Maps: at the interface between combinatorics and probability

Marie Albenque (CNRS, LIX, École Polytechnique)
Maps – Definition(s)

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Euler’s formula: for every map \( m \) (on a closed surface without boundary),

\[
|V(m)| + |F(m)| = 2 + |E(m)| - 2g(m)
\]

vertices faces edges genus
Maps – Definition(s)

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Here, the resulting surface is the sphere: this is a planar map.

We will also encounter maps on other closed orientable surfaces: torus of genus $g$, disks, ...

Euler’s formula: for every map $m$,

$$|V(m)| + |F(m)| = 2 + |E(m)| - 2g(m)$$

If all the polygons have $p$ sides, the map is called a $p$-angulation

3-angulation = triangulation, 4-angulation = quadrangulation
Maps – Definition(s)

A **planar map** is a proper embedding of a planar connected graph in the 2-dimensional sphere (considered up to orientation-preserving homeomorphisms).
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Planar map = planar graph + cyclic order of edges around each vertex.
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![Diagram showing planar maps]

This is the root corner

Planar map = planar graph + cyclic order of edges around each vertex.

To avoid dealing with symmetries: maps are **rooted** (a corner is marked).
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A **planar map** is a proper embedding of a planar connected graph in the 2-dimensional sphere (considered up to orientation-preserving homeomorphisms).

Planar map = planar graph + cyclic order of edges around each vertex.

To avoid dealing with symmetries: maps are **rooted** (a corner is marked).

A map $M$ defines a discrete **metric space**:

- points: set of vertices of $M = V(M)$.
- distance: graph distance = $d_{gr}$. 
I - Bijective enumeration of maps

II - Scaling limits of random planar maps

III - Local limit of Ising-weighted random triangulations
I - Bijective enumeration of maps

a tribute to blossoming bijections.
Bijections with blossoming trees

A blossoming tree is a plane tree where vertices can carry opening stems or closing stems:

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Via this construction, a planar map is canonically associated to a blossoming tree.
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**Can we reverse the construction?**

i.e. can we determine a canonical spanning tree?
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Yes...
Enumeration of planar maps

In the 60’s, Tutte obtained closed enumerative formulas for many families of planar maps.

\[
\text{e.g. } \# \{ \text{rooted planar maps with } n \text{ edges} \} = \frac{2 \cdot 3^n}{n+2} \text{Catalan}(n) \quad [\text{Tutte 63}]
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[Tutte 63]

Combinatorial proof ? Bijection ?

Yes ! [Cori & Vauquelin 81], [Schaeffer 97, 98]
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Map with \( n \) edges

Radial construction

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Map with \(n\) edges

4-valent map with \(n\) vertices

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\# \left\{ \text{plane trees with } n \text{ vertices} \right\} = \# \left\{ \text{binary plane trees with } n \text{ inner vertices} \right\}
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Radial construction [Tutte 63]

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As a corollary:
combinatorial proof of Tutte’s formula.
Enumeration of planar maps: a dichotomy of bijections

- Radial construction
  - [Tutte 63]

- 4-valent map with $n$ vertices

- Blossoming bijection
  - [Schaeffer 97]

- 4-valent blossoming trees with $n$ vertices

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- Label vertices by their distance to the root vertex
- 4-valent blossoming trees with $n$ vertices
- 4-valent map with $n$ vertices
- Quadrangulation with $n$ faces
- Map with $n$ edges

- duality

- [Schaeffer 98]
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4-valent map with $n$ vertices

4-valent blossoming trees with $n$ vertices

Tutte's bijection [Tutte 63]

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Label vertices by their distance to the root vertex

quadrangulation with $n$ faces

Well-labeled tree
Enumeration of planar maps: a dichotomy of bijections
Enumeration of planar maps: a dichotomy of bijections

Blossoming bijections

- Eulerian maps [Schaeffer 97]
- General maps with prescribed vertices degree sequences [Bouttier, Di Francesco, Guitter 02]
- Constellations [Bousquet-Mélou, Schaeffer 00]
- Bipartite maps [Bousquet-Mélou, Schaeffer 02]
- Simple triangulations [Poulalhon, Schaeffer 05], simple quadrangulations [Fusy 07]

(without loops nor multiple edges)
Enumeration of planar maps: a dichotomy of bijections

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**Mobile type bijections**

- Quadrangulations [Schaeffer 98]
- General maps with prescribed faces degree sequences [Bouttier, di Francesco, Guitter 04] = BDG bijection
- Maps with sources and delays [Miermont 09], [Bouttier, Fusy, Guitter 14 ]
- Extension to higher genus [Chapuy, Marcus, Schaeffer 09],
Bijections with blossoming trees

Can we unify all the blossoming bijections?
Bijections with blossoming trees

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Can we reverse the construction??
Can we unify all the blossoming bijections? 

Via this construction, an oriented planar map is canonically associated to a blossoming tree.

Can we reverse the construction? Yes, by a generic bijective scheme:

**Theorem:** [A., Poulalhon 15] (generalization of results of [Bernardi ’07])
If a planar map $M$ is endowed with a “nice orientation” of its edges, then there exists a unique blossoming tree whose closure is $M$ endowed with its orientation.
Bijections with blossoming trees

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If a planar map $M$ is endowed with a “nice orientation” of its edges, then there exists a **unique** blossoming tree whose closure is $M$ endowed with its orientation.

Combined with the general theory of $c$-orientations [Propp 03] and/or $\alpha$-orientations [Felsner 04], this allows to retrieve **all the bijections** mentioned above and to obtain **new bijections** for which no enumerative formulas are available (cf also [Bernardi, Fusy 12]).
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Often easier to “guess” the right orientations than the right families of trees.

Blossoming bijection for $d$-angulations of girth $d$ with a boundary, [A., Poulalhon 15].

= length of the smallest cycle
Blossoming bijections in higher genus

**Theorem:** [Tutte 63], bijective proof in [Schaeffer 97]

\[ M(z) = \sum_{m} z^{|E(m)|}, \quad \text{where} \quad m \in \{ \text{planar maps} \}. \]

Then:

\[ M = T^2(1 - 4T) \quad \text{where} \quad T \text{ unique formal power series defined by} \quad T = z + 3T^2 \]
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Theorem: [Bender, Canfield 91], first bijective proof in [Lepoutre 19+]

For any \( g \geq 1 \), let \( M_g(z) = \sum_m z^{|E(m)|}, \text{ where } m \in \{\text{maps of genus } g\}. \)

Then \( M_g \) is a rational function of \( T \).

Idea of proof: Generalization of Schaeffer’s blossoming bijection to higher genus.

Careful analysis of the blossoming uncellular maps
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Careful analysis of the blossoming **unicellular** maps

Result not available with the “mobile-type” bijection of [Chapuy – Marcus – Schaeffer]
Theorem: [Tutte 63], bijective proof in [Schaeffer 97]

\[ M(z_\bullet, z_\circ) = \sum_{m} z_\bullet^{\left| V(m) \right|} z_\circ^{\left| F(m) \right|}, \text{ where } m \in \left\{ \text{planar maps} \right\}. \]

Then \[ M = T_\circ T_\bullet (1 - 2T_\circ - 2T_\bullet) \] where

\[
\begin{align*}
T_\bullet &= z_\bullet + T_\bullet^2 + 2T_\circ T_\bullet \\
T_\circ &= z_\circ + T_\circ^2 + 2T_\bullet T_\circ
\end{align*}
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**Euler’s formula:**

\[ |V(m)| + |F(m)| = 2 + |E(m)| - 2g(m) \]
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**Euler’s formula:** \[ \vert V(m)\vert + \vert F(m)\vert = 2 + \vert E(m)\vert - 2g(m) \]

---

Map with \( n \) edges

Radial construction [Tutte 63]

4-valent map with \( n \) vertices
Blossoming bijections in higher genus

**Theorem:** [Tutte 63], bijective proof in [Schaeffer 97]

\[ M(\bullet,\circ) = \sum_{m} \bullet^{|V(m)|} \circ^{|F(m)|}, \text{ where } m \in \{\text{planar maps}\}. \]

Then \( M = T \circ T \bullet (1 - 2T \circ - 2T \bullet) \) where

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**Euler’s formula:** \( |V(m)| + |F(m)| = 2 + |E(m)| - 2g(m) \)

**Radial construction** [Tutte 63]

Already for planar maps, this result is not accessible with mobile-type bijections.
Blossoming bijections in higher genus

**Theorem:** [Tutte 63], bijective proof in [Schaeffer 97]

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M(z\bullet, z\circ) = \sum_m z^{|V(m)|} z\circ^{|F(m)|}, \text{ where } m \in \{\text{planar maps}\}.
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**Euler’s formula:** \(|V(m)| + |F(m)| = 2 + |E(m)| - 2g(m)\)

**Theorem:** [Bender, Canfield, Richmond 95], bijective proof in [A., Lepoutre 20+]

For any \( g \geq 1 \), let

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\]

Then \( M_g \) is a rational function of \( T\bullet \) and \( T\circ \).

**Idea of proof:** Same bijection but different proof for the analysis of the unicellular blossoming maps (gives also a simpler proof of the univariate case).
II - Scaling limits of random maps

Global point of view
(scaling limit):

Simulation by T.Budd
Scaling limit of random quadrangulations

\[ Q_n = \{\text{Quadrangulations of size } n\} = n + 2 \text{ vertices, } n \text{ faces, } 2n \text{ edges} \]

\[ Q_n = \text{Uniform random element of } Q_n \]

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When the size of the map goes to infinity, so does the typical distance between two vertices.

Idea: ”scale” the map = length of edges decreases with the size of the map.
Goal: obtain a limiting (non-trivial) compact object
Scaling limit of random quadrangulations

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Motivations:

- Natural random discretization of a continuous surface.
- Construction of a 2-dim. analogue of the Brownian motion: The Brownian Map
  [Miermont 13], [Le Gall 13].
- Link with Liouville Quantum Gravity,
  [Duplantier, Sheffield 11], [Duplantier, Miller, Sheffield 14], [Miller, Sheffield 16,16,17]
Scaling limit of uniform quadrangulations

Idea: "scale" the map = length of edges decreases with the size of the map.
Goal: obtain a limiting (non-trivial) compact object

For quadrangulations: well understood

- The bijection of Schaeffer: quadrangulations ↔ labeled trees.
  Labels in the trees = distances between the vertices and the root.
- distance between two random points \( \sim n^{1/4} + \text{law of the distance} \) [Chassaing-Schaeffer '04]
- convergence of normalized quadrangulations + properties of the limit [Marckert-Mokkadem '06], [Le Gall '07], [Le Gall, Paulin '08] [Miermont '08]

Hausdorff dimension = 4

topology of the limit = sphere

Theorem: [Miermont 13], [Le Gall 13]
Let \( (Q_n) \) be a sequence of random quadrangulations of size \( n \). Then:

\[
\left( V(Q_n), \left( \frac{9}{8n} \right)^{1/4} d_{gr} \right) \xrightarrow{(d)} \text{The Brownian Map}
\]
Universality of the scaling limit

what if quadrangulations are replaced by
triangulations, maps, simple triangulations, ...?

Idea: The Brownian map is a universal limiting object.

All "reasonable models" of maps (properly rescaled) are expected to converge towards it.
Universality of the scaling limit

Idea: The Brownian map is a **universal** limiting object.

All "reasonable models" of maps (properly rescaled) are expected to converge towards it.

**Theorem:** [Le Gall 13]

Fix $p \in \{3\} \cup 2\mathbb{N}$, let $(M_n)$ be a sequence of random $p$-angulations of size $n$. Then:

$$(V(M_n), \frac{C_p}{n^{1/4}} d_{gr}) \xrightarrow{(d)} \text{The Brownian Map}$$

Replacing Schaeffer’s bijection by the bijection of [Bouttier, Di Francesco, Guitter 04].

**Le Gall’s magic trick:**

Since uniform quadrangulations are invariant by rerooting, the fact that they converge to the Brownian map, implies that the **Brownian map is invariant by rerooting**.

→ Use this invariance to prove the convergence of others models of maps.
Universality of the scaling limit

To prove that another model of maps converges to the Brownian map:

1. encode the maps by some labeled trees,
2. study the limits of the labeled trees,
3. interpret the distance in the maps by some function of the labeling of the tree.
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To prove that another model of maps converges to the Brownian map:

1. encode the maps by some labeled trees,
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**Theorem:** [Addario–Berry, A. 20] For $p \in 2\mathbb{N} + 1$, $(M_n) = \text{random } p\text{-angulations}:
\left(V(M_n), \frac{C_p}{n^{1/4}} d_{gr}\right) \xrightarrow{(d) \text{ Gromov–Hausdorff topology}} \text{The Brownian Map}

**Difficulty:** 2. The labeled trees obtained by the BDG bijection are not “nice”.
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**Theorem:** [Addario–Berry, A. 20] For \( p \in 2\mathbb{N} + 1 \), \( (M_n) = \) random \( p \)-angulations:

\[
\left( V(M_n), \frac{C_p}{n^{1/4}} d_{gr} \right) \xrightarrow{(d)} \text{The Brownian Map}
\]

**Difficulty:** 2. The labeled trees obtained by the BDG bijection are not “nice”.

**Theorem:** [Addario–Berry, A. 15] Let \( (\Delta_n) = \) random simple triangulations:

\[
\left( V(\Delta_n), \left( \frac{3}{4n} \right)^{1/4} d_{gr} \right) \xrightarrow{(d)} \text{The Brownian Map}
\]

**Difficulty:** 3. Track distances in blossoming bijections.
Convergence of trees
Convergence of trees
Convergence of trees
Convergence of trees
Convergence of trees

\[ T \rightarrow C_n(2n \cdot t) = \text{contour process} \]
Convergence of trees

\[ C_n(2n \cdot t) = \text{contour process} \]

\( i \) and \( j \)

= same vertex of \( T \)

\[ C_n(i) = C_n(j) \]

= \( \min_{i \leq k \leq j} C_n(k) \)
Convergence of trees

$T$ = labeled tree,

$C_n(2n \cdot t) = \text{contour process}$

$(C_n(2n \cdot t), Z_n(2n \cdot t)) = \text{contour and label processes}$

$i$ and $j$

$= \text{same vertex of } T$

$\iff$

$C_n(i) = C_n(j)$

$= \min_{i \leq k \leq j} C_n(k)$
Convergence of trees

\[ C_n(2n \cdot t) = \text{contour process} \]

\[ (e_t)_{0 \leq t \leq 1} = \text{Brownian excursion} \]

Simulations by I. Kortchemski
Convergence of trees

$C_n(2n \cdot t) = \text{contour process}$

scaling limit (rescaled by $n^{-1/2}$)

$(e_t)_{0 \leq t \leq 1} = \text{Brownian excursion}$

Continuum Random Tree

$\mathcal{T}_e$, [Aldous 91]

Simulations by I. Kortchemski

$i$ and $j$

$= \text{same vertex of } T$

$\iff$

$C_n(i) = C_n(j)$

$= \min_{i \leq k \leq j} C_n(k)$
Convergence of trees of labeled trees

1st step: the Brownian tree

Simulations by I. Kortchemski
Convergence of trees of labeled trees

1st step: the Brownian tree

2nd step: Brownian snake

\[
(e_t, Z_t) = \text{Brownian snake} \quad \text{[Le Gall 93]}
\]

Theorem: [Janson, Marckert 04], [Miermont 08], ...

For a sequence \((T_n)\) of “nice” random labeled trees:

\[
\left( \frac{a C_n(2nt)}{n^{1/2}}, \frac{b Z_n(2nt)}{n^{1/4}} \right) \xrightarrow{(d)} (e_t, Z_t)
\]

for the uniform topology of \(C([0, 1], \mathbb{R})^2\),

Conditional on \(T_e\), \(Z\) a centered Gaussian process with \(Z_\rho = 0\) and \(E[(Z_s - Z_t)^2] = d_e(s, t)\).

\(Z \sim \text{Brownian motion on the tree}\)
Convergence of trees of labeled trees

1st step: the Brownian tree

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\[ Z \sim \text{Brownian motion on the tree} \]

Nice = typically Galton-Watson trees, with centered increments of labels along edges.
Convergence of odd-angulations

Illustration of the Bouttier – Di Francesco – Guitter bijection for a **non-bipartite** map.
Convergence of odd-angulations

Illustration of the Bouttier – Di Francesco – Guitter bijection for a non-bipartite map.

Labeled tree obtained = 4-type Galton-Watson tree $T +$ random label increments along edges.

**Problem:** For $e$ an edge of $T$, $\mathbb{E}[\text{label increments along } e] \neq 0$

i.e. the label increments are not centered.
Convergence of odd-angulations

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Problem: For $e$ an edge of $T$, $\mathbb{E}[\text{label increments along } e] \neq 0$

i.e. the label increments are not centered.

$\Rightarrow$ Known results of convergence do not apply.

A source of hope:

$$\sum_{i=1}^{k} \mathbb{E}[\text{label increment along } e_i] = 0$$
Convergence of odd-angulations

A source of hope:

\[ \sum_{i=1}^{k} \mathbb{E}[\text{label increment along } e_i] = 0 \]

**Theorem:** [Addario–Berry, A. 20] For \( p \in 2\mathbb{N} + 1 \), \((M_n^{(p)}) = \) random \( p \)-angulations:

Let \( T_n^{(p)} = \Phi_{\text{BDG}}(M_n^{(p)}) \), then:

\( \left( \frac{a_p C_n(2nt)}{n^{1/2}}, \frac{b_p Z_n(2nt)}{n^{1/4}} \right) \) converges to \((e_t, Z_t)\) for the uniform topology of \( C([0, 1], \mathbb{R})^2\).

[Marckert 07] convergence in this setting (with even weaker “centering assumption”) but requires **monototype** GW trees + **bounded** number of children.
Convergence of odd-angulations

A source of hope:

\[ \sum_{i=1}^{k} E[\text{label increment along } e_i] = 0 \]

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Let \( T_n^{(p)} = \Phi_{BDG}(M_n^{(p)}) \), then:

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\left( \frac{a_p C_n(2nt)}{n^{1/2}}, \frac{b_p Z_n(2nt)}{n^{1/4}} \right) \quad \xrightarrow{\text{for the uniform topology of } C([0, 1], \mathbb{R})^2} \quad (e_t, Z_t)
\]

[Marckert 07] convergence in this setting (with even weaker “centering assumption”) but requires **monototype** GW trees + **bounded** number of children.

**Strategy of proof:** Randomly shuffle “our” trees to get a coupling with a “nice” model.

in our case [Miermont 08] is the nice model but it gives a general **bootstrapping principle**.
Convergence of simple triangulations

First step: blossoming bijection of [Poulalhon, Schaeffer 05] for simple triangulations.
Convergence of simple triangulations

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Problem: How to track distances in the map in the blossoming tree?
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- Encode the blossoming trees by labeled trees
  Prove that the scaling limit of trees is the Brownian snake.
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Requires the bootstrapping principle!
Convergence of simple triangulations

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Problem: How to track distances in the map in the blossoming tree?

- Encode the blossoming trees by labeled trees
  - Prove that the scaling limit of trees is the Brownian snake.
- Requires the bootstrapping principle!
- Prove that the labels of the tree give some distance information in the map.
Convergence of simple triangulations

**Problem:** How to track distances in the map in the blossoming tree?

- Encode the blossoming trees by labeled trees
  - Prove that the scaling limit of trees is the Brownian snake.

- Prove that the labels of the tree give some distance information in the map
  - Two key combinatorial observations:
    - Labels in the tree = length of leftmost path in the map
    - Leftmost paths are almost geodesic (up to $o(n^{1/4})$ error term).
Convergence of simple triangulations

**Problem:** How to track distances in the map in the blossoming tree?

- Encode the blossoming trees by labeled trees
  
  Prove that the scaling limit of trees is the **Brownian snake**.

- Prove that the labels of the tree give some distance information in the map

→ Two key combinatorial observations:

  Labels in the tree = length of leftmost path in the map

  Leftmost paths are almost geodesic (up to $o(n^{1/4})$ error term).

**Theorem:** [Addario–Berry, A. 15] Let $(\Delta_n) = \text{random simple triangulations}$:

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\left( V(\Delta_n), \left( \frac{3}{4n} \right)^{1/4} d_{gr} \right) \xrightarrow{\text{Gromov–Hausdorff topology}} \text{The Brownian Map}
$$

Same ideas sucessfully applied to study simple maps [Bernardi, Collet, Fusy 14], simple triangulations on the torus [Beffara, Huynh, Lévêque 20], simple triangulations with a boundary [A., Holden, Sun 20]
Link with Liouville Quantum Gravity

\( \gamma \in (0, 2), \gamma\text{-Liouville Quantum Gravity} = \text{measure on a surface} \) [Duplantier, Sheffield 11].

Simulation of the Brownian map by T.Budd

Simulation of \( \sqrt{\frac{8}{3}} \text{-LQG} \) by T.Budd
Link with Liouville Quantum Gravity

\( \gamma \in (0, 2) \), \( \gamma \)-Liouville Quantum Gravity = measure on a surface [Duplantier, Sheffield 11].

Simulation of the Brownian map by T.Budd [Duplantier, Miller, Sheffield 14]
[Miller, Sheffield 16+16+17]

Construction in the continuum.
in the discrete setting?

Simulation of \( \sqrt{\frac{8}{3}} \)-LQG by T.Budd
Link with Liouville Quantum Gravity

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\[ \sqrt{\frac{8}{3}} \text{-LQG} \]

Simulation of the Brownian map by T.Budd

Construction in the continuum. in the discrete setting?

A priori, there is no canonical way to embed a planar map in the sphere.

But, for simple triangulations: the circle packing theorem gives a canonical embedding.

(Unique up to Möbius transformations.)
Link with Liouville Quantum Gravity

Simulation of a large simple triangulation embedded in the sphere by circle packing.

Software CirclePack by K.Stephenson.

Simulation of $\sqrt{\frac{8}{3}}$-LQG by T.Budd

Motivation to study simple triangulations, but so far no results for random circle packings.
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Motivation to study simple triangulations, but so far no results for random circle packings.

However, [Holden, Sun 19] proved that uniform triangulations (without multiple edges) embedded via the Cardy embedding converge towards $\sqrt{\frac{8}{3}}$-LQG.

Proof is built on many results, among which [A., Holden, Sun ’20]: the scaling limit of triangulations without multiple edges and with a boundary is the Brownian disk.
III - Local limit of Ising-weighted random triangulations

Local point of view
(Benjamini-Schramm topology):

Simulation by I. Kortchemski
Local limit of large uniformly random triangulations

Take a random triangulation with $n$ edges. What does it look like when $n \to \infty$ ?
Local limit of large uniformly random triangulations

Take a random triangulation with $n$ edges. What does it look like when $n \to \infty$?

**Local point of view:**

Look at neighborhoods of the root

The **local topology** (= Benjamini–Schramm topology) on finite maps is induced by the distance:

$$d_{loc}(m, m') = \frac{1}{1 + \max\{r \geq 0 : B_r(m) = B_r(m')\}}$$

where $B_r(m) =$ ball of radius $r$ centered at the root vertex of $m$. 

Simulation by I. Kortchemski
Local limit of large uniformly random triangulations

Take a random triangulation with \( n \) edges. What does it look like when \( n \to \infty \)?

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Simulation by I. Kortchemski

**Theorem** [Angel – Schramm, ’03]

Let \( \mathbb{P}^\Delta_n \) = uniform distribution on triangulations of size \( n \).

\[
\mathbb{P}^\Delta_n \xrightarrow{(d)} \text{UIPT}, \quad \text{for the local topology}
\]

UIPT = Uniform Infinite Planar Triangulation

\( = \) measure supported on infinite planar triangulations.
Local limit of large uniformly random triangulations

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$$\mathbb{P}^\Delta_n \xrightarrow{(d)} \text{UIPT}, \quad \text{for the local topology}$$

UIPT = Uniform Infinite Planar Triangulation

= measure supported on infinite planar triangulations.

Some properties of the UIPT:

- The UIPT has almost surely one end [Angel – Schramm, 03]

- Volume (nb. of vertices) and perimeters of balls known to some extent.

  $$\mathbb{E} [|B_r(T_\infty)|] \sim \frac{2}{7} r^4$$  
  [Angel 04, Curien – Le Gall 12]

- Simple random Walk is recurrent [Gurel-Gurevich and Nachmias 13]
Local limit of large uniformly random triangulations

**Theorem** [Angel – Schramm, ’03]

Let \( \mathbb{P}_n^\Delta \) = uniform distribution on triangulations of size \( n \).

\[ \mathbb{P}_n^\Delta \overset{(d)}{\longrightarrow} \text{UIPT}, \text{ for the local topology} \]

UIPT = Uniform Infinite Planar Triangulation
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- The UIPT has almost surely one end [Angel – Schramm, 03]
- Volume (nb. of vertices) and perimeters of balls known to some extent.
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- Simple random Walk is recurrent [Gurel-Gurevich and Nachmias 13]

**Universality:** we expect the same behavior for other “reasonable” models of maps.

  In particular, we expect the volume growth to be 4.

  (proved for quadrangulations [Krikun 05], simple triangulations [Angel 04])
Escaping universality: adding matter

First, **Ising model** on a finite deterministic planar triangulation $T$:

**Spin configuration** on $T$:

$$
\sigma : V(T) \rightarrow \{-1, +1\} = \{\circ, \triangleright\}
$$

**Ising model** on $T$: take a random spin configuration with probability:

$$
P(\sigma) \propto e^{\beta J \sum_{\nu \sim \nu'} \mathbb{1}_{\{\sigma(\nu) = \sigma(\nu')\}}}
$$

$\beta > 0$: inverse temperature.  
$J = \pm 1$: coupling constant.  
$h = 0$: no magnetic field.
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- $\beta > 0$: inverse temperature.
- $J = \pm 1$: coupling constant.
- $h = 0$: no magnetic field.

**Combinatorial formulation**: $P(\sigma) \propto \nu^{m(\sigma)}$

with $m(\sigma) = \text{number of monochromatic edges}$ ($\nu = e^{\beta J}$).
Escaping universality: adding matter

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$$

$$
h = 0: \text{no magnetic field.}
$$

**Combinatorial formulation:**

$$
P(\sigma) \propto \nu^{m(\sigma)}
$$

with $m(\sigma) = \text{number of monochromatic edges} (\nu = e^{\beta J})$.

Next step: Sample a triangulation of size $n$ **together** with a spin configuration, with probability $\propto \nu^{m(T,\sigma)}$.

$$
\mathbb{P}_n^\nu\left(\{(T,\sigma)\}\right) = \frac{\nu^{m(T,\sigma)} \delta_{|e(T)|=3n}}{\mathcal{Z}_n}.
$$

$\mathcal{Z}_n = \text{normalizing constant}$. 
Escaping universality: adding matter

First, Ising model on a finite deterministic planar triangulation $T$:

**Spin configuration** on $T$:

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$$P(\sigma) \propto e^{\beta J \sum_{\nu \sim \nu'} 1\{\sigma(\nu) = \sigma(\nu')\}}$$

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Next step: Sample a triangulation of size $n$ together with a spin configuration, with probability $\propto \nu^{m(T,\sigma)}$.

| $\mathbb{P}_n^\nu \left( \{(T, \sigma)\} \right) \frac{\nu^{m(T,\sigma)} \delta|e(T)|=3n}{\mathcal{Z}_n}$ |
| --- |
| $\mathcal{Z}_n = \text{normalizing constant.}$ |

Remark: This is a probability distribution on triangulations with spins. But, forgetting the spins gives a probability a distribution on triangulations without spins different from the uniform distribution.
Escaping universality: new asymptotic behavior

Counting exponent for undecorated maps:

\[ \text{coeff } [t^n] \text{ of generating series of (undecorated) maps } \sim \kappa \rho^{-n} n^{-5/2} \]

(e.g.: triangulations, quadrangulations, general maps, simple maps,...)

where \( \kappa \) and \( \rho \) depend on the combinatorics of the model.
Escaping universality: new asymptotic behavior

Counting exponent for undecorated maps:

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where \( \kappa \) and \( \rho \) depend on the combinatorics of the model.

Generating series of Ising-weighted triangulations:

\[
Q(\nu, t) = \sum_{T \in \mathcal{T}_f} \sum_{\sigma: V(T) \rightarrow \{-1, +1\}} \nu^{m(T, \sigma)} t^{e(T)}.
\]

Theorem [Bernardi – Bousquet-Mélou 11]

For every \( \nu > 0 \), \( Q(\nu, t) \) is algebraic and satisfies

\[
[t^{3n}] Q(\nu, t) \sim \begin{cases} 
\kappa \rho_{\nu_c}^{-n} n^{-7/3} & \text{if } \nu = \nu_c = 1 + \frac{1}{\sqrt{7}}, \\
\kappa \rho_{\nu}^{-n} n^{-5/2} & \text{if } \nu \neq \nu_c.
\end{cases}
\]

See also [Boulatov – Kazakov 1987], [Bousquet-Melou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

This suggests a different behavior of the underlying maps for \( \nu = \nu_c \).
Local convergence of triangulations with spins

**Theorem** [A. – Ménard – Schaeffer, 21]

Let $\mathbb{P}_\nu = \nu$-Ising weighted probability distribution:

$$\mathbb{P}_\nu \xrightarrow{(d)} \nu\text{-IIPT}, \quad \text{for the local topology}$$

$\nu\text{-IIPT} = \nu$-Ising Infinite Planar Triangulation

= measure supported on infinite planar triangulations.

Moreover, simple random walk is **recurrent** on the $\nu_c\text{-IIPT}$. 

Local convergence of triangulations with spins

**Theorem** [A. – Ménard – Schaeffer, 21]

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**Strategy of proof:**

- Refinement of enumerative results of [Bernardi, Bousquet-Mélou]

**Theorem** [A. – Ménard – Schaeffer 21]

For every $\nu > 0$, for every $\omega \in \{-1, +1\}^*$

$$[t^{3n}]Z_\omega(\nu, t) \sim_{n \to \infty} \begin{cases} \kappa \rho_{\nu_c}^{-n} n^{-7/3} & \text{if } \nu = \nu_c, \\ \kappa \rho_{\nu}^{-n} n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases}$$

with $Z_\omega(\nu, t)$ generating series of Ising-weighted triangulations with boundary condition given by $\omega$. 
Local convergence of triangulations with spins

**Theorem [A. – Ménard – Schaeffer, 21]**

Let \( P_n^\nu = \nu\)-Ising weighted probability distribution:

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**Strategy of proof:**

- Refinement of enumerative results of [Bernardi, Bousquet-Mélou]

\[
\begin{align*}
[t^{3n}] Z_\omega(\nu, t) & \sim \\
& \begin{cases} 
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\kappa \rho_{\nu}^{-n} n^{-5/2} & \text{if } \nu \neq \nu_c.
\end{cases}
\end{align*}
\]

with \( Z_\omega(\nu, t) \) generating series of Ising-weighted triangulations with boundary condition given by \( \omega. \)

We use the blossoming bijection of [Bousquet-Mélou, Schaeffer 02] to prove that!
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Moreover, simple random walk is **recurrent** on the $\nu_c$-IIPT.

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- Refinement of enumerative results of [Bernardi, Bousquet-Mélou]

  \[ [t^{3n}] Z_\omega(\nu, t) \sim_n \begin{cases} \kappa \rho_{\nu_c}^{-n} n^{-7/3} & \text{if } \nu = \nu_c, \\ \kappa \rho_{\nu}^{-n} n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases} \]

  with $Z_\omega(\nu, t)$ generating series of Ising-weighted triangulations with boundary condition given by $\omega$.

- Proof of the tightness: combinatorial proof by a double counting argument.
• Blossoming bijections in higher genus? Other rationality schemes to investigate?

• Track distances in blossoming bijections to study more constrained models.
  e.g. scaling limit of planar graphs?

• Extend bootstrapping principle for the convergence of trees to more general models.
  e.g. $\alpha$-stable trees?

• Study of the clusters of the $\nu$-IIPT, following [Bernardi, Curien, Miermont, 15]

• Bijectons for the Ising model, blossoming bijection by [Bousquet-Mélou, Schaeffer 02].
  Can we find a “mating-of-tree” type bijection?

• Can we say anything about the growth volume of the $\nu$-IIPT?
Thank you for your attention!

Un énorme merci à tous
mes collaborateurs et collaboratrices :