

# Growth function for a class of monoids

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Formal Power Series and Algebraic Combinatorics

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## First motivation = counting braids

braid diagram = a sequence of strand crossings.

$\sigma_{t,s} = \sigma_{s,t}$  ( $s < t$ ) = crossing of strands  $s$  and  $t$ , where strand  $s$  is above strand  $t$

braid diagram = word on the alphabet  $\{\sigma_{s,t}\}$

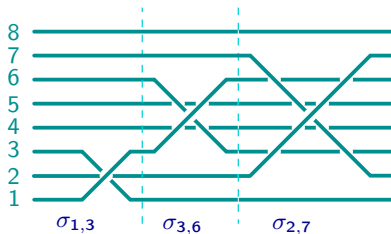
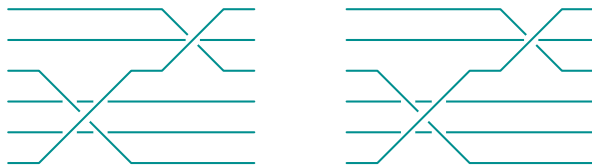


Figure: A braid diagram and the corresponding word

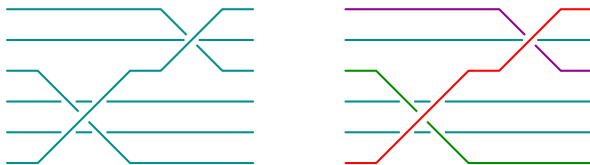
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$$\sigma_{1,4} \sigma_{4,6} \equiv \sigma_{4,6} \sigma_{1,6}$$

Braid = equivalence class of diagrams.

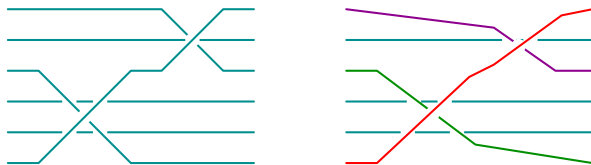
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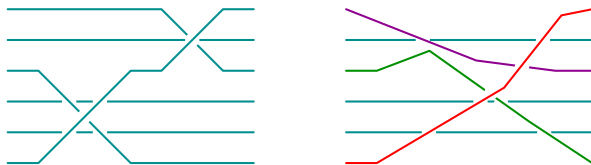
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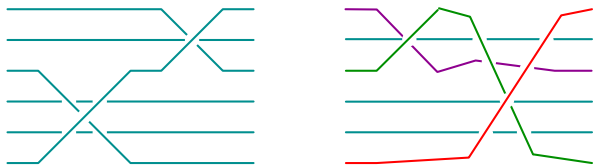
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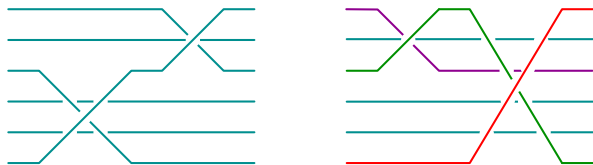
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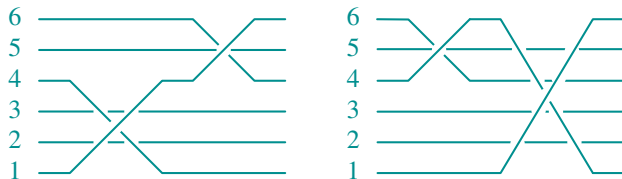


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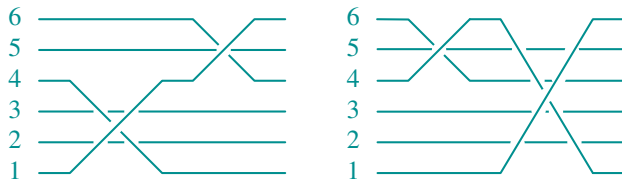
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## Presentation of the dual braid monoid.

The set of generators of  $M$  is :

$$\mathcal{S} = \{ \sigma_{s,t} = \sigma_{t,s} \text{ pour } 1 \leq s < t \leq n, \}$$

with the following equivalence relations :

$$\sigma_{s,t} \sigma_{u,v} = \sigma_{u,v} \sigma_{s,t} \text{ si } s <_s t <_s u <_s v,$$

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where  $<_s =$  cyclic order  $\mathbb{Z}/n\mathbb{Z}$  defined by :

$$s <_s s+1 <_s s+2 <_s \dots <_s s-1.$$

Length of a braid =  $|m|_{\mathcal{S}}$

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## How many braids ?

$a_k$  = number of braids of length  $k$

$$F_n(t) = \sum_{k \geq 0} a_k t^k = a_0 + a_1 t + a_2 t^2 \dots$$

Theorem (A., Nadeau '08)

*The growth function of the dual braid monoid on  $n$  strands is :*

$$F_n(t) = \left[ \sum_{k=0}^{n-1} \frac{(n-1+k)!(-t)^k}{(n-1-k)!k!(k+1)!} \right]^{-1} .$$

## Steps of the proofs

Involution  
 $\Psi$



Computation of the growth function of the monoid

Alternating generating series of lcm

## A few definition about lcm

$\sigma \prec m$  = there exists a diagram of  $m$  whose first letter is  $\sigma$

### Definition

$J \subset \mathcal{S}$  is a **clique** if it admits a common multiple.  
The set of cliques is denoted  $\mathcal{J}$

If  $J \in \mathcal{J}$ , then a **least common multiple (lcm)** exists, is unique and is denoted  $M_J$ .

We fix arbitrarily a linear ordering on  $\mathcal{S}$ , and denote a clique as

$$J = \sigma_1 \dots \sigma_n, \text{ with } \sigma_i < \sigma_{i+1}$$



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## Theorem

$$\left(\sum_{J \in \mathcal{J}} (-1)^{|J|} M_J\right) \cdot \left(\sum_{m \in M} m\right) = 1$$

## Corollary (Bronfman '05, Kraamer '05)

*The growth series of the monoid verifies then:*

$$\left[\sum_{J \in \mathcal{J}} (-1)^{|J|} t^{|M_J|}\right] F(t) = 1$$

## A large class of monoids

Our approach works for every monoid  $M$  which admits a presentation with generators and relations and which is:

- atomic,
- left-cancellable :  $a, u, v \in M, au = av \Rightarrow u = v,$
- if a subset of generators has a right common multiple then it has a least common multiple.

[Bronfman, 00], [Krammer, 04]

Trace monoids, Garside monoids, Artin-Tits monoids, ...

To get the growth series from the involution, the relations must besides preserve the length.

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## Proof of the inversion formula

$$\left(\sum_{J \in \mathcal{J}} (-1)^{|J|} M_J\right) \left(\sum_{m \in M} m\right) = \sum_{(J, m)} (-1)^{|J|} M_J m = 1$$

$\Psi$  is an **involution** with only  $(1, 1)$  as fixed point :

$$\Psi : \mathcal{J} \times M \rightarrow \mathcal{J} \times M$$

$$(J, m) \mapsto (J', m') \text{ with } M_J m = M_{J'} m' \text{ and } |J \Delta J'| = 1$$

$$\sigma_m = \max\{\sigma \text{ such that } \sigma \prec M_J m\}$$

$$\Psi(J, m) = \begin{cases} (J \cup \{\sigma_m\}, (M_{J \cup \{\sigma_m\}})^{-1} \cdot m) & \text{if } \sigma_m \notin J \\ (J \setminus \{\sigma_m\}, (M_{J \setminus \{\sigma_m\}})^{-1} M_J \cdot m) & \text{otherwise} \end{cases}$$



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# Computation of the alternating generating series of $\text{lcm}$

$(M, \prec)$  = locally finite Poset

Möbius inversion formula :  $(\sum \mu(m)m)(\sum m) = 1$

Our inversion formula is a generalization of Rota's cross-cut theorem.

Computation of the Möbius function :

- Use of NBB base with an appropriate order on  $\mathcal{S}$   
[Blass and Sagan, '96]
- Combinatorial proof

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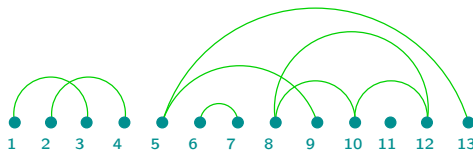
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## Common multiple of braids

Lcm of  $\{\sigma_{1,3}, \sigma_{2,4}, \sigma_{5,13}, \sigma_{5,9}, \sigma_{6,7}, \sigma_{8,12}, \sigma_{8,10}, \sigma_{10,12}\}$  ?

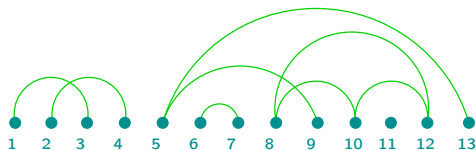


$$M_J = \sigma_{1,4} \sigma_{4,3} \sigma_{2,3} \cdot \sigma_{5,13} \sigma_{13,12} \sigma_{12,10} \sigma_{10,9} \sigma_{9,8} \cdot \sigma_{7,6}$$

$$|M_J| = \text{number of vertices} - \text{number of parts} = 13 - 4 = 9.$$

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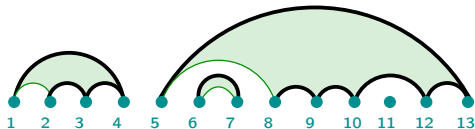
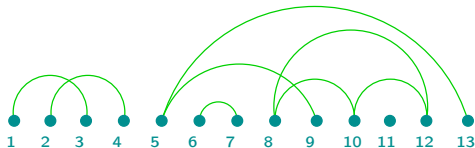


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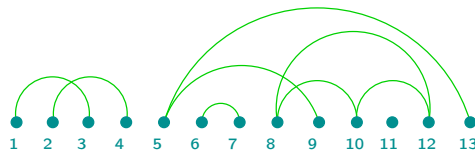


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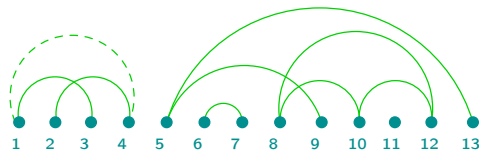
# Involution on the edge configurations



⇒ Counting non-crossing alternating forests

Length of the lcm = number of edges of the forest

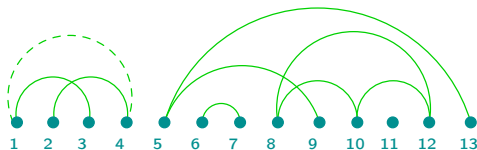
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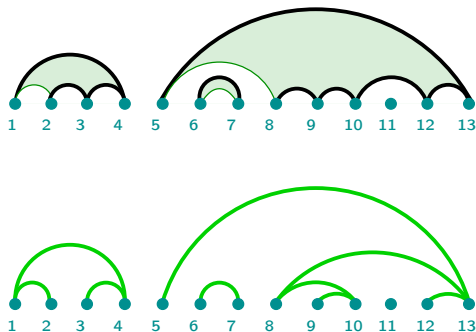


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## Order compatible cliques

### Definition

An **order compatible (OC)** clique is  $\sigma_1 \dots \sigma_n$  such that :

$$\sigma_i = \max\{\sigma \prec M_{\sigma_1 \dots \sigma_i}\}$$

### Theorem (Blass-Sagan, '96)

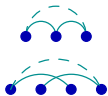
$$\mu(m) = \sum (-1)^{|J|}, \text{ where } J \text{ is an OC clique s.t. } M_J = m$$

$$\sigma_{i,j} < \sigma_{k,l} \iff [i,j] \not\subseteq [k,l],$$

the OC-cliques are exactly the noncrossing alternating forests.

# Steps of the proof

Involution  
 $\Psi$



Computation of the growth function of the monoid

Alternating generating series of lcm

Noncrossing alternating forests

?

# Noncrossing alternating forests and unary-binary trees



[Gelfand *et al.*, 97]

Bijection between the noncrossing alternating forests with  $n$  vertices and  $k$  edges and the unary binary trees with  $n + k$  nodes,  $k$  of which being binary.

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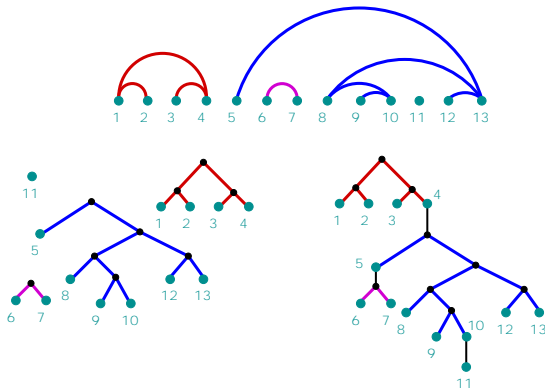


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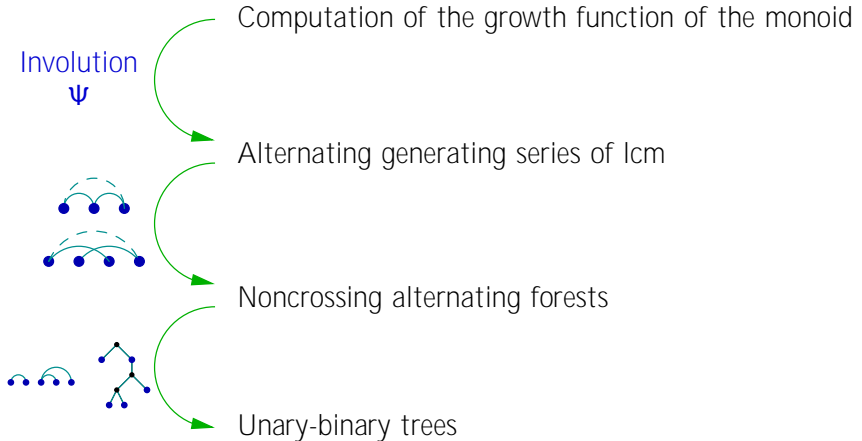
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# Steps of the proof



# Growth function of the dual braid monoid

Theorem (A., Nadeau '08)

*The growth function of the dual braid monoid on  $n$  strands is :*

$$F_n(B)$$

$$F_n(B) = \sum_{k=0}^{\infty} \sum_{\substack{J \subseteq [n] \\ |J|=k}} \binom{n}{k} F_n(B) \cdot F_n(B)$$

Resolution of  $\mathbb{Z}$ 

- $A := \mathbb{Z}M$  : monoid algebra of  $M$
- $B := \mathbb{Z}\mathcal{J}$  : free module with basis  $\mathcal{J}$   
 $B_n := \mathbb{Z}\mathcal{J}_n$  : submodule with basis  $\mathcal{J}_n$  (cliques of size  $n$ )
- $C_n := B_n \otimes_{\mathbb{Z}} A$

## Definition

$d_n : C_n \rightarrow C_{n-1}$  is a  $A$ -module homomorphism defined by:

$$d_n(\sigma_1 \dots \sigma_n \otimes 1) = \sum_{i=1}^n (-1)^{n-i} \sigma_1 \dots \hat{\sigma}_i \dots \sigma_n \otimes \delta_{\sigma_1 \dots \hat{\sigma}_i \dots \sigma_n}^{\sigma_i}$$

where  $M_{J_i} \delta_{J_i}^{\sigma_i} = M_{J_i \cup \{\sigma_i\}}$ .

## Theorem

$0 \longrightarrow C_{|S|} \xrightarrow{d_{|S|}} C_{|S|-1} \xrightarrow{d_{|S|-1}} \dots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 = A \xrightarrow{\epsilon} \mathbb{Z}$   
 is a resolution of  $\mathbb{Z}$  as an  $A$ -module (i.e.  $\text{Im}(d_n) = \text{Ker}(d_{n-1})$ ).

## Koszul Algebras

- $\tilde{\mathcal{C}}_n :=$  submodule of  $\mathcal{C}_n$  with bases OC cliques of size  $n$

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The coefficients of the matrices of the resolution for the OC cliques are  $\delta_{j_i}^{\sigma_i} = \sigma_i$  of length 1.

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## Artin-Tits monoids

$\mathcal{S}$  is a finite set,  $\mathbb{M}$  a symmetric matrix, with  $m_{s,t} \in \mathbb{N} \cup \{\infty\}$  and  $m_{s,s} = 1$ .

The **Artin-Tits monoid** associated to  $\mathcal{S}$  and  $\mathbb{M}$  is:

$$M = \langle s \in \mathcal{S} \mid \underbrace{sts\dots}_{m_{s,t} \text{ terms}} = \underbrace{tst\dots}_{m_{s,t} \text{ terms}} \text{ if } m_{s,t} \neq \infty \rangle$$

Coxeter groups associated to  $M$ :  $W = M/\{s^2 = 1\}$

An Artin-Tits monoid is **spherical** if its Coxeter group is finite.

## Artin-Tits monoids

$\mathcal{S}$  is a finite set,  $\mathbb{M}$  a symmetric matrix, with  $m_{s,t} \in \mathbb{N} \cup \{\infty\}$  and  $m_{s,s} = 1$ .

The **Artin-Tits monoid** associated to  $\mathcal{S}$  and  $\mathbb{M}$  is:

$$M = \langle s \in \mathcal{S} \mid \underbrace{sts\dots}_{m_{s,t} \text{ terms}} = \underbrace{tst\dots}_{m_{s,t} \text{ terms}} \text{ if } m_{s,t} \neq \infty \rangle$$

Coxeter groups associated to  $M$ :  $W = M/\{s^2 = 1\}$

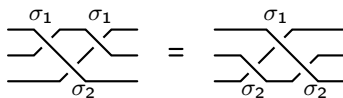
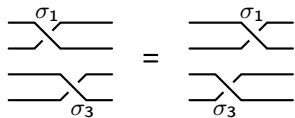
An Artin-Tits monoid is **spherical** if its Coxeter group is finite.



## Braid monoids

From the classification of finite Coxeter groups, the classical braid monoids of type A, B and D are defined.

$$\mathcal{A}(A_N) = \left\langle \sigma_1, \dots, \sigma_n \left| \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2 \end{array} \right. \right\rangle.$$



$$F_n(t) := [H_n(t)]^{-1}, \quad H_n(t) = \sum_{k=1}^n (-1)^{k+1} t^{k(k-1)/2} H_{n-k}$$

## Dual braid monoids:

$W$  a Coxeter group :

$\mathcal{T}$  = New set of generators = { reflexions } =  $\{ wsw^{-1}, s \in \mathcal{S}, w \in W \}$

Definition of a dual structure [Birman,Ko,Lee, '98],[Bessis, '03],  
where the set of lcms is a lattice.

Lattice isomorphic to some lattice of non-crossing partitions.

- Type A
- Type B [Reiner, '97]
- Type D [Athanasiadis & Reiner,'04]

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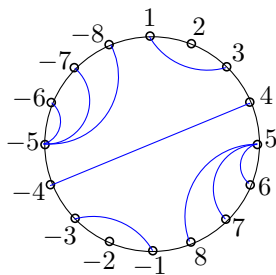
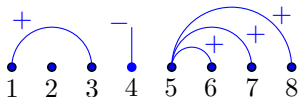
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## Dual braids of type B

Noncrossing partition of type  $B$  :

- Partition of the set  $\{1, \dots, n, -1, \dots, -n\}$
- $i, j$  in the same block  $\Rightarrow -i, -j$  also.



### Theorem

*The monoid algebra of the dual braid monoid of type B is a Koszul algebra.*

$$F_n^B(t) = \left( \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k-1}{k} t^k \right)^{-1}$$

Thank you !