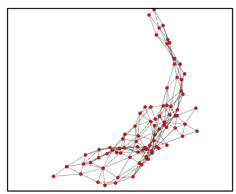
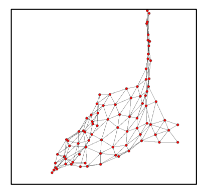


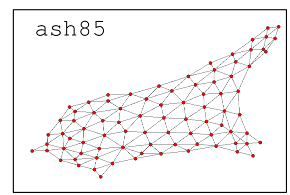
Figure 64. Three-dimensional layouts of an icosahedron variant









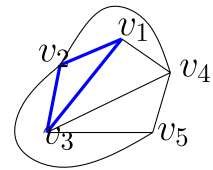


Related drawing paradigm: spectral drawing

(not covered)

$$L_G[i,j] = \begin{cases} deg(v_i) & \textbf{si } i = j \\ -A_G[i,j] & \text{otherwise} \end{cases}$$

$$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

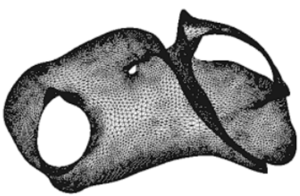


(`degree-normalized` (Koren))

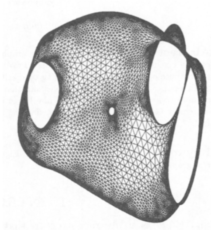
$$E(\rho) := \sum_{(ij)\in E} \|\rho(v_i) - \rho(v_j)\|^2$$

$$min_{\underline{x}}\, E(\underline{x}) := \frac{x^T L_G\, x}{x^T \Delta x}$$

$$\begin{cases} min_{\underline{x}}\, E(\underline{x}) := x\, L_G\, x \\ \text{constraint: } \underline{x}^T \cdot \underline{x} = 1 \\ x_M = \sum_i x_i = 0 \qquad \underline{x}^T \cdot 1_n = 0 \end{cases} \longrightarrow (x_1, \dots, x_d) = \left(\frac{v_2[i]}{\sqrt{\lambda_2}}, \frac{v_3[i]}{\sqrt{\lambda_3}}, \dots, \frac{v_{d+1}[i]}{\sqrt{\lambda_{d+1}}}\right)$$

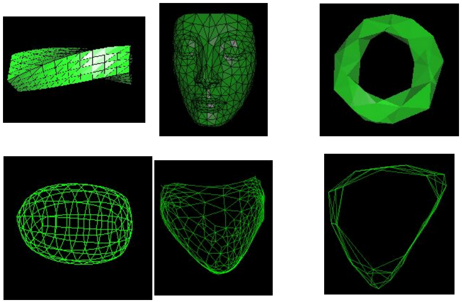


(`4elt` graph, force-directed layout)



(b) The 4elt graph [10] $|V| = 15,606$, $|E| = 45,878$.

(`4elt` graph, spectral layout)
(image from Koren, 2005)



(images from TD, INF562)

Tutte’s theorem: the proof

First: existence and uniqueness of barycentric representations

Second: the barycentric representation defines a planar drawing (no edge crossing)

Third: characterization of non-separating cycles

First: existence and uniqueness of barycentric representations

Theorem (Tutte)

Let G be a 3-connected planar graph with n vertices, and F a peripheral cycle (such that $G \setminus F$ is connected). Let P be a convex polygon, such that $\rho(F) = P$. Then the barycentric representation ρ exists (and is unique)

***Goal**: show that the two systems above admit a solution (unique)*

Let us denote $\rho(v_i) := (x_i, y_i) = \mathbf{x}_i$ the coordinates of vertex v_i

$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases} \longleftrightarrow \quad \rho(v_i) = \sum_{1}^{n} w_{ij} \rho(v_j) \qquad i = 1, \dots, (n-k)$$

(one equation for each inner vertex)

$\mathbf{x} = [x_1, x_2, \dots, x_{n-k}]$

$\mathbf{y} = [y_1, y_2, \dots, y_{n-k}]$

(coordinates of inner vertices)

$$\rho(v_i) = \sum_{j \in N(i)} w_{ij} \rho(v_j)$$

Theorem (Tutte)
Let G be a 3-connected planar graph with n vertices, and F a peripheral cycle (such that $G \setminus F$ is connected). Let P be a convex polygon, such that $\rho(F) = P$. Then the barycentric representation ρ exists (and is unique)

$$\begin{cases} (I - W) \cdot \mathbf{x} = \mathbf{b}_x \\ (I - W) \cdot \mathbf{y} = \mathbf{b}_y \end{cases} \longleftrightarrow \rho(v_i) = \sum_{1}^{n} w_{ij}\rho(v_j) \quad i = 1, \dots, (n-k)$$

($I - W$ is not symmetric)

(one equation for each inner vertex)

$\mathbf{x} = [x_1, x_2, \dots, x_{n-k}]$

$\mathbf{y} = [y_1, y_2, \dots, y_{n-k}]$

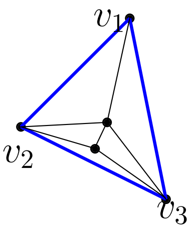
(coordinates of inner vertices)

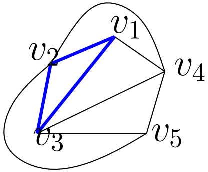
Example *(to help intuition)*

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_{4x} \\ b_{5x} \end{bmatrix} \qquad \begin{cases} \rho(v_4) = \frac{1}{4}\rho(v_1) + \frac{1}{4}\rho(v_2) + \frac{1}{4}\rho(v_3) + \frac{1}{4}\rho(v_5) \\ \rho(v_5) = \frac{1}{3}\rho(v_2) + \frac{1}{3}\rho(v_3) + \frac{1}{3}\rho(v_4) \end{cases}$$

$$N(v_4) = \{v_1, v_2, v_3, v_5\}$$
$$N(v_5) = \{v_2, v_3, v_4\}$$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} b_{4y} \\ b_{5y} \end{bmatrix} \qquad \begin{cases} \rho(v_4) - \frac{1}{4}\rho(v_5) = \frac{1}{4}\rho(v_1) + \frac{1}{4}\rho(v_2) + \frac{1}{4}\rho(v_3) \\ -\frac{1}{3}\rho(v_4) + \rho(v_5) = \frac{1}{3}\rho(v_2) + \frac{1}{3}\rho(v_3) \end{cases}$$





First: existence and uniqueness of barycentric representations

Lemma *The barycentric representation ρ exists (and is unique)*

Proof: (via energy minimization)

Let us denote $\rho(v_i) := (x_i, y_i) = \mathbf{x}_i$

spring energy for edge (v_i, v_j)

$$E(v_i, v_j) := D_{ij}\|\rho(v_i) - \rho(v_j)\|^2$$

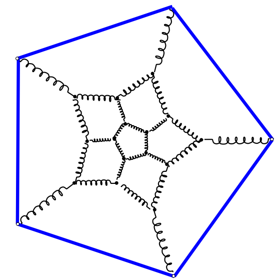
Consider the spring energy of the whole system (of all inner edges):

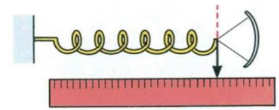
$$E(\rho) := \sum_{e=(i,j)\in E} D_{ij}\|\rho(v_i) - \rho(v_j)\|^2 = \sum_{(i,j)\in E} D_{ij}[(x_i - x_j)^2 + (y_i - y_j)^2]$$

Rewrite the sum:

$$E(\rho) := \frac{1}{2}\sum_{v_i\in V}\sum_{j\in N_i} D_{ij}\|\mathbf{x}_i - \mathbf{x}_j\|^2$$

physical analogy





(spring energy)

$$E(x) = \frac{1}{2}kx^2$$

$$F(x) = kx$$

to find the coordinates $\mathbf{x}_i$ minimizing the energy, compute the gradient of E:

$$\frac{\partial E}{\partial \mathbf{x}_i} = 0 \qquad \sum_{j\in N_i} D_{ij}(\mathbf{x}_i - \mathbf{x}_j) = \sum_{j\in N_i} D_{ij}\mathbf{x}_i - \sum_{j\in N_i} D_{ij}\mathbf{x}_j = 0$$

$$\mathbf{x}_i = \sum_{j\in N_i}[\frac{D_{ij}}{\sum_{j\in N_i} D_{ij}}]\mathbf{x}_j = \sum_{j\in N_i} w_{ij}\mathbf{x}_j$$

Remark: the solution is **not degenarate**, because of boundary constraints
(to be proved... later)

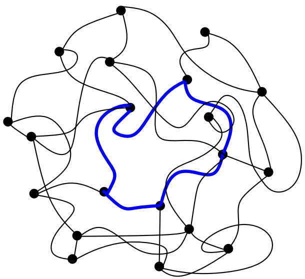
$$w_{ij} := [\frac{D_{ij}}{\sum_{j\in N_i} D_{ij}}]$$

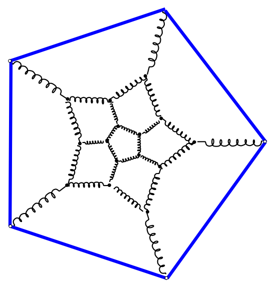
barycentric coordinates

Second: the barycentric representation defines a planar drawing

Theorem (Tutte)
Let G be a 3-connected planar graph with n vertices, and F a non-separating cycle (such that $G \setminus F$ is connected). Let P be a convex polygon, such that $\rho(F) = P$. Then the barycentric representation defines a planar drawing (no edge crossing)

Proof: (we follow the presentation given by Jeff Erickson)





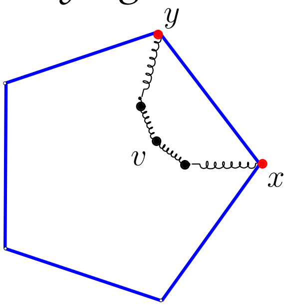
Second: the barycentric representation defines a planar drawing

Lemma (outer face)

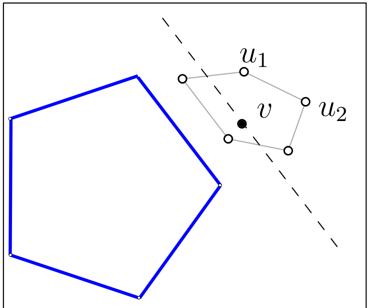
In any Tutte embedding the image of every inner vertex v is a point lying in the interior of the outer face (the convex polygon)

Proof:

Remark: (every inner vertex v is a barycentric combination of its neighbors u_i)

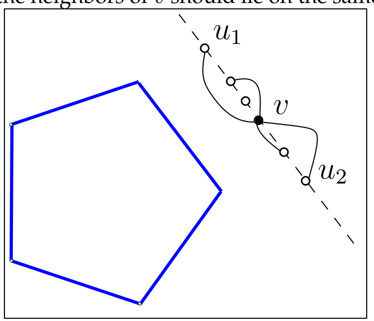


3-connectedness is important

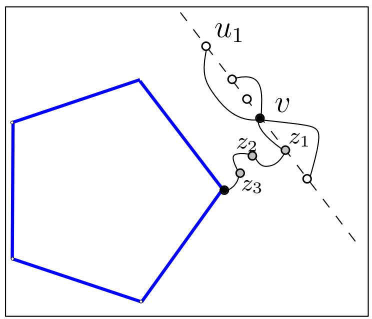


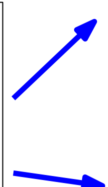
$v :=$ farthest point

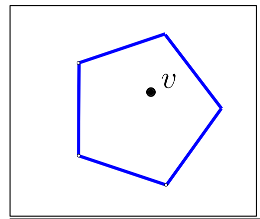
All the neighbors of v should lie on the same line



G is connected, then (by induction) there is a path $P = \{z_1, z_2, \ldots\}$

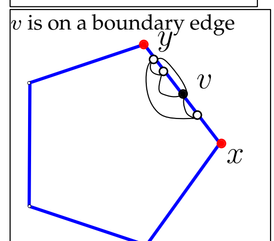






v is in the interior of C

then we are done



All the neighbors of v should lie on the same boundary between x and y, and also all the inner vertices reachable from v

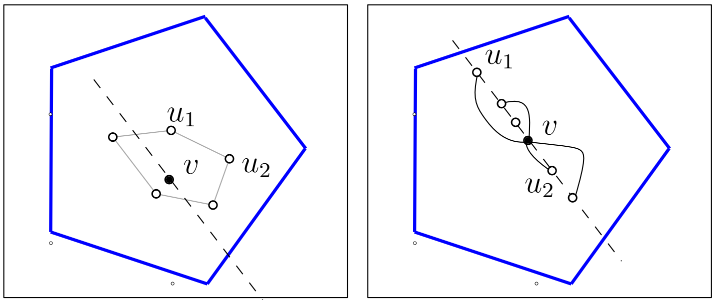
there is a cut-pair x, y contradicting the 3-conn.

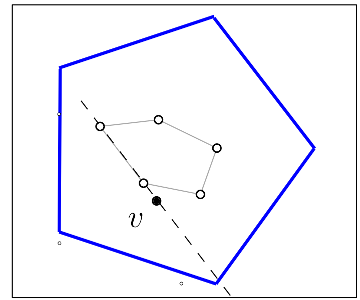
Second: the barycentric representation defines a planar drawing

Lemma (both sides)

Given an inner vertex v and a line l passing through its image $\rho(v)$ either all neighbors of v lie on l, or there are neighbors on both sides of l.

Proof:





impossible (v must be in the interior or the convex hull)

Lemma (convexity)

Every face in the Tutte embedding is a convex polygon.

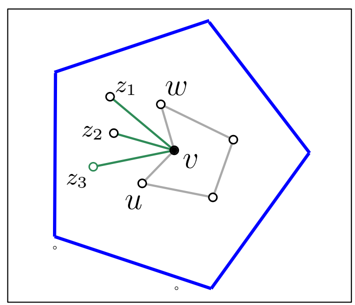
Remark: the drawing could be still degenrate (to be proved)

Proof:

By contradiction, assume f is not convex

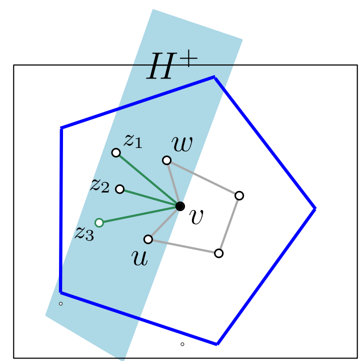
There must be a reflex angle at v

All neighbors of v must lie in the concave region between (u, v) and (w, v)



all neighbors must lie in the half-plane H^+

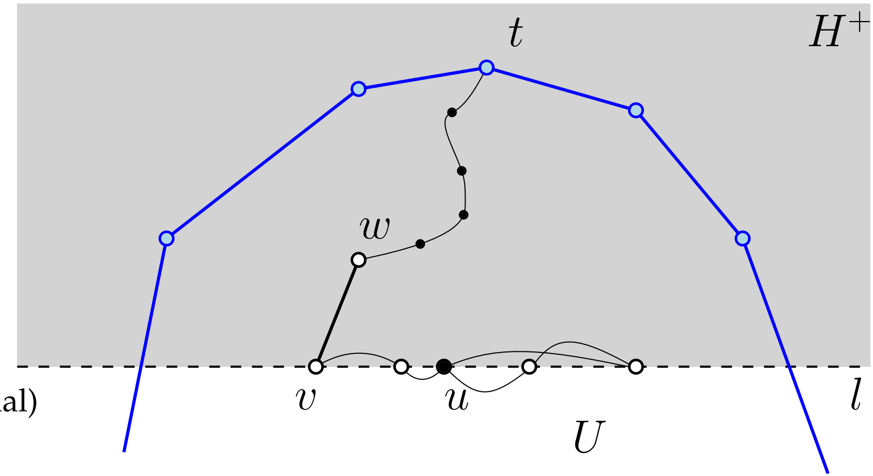
contradicting previous Lemma



Second: the barycentric representation defines a planar drawing

Lemma (half-plane)
Let H^+ be an half-plane containig at least one vertex of G. Then the sub-graph of G induced by all the vertices lying in H^+ is connected.

Proof:



$t :=$ vertex with larget y-coord
(remark: t must lie on the convex hull)

Let u be an arbitrary vertex in H^+

claim: there is a path from u to t
(with non-decreasing y-coordinates)

assume $u^y < t^y$ (otherwise the claim is trivial)

G is connected, then there is $v \in U$ with neighbors in both H^+ and H^-
(because previous Lemma)

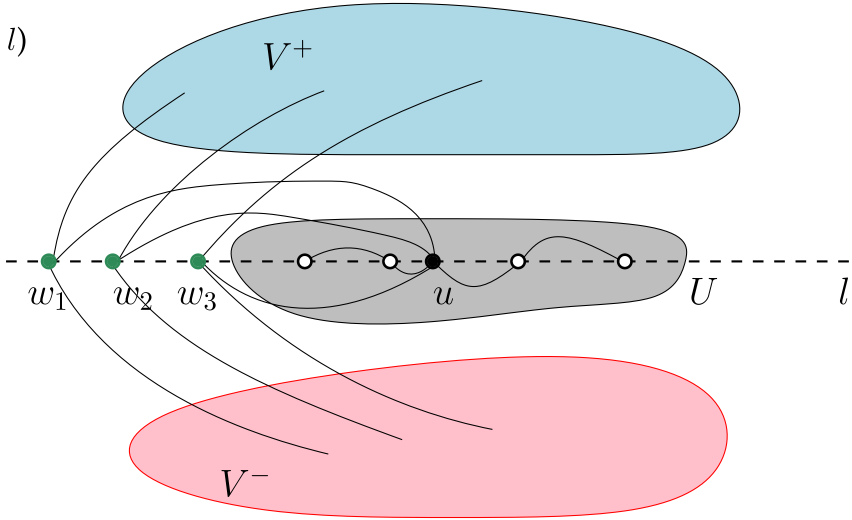
apply induction to the vertex w neighbor of v: since $w^y > v^y$ we can find a path from v to the boundary

Second: the barycentric representation defines a planar drawing

Lemma (non-degeneracy)

No vertex is collinear with all its neighbors.

Proof: (by contradiction u has all its neighbors on l)



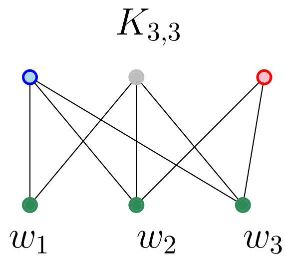
The induced graphs $G(V^+)$ and $G(V^-)$ are connected (previous lemma)

U := set of vertices reachable from u and whose neighbors all lie on l

W := set of vertices lying on l having at least one neighbor not in U

G is 3-connected, so $|W| \geq 3$

contract all edges in $G(V^+)$ and $G(V^-)$
(edge contraction preserve planarity)

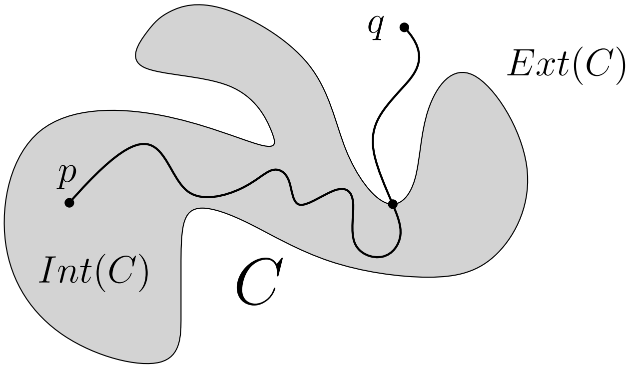


($K_{3,3}$ contradicts the planarity of G)

The Jordan curve theorem

Theorem

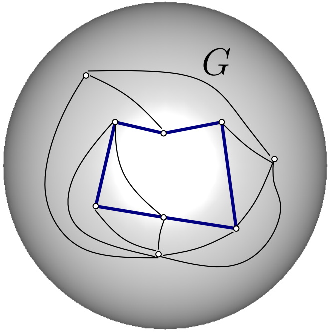
Any simple closed curve C in the plane partitions $\mathbb{R}^2$ into two disjoint arcwise-connected open sets.



($Ext(C)$ and $Int(C)$ are closed sets)

$$Ext(C) \cap Int(C) = C$$

Remark:
Any arc joining a point p in the (open) interior to a point q in the (open) exterior must meet C at least once.



Jordan curve Theorem (reformulation)

Let G a graph embedded on $\mathbb{S}^2$. Then G disconnets $\mathbb{S}^2$ if and only if it contains a circuit

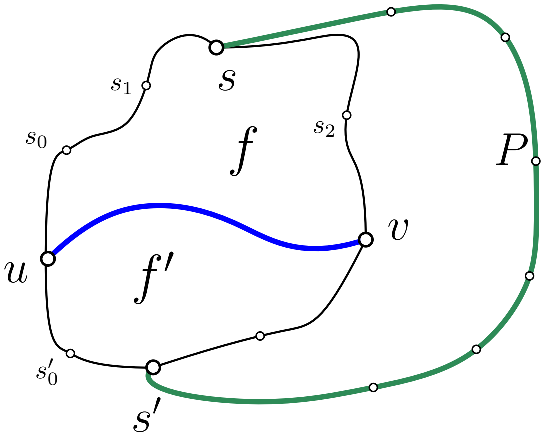
Second: the barycentric representation defines a planar drawing

Lemma (Geelen)

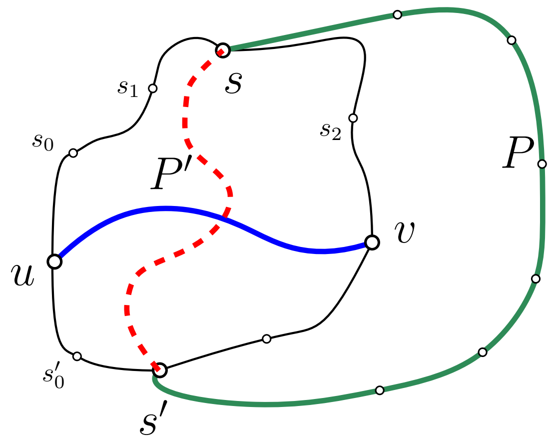
Let us consider an edge $e = (u, v)$ incident to two faces f and f', whose remaining vertices are in two sets S and S'. Consider an arbitraty path P from one vertex in S to one vertex in S'. Then every path from u to v either consists of the edge (u, v) or contains a vertex of the path P.

Proof:

Consider an arbitrary planar embedding of G



$P' :=$ a curve crossing (u, v), lying inside $f \cup f'$



The closed curve $C = P \cup P'$ separates u from v

Then every path from u to v must cross C (by Jordan curve thm)

Second: the barycentric representation defines a planar drawing

Lemma (Split Faces)

Let us consider an edge $e = (u, v)$ incident to two faces f and f', whose remaining vertices are in two sets S and S'. Consider a line l passing through u and v. Then the vertices in S and S' lie on opposite sides with respect to l (and there is no vertex on l).

Proof: (by contradiction: assume there is $s, s' \in H^-$)

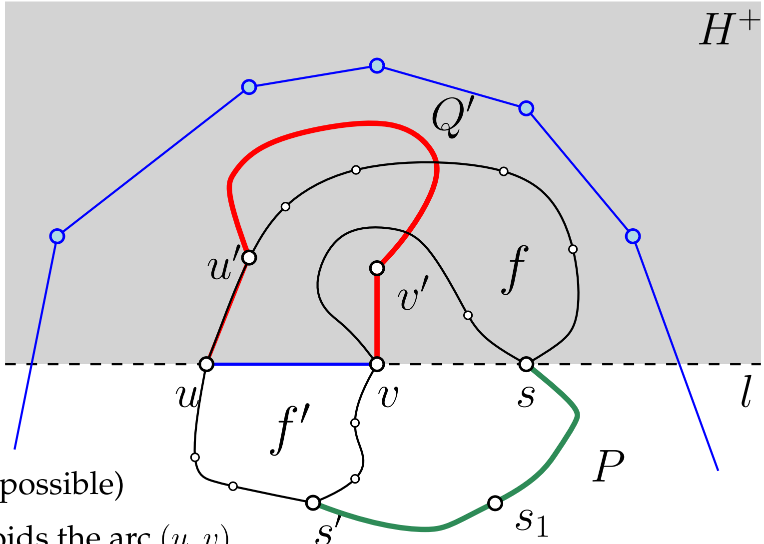
$\exists s_1, s'$ are strictly below l (degeneracy Lemma)

The graph included in H^- is connected (half-space Lemma): then there exists a path P from s to s'

$\exists u', v' \in H^+$ above l (degeneracy Lemma)

there exists a path Q' from u' to v' (above l)

consider the path $Q := Q' \cup (u, u') \cup (v, v')$ (above l)



Apply Geelen's Lemma: path Q should cross P (impossible)

contradiction: the path Q does not cross P, and avoids the arc (u, v)

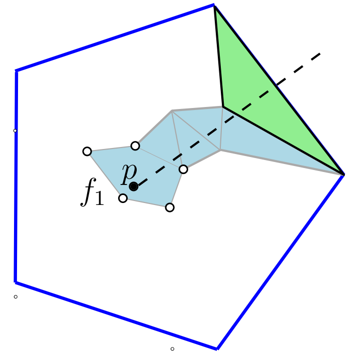
Second: the barycentric representation defines a planar drawing

Final Lemma (no edge crossings)

The Tutte embedding of G is crossing-free (faces are non-overlapping).

Proof:

Claim: a generic point cannot lie into two different faces f_1 and f_2



Strategy: draw a (generic) line from p to infinity (crossing edges only at their interior)