Asymptotic behaviour of large random stack-triangulations

Marie Albenque et Jean-François Marckert

LIAFA – LABRI

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Outline

Stack-triangulations

Convergence of planar maps

Uniform law and normalized convergence

Other types of convergence

Perpectives
Definition of planar maps

- **Planar map** = planar connected graph embedded properly in the sphere up to a direct homomorphism of the sphere.
- **Rooted planar map** = an oriented edge \((e_0, e_1)\) is marked, \(e_0 = \text{root vertex}\).

Map = Metric space with graph distance.
Definition of planar maps

- Planar map = planar connected graph embedded properly in the sphere up to a direct homomorphism of the sphere
- Rooted planar map = an oriented edge \((e_0, e_1)\) is marked, \(e_0 = \) root vertex.
Maps and faces

Faces = connected components of the sphere without the edges or the map.
Triangulation = map whose faces are all of degree 3.
Quadrangulation = map whose faces are all of degree 4.

Figure: Two quadrangulations and two triangulations
Random Apollonian networks – Stack-triangulations

Stack-triangulations = triangulations obtained recursively:

$\triangle_{2k} = \text{(finite) set of stack-triangulations with } 2k \text{ faces.}$
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Stack-triangulations vs Triangulations

\[ \{\text{Stack-triangulations}\} \subsetneq \{\text{Triangulations}\} \]
Convergence of large random planar maps

- **Large**: Number of vertices grows to infinity.
- **Random**: Which law?
- **Convergence**: Which notion of convergence?

[Angel et Schramm, 03], [Chassaing et Schaeffer, 04], [Bouttier, Di Francesco, Guitter, 04], [Chassaing et Durhuss, 06], [Marckert et Mokkadem, 06], [Miermont, 06], [Marckert et Miermont, 07], [Le Gall, 07], [Le Gall et Paulin, 08], [Miermont et Weill, 08], [Chapuy, 08], [Bouttier et Guitter, 08], [Le Gall, 08]
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Two probability distributions

$\triangle_{2k} = \text{set of stack-triangulations with } 2k \text{ faces.}$

Two natural probability distributions on $\triangle_{2k}$:

- the uniform law, denoted $\U_{2k}$,

- the “historical” law, denoted $\Q_{2k}$: the probability of each map is proportional to its number of histories.
Two probability distributions

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Two probability distributions

$\triangle_{2k} =$ set of stack-triangulations with $2k$ faces.

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Results on random stack-triangulations

According to $Q_{2k}^\triangle$, 
- Degree of a vertex and expected value of the distance between two vertices  
  [Zhou et al., 05], [Zhang et al., 06], [Zhang et al., 08]

According to $U_{2k}^\triangle$,  
- Degree of a vertex [Darasse et Soria, 07]  
- Expected value of the distance between two vertices  
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According to $\mathcal{U}_{2k}$,

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Which definition of convergence?
Two notions of convergence: local convergence

\[ B_m(r) = \text{ball of radius } r \text{ centered at the root of } m. \]

**Definition**

Let \( m \) and \( m' \) be two planar maps, the local distance between them is:

\[ d_L(m, m') = \inf \left\{ \frac{1}{1 + r} \text{ where } B_m(r) \sim B_{m'}(r) \right\}, \]

Local convergence = Convergence of the balls centered at the root.
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Two notions of convergence: overall convergence

Number of vertices grows to infinity
⇒ distance between two vertices grows to infinity.

To study the overall behavior of the map,
we have to normalize it:
Length of an edge = dependent on the number of vertices.
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The Theorem

Theorem (A., Marckert ’08)

Under the uniform law on $\triangle_{2n}$,

$$\left( m_n, \frac{D_{m_n}}{(2/11)\sqrt{3n/2}} \right) \xrightarrow{(d)} (T_{2e}, d_{2e}),$$

for the Gromov-Hausdorff topology on the set of compact metric spaces.

- Gromov-Hausdorff ?
- $(T_{2e}, d_{2e}) =$ Aldous’ Continuum Random Tree (CRT)
- $2/11$ ?
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The Theorem

**Theorem (A., Marckert ’08)**

*Under the uniform law on $\triangle_{2n}$,*

$$
\left( m_n, \frac{D_{mn}}{(2/11)\sqrt{3n/2}} \right) \xrightarrow{(d)} (T_{2e}, d_{2e}),
$$

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Gromov-Hausdorff distance

Hausdorff distance between $X$ and $Y$ two compact sets of $(E, d)$:

$$d_H(X, Y) = \max\{\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)\}$$

**Gromov-Hausdorff** distance between two compact metric spaces $E$ and $F$:

$$d_{GH}(E, F) = \inf d_H(\phi(E), \psi(F))$$

Infimum taken on :
- all the metric spaces $M$
- all the isometric embeddings $\phi : E \to M$ et $\psi : F \to M$.

\{isometric classes of compact metric spaces\}

= complete and separable (= “polish”) space.
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Triangulations and ternary trees
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Harris walk of a tree
Continuum Tree

\[ f = \text{function from } [0, 1] \text{ onto } \mathbb{R}^+ \text{ such that } f(0) = f(1) = 0. \]

- \( s \sim s' \) if and only if \( f(s) = f(s') = m_f(s, s') \)
- continuum tree = \([0, 1]/\sim\)
- distance: \( d_f(s, t) = f(s) + f(t) - 2m_f(s, t) \)
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Continuum Random Tree – CRT

A normalized brownian excursion $e = (e_t)_{t \in [0,1]}$ is a brownian motion conditioned to satisfy $B_0 = 0$, $B_1 = 0$ and $B(t) > 0$ for every $t \in ]0,1[$.

CRT = Tree obtained from a normalized brownian excursion. It is denoted $(T_{2e}, d_{2e})$. 
Uniform law on stack-triangulations with $2n$ faces
⇒ uniform law $\mathbb{U}^{\text{ter}}_{3n-2}$ on the set of ternary trees with $3n - 2$ nodes.

**Proposition (Aldous)**

*Under $\mathbb{U}^{\text{ter}}_{3n+1}$, for the Gromov-Hausdorff topology:*

$$
\left( T, \frac{d_T}{\sqrt{3n/2}} \right) \xrightarrow{(d) / n} (T_{2e}, d_{2e}).
$$
Triangulations and ternary trees
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Bijection between trees and maps

Proposition

For any $K \geq 1$, there exists a bijection

$$\Psi^\Delta_K : \triangle_{2K} \longrightarrow T^\text{ter}_{3K-2}$$

$m \mapsto t := \Psi^\Delta_K(m)$

such that:

(i) (a) Every internal node $u$ of $m$ corresponds bijectively to an internal node $v$ of $t$. $u'$ denotes the image of $u$.

(b) Each leaf of $t$ corresponds bijectively to a finite face of $m$.

(ii) For any internal node $u$ of $m$, $|\Gamma(u') - d_m(\text{root}, u)| \leq 1$.

(ii') For any pair on internal nodes $u$ and $v$ of $m$

$$|d_m(u, v) - \Gamma(u', v')| \leq 3.$$
Proposition

For any $K \geq 1$, there exists a bijection

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Neveu formalism

- A ternary tree = set of words on the alphabet \{1, 2, 3\}.
- Vertex of the tree = a word
If type($u$) = ($i$, $j$, $k$),

\[
\begin{align*}
\text{type}(u_1) &= (1 + i \land j \land k, \quad j, \quad k), \\
\text{type}(u_2) &= (i, \quad 1 + i \land j \land k, \quad k), \\
\text{type}(u_3) &= (i, \quad j, \quad 1 + i \land j \land k)
\end{align*}
\]
If \( \text{type}(u) = (i, j, k) \),

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\end{align*}
$$
A language for distances

\[ \mathcal{L}_{1,2,3} = \{ \text{words of } \{1,2,3\}^* \text{ with at least one occurrence of } 1, 2 \text{ and } 3 \} \]

Let \( u \in \{1,2,3\}^* \),

\[ \Gamma(u) = \max \{ k \text{ such that } u = u_1 \ldots u_k, \ u_i \in \mathcal{L}_{1,2,3} \text{ for } i \in \{1,2,3\} \} \]

\[ u = 12213213212232 \Rightarrow \Gamma(u) = 3. \]

Let \( u = w \cdot u_1 \ldots u_k \) et \( v = w \cdot v_1 \ldots v_l \) with \( u_1 \neq v_1 \), we denote :

\[ \Gamma(u, v) = \Gamma(u_1 \ldots u_k) + \Gamma(v_1 \ldots v_l) \]
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Convergence of stack-triangulations

Lemma

Let \((X_i)_{i \geq 1}\) be a sequence of independant random variables uniformly distributed on \(\{1, 2, 3\}\). Let \(W_n\) be the word \(X_1 \ldots X_n\) then

\[
\frac{\Gamma(W_n)}{n} \xrightarrow{(a.s.)} \Gamma_\triangle, \text{ where } \Gamma_\triangle = \frac{2}{11}
\]

Distance in the map and in the tree:

\[
|d_{mn}(u, v) - \Gamma(u', v')| \leq 3
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We show:

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P(\sup |d_{mn}(u, v) - \frac{2}{11} d_{T_n}(u', v')| \geq n^{1/3}) \xrightarrow{n \to \infty} 0
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Convergence of scaled stack-triangulations

**Theorem**

*Under the uniform law on* $\triangle_{2n}$,

$$
\left( m_n, \frac{D_{m_n}}{\Gamma_\triangle \sqrt{3n/2}} \right) \xrightarrow{(d) n} (T_{2e}, d_{2e}),
$$

*for Gromov-Hausdorff topology on the set of compact metric spaces.*
<table>
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<th>Stack-triangulations</th>
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Convergence of stack-triangulations according to $Q^\triangle$

**Theorem (A., Marckert ’08)**

Let $M_n$ a stack-triangulation according to $Q^\triangle_{2n}$. Let $k \in \mathbb{N}$ et $v_1, \ldots, v_k$, $k$ nodes $M_n$ chosen independently and uniformly amongst the internal nodes of $M_n$, then:

$$
\left(\frac{D_{M_n}(v_i, v_j)}{3\Gamma \triangle \log n}\right)_{(i,j) \in \{1, \ldots, k\}^2} \xrightarrow[n \rightarrow \infty]{\text{proba.}} (1_{i \neq j})(i,j) \in \{1, \ldots, k\}^2.
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Study of the trees under the historical law = study of increasing trees

... [Broutin, Devroye, McLeish, de la Salle 08]
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Local convergence of stack-triangulations: Uniform law

Under $\mathbb{U}_n^\Delta$:

**Theorem (A., Marckert ’08)**

The sequence $(\mathbb{U}_n^\Delta)$ weakly converges towards $P_\infty^\Delta$, for the topology of local convergence, where the support of $P_\infty^\Delta$ is a set of infinite stack-triangulations.

Ingredients:

- Local convergence of Galton-Watson trees towards a tree with a unique infinite spine.
- Definition of an infinite planar map similar to the UIPT of Angel and Schramm.
Local convergence of stack-triangulations: Historical law

Degree of the root = number of white balls in an urn

- Initially: 2 white balls and 1 black ball
- Matrix replacement: \[
\begin{pmatrix}
2 & 1 \\
0 & 3
\end{pmatrix}
\]

[Flajolet, Dumas, Puyhaubert, 06]

⇒ The degree of the root grows to infinity.
⇒ No local convergence.
Local convergence of stack-triangulations: Historical law

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Stack-quadrangulations

We managed to deal with a special case of stack-quadrangulations

but more general models resist...
Brownian Map

Convergence of scaled quadrangulations under the uniform law?

[Chassaing et Schaeffer, 04], [Marckert et Mokkadem, 06], [Marckert et Miermont, 07], [Le Gall, 07], [Le Gall et Paulin, 08]

- Universality principle? Convergence of all the “reasonable” models to the same limit?
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• Universality principle? Convergence of all the “reasonable” models to the same limit?
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Thank you!